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SOME EIGENVALUE PROBLEMS FOR VECTORIAL STURM-LIOUVILLE EQUATIONS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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ABSTRACT. We investigate the n-dimensional vectorial Sturm-Liouville equation

$$\vec{y}\,''(x) + [\lambda^2 I_n - Q(x)]\vec{y}(x) = \vec{0}$$

with eigenparameter dependent boundary conditions

$$\vec{y}(0) = \vec{0}, A\vec{y}\,'(\pi) + \lambda\vec{y}(\pi) = \vec{0}.$$

Under the assumption that $Q(x)$ is nonnegative definite in $[0,\pi]$, we prove that the eigenvalues of the n-dimensional vectorial Sturm-Liouville equation are real.

For the case $n = 2$, we show that the *algebraic multiplicity* of an eigenvalue of the problem as a zero of the *characteristic function*

$$\omega_A(\lambda; Q) = \det[AY'(\pi; \lambda^2; Q) + \lambda Y(\pi; \lambda^2; Q)]$$

is equal to its *geometric multiplicity*. By the theory of Hadamard's factorization, we also prove that the *characteristic function* $\omega_A(\lambda; Q)$ is uniquely determined by the *spectral set* of the equation. Moreover, we consider the inverse problem for the equation, i.e., how many *spectral sets* can determine the potential function $Q(x)$ uniquely, and we find that three *spectral sets* are necessary for us to determine the potential function $Q(x)$ uniquely.

1. INTRODUCTION

The purpose of this paper is to investigate a class of vectorial Sturm-Liouville equations with eigenparameter dependent boundary conditions. The spectral problems related to the Sturm-Liouville equations with eigenparameter dependent boundary conditions were first systematically studied by Hochstadt in [H]. There, Hochstadt investigated this kind of eigenvalue problem for Sturm-Liouville equations with a scalar coefficient. He considered the following eigenvalue problem:

$$\begin{cases} y''(x) + [\lambda^2 - q(x)]y(x) = 0, 0 \le x \le \pi, \\ y(0) = 0, ay'(\pi) + \lambda y(\pi) = 0, \end{cases} \tag{1.1}$$

where $a \neq 0$, $a \in \mathbb{R}$, $q(x)$ is real valued and integrable on $[0,\pi]$. Let $y(x; \nu; q)$ denote the solution of the initial-value problem

$$y''(x) + [\nu - q(x)]y(x) = 0, y(0) = 0, y'(0) = 1.$$

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Denote

$$\omega_a(\lambda; q) = ay'(\pi; \lambda^2; q) + \lambda y(\pi; \lambda^2; q),$$

and call it the *characteristic function* of equation (1.1). Then λ_* is an eigenvalue of equation (1.1) if and only if λ_* is a zero of $\omega_a(\lambda; q)$. The number of linearly independent solutions corresponding to λ_* is called the *geometric multiplicity* of λ_*, and the multiplicity of λ_* as a zero of $\omega_a(\lambda; q)$ is called the *algebraic multiplicity* of λ_*.

Under some assumptions, Hochstadt used the idea of *inverse Liouville transformation* to transform equation (1.1) into a string equation of the form

$$\begin{cases} z''(x) + \lambda^2 M^4(x) z(x) = 0, 0 \leq x \leq c, \\ z(0) = 0, az'(c) + \lambda z(c) = 0, \end{cases} \tag{1.2}$$

where $c = \int_0^\pi \frac{1}{M^2(t)} dt$. Then he transformed equation (1.2) into an integral equation and applied the properties of the integral operator corresponding to the integral equation to obtain the following result:

Theorem 1.1. *All eigenvalues of equation* (1.1) *are real, geometrically simple, and algebraically simple.*

Following Theorem 1.1, we see that the *characteristic function* $\omega_a(\lambda; q)$ of equation (1.1) is completely determined by its eigenvalues. Moreover, Hochstadt proved in [H] that the potential function of equation (1.1) is uniquely determined by the spectrum of equation (1.1) ([H, Theorem B]):

Theorem 1.2. *Consider the boundary value problems*

$$y''(x) + [\lambda^2 - q_1(x)]y(x) = 0, y(0) = ay'(\pi) + \lambda y(\pi) = 0,$$
$$u''(x) + [\lambda^2 - q_2(x)]u(x) = 0, u(0) = au'(\pi) + \lambda u(\pi) = 0,$$

where $a \neq 0$, $a \in \mathbb{R}$, $q_j(x)$ *are real valued and integrable on* $[0, \pi]$, $j = 1, 2$. *If* $\omega_a(\lambda; q_1) = \omega_a(\lambda; q_2)$, *then* $q_1 = q_2$ *almost everywhere in* $[0, \pi]$.

We are interested in extending Hochstadt's work to the n-dimensional vectorial Sturm-Liouville equations of the form

$$\begin{cases} \vec{y}\,''(x) + [\lambda^2 I_n - Q(x)]\vec{y}(x) = \vec{0}, \\ \vec{y}(0) = \vec{0}, A\vec{y}\,'(\pi) + \lambda \vec{y}(\pi) = \vec{0}, \end{cases} \tag{1.3}$$

where $Q(x)$ is an $n \times n$ real symmetric matrix-valued smooth function defined in the interval [0,π], A is an $n \times n$ nonsingular real symmetric matrix, $\vec{y}(x)$ is an $\mathbb{R}^n$-function defined in the interval [0,π], and $\vec{0}$ is the zero vector in $\mathbb{R}^n$. Let $Y(x; \nu; Q)$ denote the $n \times n$ matrix solution of the $n \times n$ matrix differential equation

$$Y''(x) + [\nu I_n - Q(x)]Y(x) = 0, Y(0) = 0, Y'(0) = I_n,$$

where 0 is the $n \times n$ zero matrix. Denote

$$\omega_A(\lambda; Q) = \det(AY'(\pi; \lambda^2; Q) + \lambda Y(\pi; \lambda^2; Q)),$$

and call it the *characteristic function* of equation (1.3). Then λ_* is an eigenvalue of equation (1.3) if and only if λ_* is a zero of $\omega_A(\lambda; Q)$. The number of linearly independent solutions corresponding to λ_* is called the *geometric multiplicity* of λ_*, and the multiplicity of λ_* as a zero of $\omega_A(\lambda; Q)$ is called the *algebraic multiplicity* of λ_*.

However, there are some major differences between the scalar case considered by Hochstadt and its vectorial analogy. We find a class of matrix potential functions for which the equation possesses nonreal eigenvalues. Consider the following example: assume $\frac{1}{4} < a < \frac{9}{4}, b \geq 0$ and consider the eigenvalue problem

$$\begin{cases} \vec{y}\,''(x) + [\lambda^2 I_2 - \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}]\vec{y}(x) = \vec{0}, \\ \vec{y}(0) = \vec{0}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{y}\,'(\pi) + \lambda \vec{y}(\pi) = \vec{0}, \end{cases} \tag{1.4}$$

where $\vec{y}(x)$ is an $\mathbb{R}^2$-function defined in the interval $[0,\pi]$, and $\vec{0}$ is the zero vector in $\mathbb{R}^2$. Denote

$$Q_1 = \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then we find

$$Y(x;\lambda^2;Q_1) = \begin{pmatrix} \frac{\sin(\sqrt{\lambda^2-b}x)}{\sqrt{\lambda^2-b}} & 0 \\ 0 & \frac{\sin(\sqrt{\lambda^2+a}x)}{\sqrt{\lambda^2+a}} \end{pmatrix}. \tag{1.5}$$

Thus the characteristic function $\omega_{A_1}(\lambda;Q_1)$ of equation (1.4) is

$$\begin{aligned} \omega_{A_1}(\lambda;Q_1) &= \det(A_1 Y'(\pi;\lambda^2;Q_1) + \lambda Y(\pi;\lambda^2;Q_1)) \\ &= \frac{\lambda^2}{\sqrt{\lambda^2-b}\sqrt{\lambda^2+a}} \sin(\sqrt{\lambda^2-b}\pi)\sin(\sqrt{\lambda^2+a}\pi) \\ &\quad - \cos(\sqrt{\lambda^2-b}\pi)\cos(\sqrt{\lambda^2+a}\pi). \end{aligned} \tag{1.6}$$

