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GL(n) CONTRAVARIANT MINKOWSKI VALUATIONS

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Abstract. A complete classification of all continuous GL(n) contravariant Minkowski valuations is established. As an application we present a family of sharp isoperimetric inequalities for such valuations which generalize the classical Petty projection inequality.

1. Introduction

The projection body of a convex body is one of the central notions that Minkowski introduced within convex geometry. Over the past four decades it has become evident that the projection operator, its range (i.e., the class of centered zonoids), and its polar are objects which arise naturally in a number of different areas; see, e.g., [8, 18, 27, 28, 35, 36, 39, 40]. The most important affine isoperimetric inequality for projection bodies is the *Petty projection inequality* [31] which is significantly stronger than the classical isoperimetric inequality. This remarkable inequality forms the geometric core of the affine Zhang–Sobolev inequality [26, 42] which strengthens and directly implies the classical Sobolev inequality.

The special role of projection bodies in affine convex geometry was demonstrated only recently by Ludwig [18, 20]: The projection operator was characterized as the unique continuous Minkowski valuation which is GL(n) contravariant *and* invariant under translations. Moreover, the assumption of translation invariance can be omitted when the domain of the valuation is restricted to convex bodies containing the origin. Through the seminal work of Ludwig, convex and star body valued valuations have become the focus of increased attention; see, e.g., [9, 10, 11, 21, 38]. A very recent development in this area explores the connections between these valuations and the theory of isoperimetric inequalities; see, e.g., [12, 13].

In this article we establish a complete classification of all continuous and GL(n) contravariant Minkowski valuations, without any further assumption on their behavior under translations or any restrictions on their domain. We show that there is a two parameter family of such valuations generated by the projection operator and the convex hull with the origin. We also obtain a Petty projection type inequality for each member of this family and identify *the* Petty projection inequality as the strongest inequality.

Let $\mathcal{K}^n$ denote the space of convex bodies (compact convex sets) in $\mathbb{R}^n$, $n \geq 3$, endowed with the Hausdorff metric, and let $\mathcal{K}_0^n$ denote the subset of $\mathcal{K}^n$ of bodies containing the origin. A convex body K is uniquely determined by its support function $h(K, x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$.

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Definition. A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Minkowski valuation if

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K, L, K \cup L \in \mathcal{K}^n$ and addition on $\mathcal{K}^n$ is Minkowski addition.

The map Φ is called $\mathrm{GL}(n)$ contravariant if there exists $q \in \mathbb{R}$ such that for all $\phi \in \mathrm{GL}(n)$ and all $K \in \mathcal{K}^n$,

$$\Phi(\phi K) = |\det \phi|^q \phi^{-\mathrm{T}} \Phi K,$$

where $\phi^{-\mathrm{T}}$ denotes the transpose of the inverse of ϕ.

A well-known classical example of a $\mathrm{GL}(n)$ contravariant Minkowski valuation is the projection operator: The *projection body* ΠK of $K \in \mathcal{K}^n$ is defined by

$$h(\Pi K, u) = \mathrm{vol}_{n-1}(K|u^\perp), \qquad u \in S^{n-1}, \tag{1.1}$$

where $K|u^\perp$ denotes the projection of K onto the hyperplane orthogonal to u. It was first proved by Petty [30] that for all $\phi \in \mathrm{GL}(n)$ and all $K \in \mathcal{K}^n$,

$$\Pi(\phi K) = |\det \phi| \phi^{-\mathrm{T}} \Pi K.$$

The notion of valuation plays a central role in geometry. It was the critical ingredient in the solution of Hilbert's Third Problem and has since been intimately tied to the dissection theory of polytopes (see [29]). Over the last decades the theory of valuations has evolved enormously and had a tremendous impact on various disciplines; see, e.g., [1, 2, 3, 4, 5, 7, 15, 16, 17, 19, 22].

First results on Minkowski valuations which are *rigid motion* equivariant were obtained by Schneider [32] in the 1970s; see [14, 34, 37, 38] for recent extensions of these results. The starting point for the systematic study of convex and star body valued valuations which are compatible with *linear transformations* were two highly influential articles by Ludwig [18, 20]. The following theorem is an example of the numerous results obtained there:

Theorem 1 (Ludwig [20])**.** *A map $\Phi : \mathcal{K}_0^n \to \mathcal{K}^n$ is a $\mathrm{GL}(n)$ contravariant continuous Minkowski valuation if and only if there exists a constant $c \geq 0$ such that for every $K \in \mathcal{K}_0^n$,*

$$\Phi K = c\, \Pi K.$$

If we consider Minkowski valuations that are defined on all convex bodies, then the projection operator is no longer the only $\mathrm{GL}(n)$ contravariant valuation. To see this consider the map $\Pi_{\mathrm{o}} : \mathcal{K}^n \to \mathcal{K}^n$ defined by

$$\Pi_{\mathrm{o}} K := \Pi(\mathrm{conv}(\{0\} \cup K)),$$

where $\mathrm{conv}(\{0\} \cup K)$ denotes the convex hull of K and the origin. It is easy to see that Π_{o} is a $\mathrm{GL}(n)$ contravariant continuous Minkowski valuation but, clearly, it is not a multiple of the projection operator.

