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ON RELATIVE PROPERTY (T) AND HAAGERUP'S PROPERTY

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ABSTRACT. We consider the following three properties for countable discrete groups Γ: (1) Γ has an infinite subgroup with relative property (T), (2) the group von Neumann algebra $L\Gamma$ has a diffuse von Neumann subalgebra with relative property (T) and (3) Γ does not have Haagerup's property. It is clear that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). We prove that both of the converses are false.

0. INTRODUCTION

In this paper, we investigate the relationship between *relative property (T)* and *Haagerup's property* in the context of countable groups and finite von Neumann algebras. An inclusion $(\Gamma_0 \subset \Gamma)$ of countable discrete groups has *relative property (T)* of Kazhdan-Margulis if any unitary representation of Γ which has almost invariant vectors necessarily has a non-zero Γ_0-invariant vector. The classical examples here are $(\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$ and $(\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{Z}))$, for $n \geq 3$ ([Ka67], [Ma82]).

The presence of relative property (T) subgroups is an obstruction to Haagerup's property. A countable group Γ is *Haagerup* if it admits a c_0 (or *mixing*) unitary representation which has almost invariant vectors. This class, which includes amenable groups and free groups, is closed under free products and, more surprisingly, wreath products ([CSV09]).

It is clear from the definitions that a group with Haagerup's property cannot have relative property (T) with respect to any infinite subgroup. In [CCJJV01, Section 7.1] the authors asked whether the converse holds true, i.e. if having an infinite subgroup with relative property (T) is the *only* obstruction to Haagerup's property. This question was answered negatively by Y. de Cornulier ([Co06a], [Co06b]). He showed that there are certain groups, e.g. $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$, for α not a cube, which do not have relative property (T) with respect to any infinite subgroup but yet have relative property (T) with respect to some infinite *subset* (see Definition 1.1). The latter property still guarantees the failure of Haagerup's property.

The notion of *relative property (T)* (or *rigidity*) for inclusions of finite von Neumann algebras was introduced by S. Popa in [Po06a] (see Definition 1.4). Since then it has found many striking applications to von Neumann algebras theory and orbit equivalence ergodic theory (see the surveys [Po07] and [Fu09]). Thus, it was used as a key ingredient in Popa's solution to the long-standing problem of finding II_1 factors with trivial fundamental group ([Po06a]). Examples of rigid inclusions

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of von Neumann algebras are provided by inclusions of groups. Precisely, the inclusion ($\Gamma_0 \subset \Gamma$) of two countable groups has relative property (T) if and only if the inclusion of group von Neumann algebras ($L\Gamma_0 \subset L\Gamma$) has it ([Po06a]). Similarly, there is a notion of Haagerup's property for finite von Neumann algebras which generalizes the corresponding notion for groups ([CJ85], [Ch83]).

Relative property (T) and Haagerup's property are also incompatible in the framework of von Neumann algebras: if a finite von Neumann algebra N has Haagerup's property, then it does not have any diffuse (i.e. non-atomic) relatively rigid subalgebra. The main goal of this paper is to show that the converse of this statement is false even in the case of von Neumann algebras N arising from countable groups. Thus, we are interested in finding a countable group Γ such that its group von Neumann algebra $L\Gamma$ neither has Haagerup's property nor any diffuse rigid subalgebra. The natural candidates are Cornulier's examples, Γ_α, since one might hope that the absence of relative property (T) subgroups is inherited by the group von Neumann algebra. However, we prove that this is not the case:

0.1 Theorem (Corollary 2.2)**.** *Let $\alpha \in \mathbb{N} \setminus \{\beta^3 | \beta \in \mathbb{N}\}$ and let $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$. Then there exists a diffuse von Neumann subalgebra B of $L(\mathbb{Z}[\sqrt[3]{\alpha}]^3)$ such that the inclusion $(B \subset L(\Gamma_\alpha))$ is rigid and $B' \cap L(\Gamma_\alpha) = L(\mathbb{Z}[\sqrt[3]{\alpha}]^3)$.*

In [Co06b], a concept of *resolutions* was introduced in order to quantify the transfer of property (T) from a locally compact group to its lattices. In particular, they can be used to locate the relative property (T) subsets of lattices Γ in Lie groups, e.g. $\Gamma = \Gamma_\alpha$. The idea behind the proof of Theorem 0.1 is that by combining resolutions with results from [Io09] we can also detect certain rigid subalgebras of $L\Gamma$.

Next, we consider another class of groups. If A and Γ are two countable groups and X is a countable Γ-set, then the *generalized wreath product* group $A \wr_X \Gamma$ is defined as $A^X \rtimes \Gamma$. If $X = \Gamma$, together with the left multiplication action, then we recover the standard wreath product $A \wr \Gamma$. Following results from [Po06a], [Po06b] and [Io07] we know that if A and Γ have Haagerup's property, then the von Neumann algebra $L(A \wr_X \Gamma)$ does not have a diffuse rigid von Neumann subalgebra, regardless of the set X. Thus, in order to get examples of groups with the desired properties, it suffices to find a suitable set X for which $A \wr_X \Gamma$ is not Haagerup.

0.2 Theorem (Corollary 3.4)**.** *Let A be a non-trivial countable Haagerup group. Let Γ be a countable Haagerup group together with a quotient group Γ_0. Assume that Γ_0 is not Haagerup and endow it with the left multiplication action of Γ. Then $A \wr_{\Gamma_0} \Gamma$ is not Haagerup. Thus, the group von Neumann algebra $N = L(A \wr_{\Gamma_0} \Gamma)$ does not have Haagerup property and does not admit any diffuse von Neumann subalgebra B such that the inclusion $(B \subset N)$ is rigid.*

The proof of Theorem 0.2 is based on a general result: a semidirect product $A \rtimes \Gamma$ is not Haagerup whenever A is abelian, and Γ acts on A through a non-Haagerup quotient group, Γ_0.

Recently, Cornulier, Stalder and Valette proved that the class of Haagerup groups is closed under standard wreath products ([CSV09]). Moreover, they showed that if A and Γ are Haagerup groups, then the generalized wreath product $A \wr_X \Gamma$ is Haagerup for certain sets X, and they conjectured that this is the case for any

X. Theorem 0.2 provides in particular a counterexample to their conjecture. For example, if Γ is a free group and Γ_0 is a property (T) quotient of Γ, then $\mathbb{Z}\wr_{\Gamma_0}\Gamma$ is not Haagerup.

1. Preliminaries

We start this section by reviewing the notion of relative property (T) for groups. Then we explain Cornulier's examples of groups which are not Haagerup but do not have any infinite subgroup with relative property (T). Finally, we recall Popa's notion of rigidity for inclusions of von Neumann algebras.

A continuous unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ has *almost invariant vectors* if for all $\varepsilon > 0$ and any compact set $F \subset G$ we can find a unit vector $\xi \in \mathcal{H}$ such that $||\pi(g)(\xi) - \xi|| \leq \varepsilon$, for all $g \in F$. If H is a closed subgroup of G, then the inclusion $(H \subset G)$ has *relative property* (T) of Kazhdan-Margulis if any unitary representation of G which has almost invariant vectors must have a non-zero H-invariant vector ([Ka67], [Ma82]). Also recall that G has *Haagerup's property* (in short, *is Haagerup*) if it admits a c_0 unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ which has almost invariant vectors. Being c_0 means that for every $\xi, \eta \in \mathcal{H}$ we have that $\lim_{g\to\infty}\langle\pi(g)(\xi), \eta\rangle = 0$.

