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SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN $\mathbb{S}^2 \times \mathbb{S}^2$ AND $\mathbb{H}^2 \times \mathbb{H}^2$

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Abstract. Two holomorphic Hopf differentials for surfaces of non-null parallel mean curvature vector in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ are constructed. A 1:1 correspondence between these surfaces and pairs of constant mean curvature surfaces of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is established. Using this, surfaces with vanishing Hopf differentials (in particular, spheres with parallel mean curvature vector) are classified and a rigidity result for constant mean curvature surfaces of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is proved.

1. Introduction

Surfaces with constant mean curvature (CMC-surfaces) in three manifolds is a classic topic in differential geometry and it has been extensively studied when the ambient manifold has constant curvature. In 2004, Abresch and Rosenberg [1] studied CMC-surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ (respectively $\mathbb{H}^2$) is the two-dimensional sphere (respectively the hyperbolic plane). They defined on such surfaces a holomorphic two-differential which generalizes the classical Hopf differential defined for CMC-surfaces of space forms. They also classified those CMC-surfaces with vanishing Hopf-differential. In particular, they classified the orientable CMC-surfaces of genus zero in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

When the codimension of the surface is larger than one, the natural generalization of these types of surfaces are the surfaces with parallel mean curvature vector (in what follows, PMC-surfaces). Although there are results for codimension larger than two, the most relevant ones are obtained when the codimension is two. In 1971, Ferus [8] proved that a genus zero orientable surface with (non-null) parallel mean curvature vector in a simply-connected space form is a round sphere. In [5] and [13], Chen and Yau independently classified all the surfaces with parallel mean curvature vector in space forms, proving that they are CMC-surfaces of three-dimensional umbilical hypersurfaces. Both results are based on the following fact: if H is the mean curvature vector of the surface, as the dimension of the normal bundle is two, it is possible to consider another parallel vector field in the normal bundle $\tilde{H}$ orthogonal to H with the same length and to define two holomorphic Hopf differentials associated to H and $\tilde{H}$.

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In 2000, Kenmotsu and Zhou [9] classified surfaces with parallel mean curvature vector in the complex projective and the complex hyperbolic planes. In this case, it is well known that there are not umbilical hypersurfaces of these 4-manifolds, therefore there is not a method as in space forms to construct surfaces of parallel mean curvature vector. The authors did not use the existence of Hopf differentials. Instead, they reduced the classification theorem, using a result by Ogata [12], to solve an O.D.E. system on the surface. Using an analytic method, they classified these surfaces, proving that there are few of them and they have a good behavior with respect to the complex structure of the ambient space.

In this paper we study surfaces with parallel mean curvature vector in $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$. In this case, although there are umbilical hypersurfaces of the ambient space, only the totally geodesic ones (up to congruences $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$) have constant mean curvature (see Proposition 1), and so CMC-surfaces of $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$ are surfaces with parallel mean curvature vector in $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$, respectively.

The most important idea in the paper is the construction of two holomorphic Hopf differentials on PMC-surfaces of $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$ (see section 3) which generalize the Abresch-Rosenberg differential in the sense that if a PMC-surface of $\mathbb{S}^2\times\mathbb{S}^2$ or $\mathbb{H}^2\times\mathbb{H}^2$ factorizes through a CMC-surface of $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$, both Hopf differentials are equal and coincide (up to a constant) with the Abresch-Rosenberg differential (see Lemma 1). To define these Hopf differentials we use the two Kähler structures that these 4-manifolds have (see section 3).

In section 4 we prove the main results of the paper. Theorem 1 proves that given a simply-connected Riemannian surface (Σ, g) there exists, up to congruences, a 1:1 correspondence between PMC-isometric immersions of (Σ, g) in $\mathbb{S}^2\times\mathbb{S}^2$ (respectively $\mathbb{H}^2\times\mathbb{H}^2$) and pairs of CMC-isometric immersions of (Σ, g) in $\mathbb{S}^2\times\mathbb{R}$ (respectively $\mathbb{H}^2\times\mathbb{R}$). Moreover, these two CMC-surfaces are congruent if and only if the corresponding PMC-immersion factorizes through a CMC-immersion of $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$. So the existence of full PMC-immersions in $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$ is deeply related to the rigidity of CMC-immersions in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$.

In Theorem 2 we classify an important family of surfaces of $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$ with parallel mean curvature vector: those that are Lagrangian surfaces with respect to some of the two Kähler structures that these manifolds have. Theorem 3 is the most important contribution of the paper—it classifies the surfaces with parallel mean curvature vector with null extrinsic normal curvature. In the classification they appear the CMC-surfaces of $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, the Lagrangian PMC-surfaces and a new family of PMC-surfaces invariant under 1-parameter groups of isometries of $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$, which are described in Proposition 5. This result allows us to classify the parallel mean curvature surfaces with vanishing Hopf-differentials (Theorem 4) and in particular the parallel mean curvature spheres of $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$ (Corollary 1).

In section 5, using Theorem 1 and the examples of Proposition 5, we construct examples of CMC-surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$. Among them it is interesting to remark on a two-parameter family of CMC-embedded tori in $\mathbb{S}^2\times\mathbb{S}^1$ (Proposition 7). Moreover, Corollary 3 is a rigidity result for CMC-surfaces of $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$. Finally, in section 6 we study general properties of compact PMC-immersions in $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$.

The product of two Riemannian surfaces with different constant curvatures is not an Einstein manifold, and this is a big problem in studying its PMC-surfaces.

Following the ideas developed in this paper, on a PMC-surface of the product of two Riemannian surfaces with constant curvatures it is possible to define a holomorphic 2-differential which coincides with the sum of the two Hopf differentials when the constant curvatures are equal. When these curvatures are opposite, this holomorphic differential was defined in [11].

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2. Preliminaries and examples

We denote by $M^2(\epsilon)$, $\epsilon = 1, -1$, the two-dimensional sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$ endowed with the canonical metric of constant curvature 1 when $\epsilon = 1$ and the hyperbolic plane $\mathbb{H}^2 = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 - x_3^2 = -1,\, x_3 > 0\}$ endowed with the canonical metric of constant curvature -1 when $\epsilon = -1$. We denote by ω the Kähler 2-form on $M^2(\epsilon)$ and by J the corresponding complex structure, i.e. $\omega(\,,\,) = \langle J\,,\rangle$, where $\langle\,,\,\rangle$ denotes the metric of $M^2(\epsilon)$.

If we consider $M^2(\epsilon) \times M^2(\epsilon)$ endowed with the product metric, which will also be denoted by $\langle,\rangle$, then it is an orientable Einstein manifold with scalar curvature 4ϵ. The orientation will be given by the 4-form $\pi_1^*\omega \wedge \pi_2^*\omega$, where π_j, $j = 1, 2$, are the projections on the factors.

Throughout the paper we will consider $M^2(\epsilon) \times M^2(\epsilon)$ embedded isometrically in $\mathbb{R}^3 \times \mathbb{R}^3 \equiv \mathbb{R}^6$ when $\epsilon = 1$ and in $\mathbb{R}^3_1 \times \mathbb{R}^3_1 \equiv \mathbb{R}^6_2$ when $\epsilon = -1$, $\mathbb{R}^3_1$ being the Lorentz-Minkowski 3-space.

Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be an immersion of an oriented surface Σ. If $T^\perp\Sigma$ is the normal bundle of Φ, then we have the orthogonal decomposition

$$\Phi^*T(M^2(\epsilon) \times M^2(\epsilon)) = T\Sigma \oplus T^\perp\Sigma.$$

Let $\bar\nabla$ be the connection on $\Phi^*T(M^2(\epsilon) \times M^2(\epsilon))$ induced by the Levi–Civita connection of $M^2(\epsilon) \times M^2(\epsilon)$ and let $\bar\nabla = \nabla + \nabla^\perp$ be the corresponding decomposition. If $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal local frame on $\Phi^*T(M^2(\epsilon) \times M^2(\epsilon))$ such that $\{e_1, e_2\}$ is an oriented frame on $T\Sigma$, then we define the *normal curvature* $K^\perp$ of the immersion Φ by

$$K^\perp = R^\perp(e_1, e_2, e_3, e_4),$$

where $R^\perp$ is the curvature tensor of the normal connection $\nabla^\perp$. Also, we define the *extrinsic normal curvature* $\bar K^\perp$ as the function on Σ given by

$$\bar K^\perp = \bar R(e_1, e_2, e_3, e_4),$$

where $\bar R$ is the curvature tensor of $\bar\nabla$.

Definition 1. Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be an immersion. We say that Φ has a non-null parallel mean curvature vector, from now on called a PMC-immersion, if $\nabla^\perp H = 0$ and H is non-null. In such a case, $|H|$ is a positive constant.

Suppose that $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ is a PMC-immersion of an orientable surface Σ. We can define another parallel normal vector field $\tilde H$ as the only one with $|\tilde H| = |H|$ and $\{\tilde H, H\}$ defining the same orientation on the normal bundle as $\{e_3, e_4\}$. Because H is parallel, $K^\perp = 0$, and hence the Ricci equation of Φ is given by

$$|H|^2 \bar K^\perp = \langle\, [A_H, A_{\tilde H}] e_1, e_2\rangle,$$

where A_ξ is the Weingarten endomorphism associated to a normal vector ξ.

In order to get examples of PMC-surfaces, we make use of the following trivial fact: *If Σ is a constant mean curvature surface of a totally umbilical hypersurface with constant mean curvature of $M^2(\epsilon)\times M^2(\epsilon)$, then Σ has parallel mean curvature vector as a surface of $M^2(\epsilon)\times M^2(\epsilon)$.* The next proposition describes the umbilical hypersurfaces with constant mean curvature of $M^2(\epsilon)\times M^2(\epsilon)$.

Proposition 1. *Let $\Psi : N \to M^2(\epsilon)\times M^2(\epsilon)$ be a totally umbilical hypersurface with constant mean curvature. Then Ψ is totally geodesic, and it is locally congruent to the totally geodesic immersion:*

$$\begin{array}{cc} \epsilon = 1, & \epsilon = -1, \\ \mathbb{S}^2\times\mathbb{R} \to \mathbb{S}^2\times\mathbb{S}^2, & \mathbb{H}^2\times\mathbb{R}\to\mathbb{H}^2\times\mathbb{H}^2, \\ (p,t)\mapsto (p,(\cos t,\sin t,0)), & (p,t)\mapsto(p,(0,\sinh t,\cosh t)). \end{array}$$

Proof. Let η be a unit normal vector field of N in $M^2(\epsilon)\times M^2(\epsilon)$, $\hat\sigma$ the second fundamental form of Ψ and $\hat H$ the mean curvature. As Ψ is totally umbilical we have that

$$\hat\sigma(v,w) = \hat H\langle v,w\rangle\eta, \quad \forall\, v,w\in TN.$$

As $\hat H$ is constant, this shows that the Codazzi equation for Ψ becomes

$$\bar R(x,v,w,\eta) = 0, \quad \forall\, x,v,w\in TN.$$

Let $p\in N$ and $\eta_p=(a,b)$. As the differential at p of every component of Ψ has rank less than or equal to 2, there exists at p an orthonormal reference $\{e_1 = (e_1^1,0), e_2, e_3\}$. Then the above equation becomes

$$0 = R^1(e_j^1, e_1^1, e_1^1, a) = \epsilon\langle e_j^1, a\rangle, \qquad j=2,3,$$

where R^1 is the curvature tensor of $M^2(\epsilon)$. If $a\neq 0$, then $\{a, e_1^1\}$ is an orthogonal reference of $T_{\Psi_1(p)}M^2(\epsilon)$, and the last equation says that $e_j^1=0$ for $j=2,3$. So $\{e_2^2, e_3^2\}$ are linearly independent, and hence $0=\langle\eta,e_j\rangle = \langle b, e_j^2\rangle$, $j=2,3$, which means $b=0$. So at every point one of the two components of η vanishes, and then locally, up to isometries of $M^2(\epsilon)\times M^2(\epsilon)$, we can take $\eta=(0,\eta_2)$.

If $\Psi=(\Psi_1,\Psi_2)$, then Ψ and $\hat\Psi=(\Psi_1,-\Psi_2)$ are an orthogonal reference in the normal bundle of $M^2(\epsilon)\times M^2(\epsilon)$ in $\mathbb{R}^6$ or $\mathbb{R}^6_2$. So for any $v\in TN$, taking into account that $\langle\hat\Psi_*(v),\eta\rangle = -\langle v,\eta\rangle = 0$, we have that

$$D_v\eta = -\hat A_\eta v = -\hat H v,$$

where D stands for the Levi–Civita connection of $\mathbb{R}^6$ or $\mathbb{R}^6_2$. So the map $\eta+\hat H\Psi : N\to\mathbb{R}^6$ is a constant $A=(A_1,A_2)\in\mathbb{R}^3\times\mathbb{R}^3\equiv\mathbb{R}^6$, and hence $\hat H\Psi_1 = A_1$ and $\eta_2+\hat H\Psi_2 = A_2$. As N is a 3-manifold and Ψ is an immersion, Ψ_1 cannot be a constant, and so $\hat H=0$, which implies that Ψ is totally geodesic. Now the second equation says that $\eta_2=A_2$, and so $\langle\Psi_2,A_2\rangle = \langle\Psi,\eta\rangle = 0$ with $|A_2|=|\eta_2|=1$. This proves that $\Psi_2(N)$ is a geodesic of $\mathbb{S}^2$ or $\mathbb{H}^2$, and the proof is complete. $\square$

As a consequence of this result we obtain that

> *CMC-surfaces of $M^2(\epsilon)\times\mathbb{R}$ are surfaces of $M^2(\epsilon)\times M^2(\epsilon)$ with parallel mean curvature vector.*

Other examples of PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$ can be constructed in the following way: given two regular curves $\alpha : I \to M^2(\epsilon)$ and $\beta : I' \to M^2(\epsilon)$, then

$$\Phi : I \times I' \to M^2(\epsilon) \times M^2(\epsilon),$$
$$\Phi(t,s) = (\alpha(t), \beta(s))$$

is an immersion of the surface $I \times I'$ whose mean curvature vector is given by

$$H = \frac{k_\alpha}{2}(J\alpha', 0) + \frac{k_\beta}{2}(0, J\dot{\beta}),$$

where $'$ (respectively $\dot{}$) stands for the derivative with respect to t (respectively s), k_α and k_β are respectively the curvatures of α and β, and we have assumed that $|\alpha'| = |\dot{\beta}| = 1$. So we obtain that Φ *has parallel mean curvature vector if and only if α and β are curves of constant curvature*. In that case, $4|H|^2 = k_\alpha^2 + k_\beta^2$, and hence Φ is minimal if and only if α and β are geodesics. It is interesting to remark that the induced metric on $I \times I'$ by Φ is flat.