The main object of this paper is to show that all $\mathrm{GL}(n)$ contravariant continuous Minkowski valuations on $\mathcal{K}^n$ are given by combinations of the projection operator and the map Π_{o}.

Theorem 2. *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a $\mathrm{GL}(n)$ contravariant continuous Minkowski valuation if and only if there exist constants $c_1, c_2 \geq 0$ such that for every $K \in \mathcal{K}^n$,*

$$\Phi K = c_1 \Pi K + c_2 \Pi_{\mathrm{o}} K.$$

Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ denote the *polar body* of a convex body K containing the origin in its interior. We write $\Phi^* K$ to denote $(\Phi K)^*$, $V(K)$ for the volume of K, and B for the Euclidean unit ball.

In the 1970s Petty established the following fundamental affine isoperimetric inequality, now known as the Petty projection inequality. (For important recent generalizations, see [25, 28].) If $K \in \mathcal{K}^n$ has nonempty interior, then

$$V(K)^{n-1} V(\Pi^* K) \leq V(B)^{n-1} V(\Pi^* B) \tag{1.2}$$

with equality if and only if K is an ellipsoid.

As an application of Theorem 2 we extend the Petty projection inequality to the entire class of GL(n) contravariant continuous Minkowski valuations which are *nontrivial*, i.e., which do not map every convex body to the origin.

Theorem 3. *Let $K \in \mathcal{K}^n$ have nonempty interior. If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a nontrivial* GL(n) *contravariant continuous Minkowski valuation, then*

$$V(K)^{n-1} V(\Phi^* K) \leq V(B)^{n-1} V(\Phi^* B).$$

If Φ is not a multiple of Π, there is equality if and only if K is an ellipsoid containing the origin; otherwise, equality holds if and only if K is an ellipsoid.

In view of Theorems 2 and 3, the natural problem arises to determine for fixed $K \in \mathcal{K}^n$ the maximum value of $V(\Phi^* K)$ among all suitably normalized (say, e.g., $\Phi B = \Pi B$) GL(n) contravariant continuous Minkowski valuations. Here, we will show that

$$V(\Phi^* K) \leq V(\Pi^* K).$$

This shows that the classical Petty projection inequality gives rise to the strongest inequality among the inequalities of Theorem 3.

2. Background material

In the following we state for quick reference some basic facts about convex bodies and the geometric inequalities needed in the proof of Theorem 3. For general reference the reader may wish to consult the book by Schneider [33].

A convex body $K \in \mathcal{K}^n$ is uniquely determined by the values of its support function $h(K, \cdot)$ on S^{n-1}. If $\phi \in \mathrm{GL}(n)$ and $K \in \mathcal{K}^n$, then for every $x \in \mathbb{R}^n$,

$$h(\phi K, x) = h(K, \phi^{\mathrm{T}} x). \tag{2.1}$$

If $K \in \mathcal{K}^n$ has nonempty interior, then K is also determined up to translations by its *surface area measure* $S_{n-1}(K, \cdot)$. Recall that for a Borel set $\omega \subseteq S^{n-1}$, $S_{n-1}(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω.

It is a well known fact that the projection function $\mathrm{vol}_{n-1}(K|\cdot^{\perp})$ of a convex body $K \in \mathcal{K}^n$ is (up to a constant) the *cosine transform* of the surface area measure of K. More precisely, we have

$$\mathrm{vol}_{n-1}(K|u^{\perp}) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS_{n-1}(K, v), \qquad u \in S^{n-1}. \tag{2.2}$$

For a compact set L in $\mathbb{R}^n$ which is star shaped with respect to the origin, its radial function is defined by $\rho(L, x) = \max\{\lambda \geq 0 : \lambda x \in L\}$, $x \in \mathbb{R}^n \backslash \{0\}$. For $\alpha, \beta \geq 0$ (not both zero) and $K, L \in \mathcal{K}^n$ containing the origin in their interiors, the

harmonic (radial) combination $\alpha \cdot K \widehat{+} \beta \cdot L$ of K and L is the convex body defined by

$$\rho(\alpha \cdot K \widehat{+} \beta \cdot L, \cdot)^{-1} = \alpha\rho(K, \cdot)^{-1} + \beta\rho(L, \cdot)^{-1}.$$