If a countable, discrete group Γ is Haagerup, then it does not have relative property (T) with respect to any infinite subgroup. Cornulier proved that the converse is false ([Co06a], [Co06b]). For example, he showed that if $\alpha \in \mathbb{N} \setminus \{\beta^3|\beta \in \mathbb{N}\}$, then $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$ neither has Haagerup property nor admits an infinite subgroup with relative property (T). To quickly see that Γ_α is not Haagerup just notice that it is measure equivalent (see [Fu09] for the definition) to the group $\Lambda = \mathbb{Z}^3 \times (\mathbb{Z}[i]^3 \rtimes SO_3(\mathbb{Z}[i]))$ which has an infinite subgroup with relative property (T) (i.e. $\mathbb{Z}[i]^3$). Indeed, both Γ_α and Λ are lattices in $G = (\mathbb{R}^3 \rtimes SO_3(\mathbb{R})) \times (\mathbb{C}^3 \rtimes SO_3(\mathbb{C}))$ ([Ma91], [Wi08]). This example shows that having an infinite subgroup with relative property (T) is not a measure equivalence invariant. To better explain the failure of Haagerup's property for Γ_α, the following two notions were introduced in [Co06b]:

1.1 Definitions ([Co06b]). (a) Let $p : \Gamma \to G$ be a morphism between two locally compact groups with dense image. We say that p is a *resolution* if for any unitary representation π of Γ which has almost invariant vectors, there exists a subrepresentation σ of π of the form $\sigma = \tilde{\sigma} \circ p$, where $\tilde{\sigma}$ is a unitary representation of G which has almost invariant vectors.

(b) Given a subset X of a locally compact group G, we say that the inclusion $(X \subset G)$ has *relative property (T)* if for any unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ which has almost invariant vectors and any $\varepsilon > 0$, we can find a unit vector $\xi \in \mathcal{H}$ such that $||\pi(g)(\xi) - \xi|| \leq \varepsilon$, for all $g \in X$.

Resolutions are useful to encode the transfer of relative property (T) from a group to its lattices. To see this, assume that Γ is a lattice in a locally compact group G, let $H \subset G$ be a normal, closed subgroup such that the inclusion $(H \subset G)$ has relative property (T) and let $p : G \to G/H$ be the projection. Under these assumptions, [Co06b, Theorem 4.3.1] asserts that the morphism $p_{|\Gamma} : \Gamma \to \overline{p(\Gamma)}$ is a resolution. In the case when G has property (T) and $H = G$, this is just saying that any lattice Γ of G has property (T), thus recovering Kazhdan's classical result ([Ka67]).

Now, if $G = (\mathbb{R}^3 \rtimes SO_3(\mathbb{R})) \times (\mathbb{C}^3 \rtimes SO_3(\mathbb{C}))$ and $H = \mathbb{C}^3$, then the inclusion $(H \subset G)$ has relative property (T) (see [Co06b, 3.3.1]). By applying the above theorem to this situation the following was deduced in [Co06b, the proof of 4.6.3]:

1.2 Corollary ([Co06b]). *The inclusion $\Gamma_\alpha \hookrightarrow \mathbb{R}^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$ is a resolution. Thus, if $\mathcal{B}$ is the unit ball of $\mathbb{R}^3$ and $X = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \cap \mathcal{B}$, then $(X \subset \Gamma_\alpha)$ has relative property (T), for every α. In particular, Γ_α is not Haagerup.*

Notice moreover that X is a normal subset of Γ_α. In relation to this, let us note that results from [Co06b] imply that *any* lattice in a connected Lie group either has Haagerup's property or admits an infinite, "weakly normal" subset with relative property (T):

1.3 Corollary. *Let G be a connected Lie group which does not have Haagerup's property. Let Γ be a lattice in G. Then there exists an infinite set $X \subset \Gamma$ such that the inclusion $(X \subset \Gamma)$ has relative property (T) and $\gamma X \gamma^{-1} \cap X$ is infinite, for all $\gamma \in \Gamma$.*

Proof. Since G is not Haagerup, by [Co06b, 3.3.1] and [CCJJV01, Chapter 4] we get that it has a non-trivial, normal, closed subgroup H such that the inclusion $(H \subset G)$ has relative property (T) and G/H has Haagerup's property. Let $p : G \to G/H$ denote the projection and set $Q = \overline{p(\Gamma)}$. By [Co06b, 4.3.1] the morphism $p_{|\Gamma} : \Gamma \to Q$ is a resolution. Thus, since the inclusion $(\{1\} \subset Q)$ has relative property (T) by [Co06b, 4.2.6], we deduce that the inclusion $((\Gamma \cap H) \subset \Gamma)$ does as well.

Since $\Gamma \cap H$ is a normal subgroup of Γ we can hereafter assume that it is finite (otherwise, we can take $X = \Gamma \cap H$). Under this assumption, we claim that $p(\Gamma)$ is a non-discrete subgroup of G/H. Indeed, if $p(\Gamma)$ is discrete, then it must have Haagerup's property, as G/H has. However, $p(\Gamma)$ is isomorphic to $\Gamma/(\Gamma \cap H)$ and, since $\Gamma \cap H$ is finite, we would get that Γ is Haagerup, a contradiction.

Next, let V be a neighborhood of $1 \in Q$ with compact closure and define $X = p^{-1}(V) \cap \Gamma$. Since the inclusion $(V \subset Q)$ has relative property (T) by [Co06b, 4.2.6], we deduce that the inclusion $(X \subset \Gamma)$ has relative property (T). To check the normality assertion, fix $\gamma \in \Gamma$ and denote $W = p(\gamma) V p(\gamma)^{-1} \cap V$. Then $\gamma X \gamma^{-1} \cap X = \{x \in \Gamma | p(x) \in W\}$. Since $p(\Gamma) \subset G/H$ is non-discrete and W is a neighborhood of $1 \in Q$, the latter set is infinite. $\square$

To recall Popa's notion of rigidity for von Neumann algebras, let M be a separable finite von Neumann algebra with a faithful, normal trace $\tau : M \to \mathbb{C}$ and let $B \subset M$ be a von Neumann subalgebra. A Hilbert space $\mathcal{H}$ is called a *Hilbert M-bimodule* if it admits commuting left and right Hilbert M-module structures. A vector $\xi \in \mathcal{H}$ is called *tracial* if $\langle x\xi, \xi \rangle = \langle \xi x, \xi \rangle = \tau(x)$, for all $x \in M$, and *B-central* if $b\xi = \xi b$, for all $b \in B$. A Hilbert M-bimodule $\mathcal{H}$ together with a unit vector $\xi \in \mathcal{H}$ is called a *pointed Hilbert M-bimodule* and is denoted $(\mathcal{H}, \xi)$.

1.4 Definition ([Po06a]). The inclusion $(B \subset M)$ is *rigid* (or has *relative property (T)*) if for every $\varepsilon > 0$ there exists $F \subset M$ finite and $\delta > 0$ such that whenever $(\mathcal{H}, \xi)$ is a pointed Hilbert M-bimodule with ξ a tracial vector verifying $||x\xi - \xi x|| \leq \delta$, for all $x \in F$, there exists a B-central vector $\eta \in \mathcal{H}$ with $||\eta - \xi|| \leq \varepsilon$.