Taking $\lambda = ci$ in equation (1.6), where $c > 0, i = \sqrt{-1}$, we have

$$\begin{aligned} \omega_{A_1}(ci;Q_1) = &-\frac{c^2}{\sqrt{b+c^2}\sqrt{a-c^2}} \sinh(\sqrt{b+c^2}\pi)\sin(\sqrt{a-c^2}\pi) \\ &- \cosh(\sqrt{b+c^2}\pi)\cos(\sqrt{a-c^2}\pi). \end{aligned} \tag{1.7}$$

Note that

$$\lim_{c\to 0^+} \omega_{A_1}(ci;Q_1) = -\cosh(\sqrt{b}\pi)\cos(\sqrt{a}\pi) > 0 \tag{1.8}$$

and

$$\lim_{c\to\sqrt{a}^-} \omega_{A_1}(ci;Q_1) = -\frac{a\pi}{\sqrt{b+a}}\sinh(\sqrt{b+a}\pi) - \cosh(\sqrt{b+a}\pi) < 0. \tag{1.9}$$

From (1.8) and (1.9), we find that there exists c_0 in $(0,\sqrt{a})$ such that $\omega_{A_1}(c_0 i;Q_1) = 0$, i.e., $c_0 i$ is an eigenvalue of equation (1.4). This example shows that there are nonpositive definite and indefinite potential functions $Q(x)$ such that equation (1.3) has a nonreal eigenvalue.

Besides, the method of the *inverse Liouville transformation* used in Hochstadt's work for proving Theorem 1.1 cannot be extended to the vectorial case. To remedy these situations, in section 3, we use a different approach to deal with the vectorial case under the assumption that the potential function $Q(x)$ is nonnegative definite for all x in $[0,\pi]$. Under this assumption, we prove that the eigenvalues of equation (1.3) are real (see Theorem 3.1). The problem about the equality between the *geometric multiplicity* and the *algebraic multiplicity* of the eigenvalues of the vectorial Sturm-Liouville equation is more complicated. To deal with such a problem, we apply an approach similar to that used in [Sh]. In [Sh], Shen considered the eigenvalue problems

$$\begin{cases}\vec{y}''(x)+[\lambda I_2-Q(x)]\vec{y}(x)=\vec{0}, 0\le x\le 1,\\ \vec{y}(0)=\vec{0}, \vec{y}(1)=\vec{0},\end{cases} \tag{1.10}$$

$$\begin{cases}\vec{z}''(x)+[\lambda I_2-Q(x)]\vec{z}(x)=\vec{0}, 0\le x\le 1,\\ \vec{z}'(0)=\vec{0}, \vec{z}(1)=\vec{0},\end{cases} \tag{1.11}$$

and

$$\begin{cases}\vec{u}''(x)+[\lambda I_2-Q(x)]\vec{u}(x)=\vec{0}, 0\le x\le 1,\\ \vec{u}'(0)-B\vec{u}(0)=\vec{0}, \vec{u}(1)=\vec{0},\end{cases} \tag{1.12}$$

where $Q(x)$ is a two-by-two real symmetric matrix-valued smooth function defined in [0,1]; B is a two-by-two nonsingular real symmetric matrix; $\vec{y}(x)$, $\vec{z}(x)$, $\vec{u}(x)$ are $\mathbb{R}^2$-functions defined in the interval [0,1]; and $\vec{0}$ is the zero vector in $\mathbb{R}^2$. He proved that if λ_* is an eigenvalue of equation (1.10) (or (1.11), (1.12)), then the *geometric multiplicity* and the *algebraic multiplicity* of λ_* are the same. On the other hand, let $\sigma_D(Q)$, $\sigma_{ND}(Q)$, and $\sigma_B(Q)$ denote the *spectral set* of equations (1.10), (1.11), and (1.12), respectively. Shen also proved that if B_1, B_2, and B_3 are three linearly independent real symmetric matrices, then $\sigma_D(Q)$, $\sigma_{ND}(Q)$, and $\sigma_{B_j}(Q)$, j=1, 2, 3, determine $Q(x)$ uniquely.

In section 4, we concentrate on the two-dimensional case in order to obtain some results about the *geometric multiplicity* and the *algebraic multiplicity* of an eigenvalue. Consider the eigenvalue problem

$$\begin{cases}\vec{y}''(x)+[\lambda^2 I_2-Q(x)]\vec{y}(x)=\vec{0}, 0\le x\le \pi,\\ \vec{y}(0)=\vec{0}, A\vec{y}'(\pi)+\lambda\vec{y}(\pi)=\vec{0},\end{cases} \tag{1.13}$$

where $Q(x)$ is a two-by-two nonnegative definite matrix-valued smooth function defined in [0,π], A is a two-by-two nonsingular real symmetric matrix, $\vec{y}(x)$ is an $\mathbb{R}^2$-function defined in the interval [0,π], and $\vec{0}$ is the zero vector in $\mathbb{R}^2$. According to the results that we obtained in section 3 and using the argument similar to that Shen used to deal with equations (1.10), (1.11), and (1.12), we find that the *geometric multiplicity* and the *algebraic multiplicity* of an eigenvalue of equation (1.13) are the same (see Theorem 4.5). Applying this result and using some asymptotic arguments, we also prove that $\omega_A(\lambda; Q)$ is uniquely determined by the *spectral set* of equation (1.13).

At the end of this paper, we study an inverse problem for the vectorial Sturm-Liouville equation with eigenparameter dependent boundary conditions: how many *spectral sets* of the eigenvalue problem of the form like equation (1.13) can determine $Q(x)$ uniquely? We find that the *spectral sets* corresponding to three linearly independent real symmetric matrices can determine $Q(x)$ uniquely (see Theorem 4.9). From the example we constructed in section 4 (see equations (4.41) and (4.42)), we find the requirement of three spectral conditions in Theorem 4.9 is optimal for the two-dimensional vectorial Hochstadt problem.

2. Preliminaries

In this section, we recall some properties which will be used later. Let $Q(x)$ be an $n\times n$ real symmetric matrix-valued smooth function defined in [0,π], and let A be an $n \times n$ nonsingular real symmetric matrix, where $n \in \mathbb{N}$. Consider the eigenvalue problem

$$\begin{cases} \vec{y}''(x) + [\lambda^2 I_n - Q(x)]\vec{y}(x) = \vec{0}, \\ \vec{y}(0) = \vec{0}, A\vec{y}'(\pi) + \lambda\vec{y}(\pi) = \vec{0}, \end{cases} \tag{2.1}$$

where $\vec{y}(x)$ is an $\mathbb{R}^n$-function defined in the interval $[0,\pi]$, and $\vec{0}$ is the zero vector in $\mathbb{R}^n$. In order to investigate equation (2.1), we apply the following $n \times n$ matrix differential equations:

$$Y''(x) + [\nu I_n - Q(x)]Y(x) = 0, Y(0) = 0, Y'(0) = I_n, \tag{2.2}$$

$$Z''(x) + [\nu I_n - Q(x)]Z(x) = 0, Z(0) = I_n, Z'(0) = 0, \tag{2.3}$$

where 0 is the $n \times n$ zero matrix. Let $Y(x;\nu;Q)$ and $Z(x;\nu;Q)$ denote the $n \times n$ matrix solutions of equations (2.2) and (2.3), respectively. Then we have (see [PT], [Sh])

Lemma 2.1. *For fixed $x \in [0,\pi]$, the matrix functions $Y(x;\lambda^2;Q)$, $Y'(x;\lambda^2;Q)$, $Z(x;\lambda^2;Q)$, and $Z'(x;\lambda^2;Q)$ are entire functions with respect to variable λ^2, and*

$$Y(x;\lambda^2;Q) = \frac{\sin\sqrt{\lambda^2}x}{\sqrt{\lambda^2}} I_n + O(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|^2}), |\lambda| \to \infty, \tag{2.4}$$

$$Y'(x;\lambda^2;Q) = (\cos\sqrt{\lambda^2}x)I_n + O(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|}), |\lambda| \to \infty, \tag{2.5}$$

$$Z(x;\lambda^2;Q) = (\cos\sqrt{\lambda^2}x)I_n + O(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|}), |\lambda| \to \infty, \tag{2.6}$$

$$Z'(x;\lambda^2;Q) = -(\sqrt{\lambda^2}\sin\sqrt{\lambda^2}x)I_n + O(e^{|Im\sqrt{\lambda^2}|x}), |\lambda| \to \infty. \tag{2.7}$$