If $K \in \mathcal{K}^n$ is a convex body containing the origin in its interior, then it follows from the definitions of support and radial functions, and the definition of the polar body of K, that

$$\rho(K^*, \cdot) = h(K, \cdot)^{-1} \qquad \text{and} \qquad h(K^*, \cdot) = \rho(K, \cdot)^{-1}.$$

Thus, we have

$$\alpha \cdot K \widehat{+} \beta \cdot L = (\alpha K^* + \beta L^*)^*, \tag{2.3}$$

which shows that the set of convex bodies containing the origin in their interiors is closed under harmonic combinations.

First results on harmonic combinations of convex bodies were obtained in the early 1960s by Firey [6]. In particular, he established the following *harmonic dual Brunn–Minkowski inequality.* If $K, L \in \mathcal{K}^n$ contain the origin in their interiors, then

$$V(K \widehat{+} L)^{-1/n} \geq V(K)^{-1/n} + V(L)^{-1/n}, \tag{2.4}$$

with equality if and only if K and L are dilates.

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called $\mathrm{SL}(n)$ *contravariant* if for all $\phi \in \mathrm{SL}(n)$ and all $K \in \mathcal{K}^n$,

$$\Phi(\phi K) = \phi^{-\mathrm{T}} \Phi K,$$

and it is called (positively) *homogeneous* of degree r, $r \in \mathbb{R}$, if for all $\lambda > 0$ and all $K \in \mathcal{K}^n$,

$$\Phi(\lambda K) = \lambda^r \Phi K.$$

If Φ is $\mathrm{SL}(n)$ contravariant and homogeneous of degree r, then we have for all $K \in \mathcal{K}^n$ that

$$\Phi(\phi K) = (\det \phi)^{(r+1)/n} \phi^{-\mathrm{T}} \Phi K, \tag{2.5}$$

for every $\phi \in \mathrm{GL}(n)$ with $\det \phi > 0$.

Since, clearly, every $\mathrm{GL}(n)$ contravariant map is $\mathrm{SL}(n)$ contravariant and homogeneous, the following result is a stronger version of Theorem 1.

Theorem 4 (Ludwig [20])**.** *A map $\Phi : \mathcal{K}_0^n \to \mathcal{K}^n$ is an $\mathrm{SL}(n)$ contravariant and homogeneous continuous Minkowski valuation if and only if there exists a constant $c \geq 0$ such that for every $K \in \mathcal{K}_0^n$,*

$$\Phi K = c\, \Pi K.$$

3. An auxiliary result

In this section we show that any $\mathrm{SL}(n)$ contravariant continuous Minkowski valuation which is homogeneous of degree $r \neq n-1$ is trivial. The proof is based on the ideas and techniques developed by Ludwig in [20].

As a first step we consider the image of convex bodies contained in a hyperplane. We denote by $\operatorname{aff} K$ the affine hull of $K \subseteq \mathbb{R}^n$.

Lemma 5. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be $\mathrm{SL}(n)$ contravariant and homogeneous of degree r. Then the following statements hold:*

(i) *If $\dim K < n-2$, then $\Phi K = \{0\}$.*

(ii) *If* $\dim K = n-2$, *then*

$$\begin{cases} \Phi K = \{0\} & \text{if } 0 \in \operatorname{aff} K, \\ \Phi K \subseteq \operatorname{aff}(\{0\} \cup K)^{\perp} & \text{if } 0 \notin \operatorname{aff} K. \end{cases}$$

(iii) *If* $\dim K = n-1$ *and* $0 \in \operatorname{aff} K$, *then* $\Phi K \subseteq (\operatorname{aff} K)^{\perp}$.

(iv) *If* K *is contained in a hyperplane through the origin and* $r \neq n-1$, *then* $\Phi K = \{0\}$.

Proof. Let $\{e_1, \dots, e_n\}$ be a fixed orthonormal basis of $\mathbb{R}^n$. For $k \geq 1$, we use the direct sum decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, where

$$\mathbb{R}^k = \operatorname{span}\{e_1, \dots, e_k\} \qquad \text{and} \qquad \mathbb{R}^{n-k} = \operatorname{span}\{e_{k+1}, \dots, e_n\}.$$

Let $\phi \in \mathrm{SL}(n)$ be the matrix defined (with respect to the chosen basis) by

$$\phi = \begin{pmatrix} I_k & B \\ 0 & A \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, B is an arbitrary $k \times (n-k)$ matrix, and $A \in \mathrm{SL}(n-k)$. A simple calculation shows that

$$\phi^{-\mathrm{T}} = \begin{pmatrix} I_k & 0 \\ -A^{-\mathrm{T}}B^{\mathrm{T}} & A^{-\mathrm{T}} \end{pmatrix}. \tag{3.1}$$