The notion of rigidity for inclusions of von Neumann algebras is analogous to and generalizes the notion of relative property (T) for groups. More precisely, given two countable groups $\Gamma_0 \subset \Gamma$, the inclusion $(L(\Gamma_0) \subset L(\Gamma))$ of their group von

Neumann algebras is rigid if and only if the inclusion $(\Gamma_0 \subset \Gamma)$ has relative property (T) ([Po06a, Proposition 5.1]). Now, if Γ is a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{Z}^2$ by matrix multiplication, then the inclusion $(\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma)$ has relative property (T) ([Bu91, section 5]), and therefore the inclusion $(L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes \Gamma))$ is rigid.

The first examples of rigid inclusions of von Neumann algebras which do not rely on relative property (T) for some inclusion of groups have recently been exhibited in [Io09]. Thus, it is shown that for *any* non-amenable subfactor N of $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$ which contains $L(\mathbb{Z}^2)$, the inclusion $(L(\mathbb{Z}^2) \subset N)$ is rigid ([Io09, Theorem 3.1]).

2. Rigid subalgebras from resolutions

The main goal of this section is to show that, in certain situations, resolutions can be used to construct rigid subalgebras of von Neumann algebras (Theorem 2.1). Thus, we employ the resolution provided by Corollary 1.2 to deduce that the group von Neumann algebra $L(\Gamma_\alpha)$ has a diffuse rigid subalgebra (Corollary 2.2). This result should be contrasted with the fact that Γ_α has no infinite subgroup with relative property (T).

2.1 Theorem. *Let Γ be a countable subgroup of $SO_n(\mathbb{R})$, for some $n \geq 3$, and consider the natural action of Γ on $H = \mathbb{R}^n$.*

- *Assume that $A \simeq \mathbb{Z}^m$ ($m \geq n+1$) is a Γ-invariant, dense subgroup of $\mathbb{R}^n$. Let $v_1, ..., v_m \in A$ such that $\theta : \mathbb{Z}^m \to A$ given by $\theta((x_i)) = \sum_{i=1}^m x_i v_i$ is an isomorphism. Identify the dual $\hat{A}$ of A with $\hat{\mathbb{Z}^m} = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ via the map $\hat{\theta}(\eta) = \eta \circ \theta$.*
- *Let $p : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $p(a) = (\langle a, v_1\rangle, ..., \langle a, v_m\rangle)$, where $\langle .,. \rangle$ is the usual scalar product on $\mathbb{R}^n$, and denote by $\pi : \mathbb{R}^m \to \mathbb{R}^m/p(\mathbb{R}^n)$ the projection.*
- *Let $i : \mathbb{T}^m \to [-\frac{1}{2}, \frac{1}{2})^m \subset \mathbb{R}^m$ be defined by $i(x + \mathbb{Z}^m) = (x + \mathbb{Z}^m) \cap [-\frac{1}{2}, \frac{1}{2})^m$, for all $x \in \mathbb{R}^m$, and set $q = \pi \circ i : \mathbb{T}^m \to \mathbb{R}^m/p(\mathbb{R}^n)$.*

Define $(Y, \nu) = (q(\mathbb{T}^m), q_*\lambda^m)$, where λ^m is the Haar measure on $\mathbb{T}^m$ and $q_*\lambda^m$ is the push-forward of λ^m through q. Then we have the following:

(1) $L^\infty(Y, \nu)$ is a diffuse von Neumann subalgebra of $L^\infty(\mathbb{T}^m, \lambda^m)$ (here, we consider the embedding $L^\infty(Y, \nu) \ni f \to f \circ q \in L^\infty(\mathbb{T}^m, \lambda^m)$).

(2) If the inclusion $A \rtimes \Gamma \to H \rtimes \Gamma$ is a resolution, then the inclusion of von Neumann algebras $L^\infty(Y, \nu) \subset M := L^\infty(\mathbb{T}^m, \lambda^m) \rtimes \Gamma$ is rigid.

(3) If $p(\mathbb{R}^n) \cap \mathbb{Z}^m = \{0\}$, then $L^\infty(Y, \nu)' \cap M = L^\infty(\mathbb{T}^m, \lambda^m)$.

In the statement of this theorem we have used the fact that if Γ is a countable group which acts by automorphisms on a countable abelian group A, then the action of Γ on $\hat{A}$ preserves the Haar measure h. Also, we note that the associated crossed product von Neumann algebra $L^\infty(\hat{A}, h) \rtimes \Gamma$ is naturally isomorphic to the group von Neumann algebra $L(A \rtimes \Gamma)$ and that this isomorphism identifies $L^\infty(\hat{A}, h)$ with $L(A)$.

2.2 Corollary. *Let $n \geq 3$, $\alpha \in \mathbb{N} \setminus \{\beta^3 | \beta \in \mathbb{N}\}$ and denote $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^n \rtimes SO_n(\mathbb{Z}[\sqrt[3]{\alpha}])$. Then there exists a diffuse von Neumann subalgebra B of $L(\mathbb{Z}[\sqrt[3]{\alpha}]^n)$ such that the inclusion $(B \subset L(\Gamma_\alpha))$ is rigid and $B' \cap L(\Gamma_\alpha) = L(\mathbb{Z}[\sqrt[3]{\alpha}]^n)$. Moreover, if $n = 3$, then $L(\Gamma_\alpha)$ is an HT factor in the sense of* [Po06a, Definition 6.1].

Proof. For every $(j,k) \in S = \{1,...,n\} \times \{0,1,2\}$, let $v_{j,k} \in \mathbb{R}^n$ be given by $v_{j,k} = (\alpha^{\frac{k}{3}}\delta_{i,j})_{1\le i\le n}$. Then $\mathbb{Z}[\sqrt[3]{\alpha}]^n = \bigoplus_{(j,k)\in S}\mathbb{Z}v_{j,k}$. Denote by $\langle .,. \rangle$ the natural scalar product on $\mathbb{R}^n$ and let $p : \mathbb{R}^n \to \mathbb{R}^{3n} = \bigoplus_{(j,k)\in S}\mathbb{R}$ be the homomorphism defined by $p(a) = (\langle a, v_{j,k}\rangle)_{j,k}$, for all $a \in \mathbb{R}^n$. Explicitly, if $a = (a_i)_{1\le i\le n}$, then $p(a) = (a_j\alpha^{\frac{k}{3}})_{j,k}$. Since $\alpha^{\frac{1}{3}}$ is irrational, it follows that $p(\mathbb{R}^n) \cap \mathbb{Z}^{3n} = \{0\}$. Also, by Corollary 1.2 (which holds for every $n \ge 3$, although we only stated it for $n = 3$) we have that the inclusion $\Gamma_\alpha \to \mathbb{R}^n \rtimes SO_n(\mathbb{Z}[\sqrt[3]{\alpha}])$ is a resolution. Altogether, Theorem 2.1 gives that there exists a subalgebra B satisfying the conclusion.

Note that $L(\mathbb{Z}[\sqrt[3]{\alpha}]^n)$ is a Cartan subalgebra of $L(\Gamma_\alpha)$. Thus, in view of the first part, in order to show that $L(\Gamma_\alpha)$ is an HT factor, it suffices to argue that $\mathrm{SO}_3(\mathbb{Z}[\sqrt[3]{\alpha}])$ has Haagerup's property and that Γ_α is ICC. The first assertion is a consequence of the following general result: every countable subgroup of $\mathrm{SO}_3(\mathbb{R})$ has Haagerup's property ([GHW05]; see [Co06a, Theorem 1.14]).