Taking into account the curves of constant curvature of $\mathbb{S}^2$ and $\mathbb{H}^2$ we have that the above examples are, up to congruences, open subsets of the following family of complete and embedded PMC-surfaces:

Example 1. When $\epsilon = 1$, the *tori product of two geodesic circles*

$$T_{a,\hat{a}} = \{(x,y) \in \mathbb{S}^2 \times \mathbb{S}^2 \,/\, x_3 = a,\, y_3 = \hat{a}\}, \quad 0 \le a \le \hat{a} < 1,\, a^2 + \hat{a}^2 > 0,$$

whose mean curvatures satisfy $4|H|^2 = \frac{a^2}{1-a^2} + \frac{\hat{a}^2}{1-\hat{a}^2}$.

When $\epsilon = -1$, we obtain three topological families of examples:

1. the *tori product of two geodesic circles*

$$\hat{T}_{a,\hat{a}} = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,/\, x_3 = a,\, y_3 = \hat{a}\}, \quad 1 < a \le \hat{a},$$

whose mean curvatures satisfy $4|H|^2 = \frac{a^2}{a^2-1} + \frac{\hat{a}^2}{\hat{a}^2-1}$ and $|H|^2 > 1/2$,

2. the *cylinders product of a geodesic circle and a hypercycle*

$$C_{a,b} = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,/\, x_3 = a,\, y_1 = b\}, \quad b \ge 0,\, a > 1,$$

whose mean curvatures satisfy $4|H|^2 = \frac{a^2}{a^2-1} + \frac{b^2}{b^2+1}$ and $|H|^2 > 1/4$, and the *cylinders product of a geodesic circle and a horocycle*

$$\hat{C}_a = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,/\, x_3 = a,\, y_1 - y_3 = 1\}, \quad a > 1,$$

whose mean curvatures satisfy $4|H|^2 = \frac{2a^2-1}{a^2-1}$ and $|H|^2 > 1/2$,

3. and finally the *planes product of two hypercycles*

$$P_{b,\hat{b}} = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,/\, x_1 = b,\, y_1 = \hat{b}\}, \quad b, \hat{b} \ge 0,\, b\hat{b} \ne 0,$$

whose mean curvatures satisfy $4|H|^2 = \frac{b^2}{b^2+1} + \frac{\hat{b}^2}{\hat{b}^2+1}$ and $|H|^2 < 1/2$, the *planes product of a hypercycle and a horocycle*

$$\hat{P}_b = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,|\, x_1 = b,\, y_1 - y_3 = 1\}, \quad b \ge 0,$$

whose mean curvatures satisfy $4|H|^2 = \frac{2b^2+1}{b^2+1}$ and $1/4 \le |H|^2 < 1/2$, and the *plane product of two horocycles*

$$\tilde{P} = \{(x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 \,|\, x_1 - x_3 = 1,\, y_1 - y_3 = 1\},$$

whose mean curvature satisfies $|H|^2 = 1/2$.

3. Hopf differentials

In order to have a deep understanding of the geometry of $M^2(\epsilon) \times M^2(\epsilon)$ and of its surfaces we need to introduce the two Kähler structures that $M^2(\epsilon) \times M^2(\epsilon)$ has. We can define two complex structures on $M^2(\epsilon) \times M^2(\epsilon)$ by

$$J_1 = (J, J), \quad J_2 = (J, -J),$$

whose Kähler two-forms are $\omega_1 = \pi_1^*\omega + \pi_2^*\omega$ and $\omega_2 = \pi_1^*\omega - \pi_2^*\omega$. Hence

$$\omega_1 \wedge \omega_1 = -\omega_2 \wedge \omega_2 = 2(\pi_1^*\omega \wedge \pi_2^*\omega),$$

and so J_1 defines the chosen orientation on $M^2(\epsilon)\times M^2(\epsilon)$ and J_2 the opposite one.

Now, $(M^2(\epsilon) \times M^2(\epsilon), \langle,\rangle, J_j)$, $j = 1, 2$, are Kähler-Einstein manifolds. It is clear that if $\mathrm{Id} : M^2(\epsilon) \to M^2(\epsilon)$ is the identity map and $F : M^2(\epsilon) \to M^2(\epsilon)$ is an anti-holomorphic isometry, then

$$(\mathrm{Id}, F) : M^2(\epsilon) \times M^2(\epsilon) \to M^2(\epsilon) \times M^2(\epsilon)$$

is a holomorphic isometry from $(M^2(\epsilon)\times M^2(\epsilon), \langle,\rangle, J_1)$ onto $(M^2(\epsilon)\times M^2(\epsilon), \langle,\rangle, J_2)$.

If $\Phi = (\phi, \psi) : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ is a PMC-immersion of an orientable surface Σ, then the *Kähler functions* on Σ, $C_1, C_2 : \Sigma \to \mathbb{R}$, associated to the complex structures J_1 and J_2 are defined by

$$\Phi^*\omega_j = C_j\omega_\Sigma, \quad j = 1, 2,$$

where ω_Σ is the area 2-form of Σ. It is clear that $C_j^2 \leq 1$ and that the points where $C_j^2 = 1$ are the complex points of Φ with respect to the J_j complex structure. Moreover, $\{p \in \Sigma \,|\, C_j^2(p) = 1\}$, $j = 1, 2$, has empty interior, because if not, its interior is a non-empty complex surface and so it is minimal, contradicting the fact that $|H|$ is a positive constant on Σ. Hence

$$\Sigma_0^j = \{p \in \Sigma \,|\, C_j^2(p) < 1\} \text{ is an open dense set in } \Sigma, \quad j = 1, 2. \tag{3.1}$$

It is interesting to remark that C_j^2 is well defined even when the surface is not orientable.

Now it is easy to check that the Jacobians of ϕ and ψ are given by

$$\mathrm{Jac}\,(\phi) = \frac{C_1 + C_2}{2}, \quad \mathrm{Jac}\,(\psi) = \frac{C_1 - C_2}{2}$$

and that the *extrinsic sectional curvature* $\bar{K} = \bar{R}(e_1, e_2, e_2, e_1)$, where $\{e_1, e_2\}$ is an orthonormal frame on $T\Sigma$, and the normal extrinsic curvature are given by

$$\bar{K} = \epsilon\frac{C_1^2 + C_2^2}{2}, \qquad \bar{K}^\perp = \epsilon\frac{C_1^2 - C_2^2}{2}.$$

We consider a local isothermal parameter $z = x + iy$ on Σ such that

$$\langle\Phi_z, \Phi_z\rangle = \langle\phi_z, \phi_z\rangle + \langle\psi_z, \psi_z\rangle = 0, \quad |\Phi_z|^2 = |\phi_z|^2 + |\psi_z|^2 = e^{2u}/2,$$

where the derivatives with respect to z and $\bar{z}$ are given by

$$\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \quad \partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

Definition 2. Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of an oriented surface Σ. We define two Hopf differentials as

$$\Theta_1(z) = \left(2\langle \sigma(\partial_z, \partial_z), H + i\tilde{H}\rangle + \frac{\epsilon}{4|H|^2}\langle J_1\Phi_z, H + i\tilde{H}\rangle^2\right)(dz)^2,$$
$$\Theta_2(z) = \left(2\langle \sigma(\partial_z, \partial_z), H - i\tilde{H}\rangle + \frac{\epsilon}{4|H|^2}\langle J_2\Phi_z, H - i\tilde{H}\rangle^2\right)(dz)^2,$$

where σ is the second fundamental form of Φ and z is a conformal parameter of Σ.

It is clear that Θ_j, $j = 1, 2$, are well defined, i.e., they are invariant by a change of conformal parameter. To prove that these Hopf differentials are holomorphic when the surface has parallel mean curvature vector, we need to study the Frenet equations of our immersion $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon) \subset \mathbb{R}^6$ (or $\mathbb{R}^6_2$).

With the chosen orientation, $\{\Phi_x, \Phi_y, \tilde{H}, H\}$ is an oriented frame on $\Phi^*T(M^2(\epsilon) \times M^2(\epsilon))$. Denoting

$$\xi = \frac{1}{\sqrt{2}|H|}(H - i\tilde{H}),$$

we have that $|\xi|^2 = 1, \langle \xi, \xi\rangle = 0$, $\nabla^\perp \xi = 0$ and $\{\xi, \bar{\xi}\}$ is a *global reference* of the complexified normal bundle. It is clear that

$$J_1\Phi_z = iC_1\Phi_z + \gamma_1\xi + \delta_1\bar{\xi}$$

and that $\gamma_1 = \delta_1 = 0$ on $\Sigma \setminus \Sigma_0^1$ (see (3.1)). On Σ_0^1 we can define a local orthonormal reference $\{e_3, e_4\}$ of the normal bundle by

$$J_1\Phi_x = C_1\Phi_y + e^u\sqrt{1 - C_1^2}\, e_4, \qquad J_1\Phi_y = -C_1\Phi_x + e^u\sqrt{1 - C_1^2}\, e_3,$$

which defines the same orientation on the normal bundle as $\{\tilde{H}, H\}$, and so $e_4 - ie_3 = e^{i\theta}(H - i\tilde{H})/|H|$ for a certain function θ. Hence the above equations become

$$J_1\Phi_z = iC_1\Phi_z + \frac{e^u\sqrt{1 - C_1^2}}{\sqrt{2}}e^{i\theta}\xi.$$

Therefore $\delta_1 = 0$ on Σ_0^1, too, and so $\delta_1 = 0$ on Σ. Making a similar reasoning for the other complex structure we finally get that

$$J_1\Phi_z = iC_1\Phi_z + \gamma_1\xi, \qquad J_1\xi = -2e^{-2u}\bar{\gamma}_1\Phi_z - iC_1\xi, \tag{3.2}$$
$$J_2\Phi_z = iC_2\Phi_z + \gamma_2\bar{\xi}, \qquad J_2\xi = -2e^{-2u}\gamma_2\Phi_{\bar{z}} + iC_2\xi, \tag{3.3}$$

for certain complex functions γ_j, $j = 1, 2$, which satisfy $|\gamma_j|^2 = e^{2u}(1 - C_j^2)/2$.

If $\hat{\Phi} := (\phi, -\psi)$, then $\{\Phi, \hat{\Phi}\}$ is an orthogonal reference along Φ of the normal bundle of $M^2(\epsilon) \times M^2(\epsilon)$ in $\mathbb{R}^6$ when $\epsilon = 1$ and in $\mathbb{R}^6_2$ when $\epsilon = -1$, with $|\Phi|^2 = |\hat{\Phi}|^2 = 2\epsilon$. Also, from (3.2) and (3.3), it follows that

$$\hat{\Phi}_z = -J_1J_2\Phi_z = C_1C_2\Phi_z + 2e^{-2u}\gamma_1\gamma_2\Phi_{\bar{z}} - iC_2\gamma_1\xi - iC_1\gamma_2\bar{\xi}.$$

Using the above information we easily get that the Frenet equations of the PMC-immersion Φ are given by

$$
\begin{aligned}
\Phi_{zz} &= 2u_z\Phi_z + f_1\xi + f_2\bar{\xi} - \epsilon\frac{\gamma_1\gamma_2}{2}\hat{\Phi},\\
\Phi_{z\bar{z}} &= \frac{|H|e^{2u}}{2\sqrt{2}}\xi + \frac{|H|e^{2u}}{2\sqrt{2}}\bar{\xi} - \epsilon\frac{e^{2u}}{4}\Phi - \epsilon\frac{e^{2u}}{4}C_1C_2\hat{\Phi},\\
\xi_z &= -\frac{|H|}{\sqrt{2}}\Phi_z - 2e^{-2u}f_2\Phi_{\bar{z}} + \epsilon\frac{iC_1\gamma_2}{2}\hat{\Phi},\\
\bar{\xi}_z &= -\frac{|H|}{\sqrt{2}}\Phi_z - 2e^{-2u}f_1\Phi_{\bar{z}} + \epsilon\frac{iC_2\gamma_1}{2}\hat{\Phi},
\end{aligned}
$$

for certain complex functions f_j, $j = 1, 2$.

We will call the *fundamental data* of the immersion Φ to the tuple $(u, C_j, \gamma_j, f_j : j = 1, 2)$. These functions satisfy some equations that we are going to obtain.

Derivating with respect to z and $\bar{z}$ in (3.2) and (3.3) and taking into account the above equations we easily get

$$
(C_j)_z = 2ie^{-2u}f_j\bar{\gamma}_j - i\frac{|H|}{\sqrt{2}}\gamma_j, \quad (\gamma_j)_{\bar{z}} = -\frac{i|H|C_je^{2u}}{\sqrt{2}}.
$$

Now, from the ξ and $\bar{\xi}$ components of $\Phi_{zz\bar{z}} = \Phi_{z\bar{z}z}$ we obtain that

$$
(f_j)_{\bar{z}} = i\epsilon\frac{e^{2u}C_j\gamma_j}{4}, \quad j = 1, 2.
$$

Conversely, we get the following result.

Proposition 2. *Let Σ be a simply connected Riemann surface, λ a positive constant, $u, C_j : \Sigma \to \mathbb{R}$ with $C_j^2 \leq 1$ and $\gamma_j, f_j : \Sigma \to \mathbb{C}$, $j = 1, 2$, functions satisfying*

$$
\left\{
\begin{array}{ll}
(C_j)_z = 2ie^{-2u}f_j\bar{\gamma}_j - i\dfrac{\lambda}{\sqrt{2}}\gamma_j, & (f_j)_{\bar{z}} = i\epsilon\dfrac{e^{2u}C_j\gamma_j}{4},\\
(\gamma_j)_{\bar{z}} = -\dfrac{i\lambda C_je^{2u}}{\sqrt{2}}, & |\gamma_j|^2 = \dfrac{e^{2u}(1-C_j^2)}{2},
\end{array}
\right. \quad j = 1, 2. \tag{3.4}
$$

Then there exists, up to congruences, a unique PMC-immersion $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ with $|H| = \lambda$ whose fundamental data are $(u, C_j, \gamma_j, f_j : j = 1, 2)$.