It is easy to see that λ_* is an eigenvalue of equation (2.1) if and only if

$$\det(AY'(\pi;\lambda_*^2;Q) + \lambda_* Y(\pi;\lambda_*^2;Q)) = 0.$$

Denote

$$\omega_A(\lambda;Q) = \det(AY'(\pi;\lambda^2;Q) + \lambda Y(\pi;\lambda^2;Q)).$$

Then $\omega_A(\lambda;Q)$ is an entire function of order one with respect to λ, which shall be called the *characteristic function* of equation (2.1). Furthermore, by (2.4) and (2.5), $\omega_A(\lambda;Q)$ has the following asymptotic formula for the case $n = 2$:

$$\begin{aligned} \omega_A(\lambda;Q) = (\cos\sqrt{\lambda^2}\pi)^2 \det A + \frac{\lambda}{\sqrt{\lambda^2}}(\cos\sqrt{\lambda^2}\pi\sin\sqrt{\lambda^2}\pi)\mathrm{trace}A \\ + \sin^2\sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}), |\lambda| \to \infty. \end{aligned} \tag{2.8}$$

The *geometric multiplicity* and the *algebraic multiplicity* of the eigenvalues are important for a vectorial Sturm-Liouville equation. For the two-dimensional Sturm-Liouville equation, Shen proved the following results in [Sh].

Consider the eigenvalue problem

$$\begin{cases} \vec{y}''(x) + [\lambda I_2 - Q(x)]\vec{y}(x) = \vec{0}, \\ \vec{y}(0) = \vec{0}, \vec{y}(\pi) = \vec{0}, \end{cases} \tag{2.9}$$

where $Q(x)$ is a two-by-two real symmetric matrix-valued smooth function defined in $[0,\pi]$, $\vec{y}(x)$ is an $\mathbb{R}^2$-function defined in the interval $[0,\pi]$, and $\vec{0}$ is the zero vector in $\mathbb{R}^2$. Denote $D_Q(\lambda)$=det $Y(\pi;\lambda;Q)$.

Theorem 2.2. *λ_* is a simple eigenvalue of equation* (2.9) *if and only if λ_* is a simple zero of $D_Q(\lambda)$.*

Theorem 2.3. *λ_* is an eigenvalue of geometric multiplicity two of equation* (2.9) *if and only if λ_* is a double zero of $D_Q(\lambda)$.*

Let λ_n be an eigenvalue of equation (2.9) and m_n be the *geometric multiplicity* of λ_n, $n \in \mathbb{N}$. We call the set $\{(\lambda_n, m_n)|n \in \mathbb{N}\}$ the *spectral set* of equation (2.9), and denote it by $\sigma_D(Q)$. Denote $K_Q(\lambda) = \widehat{Y}(\pi;\lambda;Q)Z(\pi;\lambda;Q)$, where $\widehat{Y}(\pi;\lambda;Q)$ is the adjoint matrix of $Y(\pi;\lambda;Q)$. Then Shen proved that $K_Q(\lambda)$ is real symmetric for $\lambda \in \mathbb{R}$ and

Theorem 2.4. *Suppose $\sigma_D(Q_1)$=$\sigma_D(Q_2)$ and $K_{Q_1}(\lambda)$=$K_{Q_2}(\lambda)$. Then $Q_1(x) \equiv Q_2(x)$ in $[0,\pi]$.*

3. The vectorial Hochstadt problem

The vectorial Hochstadt problem (2.1) is different from its scalar case studied by Hochstadt in [H]. One of the major differences is that if we do not require some extra assumption on the potential function $Q(x)$, then the equation (2.1) may have nonreal eigenvalues (see the example in the Introduction). To avoid the appearance of nonreal eigenvalues, we study the eigenvalue problem (2.1) under the assumption that $Q(x)$ is nonnegative definite for all $x \in [0,\pi]$ in this paper. Let λ_* be an eigenvalue of equation (2.1), and let $\vec{y}_*(x)$ be a corresponding eigenfunction. Then $\vec{y}_*(x) = Y(x;\lambda_*^2;Q)\vec{y}_*{}'(0)$, and

$$[AY'(\pi;\lambda_*^2;Q) + \lambda_* Y(\pi;\lambda_*^2;Q)]\vec{y}_*{}'(0) = \vec{0}.$$

Thus $\vec{y}_*{}'(0)$ belongs to the null space of $AY'(\pi;\lambda_*^2;Q) + \lambda_* Y(\pi;\lambda_*^2;Q)$. In other words, the number of linearly independent eigenfunctions corresponding to λ_*, i.e., the *geometric multiplicity* of λ_*, is determined by the nullity (i.e., the dimension of the null space) of $AY'(\pi;\lambda_*^2;Q) + \lambda_* Y(\pi;\lambda_*^2;Q)$. Thus we find that the *geometric multiplicity* of λ_* is less than or equal to n.

Theorem 3.1. *All eigenvalues of equation* (2.1) *are real.*

Proof. Suppose λ is an eigenvalue of equation (2.1) and $\vec{y}(x)$ is a corresponding eigenfunction. Then

$$\langle \vec{y}\,''(x), \vec{y}(x)\rangle + \langle [\lambda^2 - Q(x)]\vec{y}(x), \vec{y}(x)\rangle = 0. \tag{3.1}$$

Integrating both sides of (3.1) from 0 to π and using integration by parts, we obtain

$$[\langle \vec{y}\,'(\pi), \vec{y}(\pi)\rangle - \langle \vec{y}\,'(0), \vec{y}(0)\rangle] - \int_0^\pi \langle \vec{y}\,'(x), \vec{y}\,'(x)\rangle dx + \lambda^2 \int_0^\pi \langle \vec{y}(x), \vec{y}(x)\rangle dx = \int_0^\pi \langle Q(x)\vec{y}(x), \vec{y}(x)\rangle dx.$$

Since

$$\vec{y}(0) = \vec{0}, A\vec{y}\,'(\pi) + \lambda\vec{y}(\pi) = \vec{0},$$

and A is nonsingular, we have

$$
\begin{aligned}
(3.2)\qquad -\lambda\langle A^{-1}\vec{y}(\pi),\vec{y}(\pi)\rangle+\lambda^2\int_0^\pi\langle\vec{y}(x),\vec{y}(x)\rangle dx \\
=\int_0^\pi\langle\vec{y}\,'(x),\vec{y}\,'(x)\rangle dx+\int_0^\pi\langle Q(x)\vec{y}(x),\vec{y}(x)\rangle dx.
\end{aligned}
$$

Since $Q(x)$ is nonnegative definite in $[0,\pi]$,

$$\int_0^\pi\langle\vec{y}\,'(x),\vec{y}\,'(x)\rangle dx+\int_0^\pi\langle Q(x)\vec{y}(x),\vec{y}(x)\rangle dx\geq 0.$$

Note that (3.2) is a quardratic equation of λ with discriminant

$$
\begin{aligned}
&(\langle A^{-1}\vec{y}(\pi),\vec{y}(\pi)\rangle)^2\\
&\quad+4(\int_0^\pi\langle\vec{y}(x),\vec{y}(x)\rangle dx)(\int_0^\pi\langle\vec{y}\,'(x),\vec{y}\,'(x)\rangle dx+\int_0^\pi\langle Q(x)\vec{y}(x),\vec{y}(x)\rangle dx)\geq 0.
\end{aligned}
$$

Thus we find that λ is real. □

Lemma 3.2. 0 *is not an eigenvalue of equation* (2.1).