To prove that Γ_α is ICC it suffices to show that (1) $\{\gamma(a) - a | a \in \mathbb{Z}[\sqrt[3]{\alpha}]^n\}$ is infinite, for every $\gamma \in SO_n(\mathbb{Z}[\sqrt[3]{\alpha}]) \setminus \{I\}$ and (2) $\{\gamma(a) | \gamma \in SO_n(\mathbb{Z}[\sqrt[3]{\alpha}])\}$ is infinite, for all $a \in \mathbb{Z}[\sqrt[3]{\alpha}]^n \setminus \{0\}$. The first assertion is clear since $\mathbb{Z}[\sqrt[3]{\alpha}]^3$ is dense in $\mathbb{R}^3$. Now, since $SO_n(\mathbb{Z}[\sqrt[3]{\alpha}])$ is an irreducible lattice in the semisimple Lie group $SO_n(\mathbb{R})\times SO_n(\mathbb{C})$ and $SO_n(\mathbb{C})$ is not compact, we deduce that $SO_n(\mathbb{Z}[\sqrt[3]{\alpha}])$ is dense in $SO_n(\mathbb{R})$ (see e.g. [Ma91] or [Wi08]). This fact implies the second assertion. $\square$

For the proof of Theorem 2.1 we need two technical results. To motivate and state the first result, let us fix some notation. For a standard Borel space X (i.e. a Polish space together with its σ-algebra of Borel subsets) we denote by $\mathcal{M}(X)$ the space of regular Borel probability measures on X and by $B(X)$ the algebra of bounded Borel complex-valued functions on X. Given two measures $\mu, \nu \in \mathcal{M}(X)$, the norm $||\mu - \nu||$ is equal to $\sup_{f\in B(X),||f||_\infty\le 1} |\int_X f d\mu - \int_X f d\nu|$.

Now, if an inclusion of the form $(A \subset A \rtimes \Gamma)$ (where Γ is a countable group acting by automorphisms on a countable abelian group A) has relative property (T), then any sequence of measures $\mu_n \in \mathcal{M}(\hat{A})$ which converge weakly to δ_1 and are almost Γ-invariant must "concentrate" at the identity element $1 \in \hat{A}$, i.e. $\lim_{n\to\infty} \mu_n(\{1\}) = 0$ ([Io09, Theorem 5.1]; the converse is also true; see [Bu91]). The next proposition roughly asserts that the presence of certain resolutions for $A \rtimes \Gamma$ also guarantees that almost invariant measures on $\hat{A}$ concentrate on some subsets.

2.3 Proposition. *Let H be a locally compact abelian group together with a dense countable subgroup A, and denote by $p : \hat{H} \to \hat{A}$ the map induced by restricting characters. Let Γ be a countable group which acts by automorphisms on H and leaves A invariant. Suppose that the inclusion $A \rtimes \Gamma \to H \rtimes \Gamma$ is a resolution. Also, let $V \subset \hat{H}$ be a Γ-invariant neighborhood of $1 \in \hat{H}$.*

Then for any sequence of measures $\mu_n \in \mathcal{M}(\hat{A})$ which converge weakly to δ_1 and satisfy $\lim_{n\to\infty} ||\gamma_*\mu_n - \mu_n|| = 0$, for all $\gamma \in \Gamma$, we have that $\lim_{n\to\infty} \mu_n(p(V)) = 1$. In particular, we have that $\lim_{n\to\infty} \mu_n(p(\hat{H})) = 1$.

Proof. Given $V \subset \hat{H}$ and a sequence $\{\mu_n\}_{n\ge1} \subset \mathcal{M}(\hat{A})$ as in the hypothesis, we begin by showing:

Claim. There exists n such that $\mu_n(p(V)) > 0$.

Proof of the Claim. Let us first prove the Claim under the additional assumption that μ_n is Γ-quasi-invariant, for all n. Fix $n \ge 1$. Since μ_n is Γ-quasi-invariant, we can define $g_\gamma = (d(\gamma_*\mu_n)/d\mu_n)^{\frac{1}{2}}$, for all $\gamma \in \Gamma$, where $d(\gamma_*\mu_n)/d\mu_n$ denotes the Radon-Nikodym derivative of $\gamma_*\mu_n$ with respect to μ_n. Next, we see every $a \in A$ as a character on $\hat{A}$ and therefore as a function in $L^\infty(\hat{A}, \mu_n)$. Then the formulas

$$\pi_n(a)(f) = af, \pi_n(\gamma)(f) = g_\gamma(f \circ \gamma^{-1}),$$

for all $a \in A, \gamma \in \Gamma$ and $f \in L^2(\hat{A}, \mu_n)$, define a unitary representation $\pi_n : A \rtimes \Gamma \to \mathcal{U}(L^2(\hat{A}, \mu_n))$. Let $\xi_n = 1_{\hat{A}} \in L^2(\hat{A}, \mu_n)$. For all $a \in A$ and $\gamma \in \Gamma$, we have that

$$||\pi_n(\gamma)(\xi_n) - \xi_n|| = ||g_\gamma - 1||_2 \leq ||g_\gamma^2 - 1||_1^{\frac{1}{2}} = ||\gamma_*\mu_n - \mu_n||^{\frac{1}{2}},$$
$$||\pi_n(a)(\xi_n) - \xi_n|| = (\int_{\hat{A}} |\eta(a) - 1|^2 d\mu_n(\eta))^{\frac{1}{2}}.$$

Using the assumptions made on μ_n, it follows that the vectors ξ_n form an almost invariant sequence for the representation $\pi = \bigoplus_{n\geq 1} \pi_n : A \rtimes \Gamma \to \mathcal{U}(\mathcal{H})$, where $\mathcal{H} = \bigoplus_{n\geq 1} L^2(\hat{A}, \mu_n)$. Since the inclusion $A \rtimes \Gamma \to H \rtimes \Gamma$ is a resolution, we can find a π-invariant Hilbert subspace $\mathcal{K} \subset \mathcal{H}$ and a unitary representation $\sigma : H \rtimes \Gamma \to \mathcal{U}(\mathcal{K})$ which has almost invariant vectors and satisfies $\sigma(g) = \pi(g)_{|\mathcal{K}}$, for all $g \in A \rtimes \Gamma$. Let $\{\zeta_k\}_{k\geq 1} \subset \mathcal{K}$ be a sequence of σ-almost invariant unit vectors. For every k, let $\nu_k \in \mathcal{M}(\hat{H})$ be given by $\langle \sigma(g)\zeta_k, \zeta_k\rangle = \int_{\hat{H}} \eta(g) d\nu_k(\eta)$, for all $g \in H$. Notice that ν_k converge weakly to δ_1, as $k \to \infty$.

Next, if we set $\rho_k = p_*\nu_k \in \mathcal{M}(\hat{A})$, then for each $a \in A$ we have that

$$\text{(a)} \qquad \int_{\hat{A}} a d\rho_k = \int_{\hat{H}} (a \circ p) d\nu_k = \int_{\hat{H}} \eta(a) d\nu_k(\eta)$$
$$\langle \sigma(a)\zeta_k, \zeta_k\rangle = \langle \pi(a)(\zeta_k), \zeta_k\rangle.$$

Now, for every $k \geq 1$, decompose $\zeta_k = \sum_{n\geq 1} \zeta_k^n$, where $\zeta_k^n \in L^2(\hat{A}, \mu_n)$. Thus, for all $a \in A$, we have that

$$\text{(b)} \qquad \langle \pi(a)(\zeta_k), \zeta_k\rangle = \sum_{n\geq 1} \int_{\hat{A}} a|\zeta_k^n|^2 d\mu_n.$$

By combining (a) and (b) we deduce that $d\rho_k = \sum_{n\geq 1} |\zeta_k^n|^2 d\mu_n$, for all $k \geq 1$. Since $\rho_k(p(V)) = \nu_k(V)$ and $\nu_k \to \delta_1$ weakly, we get that $\lim_{k\to\infty} \rho_k(p(V)) = 1$. Thus, we can find n such that $\mu_n(p(V)) > 0$.