Proof. As $\{p \in \Sigma : \gamma_j(p) \neq 0\}$ is an open dense set of Σ, it is easy to deduce from (3.4) that

$$
\left\{
\begin{array}{ll}
4u_{z\bar{z}} + e^{2u}(|H|^2 + \epsilon\frac{C_1^2+C_2^2}{2}) - 4e^{-2u}(|f_1|^2 + |f_2|^2) = 0, & \text{Gauss},\\
\epsilon e^{4u}(C_1^2 - C_2^2) - 8(|f_1|^2 - |f_2|^2) = 0, & \text{Ricci},\\
(\gamma_j)_z = 2u_z\gamma_j - 2iC_jf_j, & j = 1, 2.
\end{array}
\right. \tag{3.5}
$$

From (3.4) and (3.5) we can easily check that $\Phi_{zz\bar{z}} = \Phi_{z\bar{z}z}$ and $\xi_{z\bar{z}} = \xi_{\bar{z}z}$, which are the integrability conditions of the Frenet system. □

Proposition 3. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of an orientable surface Σ. Then Θ_j, $j = 1, 2$, are holomorphic.*

Proof. Using the functions defined above, the Hopf differentials Θ_j, $j = 1, 2$, can be written as

$$
\Theta_j = \left(2\sqrt{2}|H|f_j + \frac{\epsilon}{2}\gamma_j^2\right)(dz)^2, \quad j = 1, 2.
$$

Now, from (3.4) and (3.5) we obtain that $\left(4\sqrt{2}|H|f_j + \epsilon\gamma_j^2\right)_{\bar{z}} = 0$, which proves the proposition. □

From (3.2) and (3.3) we have that $\langle J_1\Phi_z, \xi\rangle = \langle J_2\Phi_z, \bar{\xi}\rangle = 0$, and then the Hopf differentials can also be written as

$$\Theta_1(z) = \left(2\langle\sigma(\partial_z,\partial_z), H + i\tilde{H}\rangle - \frac{\epsilon}{|H|^2}\langle J_1\Phi_z, \tilde{H}\rangle^2\right)(dz)^2,$$
$$\Theta_2(z) = \left(2\langle\sigma(\partial_z,\partial_z), H - i\tilde{H}\rangle - \frac{\epsilon}{|H|^2}\langle J_2\Phi_z, \tilde{H}\rangle^2\right)(dz)^2.$$

In the following result we compute these Hopf-differentials in the examples described in section 2.

Lemma 1.

1. *Let $\Phi : \Sigma \to M^2(\epsilon)\times\mathbb{R} \hookrightarrow M^2(\epsilon)\times M^2(\epsilon)$ be a CMC-immersion. Then $\Theta_1 = \Theta_2 = 2\Theta_{AR}$, where Θ_{AR} is the Abresch-Rosenberg holomorphic differential associated to Φ (see [1]).*
2. *Let $\Phi : I \times I' \to M^2(\epsilon)\times M^2(\epsilon)$ be the product of two curves $\Phi(t,s) = (\alpha(t),\beta(s))$ of constant curvatures k_α and k_β, respectively. Then*

$$\Theta_j = \frac{\epsilon + 4|H|^2}{16|H|^2}(k_\alpha + (-1)^j i\, k_\beta)^2\,(dz)^2, \quad j = 1, 2.$$

Proof. First we prove (1). It is clear that, in this case, $\varsigma = (0,(0,0,1))$ (respectively $\varsigma = (0,(1,0,0))$) when $\epsilon = 1$ (respectively $\epsilon = -1$) is a unit normal field to the totally geodesic immersion $M^2(\epsilon)\times\mathbb{R} \hookrightarrow M^2(\epsilon)\times M^2(\epsilon)$ given in Proposition 1. So $\tilde{H} = |H|\varsigma$. If $\tilde{\sigma}$ is the second fundamental form of Σ in $M^2(\epsilon)\times\mathbb{R}$, then $\tilde{\sigma} = \sigma$ and then

$$\langle\sigma(\partial_z,\partial_z), H + i\tilde{H}\rangle = \langle\sigma(\partial_z,\partial_z), H - i\tilde{H}\rangle = \langle\tilde{\sigma}(\partial_z,\partial_z), H\rangle.$$

Also, if $\Phi = (\phi,\eta) : \Sigma \to M^2(\epsilon)\times\mathbb{R}$, then, taking into account Proposition 1, the corresponding immersion $\Phi : \Sigma \to M^2(\epsilon)\times M^2(\epsilon)$ is $\Phi = (\phi,\psi)$, where

$$\psi = (\cos\eta, \sin\eta, 0), \text{ when } \epsilon = 1, \quad \psi = (0, \sinh\eta, \cosh\eta), \text{ when } \epsilon = -1.$$

Now, from a direct computation, we have that $\langle J_1\Phi_z, \tilde{H}\rangle = |H|\eta_z$ and $\langle J_2\Phi_z, \tilde{H}\rangle = -|H|\eta_z$.

Finally, from the second expressions of Θ_j, we get that

$$\Theta_1 = \Theta_2 = (2\langle\tilde{\sigma}(\partial_z,\partial_z), H\rangle - \epsilon(\eta_z)^2)\,(dz)^2 = 2\Theta_{AR}.$$

We remark that, in this case, the functions appearing in the Frenet equations satisfy $f_1 = f_2$, $\gamma_1 = \gamma_2$, and so $C_1 = C_2$.

To prove (2) it is easy to check that, for the product of two curves, the functions appearing in the Frenet equations are given by

$$f_1 = \bar{f}_2 = \frac{1}{8\sqrt{2}|H|}(k_\alpha - ik_\beta)^2, \quad \gamma_1 = \bar{\gamma}_2 = \frac{1}{2\sqrt{2}|H|}(k_\alpha - ik_\beta), \quad C_1 = C_2 = 0.$$

Using the above equations, the proof of (2) is trivial. □

From (3.4) and (3.5) we can get some properties and formulae about PMC-surfaces which will be used in the next sections.

- First, from the Gauss and Ricci equations joint with $4u_{z\bar{z}} = -Ke^{2u}$ it is easy to deduce that

$$|f_j|^2 = \frac{e^{4u}}{8}(|H|^2 - K + \epsilon C_j^2), \quad j = 1, 2. \tag{3.6}$$

These equations say that

$$K \leq |H|^2 + 1 \quad \text{when } \epsilon = 1,$$

and the equality is attained in a point p if and only if for some $j \in \{1, 2\}$ $f_j(p) = 0$ and $C_j^2(p) = 1$. Also

$$K \leq |H|^2 \quad \text{when } \epsilon = -1,$$

and the equality is attained in a point p if and only if for some $j \in \{1, 2\}$ $f_j(p) = 0$ and $C_j(p) = 0$.

- Second, using (3.6), (3.4) and (3.5), we obtain the following relation between $|\Theta_j|^2$ and $|\nabla C_j|^2$:

$$\begin{aligned} &|\nabla C_j|^2 + 4\epsilon e^{-4u}|\Theta_j|^2 \\ &= (1 - C_j^2 + 4\epsilon|H|^2)\left(\frac{\epsilon(1 - C_j^2)}{4} + |H|^2 + \epsilon C_j^2 - K\right), \quad j = 1, 2. \end{aligned} \tag{3.7}$$

- Also, from (3.4) and (3.5), it is easy to compute the Laplacian of the Kähler functions C_j, obtaining

$$\Delta C_j = -C_j\left(4|H|^2 - 2K + \epsilon(1 + C_j^2)\right), \quad j = 1, 2. \tag{3.8}$$

This means that C_j satisfies the equation $(\Delta + F)C_j = 0$ where $F = 4|H|^2 - 2K + \epsilon(1 + C_j^2)$. Then, using classical results from elliptic theory (see [6]), we have that either $C_j = 0$ or the set $\{p \in \Sigma \,|\, C_j(p) = 0\}$ is a union of curves. In particular, its interior is empty.

- Under certain restrictions on the curvature of the surface, we can get more properties of the sets Σ_0^j, $j = 1, 2$ (see (3.1)).

Proposition 4. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion. If $K(p) \neq \epsilon$, for any $p \in \Sigma$, then $\Sigma \setminus \Sigma_0^j = \{p \in \Sigma \,|\, C_j^2(p) = 1\}$, $j = 1, 2$, are sets of isolated points.*

Proof. As the points p with $C_j^2(p) = 1$ are critical points of the function C_j, we are going to study the degeneracy of these points.

Let p_0 be a point with $C_j(p_0)^2 = 1$, with $j \in \{1, 2\}$. Then $\gamma_j(p_0) = 0$, and from equations (3.4) and (3.5) one gets that

$$\begin{aligned} (C_j)_{zz}(p_0) &= -2\sqrt{2}|H|f_j(p_0)C_j(p_0), \\ (C_j)_{z\bar{z}}(p_0) &= -\frac{C_j(p_0)e^{2u(p_0)}|H|^2}{2} - 4C_j(p_0)e^{-2u(p_0)}|f_j|^2(p_0). \end{aligned}$$

A direct computation shows that the determinant of the Hessian of C_j at p_0 is

$$e^{-4u(p_0)}\left(e^{2u(p_0)}H^2 - 8e^{-2u(p_0)}|f_j(p_0)|^2\right)^2,$$

which by (3.6) is equal to $\left(K(p_0) - \epsilon\right)^2$, and hence p_0 is degenerate if and only if $K(p_0) = \epsilon$. As the non-degenerate critical points are isolated, we complete the proof. $\square$

4. Main results

The integrability equations given in the previous section allow us to relate, at least in the simply-connected case, PMC-immersions in $M^2(\epsilon)\times M^2(\epsilon)$ with pairs of CMC-immersions in $M^2(\epsilon)\times\mathbb{R}$ with the same induced metric and the same length of the mean curvature. We affirm this relation in the following result.

Theorem 1. *Given a simply-connected Riemannian surface (Σ, g), there exists a 1:1 correspondence $[\Phi]\leftrightarrow([\Phi_1],[\Phi_2])$ between congruent classes of PMC-isometric immersions $\Phi:(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ and pairs of congruent classes of CMC-isometric immersions $\Phi_1,\Phi_2:(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ with $|H|=|H_1|=|H_2|$, where H is the mean curvature vector of Φ and H_j, $j=1,2$, are respectively the mean curvatures of Φ_j, $j=1,2$. The Abresch-Rosenberg differentials Θ^j_{AR} associated to the pair of CMC-immersions Φ_j, $j=1,2$, and the two Hopf-differentials Θ_j, $j=1,2$, associated to the PMC-immersion Φ are related by $2\Theta^j_{AR}=\Theta_j$, $j=1,2$.*

Moreover, $[\Phi_1]=[\Phi_2]$ if and only if Φ factorizes

$$\Phi:\Sigma\to M^2(\epsilon)\times\mathbb{R}\hookrightarrow M^2(\epsilon)\times M^2(\epsilon)$$

through a CMC-immersion in $M^2(\epsilon)\times\mathbb{R}$.

Remark 1. Note that, up to now, there is no known example of two isometric immersions of the same Riemannian surface with the same non-zero constant mean curvature in $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$ that, up to reparametrization, are not congruent.

Proof. In order to prove this result we are going to use the integrability equations (3.4) for PMC-conformal immersions given in the previous section and the corresponding ones for CMC-immersions in $M^2(\epsilon)\times\mathbb{R}$ given in [2]. As we work with conformal immersions we are going to use the conformal version of these equations obtained in [7], which can be described as follows.

Let $\Psi=(\psi,\eta):(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ be a CMC-isometric immersion with mean curvature H and $z=x+iy$ a local isothermal parameter such that $g=e^{2u}|\mathrm{d}z|^2$. Then the Frenet equations of $\Psi:\Sigma\longrightarrow M^2(\epsilon)\times\mathbb{R}\subset\mathbb{R}^3\times\mathbb{R}$ (or $\mathbb{R}^3_1$) are given by

$$\begin{aligned}\Psi_{zz}&=2u_z\Psi_z+pN+\epsilon\eta_z^2\hat{\Psi},\\ \Psi_{z\bar z}&=\frac{e^{2u}}{2}HN+\epsilon\left(|\eta_z|^2-\frac{e^{2u}}{2}\right)\hat{\Psi},\\ N_z&=-H\Psi_z-2e^{-2u}p\Psi_{\bar z}+\epsilon\eta_z\nu\hat{\Psi},\end{aligned}\tag{4.1}$$

where $\hat{\Psi}=(\psi,0)$, N is a unit normal vector to the immersion Ψ, p is a complex function and ν is a real function defined by $\nu=\langle N,(0,1)\rangle$. The integrability equations of this Frenet system are given by (see [7, Theorem 2.3] for more details)

$$\begin{aligned}p_{\bar z}&=\epsilon\frac{e^{2u}}{2}\nu\eta_z, &\qquad \nu_z&=-H\eta_z-2e^{-2u}p\eta_{\bar z},\\ \eta_{z\bar z}&=\frac{e^{2u}}{2}H\nu, & |\eta_z|^2&=\frac{e^{2u}}{4}(1-\nu^2).\end{aligned}\tag{4.2}$$

Now we prove the result. Let $\Phi:(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ be a PMC-isometric immersion of a simply-connected oriented surface Σ and z an isothermal parameter such that $g=e^{2u}|\mathrm{d}z|^2$. Using (3.4) it follows that $i(\gamma_j)_{\bar z}$ is a real function and so, because Σ is simply-connected, there exists a function η_j such that $i\gamma_j=\sqrt{2}(\eta_j)_z$,

$j=1,2$. The function η_j is unique only up to a constant, which corresponds to a vertical translation in $M^2(\epsilon)\times\mathbb{R}$. We consider the data

$$\big(u, H_j=|H|, \nu_j=C_j, \eta_j, p_j=\sqrt{2}f_j\big),\quad j=1,2.$$

From (3.4), it follows that these data satisfy (4.2), and so there exist two CMC-isometric immersions $\Phi_j:(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ with $|H_j|=|H|$, $j=1,2$.