Proof. Suppose 0 is an eigenvalue of equation (2.1) and $\vec{y}(x)$ is a corresponding eigenfunction. Then

$$
(3.3)\qquad \begin{cases}\vec{y}\,''(x)-Q(x)\vec{y}(x)=\vec{0},\\ \vec{y}(0)=\vec{0},\vec{y}\,'(\pi)=\vec{0}.\end{cases}
$$

Dotting equation (3.3) with $\vec{y}(x)$ and integrating it from 0 to π, we have

$$(3.4)\qquad \int_0^\pi\langle\vec{y}\,''(x),\vec{y}(x)\rangle dx=\int_0^\pi\langle Q(x)\vec{y}(x),\vec{y}(x)\rangle dx.$$

Using integration by parts, $\vec{y}(0)=\vec{0}$, and $\vec{y}\,'(\pi)=\vec{0}$, (3.4) implies that

$$(3.5)\qquad \int_0^\pi\langle\vec{y}\,'(x),\vec{y}\,'(x)\rangle dx+\int_0^\pi\langle Q(x)\vec{y}(x),\vec{y}(x)\rangle dx=0.$$

Since $Q(x)$ is nonnegative definite for all x in $[0,\pi]$, (3.5) implies that $\vec{y}(x)=\vec{0}$ for all $x\in[0,\pi]$, which is absurd. □

4. Two-dimensional vectorial Hochstadt problem

In this section, we study the relation between the *geometric multiplicity* and the *algebraic multiplicity* of an eigenvalue of equation (2.1) in the two-dimensional case. From now on, we assume that $Q(x)$ is a two-by-two nonnegative definite matrix-valued smooth function defined in $[0,\pi]$. In section 2, we have seen that the *characteristic function* $\omega_A(\lambda;Q)$ of equation (2.1) is an entire function of order one with respect to λ and is of the form $\det(AY'(\pi;\lambda^2;Q)+\lambda Y(\pi;\lambda^2;Q))$. For a two-by-two matrix M, we denote its adjoint matrix by $\widehat{M}$, which has the property that

$$M\widehat{M}=\widehat{M}M=(\det M)I_2,$$

and we denote the conjugate transpose of M by M^*.

Lemma 4.1. *Suppose λ_* is an eigenvalue of equation* (2.1) *of geometric multiplicity two. Then λ_* is a double zero of $\omega_A(\lambda;Q)$.*

Proof. Denote

$$C_Q(\lambda) = AY'(\pi;\lambda^2;Q) + \lambda Y(\pi;\lambda^2;Q). \tag{4.1}$$

Then (4.1) implies that

$$\omega_A(\lambda;Q) = \det C_Q(\lambda). \tag{4.2}$$

From (4.2), we find

$$\omega_A'(\lambda;Q)I_2 = C_Q'(\lambda)\widehat{C}_Q(\lambda) + C_Q(\lambda)\widehat{C}_Q'(\lambda). \tag{4.3}$$

Now assume that λ_* is an eigenvalue of equation (2.1) of *geometric multiplicity* two. Then

$$C_Q(\lambda_*) = \widehat{C}_Q(\lambda_*) = 0. \tag{4.4}$$

By (4.3) and (4.4), we find that

$$\omega_A'(\lambda_*;Q) = 0.$$

In order to show λ_* is a double zero of $\omega_A(\lambda;Q)$, we need to show $\omega_A''(\lambda_*;Q) \neq 0$. Differentiating (4.3) with respect to λ, we obtain

$$\omega_A''(\lambda;Q)I_2 = C_Q''(\lambda)\widehat{C}_Q(\lambda) + 2C_Q'(\lambda)\widehat{C}_Q'(\lambda) + C_Q(\lambda)\widehat{C}_Q''(\lambda). \tag{4.5}$$

Taking $\lambda = \lambda_*$ in (4.5), we find

$$\omega_A''(\lambda_*;Q)I_2 = 2C_Q'(\lambda_*)\widehat{C}_Q'(\lambda_*) = 2\det(C_Q'(\lambda_*))I_2.$$

Thus if λ_* is an eigenvalue of equation (2.1) of *geometric multiplicity* two, then

$$\omega_A''(\lambda_*;Q) = 2\det(C_Q'(\lambda_*)). \tag{4.6}$$

If we can show $\det(C_Q'(\lambda_*)) \neq 0$, then $\omega_A''(\lambda_*;Q) \neq 0$. Note that

$$C_Q'(\lambda_*) = 2\lambda_* AY_\nu'(\pi;\lambda_*^2;Q) + 2\lambda_*^2 Y_\nu(\pi;\lambda_*^2;Q) + Y(\pi;\lambda_*^2;Q). \tag{4.7}$$

Since

$$\begin{cases} Y''(x;\lambda^2;Q) + [\lambda^2 I_2 - Q(x)]Y(x;\lambda^2;Q) = 0, \\ Y(0;\lambda^2;Q) = 0, Y'(0;\lambda^2;Q) = I_2, \end{cases} \tag{4.8}$$

differentiating (4.8) with respect to λ, we obtain

$$\begin{cases} 2\lambda Y_\nu''(x;\lambda^2;Q) + 2\lambda[\lambda^2 I_2 - Q(x)]Y_\nu(x;\lambda^2;Q) + 2\lambda Y(x;\lambda^2;Q) = 0, \\ 2\lambda Y_\nu(0;\lambda^2;Q) = 0, 2\lambda Y_\nu'(0;\lambda^2;Q) = 0. \end{cases} \tag{4.9}$$

As Lemma 3.2 implies $\lambda_* \neq 0$, (4.9) can be expressed as

$$\begin{cases} Y_\nu''(x;\lambda_*^2;Q) + [\lambda_*^2 I_2 - Q(x)]Y_\nu(x;\lambda_*^2;Q) = -Y(x;\lambda_*^2;Q), \\ Y_\nu(0;\lambda_*^2;Q) = 0, Y_\nu'(0;\lambda_*^2;Q) = 0. \end{cases} \tag{4.10}$$

By (4.8), (4.10), and $\lambda_* \in \mathbb{R}$, we find

$$\begin{aligned} Y^{*\prime\prime}(x;\lambda_*^2;Q)Y_\nu(x;\lambda_*^2;Q) - Y^*(x;\lambda_*^2;Q)Y_\nu''(x;\lambda_*^2;Q) \\ = Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q). \end{aligned} \tag{4.11}$$

Integrating (4.11) from 0 to π with respect to x and using the initial conditions of (4.8) and (4.10), we have

$$\tag{4.12} Y^{*\prime}(\pi;\lambda_*^2;Q)Y_\nu(\pi;\lambda_*^2;Q) - Y^*(\pi;\lambda_*^2;Q)Y_\nu'(\pi;\lambda_*^2;Q) = \int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx.$$

Because

$$C_Q^*(\lambda_*) = Y^{*\prime}(\pi;\lambda_*^2;Q)A + \lambda_* Y^*(\pi;\lambda_*^2;Q) = 0,$$

we have

$$\tag{4.13} Y^{*\prime}(\pi;\lambda_*^2;Q) = -\lambda_* Y^*(\pi;\lambda_*^2;Q)A^{-1}.$$

Using (4.13), we rewrite (4.12) as

$$\tag{4.14} -Y^*(\pi;\lambda_*^2;Q)A^{-1}[\lambda_* Y_\nu(\pi;\lambda_*^2;Q) + AY_\nu'(\pi;\lambda_*^2;Q)] = \int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx.$$

Claim that $Y(\pi;\lambda_*^2;Q)$ is nonsingular. Otherwise there is $\vec{v}\in\mathbb{R}^2$, $\vec{v}\neq\vec{0}$, such that

$$Y(\pi;\lambda_*^2;Q)\vec{v} = \vec{0}.$$

Then by (4.4), we also have

$$Y'(\pi;\lambda_*^2;Q)\vec{v} = \vec{0}.$$

But condition $Y(\pi;\lambda_*^2;Q)\vec{v} = Y'(\pi;\lambda_*^2;Q)\vec{v} = \vec{0}$ implies that

$$Y(x;\lambda_*^2;Q)\vec{v} = \vec{0}$$

for all $x\in[0,\pi]$, which is absurd. Thus $Y(\pi;\lambda_*^2;Q)$ is nonsingular. Then (4.7) and (4.14) imply

(4.15)

$$\begin{aligned} C_Q'(\lambda_*) &= -2\lambda_* AY^{*-1}(\pi;\lambda_*^2;Q)\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx + Y(\pi;\lambda_*^2;Q) \\ &= AY^{*-1}(\pi;\lambda_*^2;Q)[Y^*(\pi;\lambda_*^2;Q)A^{-1}Y(\pi;\lambda_*^2;Q) \\ &\qquad - 2\lambda_*\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx]. \end{aligned}$$

By (4.8),

$$Y^*(x;\lambda_*^2;Q)Y''(x;\lambda_*^2;Q) + \lambda_*^2 Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q) - Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q) = 0.$$