If $K \in \mathcal{K}^n$ and $K \subseteq \mathbb{R}^k$, we have

$$\phi K = K. \tag{3.2}$$

Now let $x \in \Phi K$ and write $x = x' + x''$, where $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$. Since Φ is $\mathrm{SL}(n)$ contravariant, it follows from (3.1) and (3.2) that

$$\phi^{-\mathrm{T}}x = x' - A^{-\mathrm{T}}(B^{\mathrm{T}}x' - x'') \in \Phi K. \tag{3.3}$$

This holds for every $A \in \mathrm{SL}(n-k)$ and every $k \times (n-k)$ matrix B. Since ΦK is bounded, this implies that $x' = 0$. Thus, we have

$$\Phi K \subseteq \mathbb{R}^{n-k}. \tag{3.4}$$

Moreover, if $n-k \geq 2$, it follows from (3.3) that also $x'' = 0$, i.e., $\Phi K = \{0\}$.

Since every convex body $K \in \mathcal{K}^n$ with $\dim K < n-2$ or $\dim K = n-2$ and $0 \in \operatorname{aff} K$ is the linear image of some convex body contained in $\mathbb{R}^{n-2}$, we obtain (i) and the first assertion of (ii). The second assertion of (ii) and statement (iii) are immediate consequences of (3.4).

Finally, let $K \subseteq \mathbb{R}^{n-1}$ and $r \neq n-1$. If $\psi \in \mathrm{GL}(n)$ is defined by

$$\psi = \begin{pmatrix} I_{n-1} & 0 \\ 0 & s \end{pmatrix},$$

where $s > 0$, then, by (2.5),

$$\Phi K = \Phi(\psi K) = s^{(r-(n-1))/n}\Phi K.$$

Since this holds for every $s > 0$ and ΦK is bounded, we deduce (iv). □

Our next result reduces the proof of Theorem 2 to Minkowski valuations which are homogeneous of degree $n-1$.

Proposition 6. *If* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *is an* $\mathrm{SL}(n)$ *contravariant continuous Minkowski valuation which is homogeneous of degree* $r \neq n-1$, *then for every* $K \in \mathcal{K}^n$,

$$\Phi K = \{0\}.$$

Proof. Let $\{e_1, \dots, e_n\}$ be a fixed orthonormal basis of $\mathbb{R}^n$. For $0 < \lambda < 1$ and $1 \le i < j \le n$, we denote by $H_\lambda = H_\lambda(i,j)$ the hyperplane through the origin with normal vector $\lambda e_j - (1-\lambda)e_i$, that is,

$$H_\lambda = \operatorname{span}(\{\lambda e_i + (1-\lambda)e_j\} \cup \{e_k : k \neq i,j\}).$$

Let S be the $(n-1)$-dimensional simplex with vertices $\{e_1, \dots, e_n\}$. Then H_λ dissects S into two simplices $S \cap H_\lambda^+$ and $S \cap H_\lambda^-$, where H_λ^+, H_λ^- denote the closed halfspaces bounded by H_λ. More precisely, we have

$$\begin{aligned} S \cap H_\lambda^+ &= \operatorname{conv}(\{\lambda e_i + (1-\lambda)e_j\} \cup \{e_k : k \neq i\}), \\ S \cap H_\lambda^- &= \operatorname{conv}(\{\lambda e_i + (1-\lambda)e_j\} \cup \{e_k : k \neq j\}). \end{aligned}$$

Define linear maps $\phi_\lambda = \phi_\lambda(i,j)$ and $\psi_\lambda = \phi_\lambda(i,j)$ by

$$\phi_\lambda e_i = \lambda e_i + (1-\lambda)e_j, \quad \phi_\lambda e_k = e_k \quad \text{for } k \neq i,$$

$$\psi_\lambda e_j = \lambda e_i + (1-\lambda)e_j, \quad \phi_\lambda e_k = e_k \quad \text{for } k \neq j.$$

Then it is easy to see that

$$S \cap H_\lambda^+ = \phi_\lambda S \qquad \text{and} \qquad S \cap H_\lambda^- = \psi_\lambda S. \tag{3.5}$$

Since Φ is a Minkowski valuation, we have

$$\Phi S + \Phi(S \cap H_\lambda) = \Phi(S \cap H_\lambda^+) + \Phi(S \cap H_\lambda^-). \tag{3.6}$$

Since Φ is also $\mathrm{SL}(n)$ contravariant and homogeneous of degree $r \neq n-1$, it follows from Lemma 5 (iv) that $\Phi(S \cap H_\lambda) = \{0\}$. Therefore, (2.5), (3.5), and (3.6) yield

$$\Phi S = \lambda^q \phi_\lambda^{-\mathrm{T}} \Phi S + (1-\lambda)^q \psi_\lambda^{-\mathrm{T}} \Phi S, \tag{3.7}$$

where $q = (r+1)/n$.