Moreover, it is easy to check that if Φ is congruent to Ψ, then the corresponding Φ_j and Ψ_j are also congruent for $j=1,2$.

Conversely, let $\Phi_j=(\phi_j,\eta_j):(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ be two CMC-isometric immersions with $|H_1|=|H_2|$ and z an isothermal parameter with $g=e^{2u}|\mathrm{d}z|^2$. We may suppose, composing with an appropriate isometry if necessary, that $H_1=H_2>0$. We consider data

$$\left(u, |H|=H_1=H_2, C_j=\nu_j, \gamma_j=-i\sqrt{2}(\eta_j)_z, f_j=\frac{p_j}{\sqrt{2}} : j=1,2\right).$$

From (4.2), it follows that these data satisfy (3.4), and so there exists a PMC-isometric immersion $\Phi:(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ with $|H|=|H_1|=|H_2|$.

Moreover, it is easy to check that if Φ_j are congruent to Ψ_j, $j=1,2$, then the corresponding Φ and Ψ are also congruent.

Second, as the Abresch-Rosenberg differential for CMC-surfaces can be expressed as $\Theta^j_{AR}=(|H_j|p_j-\frac{\epsilon}{2}((\eta_j)_z)^2)(\mathrm{d}z)^2$ and the Hopf-differentials for PMC-surfaces as $\Theta_j=\left(2\sqrt{2}|H|f_j+\frac{\epsilon}{2}\gamma_j^2\right)(dz)^2$, using the above relations between the data, we obtain that $2\Theta^j_{AR}=\Theta_j$, $j=1,2$.

Finally, if $[\Phi_1]=[\Phi_2]$, then $\Phi_1,\Phi_2:(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ are two CMC-isometric immersions satisfying $\Phi_2=F\circ\Phi_1$, where F is an isometry of $M^2(\epsilon)\times\mathbb{R}$. Then, given an isothermal parameter z, and possibly up to a congruence, we can take the data of Φ_j as $|H_1|=|H_2|$, $p_1=p_2$, $\nu_1=\nu_2$ and $\eta_1=\eta_2$. Therefore the associated PMC-isometric immersion $\Phi=(\phi,\psi):(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ has $f_1=f_2$, $\gamma_1=\gamma_2$ and $C_1=C_2$. Now as $\frac{-\sqrt{2}\,i}{|H|}\tilde{H}=\xi-\bar{\xi}$, from the Frenet equations we obtain that the derivative of the function $\tilde{H}:\Sigma\to\mathbb{R}^6\,(\text{or}\,\mathbb{R}^6_2)$ is given by

$$\tilde{H}_z=\frac{i|H|}{\sqrt{2}}(\xi_z-\bar{\xi}_z)=0.$$

So $\tilde{H}=A$ for some vector $A\in\mathbb{R}^6\,(\text{or}\,\mathbb{R}^6_2)$ with $|A|=|H|>0$, and hence $0=\langle\Phi,\tilde{H}\rangle=\langle\Phi,A\rangle$ and $0=\langle\hat{\Phi},\tilde{H}\rangle=\langle\hat{\Phi},A\rangle$. Now if $\Phi=(\phi,\psi)$ and $A=(A_1,A_2)$, we finally get $\langle\phi,A_1\rangle=\langle\psi,A_2\rangle=0$. If $A_2=0$, we have that $A_1\neq0$, and so $\mathrm{Jac}(\phi)=0$, i.e., $C_1=-C_2$. Hence $C_1=C_2=0$ and $\mathrm{Jac}(\phi)=\mathrm{Jac}(\psi)=0$. So the immersion is the product of two curves α and β, and taking into account the proof of item 2 of Lemma 1 and that, in this case, $\gamma_1=\gamma_2$, we get

$$\gamma_1=\bar{\gamma}_1=\frac{1}{2\sqrt{2}|H|}(k_\alpha-ik_\beta).$$

This implies that $k_\beta=0$, i.e., ψ lies on a geodesic of $M^2(\epsilon)$. If $A_2\neq0$ as $\langle\psi,A_2\rangle=0$, ψ lies on a geodesic of $M^2(\epsilon)$, too.

Hence the immersion Φ factorizes through the totally geodesic hypersurface $M^2(\epsilon)\times\mathbb{R}$ as a CMC-surface.

Conversely, given a PMC-immersion $\Phi:(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ such that Φ factorizes through the totally geodesic hypersurface $M^2(\epsilon)\times\mathbb{R}$, then from the proof of Lemma 1 we have that the data of Φ satisfy $f_1=f_2$, $\gamma_1=\gamma_2$ and $C_1=C_2$.

Hence, the corresponding data of Φ_1 and Φ_2 are the same and so are congruent, i.e. $[\Phi_1] = [\Phi_2]$. $\square$

Remark 2. When the immersion $\Phi : I \times I' \to M^2(\epsilon) \times M^2(\epsilon)$ is the product of two curves $\Phi(t,s) = \big(\alpha(t), \beta(s)\big)$ of constant curvatures k_α and k_β and $|\alpha'| = |\dot{\beta}| = 1$, following Theorem 1, the Frenet data of Φ_1 and Φ_2 are given by $u = 0$, $H_1 = H_2$, $\nu_1 = \nu_2 = 0$, $p_2 = \bar{p}_1$ and $\eta_2(x,y) = \eta_1(-x,y) = -(k_\alpha x + k_\beta y)/\sqrt{k_\alpha^2 + k_\beta^2}$. As $G(x,y) = (-x,y)$ is an isometry of the induced metric $g = dx^2 + dy^2$, then $\Phi_1 \circ G$ is a CMC-immersion with the same Frenet data as Φ_2, and so $\Phi_1 \circ G$ and Φ_2 are congruent. Note that when $k_\alpha \neq 0$ and $k_\beta \neq 0$, the map G is not induced by an ambient isometry. Moreover, Φ_1 and Φ_2 are cylinders over curves of constant curvature $\sqrt{k_\alpha^2 + k_\beta^2}$.

The examples of product of curves of constant curvatures given in Example 1 satisfy that $C_1 = C_2 = 0$, and so in particular they are Lagrangian PMC-surfaces with respect to both complex structures. In the following result we classify (even locally) those PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$ which are Lagrangian with respect to some of the complex structures.

Theorem 2. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of a surface Σ. If Φ is Lagrangian with respect to some of the Kähler structures J_1 or J_2, then $\Phi(\Sigma)$ is an open subset of some of the examples described in Example* 1.

Remark 3. This result is a generalization of Theorem 1 in [4], where the authors proved the result when $\epsilon = 1$, i.e. when the ambient space is $\mathbb{S}^2 \times \mathbb{S}^2$ and the surface is compact.

Proof. Taking the two-fold oriented covering of Σ if necessary, we can assume that Σ is orientable. Without loss of generality we suppose that Φ is a Lagrangian immersion with respect to J_1, i.e. $C_1 = 0$. Now, it is clear that $J_1 H$ is a parallel tangent vector field to Σ and hence Σ is flat, i.e. $K = 0$.

Now we are going to prove that the other Kähler function C_2 vanishes as well, and to do that we consider the holomorphic differential Θ_2.

First, if $\Theta_2 \equiv 0$, then from (3.7) and as $K = 0$ one obtains that

$$|\nabla C_2|^2 = (1 - C_2^2 + 4\epsilon|H|^2)\left(\frac{\epsilon(1-C_2^2)}{4} + |H|^2 + \epsilon C_2^2\right).$$

As $K = 0$, (3.8) becomes

$$\Delta C_2 = -C_2(4|H|^2 + \epsilon(1 + C_2^2)).$$

Hence the last two equations say that the function C_2 is isoparametric, i.e., $|\nabla C_2|^2 = f(C_2)$ and $\Delta C_2 = g(C_2)$ for suitable real functions f, g. Now we follow a standard reasoning. We work on the open set U where $\nabla C_2 \neq 0$. We are going to prove that $U = \emptyset$, and so C_2 must be constant. As $K = 0$, the Bochner formula says that

$$\frac{1}{2}\Delta|\nabla C_2|^2 = \langle \nabla C_2, \nabla(\Delta C_2)\rangle + \sum_{i=1}^{2} |\nabla_{e_i} \nabla C_2|^2,$$

where $\{e_1, e_2\}$ is an orthonormal frame on U and where we can take $e_1 = \nabla C_2/|\nabla C_2|$. Using the last two equations, i.e. that C_2 is isoparametric, it is not difficult to check that the Bochner formula becomes

$$0 = (4|H|^2 + \epsilon(1 - C_2^2))\big(3\epsilon(\epsilon + 4|H|^2)^2 - 18(\epsilon + 4|H|^2)C_2^2 - \epsilon C_2^4\big).$$

So C_2 on U satisfies the above non-trivial polynomial, and therefore C_2 must be constant on each connected component of U, which is impossible because $\nabla C_2 \neq 0$ on U. We have proved that $U = \emptyset$. Therefore C_2 is constant. But $(C_2)_z = 0$ implies that $(1 - C_2^2) f_2 = \frac{|H|}{\sqrt{2}} \gamma_2^2$. From here and (3.6) one obtains that $C_2^2 = \epsilon K = 0$. So in this case our immersion Φ is also Lagrangian with respect to J_2.

Second, if $\Theta_2 \neq 0$, then it has isolated zeroes. In this case from the integrability equations the 1-differential

$$\Upsilon(z) = \gamma_1(z)\,(dz) = \frac{1}{\sqrt{2}|H|}\langle J_1 \Phi_z, H + i\tilde{H}\rangle \mathrm{d}z,$$

which is well defined because it is invariant by a change of conformal parameter, is also holomorphic and without zeroes. Therefore Θ_2/Υ^2 is a holomorphic function. Let p be a point with $\Theta_2(p) \neq 0$. Then in a connected neighborhood U of p we can normalize this holomorphic function as

$$\Theta_2/\Upsilon^2 = \lambda, \quad \lambda \in \mathbb{R}^*.$$

Hence $|\Theta_2|^2 = \lambda^2 |\Upsilon|^4$. Now, from (3.7), the integrability equations and the fact that $C_1 = 0$ and $K = 0$, we get

$$|\nabla C_2|^2 = (1 - C_2^2 + 4\epsilon |H|^2)\left(\frac{\epsilon(1 - C_2^2)}{4} + |H|^2 + \epsilon C_2^2\right) - \epsilon\lambda^2.$$

As $K = 0$, (3.8) becomes

$$\Delta C_2 = -C_2(4|H|^2 + \epsilon(1 + C_2^2)).$$

In this second case the last two equations say that the function C_2 is also isoparametric on U.

Then, following a similar reasoning as in the first case, we obtain that $C_2 = 0$ on U. As this can be done at any point of Σ except at the isolated zeroes of Θ_2, we conclude, in this second case, that our immersion Φ is also Lagrangian with respect to J_2.

As a consequence, $\mathrm{Jac}\,(\phi) = \mathrm{Jac}\,(\psi) = 0$ and the immersion Φ is the product of two curves. As the mean curvature is parallel we obtain the result. $\square$

As we showed in the proof of Lemma 1, PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$ coming from CMC-surfaces of $M^2(\epsilon) \times \mathbb{R}$ have $C_1 = C_2$ and in particular their extrinsic normal curvatures $\bar{K}^\perp = \epsilon(C_1^2 - C_2^2)/2$ vanish. The next theorem classifies the PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$ such that $\bar{K}^\perp = 0$. Beside the above family, an interesting family of examples appears in the classification which we describe in the next result.

Proposition 5. *Let a, b, c be real numbers with $b > 0$ and $h : I \subset \mathbb{R} \to \mathbb{R}$ a non-constant solution of the O.D.E.*

$$(h')^2(x) = (a - h^2(x))\left((a - h^2(x)) - \epsilon b(1 + (h(x) - c)^2)\right), \tag{4.3}$$

satisfying $\epsilon(a - h^2(x)) > 0$, $\forall x \in I$.

Let $\psi(x,y)=\psi(x)$ be the curve in $M^2(\epsilon)$ such that $|\psi'(x)|^2=b(1+(h(x)-c)^2)$ and with curvature $K_\psi(x)=-\frac{b\epsilon(a-h^2(x))}{|\psi'(x)|^3}$. We define $\phi: I\times\mathbb{R}\to M^2(\epsilon)$ by

1. *If $a>0$,*

$$\phi(x,y)=\frac{1}{\sqrt{a}}\Big(\sqrt{\epsilon(a-h^2(x))}\cos(\sqrt{a}y),\sqrt{\epsilon(a-h^2(x))}\sin(\sqrt{a}y),h(x)\Big).$$

2. *If $a<0$ (which implies $\epsilon=-1$),*

$$\phi(x,y)=\frac{1}{\sqrt{-a}}\Big(h(x),\sqrt{h^2(x)-a}\sinh(\sqrt{-a}y),\sqrt{h^2(x)-a}\cosh(\sqrt{-a}y)\Big).$$

3. *If $a=0$ (which implies $\epsilon=-1$),*

$$\phi(x,y)=\frac{1}{2h(x)}\left((y^2-1)h^2(x)+1,2yh^2(x),(y^2+1)h^2(x)+1\right).$$

Then $\Phi=(\phi,\psi): I\times\mathbb{R}\to M^2(\epsilon)\times M^2(\epsilon)$ is a PMC-immersion.

All the examples described above satisfy $4|H|^2=b$, $C_1=C_2$ with $C_1^2=\frac{h'^2}{(a-h^2)^2}$, and they are conformal immersions with the induced metric given by $\epsilon(a-h(x)^2)(dx^2+dy^2)$ and the Hopf differentials given by

$$\Theta_j=\frac{\epsilon b}{4}(a+1-c^2+2(-1)^j ic)(dz)^2,\quad j=1,2.$$

Remark 4.