Integrating the above equation from 0 to π, and using integration by parts, we obtain

$$\begin{aligned} &Y^*(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)\big|_0^\pi - \int_0^\pi Y^{*\prime}(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)dx \\ &+ \lambda_*^2\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx - \int_0^\pi Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q)dx = 0. \end{aligned}$$

This equation implies

$$(4.16)\quad Y^*(\pi;\lambda_*^2;Q)Y'(\pi;\lambda_*^2;Q)+\lambda_*^2\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx = \int_0^\pi Y^{*\prime}(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)dx+\int_0^\pi Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q)dx.$$

Since $C_Q(\lambda_*)=0$,

$$Y'(\pi;\lambda_*^2;Q)=-\lambda_*A^{-1}Y(\pi;\lambda_*^2;Q).$$

Equation (4.16) can be rewritten as

$$(4.17)\quad Y^*(\pi;\lambda_*^2;Q)A^{-1}Y(\pi;\lambda_*^2;Q)-\lambda_*\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx = -\frac{1}{\lambda_*}[\int_0^\pi Y^{*\prime}(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)dx+\int_0^\pi Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q)dx].$$

By (4.17), we find

$$(4.18)\quad Y^*(\pi;\lambda_*^2;Q)A^{-1}Y(\pi;\lambda_*^2;Q)-2\lambda_*\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx = -\frac{1}{\lambda_*}[\int_0^\pi Y^{*\prime}(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)dx+\int_0^\pi Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q)dx + \lambda_*^2\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx].$$

Note that

$$\int_0^\pi Y^{*\prime}(x;\lambda_*^2;Q)Y'(x;\lambda_*^2;Q)dx+\int_0^\pi Y^*(x;\lambda_*^2;Q)Q(x)Y(x;\lambda_*^2;Q)dx$$

is positive definite. Hence

$$(4.19)\quad \det[Y^*(\pi;\lambda_*^2;Q)A^{-1}Y(\pi;\lambda_*^2;Q)-2\lambda_*\int_0^\pi Y^*(x;\lambda_*^2;Q)Y(x;\lambda_*^2;Q)dx]\neq 0.$$

By (4.15) and (4.19), we find that $\det(C_Q'(\lambda_*))\neq 0$ since A and $Y^*(\pi;\lambda_*^2;Q)$ are nonsingular. Then (4.6) implies that $\omega_A''(\lambda_*;Q)\neq 0$. Hence λ_* is a zero of $\omega_A(\lambda;Q)$ of multiplicity two if it is an eigenvalue of equation (2.1) of *geometric multiplicity* two. □

The following result is an immediate consequence of Lemma 4.1.

Lemma 4.2. *If λ_* is a simple zero of $\omega_A(\lambda;Q)$, then λ_* is a simple eigenvalue of equation* (2.1)*.*

Proof. Suppose λ_* is of *geometric multiplicity* two. Then by Lemma 4.1, λ_* is a double zero of $\omega_A(\lambda;Q)$. This contradicts the assumption on λ_*. Thus λ_* is a simple eigenvalue of equation (2.1). □

Now we want to consider the converse of Lemma 4.1.

Lemma 4.3. *If λ_* is a simple eigenvalue of equation* (2.1)*, then λ_* is a simple zero of $\omega_A(\lambda;Q)$.*

Proof. Using the same notation as in Lemma 4.1, we denote

$$C_Q(\lambda) = AY'(\pi;\lambda^2;Q) + \lambda Y(\pi;\lambda^2;Q).$$

Let $R(x;\lambda) = Y(x;\lambda^2;Q)\widehat{C}_Q(\lambda)$. Then $R(x;\lambda_*)$ satisfies the equation

$$\begin{cases} R''(x;\lambda_*) + [\lambda_*^2 I_2 - Q(x)]R(x;\lambda_*) = 0, \\ R(0;\lambda_*) = 0, R'(0;\lambda_*) = \widehat{C}_Q(\lambda_*), \\ AR'(\pi;\lambda_*) + \lambda_* R(\pi;\lambda_*) = C_Q(\lambda_*)\widehat{C}_Q(\lambda_*) = 0. \end{cases} \tag{4.20}$$

We also have

$$\begin{cases} R''_\lambda(x;\lambda_*) + [\lambda_*^2 I_2 - Q(x)]R_\lambda(x;\lambda_*) = -2\lambda_* R(x;\lambda_*), \\ R_\lambda(0;\lambda_*) = 0, R'_\lambda(0;\lambda_*) = \widehat{C}'_Q(\lambda_*), \\ AR'_\lambda(\pi;\lambda_*) + \lambda_* R_\lambda(\pi;\lambda_*) \\ \qquad = C'_Q(\lambda_*)\widehat{C}_Q(\lambda_*) + C_Q(\lambda_*)\widehat{C}'_Q(\lambda_*) - R(\pi;\lambda_*). \end{cases} \tag{4.21}$$

Since Theorem 3.1 implies that $\lambda_* \in \mathbb{R}$, we can transform (4.21) into

$$\begin{cases} {R_\lambda^*}''(x;\lambda_*) + R_\lambda^*(x;\lambda_*)[\lambda_*^2 I_2 - Q(x)] = -2\lambda_* R^*(x;\lambda_*), \\ R_\lambda^*(0;\lambda_*) = 0, {R_\lambda^*}'(0;\lambda_*) = \widehat{C}_Q^{*\prime}(\lambda_*), \\ {R_\lambda^*}'(\pi;\lambda_*)A + \lambda_* R_\lambda^*(\pi;\lambda_*) \\ \qquad = \widehat{C}_Q^*(\lambda_*){C_Q^*}'(\lambda_*) + \widehat{C}_Q^{*\prime}(\lambda_*)C_Q^*(\lambda_*) - R^*(\pi;\lambda_*). \end{cases} \tag{4.22}$$

Then by (4.20) and (4.22), we obtain

$$R_\lambda^*(x;\lambda_*)R''(x;\lambda_*) - {R_\lambda^*}''(x;\lambda_*)R(x;\lambda_*) = 2\lambda_* R^*(x;\lambda_*)R(x;\lambda_*). \tag{4.23}$$

Integrating both sides of (4.23) from 0 to π, and using $R(0;\lambda_*) = R_\lambda^*(0;\lambda_*) = 0$, we find

$$R_\lambda^*(\pi;\lambda_*)R'(\pi;\lambda_*) - {R_\lambda^*}'(\pi;\lambda_*)R(\pi;\lambda_*) = 2\lambda_* \int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx. \tag{4.24}$$

Applying $AR'(\pi;\lambda_*) + \lambda_* R(\pi;\lambda_*) = 0$ in (4.20), (4.24) can be rewritten as

$$-[\lambda_* R_\lambda^*(\pi;\lambda_*)A^{-1} + {R_\lambda^*}'(\pi;\lambda_*)]R(\pi;\lambda_*) = 2\lambda_* \int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx. \tag{4.25}$$

Suppose λ_* is a zero of $\omega_A(\lambda;Q)$ of multiplicity ≥ 2. Then

$$\left.\frac{d(\det(C_Q(\lambda)))}{d\lambda}\right|_{\lambda=\lambda_*} = 0.$$

This implies

$$C'_Q(\lambda_*)\widehat{C}_Q(\lambda_*) + C_Q(\lambda_*)\widehat{C}'_Q(\lambda_*) = 0.$$

Then by (4.22), we find that

$${R_\lambda^*}'(\pi;\lambda_*)A + \lambda_* R_\lambda^*(\pi;\lambda_*) = -R^*(\pi;\lambda_*). \tag{4.26}$$

By (4.25) and (4.26), we find

$$R^*(\pi;\lambda_*)A^{-1}R(\pi;\lambda_*) = 2\lambda_* \int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx. \tag{4.27}$$

On the other hand, (4.20) implies

$$R^*(x;\lambda_*)R''(x;\lambda_*) + R^*(x;\lambda_*)[\lambda_*^2 I_2 - Q(x)]R(x;\lambda_*) = 0.$$

Integrating the above equation from 0 to π, and using the boundary conditions of $R(x;\lambda_*)$, we find

$$(4.28)\quad -\lambda_* R^*(\pi;\lambda_*)A^{-1}R(\pi;\lambda_*) + \lambda_*^2\int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx = \int_0^\pi R'^*(x;\lambda_*)R'(x;\lambda_*)dx + \int_0^\pi R^*(x;\lambda_*)Q(x)R(x;\lambda_*)dx.$$