In general, if μ_n are not necessarily quasi-invariant, let $\{\gamma_i\}_{i\geq 1}$ be an enumeration of Γ. For every n, set $\mu_n' = \sum_{i\geq 1} \frac{1}{2^i} \gamma_{i*}\mu_n$. Then μ_n' are Γ quasi-invariant measures which satisfy the hypothesis. By applying the first part of the proof, we get that $\mu_n(p(V)) = \mu_n'(p(V)) > 0$, for some n. □

Suppose by contradiction that the conclusion of the theorem is false. Then, after passing to a subsequence, we can assume that $\lim_{n\to\infty} \mu_n(p(V)) = c < 1$. Thus, for large enough n we have that $\mu_n(\hat{A} \setminus p(V)) > 0$, so can define $\mu_n' \in \mathcal{M}(\hat{A})$ by letting $\mu_n'(X) = \frac{\mu_n(X\setminus p(V))}{\mu_n(\hat{A}\setminus p(V))}$, for every Borel set $X \subset \hat{A}$. Notice that $\mu_n' \to \delta_1$ weakly, as $n \to \infty$. To see this, just remark that for every neighborhood W of $1 \in \hat{A}$ we have that $\mu_n'(\hat{A} \setminus W) \leq \frac{\mu_n(\hat{A}\setminus W)}{\mu_n(\hat{A}\setminus p(V))} \to 0$, as $n \to \infty$.

Next, it is easy to see that since V is Γ-invariant, we get that

$$\lim_{n\to\infty} ||\gamma_*\mu_n' - \mu_n'|| = 0,$$

for all $\gamma \in \Gamma$. Altogether, it follows that μ_n' satisfy the conditions of the hypothesis. Therefore, we can apply the claim and derive that $\mu_n'(p(V)) > 0$, for some n, a contradiction. $\square$

The second ingredient needed in the proof of Theorem 2.1 is the following criterion for rigidity which we derive as a consequence of results from [Io09].

2.4 Proposition. *Let $\Gamma \curvearrowright (X,\mu)$ be a measure preserving action of a countable group Γ on a standard probability space (X,μ). Let $p^i : X \times X \to X$ be the projection $p^i(x_1,x_2) = x_i$, for $i \in \{1,2\}$, and endow $X \times X$ with the diagonal action of Γ. Let (Y,ν) be another probability space together with a measurable, measure preserving onto map $q : X \to Y$. View $L^\infty(Y,\nu)$ as a von Neumann subalgebra of $L^\infty(X,\mu)$ via the embedding $L^\infty(Y,\nu) \ni f \to f \circ q \in L^\infty(X,\mu)$.*

Assume that for any sequence of measures $\{\nu_n\}_{n\geq 1} \subset \mathcal{M}(X \times X)$ such that $p^i_*(\nu_n) = \mu$, for all $i \in \{1,2\}$ and $n \geq 1$,

(i) $\lim_{n\to\infty} \int_{X\times X} f_1(x)f_2(y)d\nu_n(x,y) = \int_X f_1f_2d\mu$, for all $f_1,f_2 \in B(X)$, and

(ii) $\lim_{n\to\infty} ||\gamma_*\nu_n - \nu_n|| = 0$, for all $\gamma \in \Gamma$; we have that

$$\lim_{n\to\infty} \nu_n(\{(x,y) \in X \times X | q(x) = q(y)\}) = 1.$$

Then the inclusion of von Neumann algebras $L^\infty(Y,\nu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is rigid.

Proof. Denote $M = L^\infty(X,\mu) \rtimes \Gamma$ and let $(\mathcal{H}_n, \xi_n)$ be a sequence of pointed Hilbert M-bimodules such that $\lim_{n\to\infty} ||z\xi_n - \xi_n z|| = 0$, for all $z \in M$. To get the conclusion we have to show that there exists a sequence $\eta_n \in \mathcal{H}_n$ of $L^\infty(Y,\nu)$-central vectors such that $\lim_{n\to\infty} ||\eta_n - \xi_n|| = 0$ (see Definition 1.4).

By [Io09, Lemma 2.1] we can find a sequence $\{\nu_n\}_{n\geq 1} \subset \mathcal{M}(X\times X)$ which verifies all the conditions from the hypothesis and satisfies $\int_{X\times X} f_1(x)f_2(y)d\nu_n(x,y) = \langle f_1\xi_n f_2, \xi_n\rangle$, for all $f_1,f_2 \in B(X)$. Thus, if $\Delta_q := \{(x,y) \in X \times X | q(x) = q(y)\}$, then $\lim_{n\to\infty} \nu_n(\Delta_q) = 1$.

Next, for every $f_1,f_2 \in B(X)$, let $f_1 \otimes f_2 : X \times X \to \mathbb{C}$ be given by $(f_1 \otimes f_2)(x,y) = f_1(x)f_2(y)$. Then notice that by the way ν_n is defined, the map $L^2(X \times X, \nu_n) \ni f_1 \otimes f_2 \to f_1\xi_n f_2 \in \mathcal{H}_n$, for every $f_1,f_2 \in B(X)$, extends to an embedding of Hilbert $L^\infty(X,\mu)$-bimodules $\theta_n : L^2(X \times X, \nu_n) \to \mathcal{H}_n$. Here, on $L^2(X \times X, \nu_n)$, we consider the $L^\infty(X,\mu)$-bimodule structure given by $f_1 \cdot g \cdot f_2 = (f_1 \otimes f_2)g$, for all $f_1,f_2 \in L^\infty(X,\mu)$ and $g \in L^2(X \times X, \nu_n)$. Let $\eta_n = \theta_n(1_{\Delta_q})$. Since $1_{\Delta_q} \in L^2(X \times X, \nu_n)$ is an $L^\infty(Y,\nu)$-central vector, we get that $\eta_n \in \mathcal{H}_n$ is an $L^\infty(Y,\nu)$-central vector. Finally, notice that $||\eta_n - \xi_n|| = ||1_{\Delta_q} - 1_{X\times X}||_{L^2(X\times X,\nu_n)} = \sqrt{\nu_n((X \times X) \setminus \Delta_q)} \to 0$, as $n \to \infty$. $\square$

We are now ready to prove Theorem 2.1:

Proof of Theorem 2.1. (1) To derive that $L^\infty(Y,\nu)$ is diffuse, we only need to show that $\lambda^m(q^{-1}(\{y\})) = 0$, for every $y \in Y$. This is clear since $n < m$ and $q^{-1}(\{y\}) \subset (y + p(\mathbb{R}^n)) + \mathbb{Z}^m \subset \mathbb{T}^m$.