1. Following Proposition 5, the constant solutions of equation (4.3) satisfying $\epsilon(a-h^2)>0$ produce the PMC-surfaces of $M^2(\epsilon)\times M^2(\epsilon)$ with $C_1=C_2=0$, and so, from Theorem 1, they are the examples described in Example 1.
2. All the previous examples are invariant under the 1-parametric group of isometries $\{I(\theta)\times\mathrm{Id},\ \theta\in\mathbb{R}\}$ of $M^2(\epsilon)\times M^2(\epsilon)$, where $I(\theta): M^2(\epsilon)\to M^2(\epsilon)$ is the isometry given by

$$\begin{array}{ccc} a>0 & a<0 & a=0 \\ \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} & \begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\theta & \sinh\theta\\ 0 & \sinh\theta & \cosh\theta\end{pmatrix} & \begin{pmatrix}1-\frac{\theta^2}{2} & \theta & \frac{\theta^2}{2}\\ -\theta & 1 & \theta\\ -\frac{\theta^2}{2} & \theta & 1+\frac{\theta^2}{2}\end{pmatrix}\end{array}.$$

Proof. First it is easy to check that, in the three cases,

$$|\phi_x|^2=\epsilon[(a-h^2)-\epsilon b(1+(h-c)^2)],\quad |\phi_y|^2=\epsilon(a-h^2),$$
$$\langle\phi_x,\phi_y\rangle=0,\quad\langle\phi_x,\phi_{xy}\rangle=0,\quad\langle\phi_y,\phi_{xy}\rangle=-\epsilon hh',$$

and hence $|\Phi_x|^2=|\Phi_y|^2=\epsilon(a-h^2)$ and $\langle\Phi_x,\Phi_y\rangle=0$, which say that Φ is a conformal immersion. Now from a direct computation we have that

$$\phi_{xx}+\phi_{yy}=-\frac{bh'(h-c)}{|\phi_x|^2}\phi_x-\epsilon(|\phi_x|^2+|\phi_y|^2)\phi.$$

Also, the definition of the curve ψ means that

$$\psi_{xx}+\psi_{yy}=\psi_{xx}=\frac{bh'(h-c)}{|\psi_x|^2}\psi_x-\frac{\epsilon b(a-h^2)}{|\psi_x|^2}J\psi_x-\epsilon|\psi_x|^2\psi.$$

Therefore, as Φ is a conformal immersion, $H=(\Phi_{xx}+\Phi_{yy})^T/2\epsilon(a-h^2)$, where $(\,)^T$ denotes the tangential component to $M^2(\epsilon)\times M^2(\epsilon)$. Using the above formulae we get

$$H = \frac{1}{2\epsilon(a-h^2)}\left(-\frac{bh'(h-c)}{|\phi_x|^2}\phi_x, \frac{bh'(h-c)}{|\psi_x|^2}\psi_x - \frac{\epsilon b(a-h^2)}{|\psi_x|^2}J\psi_x\right).$$

From this equation the length of H is $|H|^2 = b/4$, and after a long straightforward computation we obtain

$$\bar{\nabla}_{\partial_x} H = -\frac{b(a-ch)}{2(a-h^2)}\Phi_x, \quad \bar{\nabla}_{\partial_y} H = \frac{bh(h-c)}{2(a-h^2)}\Phi_y,$$

which proves that H is parallel in the normal bundle.

Finally, in order to compute the Hopf differentials we only need to know that

$$\tilde{H} = \frac{1}{2\epsilon(a-h^2)}\left(\frac{bh'}{|\phi_x|^2}\phi_x, -\frac{bh'}{|\psi_x|^2}\psi_x - \frac{b\epsilon(a-h^2)(h-c)}{|\psi_x|^2}J\psi_x\right). \qquad \square$$

Now, we are going to analyze the solutions of equation (4.3). As the degree of the polynomial appearing in it is less than 5, the solutions are elliptic functions which can be obtained by knowing the roots of the polynomial. It is clear that every solution h of equation (4.3) does not have to satisfy the condition $\epsilon(a-h^2) > 0$, which is necessary to define a PMC-surface (without singularities).

If we denote $p(t) = a - t^2$ and $q(t) = -(1+\epsilon b)t^2 + 2\epsilon bct - \epsilon b(1+c^2) + a$, equation (4.3) becomes $(h')^2 = p(h)q(h)$. The condition $\epsilon(a-h^2) > 0$ means that $\epsilon p(h) > 0$, and so we obtain that $\epsilon q(h) \geq 0$ on a certain interval of $\mathbb{R}$. This inequality of the two degree polynomial $q(h)$ gives us the restrictions

$$\begin{aligned} (1+b)(a-b) &\geq bc^2 && \text{if } \epsilon = +1, \\ bc^2 &\geq (b-1)(a+b) && \text{if } \epsilon = -1 \text{ and } 4|H|^2 = b > 1, \\ c \neq 0 \text{ or } a &\leq -1 && \text{if } \epsilon = -1 \text{ and } 4|H|^2 = b = 1 \end{aligned} \tag{4.4}$$

about the parameters a, b and c. On the other hand, it is possible to obtain all the solutions of equation (4.3) in terms of Jacobi elliptic functions (see [3]), and a deep analysis of them shows that the conditions appearing in (4.4) are also sufficient in order for the solutions of equation (4.3) to satisfy $\epsilon(a-h^2) > 0$. So

> *The solutions h of equation* (4.3) *verify $\epsilon(a-h^2) > 0$ if and only if the parameters a, b and c of the equation satisfy the restrictions of* (4.4).

The integration of equation (4.3) is not complicated, but it is very long because the roots of the polynomial appearing in the equation are of a different nature depending on the values of the parameters a, b and c, and hence the solutions of the equation are also of a different nature. To illustrate the integration, we are going to integrate it in a particular case because the solution will produce a nice 1-parameter family of PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$.

Example 2. We consider, in equation (4.3), $\epsilon = -1$, $c = 0$, $b = 1$, and from (4.4), $a \leq -1$. In this case the equation becomes

$$(h')^2(x) = (a+1)(a-h^2(x))$$

and the solution h with $h(0) = 0$ is given by

$$h(x) = \sqrt{-a}\sinh(\sqrt{-(1+a)}x).$$

Hence $\epsilon(a-h^2) = -a\cosh^2(\sqrt{-(1+a)}x)$, and denoting $\lambda = \sqrt{-(1+a)}$ we get, from Proposition 5, that for all $\lambda \geq 0$, $\Phi_\lambda = (\phi, \psi) : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$ given by

$$\phi(x,y) = \Big(\sinh(\lambda x), \cosh(\lambda x)\sinh(\sqrt{1+\lambda^2}y), \cosh(\lambda x)\cosh(\sqrt{1+\lambda^2}y)\Big)$$

and $\psi(x,y) = \psi(x)$ the curve in $\mathbb{H}^2$ parametrized by $|\psi'|^2 = 1 + (1+\lambda^2)\sinh^2(\lambda x)$ and with curvature $k_\psi = \frac{-\sqrt{1+\lambda^2}\cosh^2(\lambda x)}{|\psi'|^3}$ is a *PMC-conformal embedding of the complete surface* $(\mathbb{R}^2, (1+\lambda^2)\cosh^2(\lambda x)(dx^2+dy^2))$ *with* $4|H|^2 = 1$, $\Theta_1 = \Theta_2 = \frac{\lambda^2}{4}(dz)^2$. The Gauss curvature of this metric is given by $K(x) = \frac{-\lambda^2}{\cosh^4(\lambda x)}$. When $\lambda = 0$, that is, $a = -1$, Φ_0 is the product of a geodesic and a horocycle, i.e. $\hat{P}_0$ in Example 1.

Theorem 3. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of a surface Σ. Then the extrinsic normal curvature vanishes, $\bar{K}^\perp = 0$, if and only if Φ is locally congruent to*

1. *a CMC-surface of $M^2(\epsilon) \times \mathbb{R}$,*
2. *one of the examples described in Example* 1*,*
3. *one of the examples described in Proposition* 5*.*

Remark 5. Although $\bar{K}^\perp$ is well defined only for orientable surfaces, the equation $\bar{K}^\perp = 0$, which means $C_1^2 = C_2^2$, makes sense even for non-orientable surfaces.

Proof. First, it is clear that the examples given in items 1 and 2 satisfy $\bar{K}^\perp = 0$. Also, from Proposition 5, the examples given in item 3 satisfy $\bar{K}^\perp = 0$.

Suppose now that $\bar{K}^\perp = 0$, i.e. $C_1^2 = C_2^2$. Taking the two-fold oriented covering of Σ if necessary, we can assume that Σ is orientable.

From (3.8) we have that

$$(\Delta + F)(C_1 - C_2) = 0,$$

where $F = 4|H|^2 - 2K + \epsilon(1+C_1^2) = 4|H|^2 - 2K + \epsilon(1+C_2^2)$. Now using classical results from elliptic theory (see [6]), we obtain that either $C_1 = C_2$ or $A = \{p \in \Sigma \,|\, C_1(p) = C_2(p)\}$ is a set of curves in Σ. Then as $C_1^2 = C_2^2$ we have that $C_1 + C_2 = 0$ on $\Sigma \setminus A$ and hence on Σ. So we have two possibilities, $C_1 = C_2$ or $C_1 = -C_2$. It is clear that the surfaces with $C_1 = -C_2$ can be obtained as the images of the surfaces with $C_1 = C_2$ under the isometry $F : M^2(\epsilon) \times M^2(\epsilon) \to M^2(\epsilon) \times M^2(\epsilon)$ given by $F(p,q) = (q,p)$.

Hence, we can assume that $C_1 = C_2$. Then using the integrability equations (3.4) we have that the 1-differential

$$\begin{aligned}\Omega(z) &= (\gamma_2(z) - \gamma_1(z))(\mathrm{d}z)\\ &= \frac{1}{\sqrt{2}|H|}\Big(\langle (J_2 - J_1)\Phi_z, H\rangle - i\langle (J_2 + J_1)\Phi_z, \tilde{H}\rangle\Big)(\mathrm{d}z),\end{aligned}$$

which is globally well defined because it is invariant by a conformal change of parameter, is holomorphic. So either $\Omega \equiv 0$ or Ω has isolated zeroes. In the first case we have that $\gamma_1 = \gamma_2$, and using the fact that $(C_1)_z = (C_2)_z$ and (3.1), we obtain that $f_1 = f_2$. Now, as in the proof of Theorem 1, it follows that Φ factorizes through a CMC-immersion of $M^2(\epsilon) \times \mathbb{R}$, and we obtain case 1.

Now we study the case in which the holomorphic differential Ω is non-zero. Outside its zeroes we can normalize it as $\gamma_2 - \gamma_1 = 2\sqrt{2}|H|$. As $C_1 = C_2$ we have that $|\gamma_1|^2 = |\gamma_2|^2$, and so $\Re(\gamma_1) = -\sqrt{2}|H|$. Hence

$$\gamma_1 = -\sqrt{2}|H| + ig, \quad \gamma_2 = -\bar{\gamma}_1,$$

for a certain function $g : \Sigma \to \mathbb{R}$. Now, using the integrability equations (3.4) we obtain that $g_{\bar{z}} = -\frac{e^{2u}}{\sqrt{2}}|H|C_1$, which implies that $g(x,y) = g(x)$ and it satisfies $g' = -\sqrt{2}e^{2u}|H|C_1$, where $'$ stands for ∂/∂_x. So from this equation and $e^{2u}(1 - C_1^2) = 2(2|H|^2 + g^2)$ we deduce that u and C_1 also satisfy $u(x,y) = u(x)$ and $C_1(x,y) = C_1(x)$.

Now, as $(\gamma_j)_z = (\gamma_j)_{\bar{z}}$, $j = 1,2$, again using the integrability equations we have that $u'\gamma_j - 2iC_jf_j = -\frac{ie^{2u}}{\sqrt{2}}|H|C_j$, $j = 1,2$. As $C_1 = C_2$ and $\gamma_2 = -\bar{\gamma}_1$, the above equations imply that $C_1(\bar{f}_1 - f_2) = 0$. Hence we have that either $C_1 = C_2 = 0$ and Φ is the product of two curves of constant curvature and we prove case 2 or $C_1^{-1}(\{0\})$ is a set of curves and so $\bar{f}_1 = f_2$ on $\Sigma - C_1^{-1}(\{0\})$ and then on Σ.

Now we study this third case: $C_1 = C_2$ non-null and $\bar{f}_1 = f_2$. As C_1 is a function of x and $(\Delta + F)(C_1) = 0$, the zeroes of C_1 are isolated. As $\bar{\gamma}_1^2 = \gamma_2^2$, $\bar{f}_1 = f_2$ and the Hopf differentials are holomorphic, we obtain that $\Theta_1 = \mu(dz)^2$ and $\Theta_2 = \bar{\mu}(dz)^2$ for a certain complex number μ. This says that

$$2\sqrt{2}|H|f_1 + \frac{\epsilon}{2}\gamma_1^2 = \mu.$$

In this situation, it is not difficult to see that from the third equations in (3.5) we have that

$$\begin{aligned} u' &= C_1\left(\frac{\Im(\mu)}{2|H|^2} + \frac{\epsilon g}{\sqrt{2}|H|}\right), \\ e^{2u}(1 - C_1^2) &= 4|H|^2 + 2g^2, \quad g' = -\sqrt{2}e^{2u}|H|C_1. \end{aligned} \tag{4.5}$$

We are going to integrate the Frenet equations. First of all, from (3.2) and (3.3) we obtain that $J_1\Phi_z - J_2\Phi_z = 2\Re(\gamma_1\xi)$ and $J_1\Phi_z + J_2\Phi_z = 2iC_1\Phi_z + 2i\Im(\gamma_1\xi)$. So, taking into account the definitions of J_j, we get that $(0, J\psi_z) = \Re(\gamma_1\xi)$ and $(J\phi_z, 0) = iC_1\Phi_z + i\Im(\gamma_1\xi)$. Hence

$$J\psi_y = 0, \text{ i.e. } \quad \psi(x,y) = \psi(x), \quad \text{and} \quad J\phi_x = C_1\phi_y. \tag{4.6}$$

On the other hand, as $\bar{f}_1 = f_2$, from the Frenet equations we have that

$$\Phi_{zz} = u'\Phi_z + 2\Re(f_1\xi) + \epsilon\frac{e^{2u}(1 - C_1^2)}{4}\hat{\Phi},$$

which implies, considering the imaginary part of this equation, that $\Phi_{xy} = u'\Phi_y$. This equation is irrelevant for the component ψ, but for the other component ϕ, the equation $\phi_{xy} = u'\phi_y$ can be integrated to obtain that

$$\phi(x,y) = e^{u(x)}F(y) + G(x), \tag{4.7}$$

for certain vectorial functions F and G.