Since $\lambda_* \neq 0$, (4.28) implies that

$$(4.29)\quad R^*(\pi;\lambda_*)A^{-1}R(\pi;\lambda_*) = \lambda_*\int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx - \frac{1}{\lambda_*}[\int_0^\pi R'^*(x;\lambda_*)R'(x;\lambda_*)dx + \int_0^\pi R^*(x;\lambda_*)Q(x)R(x;\lambda_*)dx].$$

Comparing (4.29) with (4.27), we obtain

$$(4.30)\quad \lambda_*^2\int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx = -[\int_0^\pi R'^*(x;\lambda_*)R'(x;\lambda_*)dx + \int_0^\pi R^*(x;\lambda_*)Q(x)R(x;\lambda_*)dx].$$

Since $\int_0^\pi R^*(x;\lambda_*)R(x;\lambda_*)dx$, $\int_0^\pi R^*(x;\lambda_*)Q(x)R(x;\lambda_*)dx$, $\int_0^\pi R'^*(x;\lambda_*)R'(x;\lambda_*)dx$ are nonnegative definite, the right-hand side of (4.30) is nonpositive definite, while the left-hand side of (4.30) is nonnegative definite, which implies $R(x;\lambda_*)$ is the zero matrix for all x in $[0,\pi]$. Since $R'(0;\lambda_*) = \widehat{C}_Q(\lambda_*)$, we find $\widehat{C}_Q(\lambda_*)$ is the zero matrix, which is absurd. Thus λ_* is a simple zero of $\omega_A(\lambda;Q)$. □

Applying Lemma 4.3, we have the following result.

Lemma 4.4. *Suppose λ_* is a double zero of $\omega_A(\lambda;Q)$. Then λ_* is also an eigenvalue of equation* (2.1) *of geometric multiplicity two.*

Proof. Suppose λ_* is a simple eigenvalue of equation (2.1). Then by Lemma 4.3, λ_* is a simple zero of $\omega_A(\lambda : Q)$, which is absurd. Thus λ_* is of *geometric multiplicity* two. □

Combining Lemmas 4.1–4.4, we obtain the following conclusion.

Theorem 4.5. *Let λ_* be an eigenvalue of equation* (2.1)*. Then the algebraic multiplicity of λ_* is equal to the geometric multiplicity of λ_*.*

We know that $\omega_A(\lambda;Q)$ is an entire function of order one with respect to λ. Following the theory of Hadamard's factorization, $\omega_A(\lambda;Q)$ can be expressed as an infinite product as

$$\omega_A(\lambda;Q) = ce^{a\lambda}\prod_{n=-\infty}^{n=\infty}(1-\frac{\lambda}{\lambda_n})^{m_n}e^{m_n[\frac{\lambda}{\lambda_n}+\frac{1}{2}(\frac{\lambda}{\lambda_n})^2+\cdots+\frac{1}{p}(\frac{\lambda}{\lambda_n})^p]},$$

where λ_n is the eigenvalue of equation (2.1) with respect to A and $Q(x)$, m_n is the *geometric multiplicity* of λ_n, $n \in \mathbb{N}$, p is the *genus* of $\omega_A(\lambda;Q)$, c and a are constants. Note that the *genus* p is the smallest integer such that $\sum_{n=0}^{\infty}(\frac{1}{|\lambda_n|})^{p+1} < \infty$ (see [A]). Since for the order ρ of $\omega_A(\lambda;Q)$, $p \leq \rho \leq p+1$ (see [A]), and $\omega_A(\lambda;Q)$ is an entire function of order one, we find that the *genus* of $\omega_A(\lambda; Q)$ is 0 or 1. Thus $\omega_A(\lambda; Q)$ can be rewritten by

$$\omega_A(\lambda; Q) = ce^{a\lambda} \prod_{n=-\infty}^{n=\infty} (1 - \frac{\lambda}{\lambda_n})^{m_n} e^{p(\frac{m_n \lambda}{\lambda_n})}.$$

As stated in section 2, we call the set $\{(\lambda_n, m_n)|n \in \mathbb{N}\}$ the *spectral set* of equation (2.1), and we denote it by $\sigma_A(Q)$.

Theorem 4.6. *The characteristic function $\omega_A(\lambda; Q)$ of equation* (2.1) *is uniquely determined by the spectral set of the equation.*

Proof. Suppose there are two functions $Q_1(x)$ and $Q_2(x)$ with $\sigma_A(Q_1) = \sigma_A(Q_2)$. Then we have

$$\omega_A(\lambda; Q_1) = c_1 e^{a_1 \lambda} \prod_{n=-\infty}^{n=\infty} (1 - \frac{\lambda}{\lambda_n})^{m_n} e^{p_1(\frac{m_n \lambda}{\lambda_n})}, \tag{4.31}$$

$$\omega_A(\lambda; Q_2) = c_2 e^{a_2 \lambda} \prod_{n=-\infty}^{n=\infty} (1 - \frac{\lambda}{\lambda_n})^{m_n} e^{p_2(\frac{m_n \lambda}{\lambda_n})}. \tag{4.32}$$

Since $\sigma_A(Q_1) = \sigma_A(Q_2)$, by the definition of *genus*, we find $p_1 = p_2$. Thus (4.31) and (4.32) imply that for all $\lambda \in \mathbb{C}$,

$$\frac{\omega_A(\lambda; Q_1)}{\omega_A(\lambda; Q_2)} = \frac{c_1}{c_2} e^{(a_1 - a_2)\lambda}. \tag{4.33}$$

By the asymptotic formula (2.8), as $|\lambda| \to \infty$,

$$\begin{aligned}
\omega_A(\lambda; Q_1) &= (\cos \sqrt{\lambda^2}\pi)^2 \det A + \frac{\lambda}{\sqrt{\lambda^2}} (\cos \sqrt{\lambda^2}\pi \sin \sqrt{\lambda^2}\pi) \mathrm{trace} A \\
&\quad + \sin^2 \sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}), \\
\omega_A(\lambda; Q_2) &= (\cos \sqrt{\lambda^2}\pi)^2 \det A + \frac{\lambda}{\sqrt{\lambda^2}} (\cos \sqrt{\lambda^2}\pi \sin \sqrt{\lambda^2}\pi) \mathrm{trace} A \\
&\quad + \sin^2 \sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}).
\end{aligned}$$

This implies, for $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\omega_A(n; Q_1)}{\omega_A(n; Q_2)} = 1.$$

Hence $c_1 = c_2$ and $a_1 = a_2$, i.e., $\omega_A(\lambda; Q_1) = \omega_A(\lambda; Q_2)$. □

Let $y_*(x)$ be an eigenfunction of equation (2.1) corresponding to the eigenvalue λ_*. Denote

$$\widetilde{y_*}(x) = y_*(\pi - x), \widetilde{Q}(x) = Q(\pi - x). \tag{4.34}$$

Then we have

$$\begin{cases} \widetilde{y_*}''(x) + [\lambda_*^2 I_2 - \widetilde{Q}(x)]\widetilde{y_*}(x) = 0, \\ A\widetilde{y_*}'(0) - \lambda_* \widetilde{y_*}(0) = 0, \widetilde{y_*}(\pi) = 0. \end{cases} \tag{4.35}$$

On the other hand, consider the eigenvalue problem

$$\begin{cases} \vec{u}''(x) + [\lambda^2 I_2 - \widetilde{Q}(x)]\vec{u}(x) = 0, \\ A\vec{u}'(0) - \lambda \vec{u}(0) = 0, \vec{u}(\pi) = 0. \end{cases} \tag{4.36}$$

Let $\vec{u}_0(x;\lambda^2)$ be a solution of the equation

$$\begin{cases} \vec{w}''(x) + [\lambda^2 I_2 - \widetilde{Q}(x)]\vec{w}(x) = 0, \\ A\vec{w}'(0) - \lambda \vec{w}(0) = 0. \end{cases} \tag{4.37}$$

Then

$$\vec{u}_0(x;\lambda^2) = Y(x;\lambda^2;\widetilde{Q})\vec{v}_1 + Z(x;\lambda^2;\widetilde{Q})\vec{v}_2.$$