(2) To prove the rigidity assertion, let $\nu_k \in \mathcal{M}(\mathbb{T}^m \times \mathbb{T}^m)$ be a sequence of measures such that

$$\text{(a)} \qquad \lim_{k\to\infty} \int_{\mathbb{T}^m\times\mathbb{T}^m} f_1(x)f_2(y)d\nu_k(x,y) = \int_{\mathbb{T}^m} f_1f_2d\lambda^m, \forall f_1,f_2 \in B(\mathbb{T}^m)$$

and

(b) $$\lim_{k\to\infty} ||\gamma_* \nu_k - \nu_k|| = 0, \forall \gamma \in \Gamma.$$

Denote $\Delta_q = \{(x,y) \in \mathbb{T}^m \times \mathbb{T}^m | q(x) = q(y)\}$. Following Proposition 2.4 in order to get the conclusion, it suffices to argue that $\lim_{k\to\infty} \nu_k(\Delta_q) = 1$. To this end, notice that (a) gives that for every bounded Borel function f on $\mathbb{T}^m$ we have that $\lim_{k\to\infty} \int_{\mathbb{T}^m\times\mathbb{T}^m} |f(x) - f(y)|^2 d\nu_k(x,y) = 0$. This implies that

$$\lim_{k\to\infty} \int_{\mathbb{T}^m\times\mathbb{T}^m} ||i(x) - i(y)||^2 d\nu_k(x,y) = 0,$$

where $||.||$ denotes the Euclidian norm on $\mathbb{R}^m$. Thus, we deduce that

(c) $$\lim_{k\to\infty} \nu_k(\{(x,y) \in \mathbb{T}^m \times \mathbb{T}^m | \, ||i(x) - i(y)|| \leq \frac{1}{2}\}) = 1.$$

Next, assume that $x, y \in \mathbb{T}^m$ satisfy $||i(x)-i(y)|| \leq \frac{1}{2}$. Since $i(x)-i(y) \in (x-y)+\mathbb{Z}^m$, we deduce that $i(x) - i(y) = i(x-y)$ and therefore that $q(x) - q(y) = q(x-y)$. By combining this fact with (c), we get that

$$\lim_{k\to\infty} \nu_k(\{(x,y) \in \mathbb{T}^m \times \mathbb{T}^m | q(x) - q(y) = q(x-y)\}) = 1.$$

Thus, showing that $\lim_{k\to\infty} \nu_k(\Delta_q) = 1$ is equivalent to proving that

(d) $$\lim_{k\to\infty} \nu_k(\{(x,y) \in \mathbb{T}^m \times \mathbb{T}^m | q(x-y) = 0\}) = 1.$$

If we let $r : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{T}^m$ be given by $r(x,y) = x - y$ and, for every k, define $\mu_k = r_*\nu_k \in \mathcal{M}(\mathbb{T}^m)$, then (d) can be rewritten as

$$\lim_{k\to\infty} \mu_k(\{x \in \mathbb{T}^m | q(x) = 0\}) = 1.$$

Now, note that the inclusion $A = \mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_m \subset H = \mathbb{R}^n$ gives rise to a homomorphism $p' : \hat{H} \simeq \mathbb{R}^n \to \hat{A} \simeq \mathbb{T}^m$ given by $p'(a) = (\langle a, v_1\rangle + \mathbb{Z}, ..., \langle a, v_m\rangle + \mathbb{Z})$, for all $a \in \mathbb{R}^n$. In other words, p' is the composition between p and the projection $\mathbb{R}^m \to \mathbb{T}^m$. Let $\varepsilon > 0$ such that $V = \{a \in \mathbb{R}^n | \, ||a|| \leq \varepsilon\}$ satisfies $p(V) \subset [-\frac{1}{2}, \frac{1}{2})^m$. We claim that $p'(V) \subset \{x \in \mathbb{T}^m | q(x) = 0\}$. Indeed, if $a \in V$, then $p(a) \in [-\frac{1}{2}, \frac{1}{2})^m$. Thus $i(p'(a)) = p(a)$, and hence $q(p'(a)) = (\pi \circ i)(p'(a)) = \pi(p(a)) = 0$.

On the other hand, V is a Γ-invariant neighborhood of $0 \in \mathbb{R}^n$. Also, remark that (a) implies that μ_k converge weakly to δ_0 while (b) implies that

$$\lim_{k\to\infty} ||\gamma_*\mu_k - \mu_k|| = 0,$$

for all $\gamma \in \Gamma$. By applying Proposition 2.3 we deduce that $\lim_{k\to\infty} \mu_k(p'(V)) = 1$. Since $p'(V) \subset \{x \in \mathbb{T}^m | q(x) = 0\}$, we get that $\lim_{k\to\infty} \mu_k(\{x \in \mathbb{T}^m | q(x) = 0\}) = 1$. This proves (d) and thus the conclusion.

(3) First, it is easy to see that the conclusion is equivalent to

$$\lambda^m(\{x \in \mathbb{T}^m | q(\gamma x) = q(x)\}) = 0,$$

for all $\gamma \in \Gamma \setminus \{I\}$ (where I denotes the identity matrix). Assuming that this is not the case we can find $\gamma \in \Gamma \setminus \{I\}$ such that $\lambda^m(\{x \in \mathbb{T}^m | q(\gamma x) = q(x)\}) > 0$. By using the definition of q, this implies that

(e) $$\lambda^m(\{x \in \mathbb{T}^m | \gamma x - x \in p(\mathbb{R}^n) + \mathbb{Z}^m\}) > 0.$$

Second, notice that the action of Γ on $\mathbb{Z}^m \simeq A$ is realized through a homomorphism $\rho : \Gamma \to \mathrm{GL}_m(\mathbb{Z})$. The (dual) action of Γ on $\mathbb{T}^m \simeq \hat{A}$ is then given by $\gamma(x + \mathbb{Z}^m) = (\rho(\gamma)^{-1})^t(x) + \mathbb{Z}^m$. Altogether, (e) implies that if μ^m denotes the Lebesgue measure on $\mathbb{R}^m$, then $\mu^m(\{x \in \mathbb{R}^m | (\rho(\gamma)^{-1})^t(x) - (x) \in p(\mathbb{R}^n) + \mathbb{Z}^m\}) > 0$. This easily implies that

$$(\rho(\gamma^{-1})^t - I)(\mathbb{R}^m) \subset p(\mathbb{R}^n). \tag{f}$$

Finally, since $\rho(\gamma^{-1}) \in \mathrm{GL}_m(\mathbb{Z})$, we get that $(\rho(\gamma^{-1})^t - I)(\mathbb{Z}^m) \subset \mathbb{Z}^m$. By combining this with (f) and the fact that $p(\mathbb{R}^n) \cap \mathbb{Z}^m = \{0\}$, we deduce that $(\rho(\gamma^{-1})^t - I)(\mathbb{Z}^m) = \{0\}$. This means that $\rho(\gamma) = I$, or, equivalently, that γ acts trivially on A. Since A is dense in $\mathbb{R}^n$, we further get that γ acts trivially on $\mathbb{R}^n$ which implies that $\gamma = I$, a contradiction. □

3. Relative property (T) subsets of semidirect product groups

In this section we show that Haagerup's property is not preserved under generalized wreath products. Using this fact we give the first examples of von Neumann algebras which neither have Haagerup's property nor admit any diffuse rigid von Neumann subalgebras. We start with the following result which asserts that if a group Γ acts on an abelian group A through a quotient group Γ_0, then the presence of a relative property (T) subset in Γ_0 (or the lack of Haagerup's property) is inherited by the semidirect product $A \rtimes \Gamma$.