Now let $1 \le k \le n$ and choose $1 \le i < j \le n$ such that $k \neq i,j$. This is possible since $n \ge 3$. By (2.1) and (3.7), we have

$$h(\Phi S, e_k) = \lambda^q h(\Phi S, e_k) + (1-\lambda)^q h(\Phi S, e_k),$$

for every $0 < \lambda < 1$. Since Φ is homogeneous of degree $r \neq n-1$, we have $q \neq 1$ which implies $h(\Phi S, e_k) = 0$. Similarly, we obtain $h(\Phi S, -e_k) = 0$. Since this holds for every $1 \le k \le n$, we must have

$$\Phi S = \{0\}. \tag{3.8}$$

Let T be an arbitrary $(n-1)$-dimensional simplex. If T is contained in a hyperplane through the origin, then, by Lemma 5 (iv), $\Phi T = \{0\}$. Otherwise, there exists a linear transformation $\phi \in \mathrm{GL}(n)$ with $\det \phi > 0$ such that $\phi S = T$. Thus, by (2.5) and (3.8), we also obtain $\Phi T = \{0\}$.

Now let $F \in \mathcal{K}^n$ be an $(n-1)$-dimensional polytope. We dissect F into $(n-1)$-dimensional simplices S_i, $i = 1, \dots, m$, that is, $F = S_1 \cup \dots \cup S_m$ and $\dim S_i \cap S_j \le n-2$ whenever $i \neq j$. For $x \in \mathbb{R}^n$ fixed, we define a real valued valuation $\varphi_x : \mathcal{K}^n \to \mathbb{R}$ by

$$\varphi_x(K) = h(\Phi K, x).$$

From the continuity of Φ, it follows that φ_x is also continuous. Therefore, φ_x satisfies the inclusion-exclusion principle

$$\varphi_x(F) = \sum_I (-1)^{|I|-1} \varphi_x(S_I),$$

where the sum is taken over all ordered k-tuples $I = (i_1, \dots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le n$ and $k = 1, \dots, m$. Here $|I|$ denotes the cardinality of I and $S_I = S_{i_1} \cap \cdots \cap S_{i_k}$ (cf. [17, p. 7]). By Lemma 5 and the fact that $\Phi S_i = \{0\}$ for $i = 1, \dots, m$, it follows that $\varphi_x(F) = 0$ for every $x \in \mathbb{R}^n$. Consequently,

$$\Phi F = \{0\} \tag{3.9}$$

for every $(n-1)$-dimensional polytope F.

Next, let $P \in \mathcal{K}^n$ be an n-dimensional polytope. If $0 \in P$, then $\Phi P = \{0\}$ by Theorem 4. Therefore, suppose that $0 \notin P$. We call a facet F of P *visible* if $[0, x) \cap P = \emptyset$ for every $x \in F$. If P has exactly one visible facet F, then, since Φ is a Minkowski valuation, we have

$$\Phi P_{\mathrm{o}} + \Phi F = \Phi F_{\mathrm{o}} + \Phi P,$$

where we use K_{o} to denote $\operatorname{conv}(\{0\} \cup K)$. Since, by Theorem 4, $\Phi K = \{0\}$ for all $K \in \mathcal{K}_0^n$ and $\Phi F = \{0\}$ by (3.9), we obtain $\Phi P = \{0\}$. If P has $m > 1$ visible facets $F_1, \dots, F_m$, we define polytopes $C_i \in \mathcal{K}^n$, $i = 1, \dots, m$, by

$$C_i = P \cap \bigcup_{t \ge 0} t F_i.$$

Clearly, $P = C_1 \cup \cdots \cup C_m$ and $\dim C_i \cap C_j \le n-1$ for $i \ne j$. Moreover, each C_i, $i = 1, \dots, m$, has exactly one visible facet. Hence, $\Phi\, C_i = \{0\}$ for $i = 1, \dots, m$. Thus, as before, the inclusion-exclusion principle and (3.9) imply

$$\Phi P = \{0\} \tag{3.10}$$

for every n-dimensional polytope P.