From (4.6) and as Φ is a conformal map it follows that

$$|\phi_x|^2 = C_1^2e^{2u}, \quad |\phi_y|^2 = e^{2u} \quad \text{and} \quad \langle\phi_x, \phi_y\rangle = 0.$$

Now taking into account that $\phi_{xy} = u'\phi_y$, it is easy to obtain that

$$\phi_{yy} + \frac{u'}{C_1^2}\phi_x + \epsilon e^{2u}\phi = 0.$$

This equation joint with (4.7) says that the function F satisfies the following O.D.E.:

$$F''(y) + (\frac{u'(x)^2}{C_1(x)^2} + \epsilon e^{2u(x)})F(y) + \tilde{G}(x) = 0,$$

for a certain vectorial function $\tilde{G}$. From this, and by taking derivatives with respect to y and x we get that $0 = (\frac{u'(x)^2}{C_1(x)^2} + \epsilon e^{2u(x)})'F'(y)$. But $e^{2u} = |\Phi_y|^2 = |\phi_y|^2 = e^{2u}|F'|^2$, which implies that $|F'|^2 = 1$. So from the above equations we finally obtain that

$$\frac{u'(x)^2}{C_1(x)^2} + \epsilon e^{2u(x)} = a \in \mathbb{R}, \quad \tilde{G}(x) = -\tilde{G}_0 \in \mathbb{R}^3\,(\mathbb{R}^3_1), \quad \forall (x,y) \in \Sigma, \tag{4.8}$$

and so finally F satisfies the following O.D.E.:

$$F''(y) + aF(y) - \tilde{G}_0 = 0.$$

The solution of this equation is given by

$$\begin{aligned} F(y) &= \cos(\sqrt{a}y)H_1 + \sin(\sqrt{a}y)H_2 + \frac{\tilde{G}_0}{a}, \quad a > 0, \\ F(y) &= \cosh(\sqrt{-a}y)H_1 + \sinh(\sqrt{-a}y)H_2 + \frac{\tilde{G}_0}{a}, \quad a < 0, \\ F(y) &= \frac{y^2}{2}\tilde{G}_0 + yH_1 + H_2, \quad a = 0 \end{aligned}$$

with

$$\begin{aligned} |H_1|^2 = |H_2|^2 = 1/a, &\quad \langle H_1, H_2\rangle = 0, \quad a > 0, \\ |H_1|^2 = -|H_2|^2 = 1/a, &\quad \langle H_1, H_2\rangle = 0, \quad a < 0, \\ |H_1|^2 = 1, |\tilde{G}_0| = 0, &\quad \langle H_1, \tilde{G}_0\rangle = 0, \quad a = 0. \end{aligned}$$

Let us observe that this implies that the case $a < 0$ is possible only when $\epsilon = -1$. Using this information in (4.7) we obtain that

$$\begin{aligned} \phi(x,y) &= e^{u(x)}\cos(\sqrt{a}y)H_1 + e^{u(x)}\sin(\sqrt{a}y)H_2 + \hat{G}(x), \quad a > 0, \\ \phi(x,y) &= e^{u(x)}\cosh(\sqrt{-a}y)H_1 + e^{u(x)}\sinh(\sqrt{-a}y)H_2 + \hat{G}(x), \quad a < 0, \\ \phi(x,y) &= e^{u(x)}\frac{y^2}{2}\tilde{G}_0 + e^{u(x)}yH_1 + \hat{G}(x), \quad a = 0, \end{aligned}$$

for a certain vectorial function $\hat{G}$.

As $\langle \phi, \phi_y\rangle = 0$, we deduce from the above equations that:

* $\langle \hat{G}(x), H_j\rangle = 0$, $j = 1, 2$ when $a \neq 0$,
* $\langle \hat{G}(x), H_1\rangle = 0$ and $\langle \hat{G}(x), \tilde{G}_0\rangle = -e^{u(x)}$ when $a = 0$.

Let us observe that the case $a = 0$ is possible only when $\epsilon = -1$ since $|\tilde{G}_0|^2 = 0$ and $\langle \hat{G}(x), \tilde{G}_0\rangle \neq 0$.

Now, up to an isometry in $\mathbb{R}^3$ or $\mathbb{R}^3_1$ we can choose $H_1 = (1/\sqrt{a},0,0)$, $H_2 = (0,1/\sqrt{a},0)$ and $\hat{G} = h(x)(0,0,1/\sqrt{a})$ when $a>0$, choose $H_1 = (0,0,1/\sqrt{-a})$, $H_2 = (0,1/\sqrt{-a},0)$ and $\hat{G} = h(x)(1/\sqrt{-a},0,0)$ when $a<0$, and choose $H_1 = (0,1,0)$, $\tilde{G}_0 = (1,0,1)$ and $\hat{G} = (\frac{1-h^2(x)}{2h(x)},0,\frac{1-h^2(x)}{2h(x)}+e^{u(x)})$ when $a=0$, for a certain function h. Therefore, the above equations become

$$\phi(x,y) = \frac{1}{\sqrt{a}}\Big(e^{u(x)}\cos(\sqrt{a}y), e^{u(x)}\sin(\sqrt{a}y), h(x)\Big), \qquad a>0,$$

$$\phi(x,y) = \frac{1}{\sqrt{-a}}\Big(h(x), e^{u(x)}\sinh(\sqrt{-a}y), e^{u(x)}\cosh(\sqrt{-a}y)\Big), \qquad a<0,$$

$$\phi(x,y) = \left(\frac{e^{u(x)}}{2}y^2+\frac{1-h^2(x)}{2h(x)}, e^{u(x)}y, \frac{e^{u(x)}}{2}y^2+\frac{1-h^2(x)}{2h(x)}+e^{u(x)}\right), \quad a=0,$$

where $h(x)^2+\epsilon e^{2u(x)} = a$.

To study the curve $\psi(x)$, from (4.6) and as $|\phi_y|^2 = e^{2u}$ we have that

$$|\psi_x|^2 = e^{2u}-|\phi_x|^2 = e^{2u}-|J\phi_x|^2 = e^{2u}-C_1^2|\phi_y|^2 = e^{2u}(1-C_1^2) = 4|H|^2+2g^2.$$

Moreover, taking into account the Frenet equation for $\Phi_{z\bar{z}}$, (3.2) and (3.3) we have

$$\begin{aligned}\langle\psi_{xx},J\psi_x\rangle &= \langle 4\Phi_{z\bar{z}},(0,J\psi_x)\rangle = 2e^{2u}\langle H,(0,J\psi_x)\rangle\\ &= \sqrt{2}e^{2u}|H|\langle\xi+\bar{\xi},(0,J\psi_x)\rangle = \frac{e^{2u}|H|}{\sqrt{2}|\gamma_1|^2}(\gamma_1+\bar{\gamma}_1)\langle J\psi_x,J\psi_x\rangle\\ &= \frac{1}{|\gamma_1|^2}(-2|H|^2e^{2u})|\psi_x|^2 = -\frac{4|H|^2}{1-C_1^2}|\psi_x|^2 = -4|H|^2e^{2u},\end{aligned}$$

and so $k_\psi(x) = -4|H|^2e^{2u}/|\psi_x|^3$.

To check that these examples are those given in Proposition 5, we only need to obtain the O.D.E. that h satisfied. From (4.8) and as $h^2+\epsilon e^{2u} = a$ we have that $h = \pm\frac{\epsilon u'}{C_1}$, and so (4.5) implies that $h = \pm\left(\frac{\epsilon\Im(\mu)}{2|H|^2}+\frac{g}{\sqrt{2}|H|}\right)$. From (4.5) again we obtain that $h' = \mp e^{2u}C_1$, and then using (4.5) one more time we obtain

$$\begin{aligned}(h')^2 &= C_1^2e^{4u} = e^{2u}(e^{2u}-4|H|^2-2g^2)\\ &= (a-h^2)(a-h^2-\epsilon 4|H|^2(1+(h\mp\epsilon\frac{\Im(\mu)}{2|H|^2})^2).\end{aligned}$$

Now, if we define $b = 4|H|^2$ and $c = \pm\epsilon\frac{\Im(\mu)}{2|H|^2}$, we obtain that h satisfies the equation of Proposition 5 and the curve $\psi(x)$ satisfies

$$|\psi'|^2 = |\psi_x|^2 = b(1+(h-c)^2) \quad\text{and}\quad k_\psi = -\frac{\epsilon b(a-h^2)}{|\psi'|^3}.$$

So, in this third case our surface is one of the examples described in Proposition 5 and we have completed the proof. □

Theorem 4. *Let $\Phi:\Sigma\to M^2(\epsilon)\times M^2(\epsilon)$ be a PMC-immersion of an orientable surface Σ. The Hopf differentials vanish, i.e. $\Theta_1 = \Theta_2 = 0$, if and only if one of the three following possibilities happens:*

1. *$\Phi(\Sigma)$ lies in $M^2(\epsilon)\times\mathbb{R}$ as a CMC-surface with a vanishing Abresch-Rosenberg differential,*
2. *$\epsilon = -1$, $4|H|^2 = 1$, and locally Φ is the product of two hypercycles α and β of $\mathbb{H}^2$ with curvatures $k_\alpha^2+k_\beta^2 = 1$,*

3. $\epsilon = -1$, $4|H|^2 < 1$ *and locally* Φ *is* $\Phi_0 = (\phi_0, \psi_0) : (-\pi/2, \pi/2) \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{H}^2$, *where*

$$\phi_0(x,y) = \frac{1}{\cos x}\left(\sin x, \sinh\frac{y}{\sqrt{1-4|H|^2}}, \cosh\frac{y}{\sqrt{1-4|H|^2}}\right),$$

and ψ_0 *is the curve in* $\mathbb{H}^2$ *given by* $|\psi_0'(x)| = \frac{2|H|}{\sqrt{1-4|H|^2}\cos x}$ *and with curvature* $k_0(x) = -\frac{\cos x}{2|H|}$.

Remark 6. Here we see that Φ_0 is a conformal embedding and the induced metric $\frac{1}{(1-4|H|^2)\cos^2 x}(dx^2+dy^2)$ is complete and with constant curvature $4|H|^2-1$. Moreover, $C_1^2 = C_2^2 = 1-4|H|^2$.

In [1, 10] Abresch, Rosenberg and Leite describe, for $|H|^2 < 1/4$, a CMC-isometric embedding Φ_{ARL} of a simply-connected complete surface with constant curvature $4|H|^2-1$ in $\mathbb{H}^2\times\mathbb{R}$ and with a vanishing Abresch-Rosenberg differential. So Φ_0 and Φ_{ARL}, considered as a PMC-surface in $\mathbb{H}^2\times\mathbb{H}^2$, are two non-congruent PMC-isometric embeddings of a simply-connected complete surface with constant curvature $4|H|^2-1$ into $\mathbb{H}^2\times\mathbb{H}^2$.

Proof. Suppose that $\Theta_1 = \Theta_2 = 0$. Then we have that $16|H|^2|f_j|^2 = \frac{|\gamma_j|^4}{2}$, $j = 1,2$, which means that

$$|H|^2 + \epsilon C_j^2 - K = \frac{(1-C_j^2)^2}{16|H|^2}, \quad j = 1,2. \tag{4.9}$$

From this equation we easily get that

$$(C_1^2 - C_2^2)\left(16\epsilon|H|^2 + (1-C_1^2) + (1-C_2^2)\right) = 0.$$

If $\epsilon = 1$, from the above equation we obtain that $C_1^2 = C_2^2$. If $\epsilon = -1$, on the open set $O = \{p \in \Sigma \,|\, C_1^2(p) \neq C_2^2(p)\}$ we have that

$$C_1^2 + C_2^2 = 2(1-8|H|^2). \tag{4.10}$$

But on O, $C_1\nabla C_1 = -C_2\nabla C_2$, and then using (3.7) and (4.9), we obtain that

$$C_1^2(1-C_1^2)(1-C_1^2-4|H|^2)^2 = C_2^2(1-C_2^2)(1-C_2^2-4|H|^2)^2.$$

Using (4.10) we obtain that C_j, $j = 1,2$, are roots of a non-trivial polynomial of degree 8, which implies that C_j, $j = 1,2$, are constant on each connected component of O. But using (3.7) again we get that either $C_j^2 = 1$ or $1 - C_j^2 = 4|H|^2$ on each connected component of O. This contradicts (4.10) on O, and so $O = \emptyset$, and hence in this case ($\epsilon = -1$) $C_1^2 = C_2^2$, too.

Therefore $\bar{K}^\perp = 0$, and from Theorem 3 we have three possibilities. In the first case, $\Phi(\Sigma)$ lies as a CMC-surface in $M^2(\epsilon)\times\mathbb{R}$ and, from Lemma 1, it has a vanishing Abresch-Rosenberg differential.

In the second case, $C_j = 0$, $j = 1,2$, and Φ is locally one of the examples of Example 1. As $\Theta_1 = \Theta_2 = 0$, Lemma 1.2 says that $\epsilon = -1$ and $4|H|^2 = 1$. This fact only happens for the product of two suitable hypercycles or for the product of a horocycle and a geodesic, but the latter is a particular case of case 1. So we have proved case 2.

Finally, in the third case we have a PMC-surface described in Proposition 5 with $\Theta_j = 0$, $j = 1, 2$. But then $a = -1$ and $c = 0$, which implies that $\epsilon = -1$. In this case, equation (4.3) becomes

$$(h')^2 = (1 - 4|H|^2)(1 + h^2)^2,$$

which implies that $4|H|^2 \leq 1$.

If $4|H|^2 = 1$, then h is constant and Φ is congruent to either the product of a geodesic and a horocycle, when $h = 0$, or the product of two suitable hypercycles, when $h \neq 0$. The first case, up to a congruence, is included in case 1 and the second one is included in case 2.