Following the boundary condition of $u_0(x;\lambda^2)$ at $x = 0$ in (4.37), we have

$$A\vec{v}_1 - \lambda \vec{v}_2 = 0. \tag{4.38}$$

Equation (4.38) implies that

$$\vec{u}_0(x;\lambda^2) = [\lambda Y(x;\lambda^2;\widetilde{Q})A^{-1} + Z(x;\lambda^2;\widetilde{Q})]\vec{v}_2.$$

Thus the eigenvalues and their multiplicity of equation (4.36) are determined by the zeros of the following entire function:

$$\widetilde{\omega}_A(\lambda;\widetilde{Q}) = \det[\lambda Y(\pi;\lambda^2;\widetilde{Q})A^{-1} + Z(\pi;\lambda^2;\widetilde{Q})].$$

Since A is nonsingular, denoting

$$\widehat{\omega}_A(\lambda;\widetilde{Q}) = \det[\lambda Y(\pi;\lambda^2;\widetilde{Q}) + Z(\pi;\lambda^2;\widetilde{Q})A],$$

we shall call $\widehat{\omega}_A(\lambda;\widetilde{Q})$ the *characteristic function* of equation (4.36). Denote the *spectral set* of equation (4.36) by $\widehat{\sigma}_A(\widetilde{Q})$, then by the previous observation, we find

$$\sigma_A(Q) = \widehat{\sigma}_A(\widetilde{Q}).$$

For the relation between $\omega_A(\lambda;Q)$ and $\widehat{\omega}_A(\lambda;\widetilde{Q})$, we have the following lemma.

Lemma 4.7. $\omega_A(\lambda;Q) = \widehat{\omega}_A(\lambda;\widetilde{Q})$.

Proof. Since $\omega_A(\lambda;Q)$ and $\widehat{\omega}_A(\lambda;\widetilde{Q})$ are entire functions of order one with respect to λ, and $\sigma_A(Q) = \widehat{\sigma}_A(\widetilde{Q})$, Hadamard's factorization formulae for $\omega_A(\lambda;Q)$ and $\widehat{\omega}_A(\lambda;\widetilde{Q})$ imply that for all $\lambda \in \mathbb{C}$,

$$\frac{\omega_A(\lambda;Q)}{\widehat{\omega}_A(\lambda;\widetilde{Q})} = \frac{ce^{a\lambda}}{\widehat{c}e^{\widehat{a}\lambda}}, \tag{4.39}$$

where c, $\widehat{c}$, a, $\widehat{a}$ are constants. On the other hand, by (2.4) and (2.6),

$$\begin{aligned} \widehat{\omega}_A(\lambda;\widetilde{Q}) = (\cos\sqrt{\lambda^2}\pi)^2 \det A + \frac{\lambda}{\sqrt{\lambda^2}}(\cos\sqrt{\lambda^2}\pi \sin\sqrt{\lambda^2}\pi)\mathrm{trace}A \\ + \sin^2\sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}), |\lambda| \to \infty. \end{aligned} \tag{4.40}$$

Using (4.40) and (2.8), for $n \in \mathbb{N}$, we find that

$$\lim_{n\to\infty} \frac{\omega_A(n;Q)}{\widehat{\omega}_A(n;\widetilde{Q})} = \lim_{n\to\infty} \frac{ce^{an}}{\widehat{c}e^{\widehat{a}n}} = 1.$$

This leads to $c = \widehat{c}$ and $a = \widehat{a}$. Hence

$$\omega_A(\lambda;Q) = \widehat{\omega}_A(\lambda;\widetilde{Q}). \qquad \square$$

Next we study the problem about how many spectra of equation (2.1) can determine $Q(x)$ uniquely. Let A_1, A_2, and A_3 be three two-by-two nonsingular real symmetric matrices, writing

$$A_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j & \gamma_j \end{pmatrix}, j = 1, 2, 3.$$

We say that A_1, A_2, and A_3 are linearly independent if the column vectors $(\alpha_j, \beta_j, \gamma_j)$, $j = 1, 2, 3$, are linearly independent over $\mathbb{R}$. Recall that in [Sh], Shen introduced a function $K_Q(\lambda) = \widehat{Y}(\pi;\lambda;Q)Z(\pi;\lambda;Q)$ and showed that $K_Q(\lambda)$ and $\sigma_D(Q)$ determine $Q(x)$ uniquely. To obtain our goal, we only need find how many boundary conditions can determine $K_Q(\lambda)$ and $\sigma_D(Q)$ uniquely.

Consider the following example: Let $Q_0(x)$ be a two-by-two nonnegative definite matrix-valued function defined in $[0, \pi]$ and

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, A_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

where $a, b \neq 0$. Consider the eigenvalue problem

$$\text{(4.41)} \qquad \begin{cases} \vec{y}''(x) + [\lambda^2 I_2 - Q_0(x)]\vec{y}(x) = \vec{0}, \\ \vec{y}(0) = \vec{0}, A_j\vec{y}'(\pi) + \lambda\vec{y}(\pi) = \vec{0}, j = 1, 2, \end{cases}$$

where

$$Q_0(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}.$$

Denote

$$\check{Q}_0(x) = \begin{pmatrix} q(x) & r(x) \\ r(x) & p(x) \end{pmatrix}.$$

Suppose $\vec{y}_*(x)$ is an eigenfunction of (4.41) corresponding to λ_*. Denoting

$$\check{\vec{y}}_*(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{y}_*(x),$$

we find

$$\text{(4.42)} \qquad \begin{cases} \check{\vec{y}}_*{}''(x) + [\lambda_*^2 I_2 - \check{Q}_0(x)]\check{\vec{y}}_*(x) = \vec{0}, \\ \check{\vec{y}}_*(0) = \vec{0}, A_j\check{\vec{y}}_*{}'(\pi) + \lambda\check{\vec{y}}_*(\pi) = \vec{0}, j = 1, 2. \end{cases}$$

From (4.41) and (4.42), we obtain that $\sigma_{A_1}(Q_0) = \sigma_{A_1}(\check{Q}_0)$ and $\sigma_{A_2}(Q_0) = \sigma_{A_2}(\check{Q}_0)$. Thus two spectra of equation (2.1) corresponding to two linearly independent nonsingular real symmetric matrices A_1 and A_2 cannot determine the potential function $Q(x)$ uniquely. In order to determine the potential function $Q(x)$ uniquely, we need more than two spectra.

Theorem 4.8. *Let $Q_1(x)$ and $Q_2(x)$ be two two-by-two nonnegative definite matrix-valued smooth functions defined in $[0,\pi]$, and let A_1, A_2, and A_3 be two-by-two non-singular real symmetric matrices. Suppose A_1, A_2, and A_3 are linearly independent, and*

$$\widehat{\omega}_{A_j}(\lambda;Q_1)=\widehat{\omega}_{A_j}(\lambda;Q_2), j=1,2,3.$$

Then $\det(Y(\pi;\lambda^2;Q_1))=\det(Y(\pi;\lambda^2;Q_2))$, $K_{Q_1}(\lambda^2)=K_{Q_2}(\lambda^2)$, *and* $Q_1(x)=Q_2(x)$.

Proof. Denote

$$D_1(\lambda^2)=\det(Y(\pi;\lambda^2;Q_1))=\det(\widehat{Y}(\pi;\lambda^2;Q_1)),$$
$$D_2(\lambda^2)=\det(Y(\pi;\lambda^2;Q_2))=\det(\widehat{Y}(\pi;\lambda^2;Q_2)).$$

By the assumptions, we find

$$D_1(\lambda^2)D_2(\lambda^2)\widehat{\omega}_{A_j}(\lambda;Q_1)=D_1(\lambda^2)D_2(\lambda^2)\widehat{\omega}_{A_j}(\lambda;Q_2), j=1,2,3.$$

Since for $k=1,2$,

$$\begin{aligned}D_k(\lambda^2)\widehat{\omega}_{A_j}(\lambda;Q_k)&=\det(\widehat{Y}(\pi;\lambda^2;Q_k))\widehat{\omega}_{A_j}(\lambda;Q_k)\\&=\det[\lambda D_k(\lambda^2)I_2+K_{Q_k}(\lambda^2)A_j],\end{aligned}\tag{4.43}$$

from (4.43), we find for $j=1,2,3$,

$$D_2(\lambda^2)\det[\lambda D_1(\lambda^2)I_2+K_{Q_1}(\lambda^2)A_j]=D_1(\lambda^2)\det[\lambda D_2(\lambda^2)I_2+K_{Q_2}(\lambda^2)A_j].$$