Finally, a combination of Lemma 5, (3.9), and (3.10) shows that the map Φ vanishes on all polytopes. By the continuity of Φ, the same holds true for all convex bodies. □

4. Proof of the main result

After these preparations, we are now in a position to prove Theorem 2. In fact, since GL(n) contravariant Minkowski valuations are SL(n) contravariant and homogeneous, we prove a slightly stronger result, analogous to Theorem 4.

Theorem 7. *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an* SL(n) *contravariant and homogeneous continuous Minkowski valuation if and only if there exist constants $c_1, c_2 \ge 0$ such that for every $K \in \mathcal{K}^n$,*

$$\Phi K = c_1 \Pi K + c_2 \Pi_{\mathrm{o}} K.$$

Proof. By Proposition 6, it is sufficient to consider Φ that are homogeneous of degree $n-1$. We first show that in this case, Φ is already determined by its values on convex bodies of dimension $n-1$. To this end, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and define $\phi_s \in \mathrm{GL}(n)$ by

$$\phi_s = \begin{pmatrix} s\, I_{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $s > 0$ and, as before, I_k denotes the $k \times k$ identity matrix. Since Φ is homogeneous of degree $n-1$, (2.5) yields

$$\phi_s \Phi K = \Phi(s \phi_s^{-\mathrm{T}} K).$$

Hence, by the continuity of Φ, letting $s \to 0$, we have

$$\phi_0 \Phi K = \Phi(K|e_n^\perp).$$

Since $h(\phi_0 L, e_n) = h(L, e_n)$ for every $L \in \mathcal{K}^n$, we obtain

$$h(\Phi K, e_n) = h(\Phi(K|e_n^\perp), e_n). \tag{4.1}$$

Now let $u \in S^{n-1}$ and choose $\vartheta \in \mathrm{SO}(n)$ such that $\vartheta e_n = u$. Using (2.1), (2.5), and (4.1), we conclude that

$$h(\Phi K, u) = h(\Phi((\vartheta^{-1}K)|e_n^\perp), e_n) = h(\Phi(K|u^\perp), u). \tag{4.2}$$

Next, we show that there exists a constant $c \geq 0$ such that

$$\Phi K = c\,\mathrm{vol}_{n-1}(K_\mathrm{o})\,[-u, u] \tag{4.3}$$

for every $K \in \mathcal{K}^n$ contained in $u^\perp$, $u \in S^{n-1}$, with $\dim K \leq n-2$. Here, $[-u, u]$ denotes the segment with endpoints $-u, u$ and $K_\mathrm{o} = \mathrm{conv}\,(\{0\} \cup K)$.

If $\dim K < n-2$ or $0 \in \mathrm{aff}\,K$, (4.3) holds by Lemma 5. Therefore, we may assume that $\dim K = n-2$ and $0 \notin \mathrm{aff}\,K$. Moreover, by (2.5), we may also assume that $K \subseteq e_n^\perp$.

Let $T \subseteq e_n^\perp$ be an $(n-2)$-dimensional simplex with $0 \notin \mathrm{aff}\,T$. Choose $\psi \in \mathrm{GL}(n-1)$ with $\det \psi > 0$ such that the linear map

$$\phi = \begin{pmatrix} \psi & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $T = \phi S$, where $S = \mathrm{conv}\{e_1, \ldots, e_{n-1}\}$. A simple calculation shows that $\det \psi = (n-1)!\,\mathrm{vol}_{n-1}(T_\mathrm{o})$. Since $\Phi S \subseteq \mathrm{span}\{e_n\}$ by Lemma 5 (ii), we obtain from (2.5) that

$$\Phi\, T = (\det \phi)\phi^{-t}\Phi S = (n-1)!\,\mathrm{vol}_{n-1}(T_\mathrm{o})\,[-a\,e_n, b\,e_n],$$

with constants $a, b \in \mathbb{R}$ depending only on Φ. Choosing $\vartheta \in \mathrm{SO}(n)$ such that $\vartheta e_n = -e_n$ and $\vartheta S = S$ (this is possible since $n \geq 3$), it follows from (2.5) that $\Phi S = \vartheta \Phi S$. Hence, $a = b \geq 0$ and (4.3) holds for T.

Now suppose that $P \subseteq e_n^\perp$ is a polytope of dimension $n-2$ such that $0 \notin \mathrm{aff}\,P$. We dissect P into $(n-2)$-dimensional simplices $P = S_1 \cup \cdots \cup S_m$ such that $\dim S_i \cap S_j < n-2$. As in the proof of Proposition 6, an application of the inclusion-exclusion principle and Lemma 5 (i) shows that

$$\Phi P = \Phi S_1 + \cdots + \Phi S_m = c\,\mathrm{vol}_{n-1}(P_\mathrm{o})\,[-e_n, e_n]$$

for some constant $c \geq 0$ depending only on Φ. Since Φ is continuous, we conclude that (4.3) holds for all convex bodies $K \subseteq e_n^\perp$ of dimension $n-2$.