3.1 Theorem. *Let Γ_0 be a countable group and let $\rho : \Gamma_0 \to Aut(A)$ be an action by automorphisms on a countable abelian group A. Let Γ be a countable group together with a surjective homomorphism $p : \Gamma \to \Gamma_0$, and consider the action of Γ on A given by $\tilde{\rho} = \rho \circ p : \Gamma \to Aut(A)$.*

(1) Suppose that X is a subset of Γ_0 such that the inclusion $(X \subset \Gamma_0)$ has relative property (T). For $a \in A$, let $X_a = \{\rho(\gamma)(a) | \gamma \in X\} \subset A$. Then the semidirect product $A \rtimes_{\tilde{\rho}} \Gamma$ has relative property (T) with respect to X_a, for every $a \in A$.

(2) Assume that there is $a \in A$ such that its stabilizer $\{\gamma \in \Gamma_0 | \rho(\gamma)(a) = a\}$ in Γ_0 is finite. If Γ_0 does not have Haagerup's property, then $A \rtimes_{\tilde{\rho}} \Gamma$ does not have Haagerup's property.

Proof. (1) Fix $a \in A$ and let $\pi : A \rtimes_{\tilde{\rho}} \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation which admits a sequence $\{\xi_n\}_{n\geq 1} \subset \mathcal{H}$ of almost invariant, unit vectors. To get the conclusion we have to show that ξ_n are uniformly $\pi(X_a)$-almost invariant, i.e. $\lim_{n\to\infty} \sup_{\gamma \in X_a} ||\pi(\rho(\gamma)(a))(\xi_n) - \xi_n|| = 0$.

First, for every $n \geq 1$, let $\mu_n \in \mathcal{M}(\hat{A})$ such that $\langle \pi(a)\xi_n, \xi_n\rangle = \int_{\hat{A}} a d\mu_n$, for each $a \in A$. By the proof of [Bu91, Proposition 7] we have that $||\gamma_*\mu_n - \mu_n|| \leq 2||\pi(\gamma)(\xi_n) - \xi_n||$, for all $\gamma \in \Gamma$. Here, on $\hat{A}$ we consider the natural actions of Γ and Γ_0 induced by $\tilde{\rho}$ and ρ, respectively. For every $\gamma \in \Gamma_0$, fix $\tilde{\gamma} \in \Gamma$ such that $p(\tilde{\gamma}) = \gamma$. Then the above implies that

$$||\gamma_*\mu_n - \mu_n|| = ||\tilde{\gamma}_*\mu_n - \mu_n|| \leq 2||\pi(\tilde{\gamma})(\xi_n) - \xi_n||, \forall \gamma \in \Gamma_0. \tag{a}$$

Second, let $\{\gamma_i\}_{i\geq 1}$ be an enumeration of $\Gamma_0 \setminus \{e\}$. For every $n \geq 1$, let $\nu_n \in \mathcal{M}(\hat{A})$ be given by $\nu_n = (1 - \frac{1}{2^n})\mu_n + \sum_{i\geq 1} \frac{1}{2^{i+n}} {\gamma_i}_*\mu_n$. Then we have that $||\nu_n - \mu_n|| \leq \frac{1}{2^{n-1}}$, for all $n \geq 1$, and thus (a) implies that

$$\text{(b)} \qquad \begin{aligned} ||\gamma_* \nu_n - \nu_n|| &\leq 2||\nu_n - \mu_n|| + ||\gamma_* \mu_n - \mu_n|| \\ &\leq \frac{1}{2^{n-2}} + 2||\pi(\tilde{\gamma})(\xi_n) - \xi_n||, \forall \gamma \in \Gamma_0. \end{aligned}$$

Next, fix $n \geq 1$. Notice that ν_n is a Γ_0-quasi-invariant measure and let $g_\gamma = (d(\gamma_* \nu_n)/d\nu_n)^{\frac{1}{2}} \in L^2(\hat{A}, \nu_n)$, for every $\gamma \in \Gamma_0$. The formula $\sigma_n(\gamma)(f) = g_\gamma(f \circ \gamma^{-1})$, for all $f \in L^2(\hat{A}, \nu_n)$ and $\gamma \in \Gamma_0$, defines a unitary representation $\sigma_n : \Gamma_0 \to \mathcal{U}(L^2(\hat{A}, \nu_n))$. If $\eta_n = 1_{\hat{A}} \in L^2(\hat{A}, \nu_n)$, then, as in the proof of Proposition 2.3, we have that

$$\text{(c)} \qquad ||\sigma_n(\gamma)(\eta_n) - \eta_n|| \leq ||\gamma_* \nu_n - \nu_n||^{\frac{1}{2}}, \forall \gamma \in \Gamma_0.$$

Since the vectors ξ_n are $\pi(\Gamma)$-almost invariant, by combining (b) and (c) we deduce that $\lim_{n\to\infty} ||\sigma_n(\gamma)(\eta_n) - \eta_n|| = 0$, for each $\gamma \in \Gamma_0$. Since the inclusion $(X \subset \Gamma_0)$ has relative property (T), by [Co06b, Theorem 1.1] we get that $\varepsilon_n := \sup_{\gamma \in X} ||\sigma_n(\gamma)(\eta_n) - \eta_n|| \to 0$, as $n \to \infty$. Now, for $\gamma \in X$ we have that

$$\begin{aligned} ||\gamma_* \nu_n - \nu_n|| &= ||g_\gamma^2 - 1||_1 \leq ||g_\gamma + 1||_2 ||g_\gamma - 1||_2 \leq 2||g_\gamma - 1||_2 \\ &= 2||\sigma_n(\gamma)(\eta_n) - \eta_n|| \leq 2\varepsilon_n, \forall n \geq 1. \end{aligned}$$

Thus, we get that $||\gamma_* \mu_n - \mu_n|| \leq 2||\nu_n - \mu_n|| + ||\gamma_* \nu_n - \nu_n|| \leq \frac{1}{2^{n-1}} + 2\varepsilon_n$, for all $\gamma \in X$. This implies that for every $\gamma \in X$ we have that

$$\text{(d)} \qquad \begin{aligned} &|\langle \pi(\rho(\gamma)(a))(\xi_n), \xi_n \rangle - \langle \pi(a)(\xi_n), \xi_n \rangle| = |\int_{\hat{A}} \rho(\gamma)(a) d\mu_n - \int_{\hat{A}} a d\mu_n| \\ &= |\int_{\hat{A}} (a \circ \gamma^{-1}) d\mu_n - \int_{\hat{A}} a d\mu_n| \leq ||\gamma_* \mu_n - \mu_n|| \leq \frac{1}{2^{n-1}} + 2\varepsilon_n. \end{aligned}$$

Finally, (d) together with a standard calculation gives that

$$||\pi(\rho(\gamma)(a))(\xi_n) - \xi_n||^2 \leq ||\pi(a)(\xi_n) - \xi_n||^2 + 2(\frac{1}{2^{n-1}} + 2\varepsilon_n), \forall \gamma \in X,$$

which proves the conclusion.

(2) Assume by contradiction that $A \rtimes_{\tilde{\rho}} \Gamma$ has Haagerup's property, while Γ_0 does not have it. Thus we can find a c_0 unitary representation $\pi : A \rtimes_{\tilde{\rho}} \Gamma \to \mathcal{U}(\mathcal{H})$ which admits a sequence $\{\xi_n\}_{n \geq 1} \subset \mathcal{H}$ of almost invariant unit vectors. Let $\sigma_n : \Gamma_0 \to \mathcal{U}(L^2(\hat{A}, \nu_n))$ and $\eta_n \in L^2(\hat{A}, \nu_n)$ be constructed as in the proof of part (1). Recall that η_n are almost invariant unit vectors, i.e. $\lim_{n\to\infty} ||\sigma_n(\gamma)(\eta_n) - \eta_n|| = 0$, for each $\gamma \in \Gamma_0$.