Calculating the determinants, we obtain

$$\begin{aligned}&D_2(\lambda^2)[\lambda^2D_1^2(\lambda^2)+\det(K_{Q_1}(\lambda^2)A_j)+\lambda D_1(\lambda^2)\mathrm{trace}(K_{Q_1}(\lambda^2)A_j)]\\&\quad=D_1(\lambda^2)[\lambda^2D_2^2(\lambda^2)+\det(K_{Q_2}(\lambda^2)A_j)+\lambda D_2(\lambda^2)\mathrm{trace}(K_{Q_2}(\lambda^2)A_j)],\end{aligned}\tag{4.44}$$

where $j=1,2,3$. Comparing the odd part (resp., the even part) of both sides of equation (4.44) with respect to λ, we find

$$\begin{aligned}&\lambda^2D_1^2(\lambda^2)D_2(\lambda^2)+D_2(\lambda^2)\det(K_{Q_1}(\lambda^2)A_j)\\&\quad=\lambda^2D_1(\lambda^2)D_2^2(\lambda^2)+D_1(\lambda^2)\det(K_{Q_2}(\lambda^2)A_j),\end{aligned}\tag{4.45}$$

and

$$\lambda D_1(\lambda^2)D_2(\lambda^2)\mathrm{trace}(K_{Q_1}(\lambda^2)A_j)=\lambda D_1(\lambda^2)D_2(\lambda^2)\mathrm{trace}(K_{Q_2}(\lambda^2)A_j).\tag{4.46}$$

Denote

$$A_j=\begin{pmatrix}\alpha_j&\beta_j\\\beta_j&\gamma_j\end{pmatrix}, j=1,2,3.$$

Noting that for $\lambda\in\mathbb{R}$, $K_{Q_1}(\lambda^2)$ and $K_{Q_2}(\lambda^2)$ are real symmetric (see [Sh]), we write

$$K_{Q_1}(\lambda^2)=\begin{pmatrix}k_{11}^{(1)}(\lambda^2)&k_{12}^{(1)}(\lambda^2)\\k_{12}^{(1)}(\lambda^2)&k_{22}^{(1)}(\lambda^2)\end{pmatrix}, K_{Q_2}(\lambda^2)=\begin{pmatrix}k_{11}^{(2)}(\lambda^2)&k_{12}^{(2)}(\lambda^2)\\k_{12}^{(2)}(\lambda^2)&k_{22}^{(2)}(\lambda^2)\end{pmatrix}.$$

Since (4.46) implies that

$$\mathrm{trace}(K_{Q_1}(\lambda^2)A_j)=\mathrm{trace}(K_{Q_2}(\lambda^2)A_j),\tag{4.47}$$

from (4.47), we find for $\lambda\in\mathbb{R}$,

$$k_{11}^{(1)}(\lambda^2)\alpha_j+2k_{12}^{(1)}(\lambda^2)\beta_j+k_{22}^{(1)}(\lambda^2)\gamma_j=k_{11}^{(2)}(\lambda^2)\alpha_j+2k_{12}^{(2)}(\lambda^2)\beta_j+k_{22}^{(2)}(\lambda^2)\gamma_j.\tag{4.48}$$

Since A_1, A_2, and A_3 are linearly independent, the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$$

is invertible. Then by (4.48), we find for all $\lambda \in \mathbb{R}$,

$$k_{11}^{(1)}(\lambda^2) = k_{11}^{(2)}(\lambda^2), k_{12}^{(1)}(\lambda^2) = k_{12}^{(2)}(\lambda^2), k_{22}^{(1)}(\lambda^2) = k_{22}^{(2)}(\lambda^2). \tag{4.49}$$

Hence for all $\lambda \in \mathbb{R}$,

$$K_{Q_1}(\lambda^2) = K_{Q_2}(\lambda^2).$$

As $k_{11}^{(l)}, k_{12}^{(l)}, k_{22}^{(l)}$ are entire functions in λ^2, $l = 1, 2$, we conclude that $K_{Q_1}(\lambda^2) = K_{Q_2}(\lambda^2)$ for all $\lambda \in \mathbb{C}$.

Now denote

$$k(\lambda^2) = \det(K_{Q_1}(\lambda^2)) = \det(K_{Q_2}(\lambda^2)), a_j = \det A_j.$$

Then (4.45) can be expressed as

$$\lambda^2 D_1^2(\lambda^2) D_2(\lambda^2) + D_2(\lambda^2) \cdot a_j \cdot k(\lambda^2) = \lambda^2 D_1(\lambda^2) D_2^2(\lambda^2) + D_1(\lambda^2) \cdot a_j \cdot k(\lambda^2). \tag{4.50}$$

From (4.50), we find that

$$[\lambda^2 D_1(\lambda^2) D_2(\lambda^2) - a_j k(\lambda^2)][D_1(\lambda^2) - D_2(\lambda^2)] = 0. \tag{4.51}$$

Since $D_1(\lambda^2), D_2(\lambda^2), k(\lambda^2)$ are entire functions in λ^2, (4.51) tells us either

$$\lambda^2 D_1(\lambda^2) D_2(\lambda^2) - a_j k(\lambda^2) \equiv 0$$

or

$$D_1(\lambda^2) - D_2(\lambda^2) \equiv 0.$$

Suppose

$$\lambda^2 D_1(\lambda^2) D_2(\lambda^2) - a_j k(\lambda^2) \equiv 0. \tag{4.52}$$

Then as $k(\lambda^2) = D_1(\lambda^2) \det(Z(\pi; \lambda^2; Q_1))$, (4.52) implies that

$$\lambda^2 D_2(\lambda^2) - a_j \det(Z(\pi; \lambda^2; Q_1)) \equiv 0. \tag{4.53}$$

As (2.4) and (2.6) imply that

$$\lambda^2 D_2(\lambda^2) = \sin^2 \sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}) \tag{4.54}$$

and

$$\det(Z(\pi; \lambda^2; Q_1)) = \cos^2 \sqrt{\lambda^2}\pi + O(\frac{e^{2|Im\sqrt{\lambda^2}|\pi}}{|\lambda|}), \tag{4.55}$$

using (4.54) and (4.55), we find for $j = 1, 2, 3$, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} [n^2 D_2(n^2) - a_j \det(Z(\pi; n^2; Q_1))] = -a_j \neq 0,$$

which contradicts (4.53). Thus we find that $D_1(\lambda^2) - D_2(\lambda^2) \equiv 0$. Since $D_1(\lambda^2) = D_2(\lambda^2)$ and $K_{Q_1}(\lambda^2) = K_{Q_2}(\lambda^2)$, by Theorem 2.5, $Q_1(x) = Q_2(x)$. □

Applying Theorem 4.8, we obtain the following result.

Theorem 4.9. *Let $Q_1(x)$ and $Q_2(x)$ be two two-by-two nonnegative definite matrix-valued smooth functions defined in $[0, \pi]$, and let A_1, A_2, and A_3 be two-by-two non-singular real symmetric matrices. Suppose A_1, A_2, and A_3 are linearly independent, and $\sigma_{A_j}(Q_1) = \sigma_{A_j}(Q_2), j = 1, 2, 3$. Then $Q_1(x) \equiv Q_2(x)$ in $[0, \pi]$.*

Proof. Since $\sigma_{A_j}(Q_1) = \sigma_{A_j}(Q_2), j = 1, 2, 3$, we know

$$\omega_{A_j}(\lambda; Q_1) = \omega_{A_j}(\lambda; Q_2).$$

Then Lemma 4.7 implies that for $j = 1, 2, 3$,

$$\widehat{\omega}_{A_j}(\lambda; \widetilde{Q}_1) = \widehat{\omega}_{A_j}(\lambda; \widetilde{Q}_2). \tag{4.56}$$

By (4.56), Theorem 4.8 implies that

$$\det(Y(\pi; \lambda^2; \widetilde{Q}_1)) = \det(Y(\pi; \lambda^2; \widetilde{Q}_2)),$$
$$K_{\widetilde{Q}_1}(\lambda^2) = K_{\widetilde{Q}_2}(\lambda^2),$$

and

$$\widetilde{Q}_1(x) = \widetilde{Q}_2(x).$$

Hence $Q_1(x) \equiv Q_2(x)$ in $[0,\pi]$. □

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