Now let $K \in \mathcal{K}^n$ be an arbitrary convex body contained in $u^\perp$, $u \in S^{n-1}$. We want to show that there exist constants $a_1, a_2 \geq 0$ depending only on Φ such that

$$\Phi K = (a_1 \mathrm{vol}_{n-1}(K) + a_2 \mathrm{vol}_{n-1}(K_\mathrm{o} \setminus K))\,[-u, u]. \tag{4.4}$$

If $\dim K < n-1$, then (4.4) holds by Lemma 5 and (4.3). Therefore, we may assume that $\dim K = n-1$. Moreover, by (2.5), we may also assume that $K \subseteq e_n^\perp$.

Let $P \subseteq e_n^\perp$ be an $(n-1)$-dimensional polytope. Recall that a face F of P is called visible if $[0, x) \cap P = \emptyset$ for every $x \in F$. Suppose that P has exactly one visible $(n-2)$-face F. Since Φ is a valuation, we have

$$\Phi F_\mathrm{o} + \Phi P = \Phi P_\mathrm{o} + \Phi F. \tag{4.5}$$

By Theorem 4, there exists a constant $a_1 \geq 0$ such that Φ restricts to $a_1\Pi$ on $\mathcal{K}_{\mathrm{o}}^n$. Hence, $\Phi F_{\mathrm{o}} = a_1 \Pi F_{\mathrm{o}}$ and $\Phi P_{\mathrm{o}} = a_1 \Pi P_{\mathrm{o}}$. Moreover, from (4.3), it follows that there exists a constant $a_2 \geq 0$ such that

$$\Phi F = a_2 \mathrm{vol}_{n-1}(F_{\mathrm{o}})[-e_n, e_n] = a_2 \Pi F_{\mathrm{o}}.$$

Thus, we can rewrite (4.5) in the following way

$$a_1 \Pi(F_{\mathrm{o}}) + \Phi P = a_1 \Pi P_{\mathrm{o}} + a_2 \Pi F_{\mathrm{o}}.$$

Since Π is a Minkowski valuation and $\Pi F = \{0\}$, we have $\Pi P_{\mathrm{o}} = \Pi P + \Pi F_{\mathrm{o}}$. Thus, using the cancelation law for Minkowski addition, we deduce that

$$\Phi P = a_1 \Pi P + a_2 \Pi F_{\mathrm{o}} = (a_1 \mathrm{vol}_{n-1}(P) + a_2 \mathrm{vol}_{n-1}(P_{\mathrm{o}} \setminus P))[-e_n, e_n].$$

In order to obtain (4.4) for an $(n-1)$-dimensional polytope $P \subseteq e_n^{\perp}$ with $m > 1$ visible $(n-2)$-faces $F_1, \ldots, F_m$, we dissect P as in the proof of Proposition 6. Finally, since Φ is continuous, we conclude that (4.4) holds for all convex bodies $K \subseteq e_n^{\perp}$.

In the last step of the proof, we combine (4.2) and (4.4) to obtain

$$h(\Phi K, u) = a_1 \mathrm{vol}_{n-1}(K|u^{\perp}) + a_2 \mathrm{vol}_{n-1}((K_{\mathrm{o}}|u^{\perp}) \setminus (K|u^{\perp})) \tag{4.6}$$

for every K in $\mathcal{K}^n$ and $u \in S^{n-1}$. Suppose that $a_1 < a_2$. Then, for $u \in S^{n-1}$,

$$h(\Phi K, u) = c_1 \mathrm{vol}_{n-1}(K_{\mathrm{o}}|u^{\perp}) - c_2 \mathrm{vol}_{n-1}(K|u^{\perp}),$$

where $c_1, c_2 > 0$. By (2.2), this is equivalent to

$$h(\Phi K, u) = \int_{S^{n-1}} |u \cdot v| \, d\rho(v), \qquad u \in S^{n-1}, \tag{4.7}$$

where ρ is the signed Borel measure defined by

$$\rho = \frac{c_1}{2} S_{n-1}(K_{\mathrm{o}}, \cdot) - \frac{c_2}{2} S_{n-1}(K, \cdot). \tag{4.8}$$

It follows from (4.7) that for every polytope P there exist polytopes Q_1 and Q_2 such that

$$\Phi P + Q_1 = Q_2.$$

Hence, for every polytope P, ΦP is also a polytope. Since, by (4.7), ΦP is also a generalized zonoid, we conclude that $\rho \geq 0$, whenever P is a polytope (cf. [33, Corollary 3.5.6]). But this is a contradiction, as can be seen by calculating the surface area measures in (4.8) for $P = \mathrm{conv}\{e_1, \ldots, e_n\}$.