Since Γ_0 does not have Haagerup's property, by [Pe09, Theorem 2.6] we can find an infinite subset X of Γ_0 and an increasing sequence $\{k_n\}_{n \geq 1}$ of natural numbers such that $\lim_{n\to\infty} \sup_{\gamma \in X} ||\sigma_{k_n}(\gamma)(\eta_{k_n}) - \eta_{k_n}|| = 0$. Let $a \in A$ such that its stabilizer in Γ_0 is finite. The last part of the proof of (1) implies that $\lim_{n\to\infty} \sup_{\gamma \in X} ||\pi(\rho(\gamma)(a))(\xi_{k_n}) - \xi_{k_n}|| = 0$. On the other hand, since the stabilizer of a in Γ_0 is finite and π is a c_0 representation, we get that

$$\lim_{\gamma \to \infty} \langle \pi(\rho(\gamma)(a))(\xi_n), \xi_n \rangle = 0.$$

Altogether, this gives a contradiction, as X is infinite. □

3.2 *Remarks.* (a) We note that the proof of part (1) in fact shows more: if $X \subset A$ is a set such that the inclusion $(X \subset A \rtimes_\rho \Gamma_0)$ has relative property (T), then the inclusion $(X \subset A \rtimes_{\tilde{\rho}} \Gamma)$ has relative property (T).

(b) Let us also remark that the proof of (2) can be adapted to show that if $A \rtimes_\rho \Gamma_0$ is not Haagerup, then $A \rtimes_{\tilde{\rho}} \Gamma$ is not Haagerup, provided that the stabilizer of some $a \in A$ in Γ_0 is finite. Indeed, in the notation from above, if $A \rtimes_\rho \Gamma_0$ is not Haagerup, then we can find an infinite set $X \subset A \rtimes_\rho \Gamma_0$ and a sequence $\{k_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} \sup_{\gamma\in X} ||\sigma_{k_n}(\gamma)(\eta_{k_n}) - \eta_{k_n}|| = 0$. Thus, $\lim_{n\to\infty} \sup_{\gamma\in X} ||\sigma_{k_n}(\gamma a\gamma^{-1})(\eta_{k_n}) - \eta_{k_n}|| = 0$. If the projection of X onto Γ_0 is infinite, then $\{\gamma a \gamma^{-1} | \gamma \in X\}$ is an infinite subset of A and a contradiction is reached as in the above proof. Otherwise, the set Y of all $a \in A$ such that $(a, \gamma) \in X$, for some $\gamma \in \Gamma_0$, is infinite. Since the projection of X onto Γ_0 is finite, it is clear that $\lim_{n\to\infty} \sup_{a\in Y} ||\sigma_{k_n}(a)(\eta_{k_n}) - \eta_{k_n}|| = 0$. Again, we obtain a contradiction as in the end of the proof of part (1).

Recently, Y. de Cornulier, Y. Stalder and A. Valette have shown that if two countable groups A and Γ have Haagerup's property, then so does their *wreath product* $A \wr \Gamma = A^\Gamma \rtimes \Gamma$ ([CSV09]). More generally, they consider *generalized wreath product* groups $A \wr_X \Gamma = A^X \rtimes \Gamma$, where X is a countable Γ-set and Γ acts on $A^X = \bigoplus_{x\in X} A$ by shifting indices. In this context, they show that if A and Γ are Haagerup, then so are certain wreath products $A \wr_X \Gamma$ (e.g. when X is a Haagerup quotient group of Γ) ([CSV09, Theorem 6.2]). Furthermore, it is conjectured in [CSV09] that all such wreath products are Haagerup, regardless of the set X. As a consequence of Theorem 3.1, we disprove this conjecture by showing, for example, that if X is an infinite quotient of Γ with property (T), then $A \wr_X \Gamma$ is not Haagerup.

3.3 Corollary. *Let A be a non-trivial countable group. Let Γ be a countable group together with a quotient group Γ_0. Endow Γ_0 with the left multiplication action of Γ. Then the generalized wreath product $A \wr_{\Gamma_0} \Gamma$ is Haagerup if and only if A, Γ and Γ_0 are Haagerup.*

Proof. The *if* part is a particular case of [CSV09, Theorem 6.2]. To prove the *only if* part, assume that $A \wr_{\Gamma_0} \Gamma$ is Haagerup. Since both A and Γ are subgroups of $A \wr_{\Gamma_0} \Gamma$, they must be Haagerup. Let $a \in A \setminus \{e\}$ and let A_0 be the cyclic group generated by a. Since $A_0 \wr_{\Gamma_0} \Gamma$ is Haagerup (being a subgroup of $A \wr_{\Gamma_0} \Gamma$) and A_0 is abelian, Theorem 3.1 (2) implies that Γ_0 is Haagerup. $\square$

The above corollary gives new examples of countable groups which are not Haagerup and yet do not admit any infinite subgroups with relative property (T). More precisely, in the above context, $G = A \wr_{\Gamma_0} \Gamma$ is such a group, whenever A, Γ are Haagerup and Γ_0 is not. Indeed, by Corollary 3.3, G is not Haagerup, while by [CSV09, Theorem 6.7] (or by [Io07, Theorem 3.6]), G does not have relative property (T) with respect to any infinite subgroup. Moreover, we can use results from [Po06a], [Po06b] and [Io07] to deduce that the group von Neumann algebra of G has no rigid diffuse von Neumann subalgebra:

3.4 Corollary. *Let A be a non-trivial countable Haagerup group. Let Γ be a countable Haagerup together with a quotient group Γ_0. Assume that Γ_0 is not Haagerup and endow it with the left multiplication action of Γ. Then the group von Neumann algebra $N = L(A \wr_{\Gamma_0} \Gamma)$ does not have Haagerup property and does not admit any diffuse von Neumann subalgebra B such that the inclusion $(B \subset N)$ is rigid.*

Proof. By Corollary 3.3, $A \wr_{\Gamma_0} \Gamma$ does not have Haagerup's property, and therefore its group von Neumann algebra does not have it either (see e.g. [Po06a]). Now, assume by contradiction that the inclusion $(B \subset N)$ is rigid, for some diffuse von Neumann subalgebra B of N. Since Γ has Haagerup's property, the proof of [Po06a, Theorem 6.2] gives that a corner of B embeds into $L(A^{\Gamma_0})$, in the sense of [Po06b, Section 2].

To get a contradiction, we apply results from [Io07], leaving the details to the reader. Notice first that N can be written as $(\overline{\bigotimes}_{\Gamma_0} L(A)) \rtimes \Gamma$, where Γ acts on $\overline{\bigotimes}_{\Gamma_0} L(A)$ by Bernoulli shifts. Using this observation and the rigidity of the inclusion $(B \subset N)$, the proof of [Io07, Theorem 3.6] implies that a corner of B can be embedded into $L(A^F)$, for some finite subset F of Γ_0. Finally, the proof of [Io07, Corollary 3.7] shows that since $L(A)$ has Haagerup's property, B cannot be diffuse, a contradiction. $\square$

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