If $4|H|^2 < 1$, the solution of the above equation is given by $h(x) = \tan(\sqrt{1-4|H|^2}x)$ with $-\frac{\pi}{2} < \sqrt{1-4|H|^2}x < \frac{\pi}{2}$. Now, reparametrizing the immersion by $(x, y) \to \sqrt{1-4|H|^2}(x, y)$, the PMC-immersion associated to h in Proposition 5 is Φ_0. Hence we get case 3.

The converse is clear. $\square$

Corollary 1. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of a sphere Σ. Then, up to congruences, Φ is a CMC-sphere in $M^2(\epsilon) \times \mathbb{R}$.*

The examples described in possibility 3 of Theorem 4 and the examples obtained by Leite in [10] can be characterized in the following way.

Corollary 2. *Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of an orientable surface Σ. Then the extrinsic and normal extrinsic curvatures $\bar{K}$ and $\bar{K}^{\perp}$ are constant if and only if one of the two following possibilities happens:*

1. *$\bar{K} = \bar{K}^{\perp} = 0$ and Φ is locally congruent to some of the examples described in Example* 1*,*
2. *$\bar{K} = 4|H|^2 - 1$, $\bar{K}^{\perp} = 0$, and Φ is locally congruent either the example given in possibility 3 of Theorem* 4 *or the example described by Leite in* [10].

Proof. First $\bar{K}$ and $\bar{K}^{\perp}$ are constant if and only if C_j, $j = 1, 2$, are constant. Also, the examples of Example 1 satisfy $C_j = 0$, $j = 1, 2$, and the examples of possibility 3 of Theorem 4 and the one given by Leite satisfy $C_j^2 = 1 - 4|H|^2$ and $\epsilon = -1$.

On the other hand, if C_j, $j = 1, 2$ are constant, from the integrability equations (3.4) we have that $(1 - C_j^2)f_j = \frac{|H|}{\sqrt{2}}\gamma_j^2$, $j = 1, 2$, and hence, computing their lengths and using (3.6), $C_1^2 = C_2^2 = \epsilon K$. So either $C_j = 0$, $j = 1, 2$, and we obtain possibility 1 or $C_1 = C_2$ is a non-null constant. In the latter, from (3.8) we obtain that $\epsilon = -1$ and $C_j^2 = 1 - 4|H|^2$, $j = 1, 2$. Using all this information we can check that $\Theta_j = 0$, $j = 1, 2$. The result is now a consequence of Theorem 4 and [10]. $\square$

5. Examples of CMC-surfaces in $M^2(\epsilon) \times \mathbb{R}$

Following Theorem 1, the examples of PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$ described in Proposition 5 have associated pairs of CMC-surfaces of $M^2(\epsilon) \times \mathbb{R}$. As these PMC-surfaces do not factorize through CMC-surfaces of $M^2(\epsilon) \times \mathbb{R}$, the pairs of CMC-surfaces are not congruent.

Let $\Phi : I \times \mathbb{R} \to M^2(\epsilon) \times M^2(\epsilon)$ be a CMC-surface associated to a solution h of (4.3) in Proposition 5. Following the proof of Theorem 3, the Frenet data associated to this immersion are given by

$$u(z) = \log\sqrt{\epsilon(a-h^2(x))},\ C_1(z) = C_1(\bar{z}) = C_2(z),\ f_2(z) = f_2(\bar{z}) = \bar{f}_1(z),$$
$$\gamma_2(z) = -\bar{\gamma}_1(z), \quad \gamma_1(z) = \sqrt{2}|H|(1+i(h(x)-c)).$$

Hence the Frenet data associated to the pair (Φ_1, Φ_2) of CMC-surfaces of $M^2(\epsilon)\times\mathbb{R}$ (see the proof of Theorem 1) are given by

$$u(z),\ \nu_1(z) = \nu_1(\bar{z}) = \nu_2(z),\ p_2(z) = p_2(\bar{z}) = \bar{p}_1(z),$$
$$\eta_1(z) = -2|H|(y + \int_{x_0}^{x}(h(t)-c)dt),\ \eta_2(z) = -2|H|(-y + \int_{x_0}^{x}(h(t)-c)dt).$$

As the map $G : I\times\mathbb{R} \to I\times\mathbb{R}$ given by $G(z) = \bar{z}$ is an isometry of the induced metric $g = \epsilon(a-h^2(x))(\mathrm{d}x^2+\mathrm{d}y^2)$, it is easy to check that $\Phi_1\circ G$ is a CMC-surface with the same Frenet data as Φ_2, and so $\Phi_1\circ G$ and Φ_2 are congruent immersions, i.e. Φ_1 and Φ_2 are weakly congruent. So there really is only a CMC-immersion associated to each PMC-immersion of Proposition 5. In this case we can also integrate the Frenet equations of these immersions, obtaining the following family of examples.

Proposition 6. *Let a, b, c be real numbers with $b > 0$ and $h : I \subset \mathbb{R} \to \mathbb{R}$ a non-constant solution of the O.D.E.* (4.3) *satisfying $\epsilon(a-h^2(x)) > b$, $\forall x \in I$.*

Let $\eta(x,y) = \sqrt{b}\left(y + \int_{x_0}^{x}\left(h(t)-c\right)\mathrm{d}t\right)$ and $\psi : I\times\mathbb{R} \to M^2(\epsilon)$ be given by

(1) *If $E = a - \epsilon b > 0$,*

$$\psi(x,y) = \frac{1}{\sqrt{E}}\left(\sqrt{\epsilon(E-h(x)^2)}\cos(\sqrt{E}f),\ \sqrt{\epsilon(E-h(x)^2)}\sin(\sqrt{E}f),\ h(x)\right).$$

(2) *If $E < 0$ (which implies $\epsilon = -1$),*

$$\psi(x,y) = \frac{1}{\sqrt{-E}}\left(h(x),\ \sqrt{h(x)^2-E}\sinh(\sqrt{-E}f),\ \sqrt{h(x)^2-E}\cosh(\sqrt{-E}f)\right).$$

(3) *If $E = 0$ (which implies $\epsilon = -1$),*

$$\psi(x,y) = h(x)\left(f^2 - \frac{1}{4} + \frac{1}{h(x)^2},\ f,\ f^2 + \frac{1}{4} + \frac{1}{h(x)^2}\right).$$

Then $\Psi = (\psi,\eta) : I\times\mathbb{R} \to M^2(\epsilon)\times\mathbb{R}$ is a CMC-immersion where in the three cases

$$f(x,y) = y + \int_{x_0}^{x}\frac{b(c-h(t))}{\epsilon(E-h^2(t))}\mathrm{d}t.$$

All the examples described above satisfy $4|H|^2 = b$, and they are conformal immersions with the induced metric given by $\epsilon(a-h^2(x))(\mathrm{d}x^2+\mathrm{d}y^2)$ and their Abresch-Rosenberg differentials given by

$$\Theta_{AR} = \frac{\epsilon b}{8}\left(a+1-c^2-2ic\right)(\mathrm{d}z)^2.$$

Remark 7.

(1) Because $\epsilon(a-h^2(x)) > b > 0$ the parameters a, b and c have to satisfy (4.4). Reciprocally, if a, b and c satisfy (4.4), then there exists a non-constant solution $h : I \to \mathbb{R}$ with $\epsilon(a-h^2(x)) > 0$. Now from (4.3) $\epsilon(a-h^2(x)) - b \geq 0$, so, as h is non-constant, there exists $I' \subseteq I$ such that $\epsilon(a-h^2(x)) - b > 0$.

Therefore $\epsilon(a-h^2(x))>b$ (for a suitable interval I') if and only if a, b and c satisfy (4.4).

(2) All these examples are invariant under the 1-parametric group of isometries $\{I(\theta)\times\tau_\theta,\ \theta\in\mathbb{R}\}$ of $M^2(\epsilon)\times\mathbb{R}$, where $\tau_\theta:\mathbb{R}\to\mathbb{R}$ is $\tau_\theta(t)=t+\theta\sqrt{b}$ (for $a\neq 0$), $\tau_\theta(t)=t+\theta\frac{\sqrt{b}}{2}$ and $I(\theta):M^2(\epsilon)\to M^2(\epsilon)$ is the isometry given in item 2 of Remark 4.

Proof. First it is easy to check that, in the three cases,

$$|\psi_x|^2=\epsilon(a-h^2(x))-b(h(x)-c)^2,\quad |\psi_y|^2=\epsilon(a-h^2(x))-b,$$
$$\langle\psi_x,\psi_y\rangle=-b(h(x)-c),\quad \langle\psi_x,\psi_{xy}\rangle=0,\quad \langle\psi_y,\psi_{xy}\rangle=-\epsilon h(x)h'(x).$$

So taking into account the definition of η and that $\Psi=(\psi,\eta)$, we get

$$|\Psi_x|^2=|\Psi_y|^2=\epsilon(a-h^2(x)),\quad \langle\Psi_x,\Psi_y\rangle=0,$$

that is, Ψ is a conformal immersion with conformal factor $\epsilon(a-h^2(x))$. Then its mean curvature vector field is given by $H=(\Psi_{xx}+\Psi_{yy})^T/2\epsilon(a-h^2(x))$, where $()^T$ denotes the tangential component to $M^2(\epsilon)\times\mathbb{R}$. So, by a direct computation we get

$$H(x,y)=\frac{\sqrt{b}}{2}\left(\frac{\sqrt{b}}{h'(x)}\big((c-h(x))\psi_x-\psi_y\big),\frac{h'(x)}{\epsilon(a-h^2(x))}\right).$$

From this we have that Ψ is a CMC-immersion with $|H|^2=b/4$, and it is straightforward to check that the associated Abresch-Rosenberg differential is

$$2\Theta_{AR}(z)=\frac{\epsilon b}{4}\big(a+1-c^2-2ic\big)(\mathrm{d}z)^2.$$

□

From Theorem 1 and Theorem 3 we can obtain the following rigidity result for CMC-surfaces of $M^2(\epsilon)\times\mathbb{R}$.

Corollary 3. *Let $\Phi_1,\Phi_2:(\Sigma,g)\to M^2(\epsilon)\times\mathbb{R}$ be two non-congruent CMC-isometric immersions of a simply-connected surface Σ with the same mean curvatures $H_1=H_2$ and the same extrinsic sectional curvatures $\bar K_1=\bar K_2$. Then Φ_1 and Φ_2 are weakly congruent, i.e. there exists an isometry G of (Σ,g) such that $\Phi_1\circ G$ and Φ_2 are congruent, and Φ_1 is either one of the examples of Proposition* 6 *or is a cylinder over a curve of constant curvature of $M^2(\epsilon)$.*

Proof. From Theorem 1, let $\Phi:(\Sigma,g)\to M^2(\epsilon)\times M^2(\epsilon)$ be the PMC-isometric immersion associated to the pair (Φ_1,Φ_2). Then its extrinsic normal curvature is given by $\bar K^\perp=\epsilon\frac{C_1^2-C_2^2}{2}=\frac{\bar K_1-\bar K_2}{2}=0$. So, as Φ_1 and Φ_2 are not congruent, Theorem 3 says that Φ is either one of the examples of Proposition 5 or the product of two curves of constant curvature. At the beginning of this section it was proved that, in the first case, Φ_1 and Φ_2 are weakly congruent and Φ_1 is one of the examples of Proposition 6. In the second case, Remark 2 completes the proof. □

Among the examples described in Proposition 6 there are some of particular interest which we will describe.

Example 3. We consider the 1-parameter family of CMC-immersions in Proposition 6 associated to the PMC-immersions given in Example 2 ($\epsilon = -1$, $c = 0$, $b = 1$). Following the notation, for each $\lambda > 0$, $\Psi_\lambda = (\psi_\lambda, \eta_\lambda) : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$, where

$$\psi_\lambda(x,y) = \frac{\sqrt{1+\lambda^2}}{\lambda}\left(\sinh x, \cosh x \sinh y + \frac{\cosh y}{\sqrt{1+\lambda^2}}, \cosh x \cosh y + \frac{\sinh y}{\sqrt{1+\lambda^2}}\right),$$
$$\eta_\lambda(x,y) = \frac{1}{\lambda}(y + \sqrt{1+\lambda^2}\cosh x)$$

is a *CMC-conformal isometric embedding of the complete surface* $\left(\mathbb{R}^2, \frac{1+\lambda^2}{\lambda^2}\cosh^2 x(dx^2+dy^2)\right)$ *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $H = 1/2$. Its Abresch-Rosenberg differential is given by $(dz)^2/8$.

Example 4. We consider the CMC-immersion in Proposition 6 associated to the example Φ_0 of Theorem 4. Following the notation, for each real number $0 < H < 1/2$, $\Psi_0 = (\psi_0, \eta_0) :]-\pi/2, \pi/2[\times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$, given by

$$\psi_0(x,y) = \frac{1}{\sqrt{1-4H^2}}\left(\tan x, \frac{\sinh y}{\cos x} + 2H^2 e^{-y}\cos x, \frac{\cosh y}{\cos x} - 2H^2 e^{-y}\cos x\right),$$
$$\eta_0(x,y) = \frac{2H}{\sqrt{1-4H^2}}\left(y - \log\cos x\right),$$

is a *CMC-conformal isometric embedding with mean curvature* H *of the hyperbolic plane* $\left(]-\pi/2,\pi/2[\times\mathbb{R}, 1/(1-4H^2)\cos^2 x\right)$ *with curvature* $4H^2-1$ *in* $\mathbb{H}^2\times\mathbb{R}$. Its Abresch-Rosenberg differential vanishes, and it is a conformal reparametrization of the Leite example [10].

Example 5. Now we are going to obtain examples of CMC-tori in $\mathbb{S}^2 \times \mathbb{S}^1$. To do that, first we need to get periodic solutions of the O.D.E. (4.3). We consider $\epsilon = 1$, $c = 0$ and from (4.4), $a > b$, and then equation (4.3) becomes

$$(h')^2(x) = (a - h^2(x))\left(a - b - (1+b)h^2(x)\right) = q(h).$$

As the roots of the polynomial q are $\pm\sqrt{a}, \pm\sqrt{(a-b)/(1+b)}$, formula 219.00 in [3] says that the solution $h : \mathbb{R} \to \mathbb{R}$ of the above equation with $h(0) = 0$ is given by

$$h(x) = \sqrt{\frac{a-b}{1+b}}\,\mathrm{sn}(\sqrt{a(1+b)}x),$$

where sn is the sine amplitude Jacobi function with modulus $\kappa^2 = (a-b)/a(1+b)$. These solutions are periodic with period $4K(\kappa)/\sqrt{a(1+b)}$, where $K(\kappa)$ is the complete elliptic integral of the first kind.