We conclude that $a_1 \geq a_2$ in (4.6). Consequently, there are constants $c_1, c_2 \geq 0$ such that (4.6) is equivalent to

$$h(\Phi K, u) = c_1 h(\Pi K, u) + c_2 h(\Pi_{\mathrm{o}} K, u), \qquad u \in S^{n-1},$$

which completes the proof of Theorem 7. □

5. A Petty projection-type inequality

The fundamental affine isoperimetric inequalities for projection bodies are the Petty [31] projection inequality and the Zhang [41] projection inequality: Among bodies of a given volume, polar projection bodies have maximal volume precisely for ellipsoids and they have minimal volume precisely for simplices. It is a major open problem to determine the corresponding results for the volume of the projection body itself (see, e.g., [23, 24]).

Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an $\mathrm{SL}(n)$ contravariant and homogeneous continuous Minkowski valuation. It follows from Theorem 7 that among bodies K of given volume, the quantity $V(\Phi^* K)$ does not, in general, attain a minimum. However, it always attains a maximum. The following result provides a generalization of Petty's projection inequality. In view of Theorem 7 it is equivalent to Theorem 3.

Theorem 8. *Let $K \in \mathcal{K}^n$ have nonempty interior. If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a nontrivial $\mathrm{SL}(n)$ contravariant and homogeneous continuous Minkowski valuation, then*

$$V(K)^{n-1} V(\Phi^* K) \leq V(B)^{n-1} V(\Phi^* B). \tag{5.1}$$

If Φ is not a multiple of Π, there is equality if and only if K is an ellipsoid containing the origin; otherwise equality holds if and only if K is an ellipsoid.

Proof. Since Φ is $\mathrm{SL}(n)$ contravariant, we have $\Phi(\vartheta K) = \vartheta \Phi K$ for every $\vartheta \in \mathrm{SO}(n)$ and every $K \in \mathcal{K}^n$. Thus, there exists an $r_\Phi \geq 0$ such that $\Phi B = r_\Phi B$. Since inequality (5.1) is homogeneous and Φ is nontrivial, we may assume that $\Phi B = \Pi B = \kappa_{n-1} B$. Here, κ_m denotes the m-dimensional volume of the Euclidean unit ball in $\mathbb{R}^m$.

Since $\Pi_{\mathrm{o}} B = \Pi B$, it follows from Theorem 7 that there exists a constant $0 \leq \lambda \leq 1$ such that for every $K \in \mathcal{K}^n$,

$$\Phi K = \lambda \, \Pi K + (1 - \lambda) \, \Pi_{\mathrm{o}} K.$$

Thus, using (2.3), we obtain for convex bodies K with nonempty interior,

$$\Phi^* K = \lambda \cdot \Pi^* K \mathbin{\widehat{+}} (1 - \lambda) \cdot \Pi_{\mathrm{o}}^* K.$$

An application of the harmonic dual Brunn–Minkowski inequality (2.4) now yields

$$V(\Phi^* K)^{-1/n} \geq \lambda V(\Pi^* K)^{-1/n} + (1 - \lambda) V(\Pi_{\mathrm{o}}^* K)^{-1/n}, \tag{5.2}$$

with equality, for $0 < \lambda < 1$, if and only if $\Pi^* K$ and $\Pi_{\mathrm{o}}^* K$ are dilates.

Since $K \subseteq \mathrm{conv}(\{0\} \cup K)$, definition (1.1) of the projection operator shows that $\Pi K \subseteq \Pi_{\mathrm{o}} K$, with equality if and only if K contains the origin. Consequently,

$$V(\Pi_{\mathrm{o}}^* K) \leq V(\Pi^* K), \tag{5.3}$$

with equality if and only if K contains the origin. Combining (5.2) and (5.3) yields

$$V(\Phi^* K) \leq V(\Pi^* K).$$

If $\Phi \neq \Pi$, there is equality if and only if K contains the origin. Finally, an application of the Petty projection inequality (1.2), yields (5.1) along with the equality conditions. □

The proof of Theorem 8 also shows that the Petty projection inequality is the strongest inequality in the family of inequalities (5.1). More precisely:

Corollary 9. *Let $K \in \mathcal{K}^n$ have nonempty interior. If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an $\mathrm{SL}(n)$ contravariant and homogeneous continuous Minkowski valuation such that $\Phi B = \Pi B$, then*

$$V(\Phi^* K) \leq V(\Pi^* K).$$

If $\Phi \neq \Pi$, there is equality if and only if K contains the origin.

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