In this case

$$\epsilon(a - h^2(x)) = a(1 - \kappa^2\,\mathrm{sn}^2(\sqrt{a(1+b)}x)) = a\,\mathrm{dn}^2(\sqrt{a(1+b)}x) > 0, \quad \forall x \in \mathbb{R},$$

where dn is the delta amplitude Jacobi function. Furthermore, $\epsilon(a - h^2(x)) > b$ because the minimum for the function dn is $\sqrt{1-\kappa^2}$, and it is easy to see that $a(1-\kappa^2) > b$ if and only if $a > b$.

Now the function f appearing in Proposition 6 is given by

$$f(x,y) = y + \frac{1}{\sqrt{a-b}}\arctan\left(\frac{\mathrm{cn}(\sqrt{a(1+b)}x)}{\sqrt{a}\,\mathrm{dn}(\sqrt{a(1+b)}x)}\right),$$

where cn is the cosine amplitude Jacobi function. Then, up to the reparametrization $(x,y) \mapsto \frac{1}{\sqrt{a(1+b)}}(x,y)$, the associated CMC-immersion $\Phi_{a,b} = (\phi_{a,b}, \eta_{a,b}) : \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R}$ is given by

$$\phi_{a,b}(x,y) = \left(\frac{\sqrt{a}\,\mathrm{dn}\,x \cos(\kappa y) - \mathrm{cn}\,x \sin(\kappa y)}{\sqrt{1+a}}, \frac{\sqrt{a}\,\mathrm{dn}\,x \sin(\kappa y) + \mathrm{cn}\,x \cos(\kappa y)}{\sqrt{1+a}}, \frac{\mathrm{sn}\,x}{\sqrt{1+b}} \right),$$

$$\eta_{a,b}(x,y) = \frac{\sqrt{b}}{\sqrt{1+b}} \log(\mathrm{dn}\,x - \kappa\,\mathrm{cn}\,x) + \frac{\sqrt{b}}{\sqrt{a(1+b)}} y.$$

We consider the local isometry $t \in \mathbb{R} \mapsto \frac{\sqrt{b}}{\sqrt{a-b}} e^{i \frac{\sqrt{a-b}}{\sqrt{b}} t} \in \mathbb{S}^1(\frac{\sqrt{b}}{\sqrt{a-b}})$ and the CMC-immersion

$$\hat{\Phi}_{a,b} = (\phi_{a,b}, \hat{\eta}_{a,b}) : \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{S}^1(\frac{\sqrt{b}}{\sqrt{a-b}}),$$

where $\hat{\eta}_{a,b}(x,y) = \frac{\sqrt{b}}{\sqrt{a-b}} e^{i \frac{\sqrt{a-b}}{\sqrt{1+b}} \log(\mathrm{dn}\,x - \kappa\,\mathrm{cn}\,x)} e^{i\kappa y}$.

It is clear that $\hat{\Phi}$ is invariant under the group $G_{a,b}$ of transformations of $\mathbb{R}^2$ generated by

$$(x,y) \mapsto (x + 4K(\kappa), y), \quad (x,y) \mapsto \left(x, y + \frac{2\pi}{\kappa}\right).$$

If $T_{a,b} = \mathbb{R}^2/G_{a,b}$ is the associated torus and $P : \mathbb{R}^2 \to T_{a,b}$ the projection, then the induced immersion

$$\tilde{\Phi}_{a,b} : T_{a,b} \to \mathbb{S}^2 \times \mathbb{S}^1(\frac{\sqrt{b}}{\sqrt{a-b}}), \quad P(x,y) \mapsto \hat{\Phi}_{a,b}(x,y),$$

defines a CMC-conformal immersion of the torus $T_{a,b}$ into $\mathbb{S}^2 \times \mathbb{S}^1(\frac{\sqrt{b}}{\sqrt{a-b}})$.

We are going to see that $\tilde{\Phi}_{a,b}$ is an embedding. In fact, if $\tilde{\Phi}_{a,b}(P(x,y)) = \tilde{\Phi}_{a,b}(P(\hat{x},\hat{y}))$, with $x, \hat{x} \in [0, 4K(\kappa)[$, $y, \hat{y} \in [0, 2\pi/\kappa[$, then we have that $\mathrm{sn}\,x = \mathrm{sn}\,\hat{x}$ and then either $x = \hat{x}$ or $x, \hat{x} \in [0, 2K(\kappa)]$ and $x + \hat{x} = 2K(\kappa)$ or $x, \hat{x} \in [2K(\kappa), 4K(\kappa)]$ and $x + \hat{x} = 6K(\kappa)$. In the first case, looking at the immersion we obtain that $y = \hat{y}$. In the other two cases, $\mathrm{cn}\,\hat{x} = -\mathrm{cn}\,x$ and $\mathrm{dn}\,\hat{x} = \mathrm{dn}\,x$. So, looking again at the immersion we easily get that

$$\cos(\kappa\hat{y} - \kappa y) = \frac{a\,\mathrm{dn}^2 x - \mathrm{cn}^2 x}{a\,\mathrm{dn}^2 x + \mathrm{cn}^2 x}, \ \cos(\kappa\hat{y} - \kappa y) = \cos\log\left(\left(\frac{\mathrm{dn}\,x - k\,\mathrm{cn}\,x}{\mathrm{dn}\,x + k\,\mathrm{cn}\,x} \right)^{\frac{\sqrt{a-b}}{\sqrt{1+b}}} \right).$$

From these equations we obtain that $x = K(\kappa)$ or $x = 3K(\kappa)$, which implies that $\hat{x} = x$. Again, $y = \hat{y}$, and so our immersion is an embedding.

We can summarize the above reasoning in the following result.

Proposition 7. *For each pair of real numbers $0 < b < a$, the immersion $\hat{\Phi}_{a,b} : T_{a,b} \to \mathbb{S}^2 \times \mathbb{S}^1(\frac{\sqrt{b}}{\sqrt{a-b}})$ described above is a CMC-conformal embedding of the rectangular torus $T_{a,b}$ with mean curvature $H = \sqrt{b}/2$. Its Abresch-Rosenberg differential is $\Theta_{AR} = (b(1+a)/8a(1+b))(dz)^2$.*

Remark 8. It is clear that $\tau : T_{a,b} \to T_{a,b}$, defined by

$$\tau(P(x,y)) = P\left(-x, y + \frac{\pi}{k}\right),$$

is an isometry of $T_{a,b}$ with $\tau^2 = \mathrm{Id}$. Because $\tilde{\Phi}_{a,b}\big(\tau P(x,y)\big) = -\tilde{\Phi}_{a,b}\big(P(x,y)\big)$ for all $(x,y) \in \mathbb{R}^2$, $\tilde{\Phi}_{a,b}$ induces a CMC-embedding of the Klein bottle $B_{a,b} = T_{a,b}/\langle\tau\rangle$ in $\mathbb{RP}^2 \times \mathbb{RP}^1(\sqrt{b}/\sqrt{a-b})$, where $\mathbb{RP}^2$ denotes the real projective plane with constant curvature 1 and $\mathbb{RP}^1(\sqrt{b}/\sqrt{a-b})$ denotes the real projective line with constant curvature $\sqrt{a-b}/\sqrt{b}$.

6. Compact PMC-surfaces

In this section we are going to prove some properties of *compact* PMC-surfaces of $M^2(\epsilon) \times M^2(\epsilon)$. Let $\Phi : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be a PMC-immersion of an orientable surface Σ. We define two vector fields X_j, $j = 1,2$, tangent to Σ as the tangential components of $J_j\tilde{H}$,

$$J_1\tilde{H} = X_1 + C_1 H, \quad J_2\tilde{H} = X_2 - C_2 H.$$

In particular, we have that $|X_j|^2 = |H|^2(1 - C_j^2)$, $j = 1,2$. Differentiating these equations and taking tangential components we obtain that

$$\nabla_v X_1 = C_1\, A_H v - C_1 J^\Sigma A_{\tilde{H}} v, \quad \nabla_v X_2 = -C_2\, A_H v - C_2 J^\Sigma A_{\tilde{H}} v$$

for any tangent vector v, where J^Σ is the complex structure of the Riemann surface Σ. From here we obtain that the divergence of X_j and the differential of the 1-forms $\alpha^{X_j}(v) = \langle X_j, v\rangle$ are given by

$$\operatorname{div} X_j = (-1)^{j+1} 2C_j|H|^2, \quad d\alpha^{X_j} = 0,\ j = 1,2. \tag{6.1}$$

Now, using the above properties and the fact that $|X_j|^2 = |H|^2(1-C_j^2)$, the Bochner formula becomes

$$\frac{1}{2}\Delta(1 - C_j^2) = K(1 - C_j^2) + (-1)^{j+1}2\langle\nabla C_j, X_j\rangle + \frac{|\nabla X_j|^2}{|H|^2}, \quad j = 1,2.$$

Now using the expression of the covariant derivative of X_j we finally get that

$$\frac{1}{2}\Delta(1 - C_j^2) = K(1 - C_j^2) + (-1)^{j+1}2\langle\nabla C_j, X_j\rangle + 2C_j^2(\epsilon C_j^2 + 2|H|^2 - K). \tag{6.2}$$

On the other hand, as $\Delta(1 - C_j^2) = -2C_j\Delta C_j - 2|\nabla C_j|^2$, $j = 1,2$, from equations (6.2) and (3.8), we obtain that

$$|\nabla C_j|^2 = (1 - C_j^2)(\epsilon C_j^2 - K) + (-1)^j 2\langle\nabla C_j, X_j\rangle, \quad j = 1,2. \tag{6.3}$$

All these formulae have some consequences when the surface is compact.

Proposition 8. *Let $\Phi = (\phi, \psi) : \Sigma \to M^2(\epsilon) \times M^2(\epsilon)$ be an immersion of a* **compact** *orientable surface with parallel mean curvature vector. Then:*

1. $\int_\Sigma C_j\, dA = 0, \quad j = 1,2.$
2. *If $\epsilon = 1$, then the degrees of ϕ and ψ are zero.*
3. *If $K \geq 0$, then either $\Phi(\Sigma)$ is a CMC-sphere of $M^2(\epsilon) \times \mathbb{R}$ with $4|H|^2 \geq 1$ when $\epsilon = 1$ and $|H|^2 \geq 1$ when $\epsilon = -1$, or $\Phi(\Sigma)$ is a torus of Example 1.*
4. *There exists a point p with $K(p) \geq 0$ when $\epsilon = 1$ and $K(p) \geq -1$ when $\epsilon = -1$.*
5. *If one of the two holomorphic differentials $\{\Theta_j : j = 1,2\}$ vanishes, then the other also vanishes, and so $\Phi(\Sigma)$ is a CMC-sphere of $M^2(\epsilon) \times \mathbb{R}$.*

Proof. Integrating the first equation of (6.1) we prove (1). If $\epsilon = 1$, then $\phi, \psi : \Sigma \to \mathbb{S}^2$ are maps such that (see section 3)

$$\phi^*\omega = \frac{C_1 + C_2}{2}\omega_\Sigma, \quad \psi^*\omega = \frac{C_1 - C_2}{2}\omega_\Sigma,$$

which proves case 2 making use of case 1.

If $K \geq 0$, then either Σ is a sphere and Corollary 1 proves that it is a CMC-sphere of $M^2(\epsilon) \times \mathbb{R}$, or Σ is a flat torus. In the first case (4.9) becomes

$$K = H^2 + \epsilon\nu^2 - \frac{1}{16H^2}(1 - \nu^2)^2.$$

But from (4.2) and using the fact that η has a maximum and a minimum, we get that ν always takes the values 1 and -1. If $\epsilon = 1$ and p a point with $\nu(p) = 0$, then, taking into account the above equation, $K(p) \geq 0$ implies that $4H^2 \geq 1$. If $\epsilon = -1$ and p a point with $\nu^2(p) = 1$, then, taking into account the above equation, $K(p) \geq 0$ implies that $H^2 \geq 1$. Conversely, if $4H^2 \geq 1$ when $\epsilon = 1$ and $H^2 \geq 1$ when $\epsilon = -1$, the previous equation says that $K \geq 0$.

In the second case, from (6.1) we have that

$$0 = \int_\Sigma \operatorname{div}(C_j X_j)\, dA = \int_\Sigma \langle \nabla C_j, X_j \rangle\, dA + (-1)^{j+1} 2|H|^2 \int_\Sigma C_j^2\, dA, \tag{6.4}$$

which, joint with the integration of equation (6.2), gives us

$$0 = \int_\Sigma (K(1 - 3C_j^2) + 2\epsilon C_j^4)\, dA.$$

As Σ is flat, we obtain that $C_j = 0$, $j = 1, 2$, and so $\Phi(\Sigma)$ is a torus of Example 1.

Now we prove (4). From (6.3), if p is a critical point of C_j, then either $C_j^2(p) = 1$ or $K(p) = \epsilon C_j^2(p)$. If $\epsilon = 1$ and $K < 0$ or $\epsilon = -1$ and $K < -1$ the second possibility cannot happen, and hence all the critical points satisfy $C^2(p) = 1$. Taking into account Proposition 4, the function C_j is a Morse function with only maximum and minimum as critical points. Therefore the surface must be a sphere, but the Gauss-Bonnet theorem gives a contradiction. This proves case 4.

Finally, if some of the holomorphic differentials vanish, i.e. $\Theta_1 = 0$, then from (3.7), (6.3) and (6.4) we obtain that

$$16|H|^2 \int_\Sigma K\, dA = \int_\Sigma (4|H|^2 + \epsilon(1 - C_1^2))^2\, dA.$$

In particular, $\int_\Sigma K\, dA \geq 0$, and again either Σ is a sphere and so $\Theta_2 = 0$ or Σ is a torus in Example 1 with $\Theta_1 = 0$, which is impossible looking at Lemma 1. $\square$

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