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CLASSIFICATION OF MINIMAL ALGEBRAS OVER ANY FIELD UP TO DIMENSION 6

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ABSTRACT. We give a classification of minimal algebras generated in degree 1, defined over any field $\mathbf{k}$ of characteristic different from 2, up to dimension 6. This recovers the classification of nilpotent Lie algebras over $\mathbf{k}$ up to dimension 6. In the case of a field $\mathbf{k}$ of characteristic zero, we obtain the classification of nilmanifolds of dimension less than or equal to 6, up to $\mathbf{k}$-homotopy type. Finally, we determine which rational homotopy types of such nilmanifolds carry a symplectic structure.

1. INTRODUCTION AND MAIN RESULTS

Let X be a nilpotent space of the homotopy type of a CW-complex of finite type over $\mathbb{Q}$ (all spaces considered hereafter are of this kind). A space is nilpotent if $\pi_1(X)$ is a nilpotent group and it acts in a nilpotent way on $\pi_k(X)$ for $k > 1$. The rationalization of X (see [3], [6]) is a rational space $X_\mathbb{Q}$ (i.e., a space whose homotopy groups are rational vector spaces) together with a map $X \to X_\mathbb{Q}$ inducing isomorphisms $\pi_k(X) \otimes \mathbb{Q} \stackrel{\cong}{\to} \pi_k(X_\mathbb{Q})$ for $k \geq 1$ (recall that the rationalization of a nilpotent group is well defined [6]). Two spaces X and Y have the same rational homotopy type if their rationalizations $X_\mathbb{Q}$ and $Y_\mathbb{Q}$ have the same homotopy type; i.e., if there exists a map $X_\mathbb{Q} \to Y_\mathbb{Q}$ inducing isomorphisms in homotopy groups.

The theory of minimal models developed by Sullivan [15] allows us to classify rational homotopy types algebraically. In fact, Sullivan constructed a one-to-one correspondence between nilpotent rational spaces and isomorphism classes of minimal algebras over $\mathbb{Q}$:

$$X \leftrightarrow (\wedge V_X, d)\,. \tag{1}$$

Recall that, in general, a minimal algebra is a commutative differential graded algebra (CDGA henceforth) $(\wedge V, d)$ over a field $\mathbf{k}$ of characteristic different from 2 in which

(1) $\wedge V$ denotes the free commutative algebra generated by the graded vector space $V = \bigoplus V^i$;
(2) there exists a basis $\{x_\tau, \tau \in I\}$, for some well-ordered index set I, such that $\deg(x_\mu) \leq \deg(x_\tau)$ if $\mu < \tau$, and each dx_τ is expressed in terms of preceding x_μ ($\mu < \tau$). This implies that dx_τ does not have a linear part.

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In the above formula (1), $(\wedge V_X, d)$ is known as the minimal model of X. Hence, X and Y have the same rational homotopy type if and only if they have isomorphic minimal models (as CDGAs over $\mathbb{Q}$).

The notion of *real* or *complex* homotopy type already appears in the literature (cf. [2] and [11]): two manifolds M_1, M_2 have the same real (resp. complex) homotopy type if the corresponding CDGAs of real (resp. complex) differential forms $(\Omega^*(M_1), d)$ and $(\Omega^*(M_2), d)$ have the same homotopy type, i.e., they can be joined by a chain of morphisms inducing isomorphisms on cohomology (quasi-isomorphisms henceforth). This is equivalent to saying that the two CDGAs have the same real (resp. complex) minimal model. It is convenient to remark ([3], §11(d)) that, if $(\wedge V, d)$ is the rational minimal model of M, then $(\wedge V \otimes_{\mathbb{Q}} \mathbb{R}, d)$ is the real minimal model of M. Recall that, given a CDGA A over a field $\mathbf{k}$, a minimal model of A is a minimal $\mathbf{k}$-algebra $(\wedge V, d)$ together with a quasi-isomorphism $(\wedge V, d) \xrightarrow{\simeq} A$. While the minimal model of a CDGA over a field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 0$ is unique up to isomorphism, the same result for arbitrary characteristic is unknown (see the appendix in which we prove uniqueness for the special case of minimal algebras treated in this paper).

We generalize this notion to an arbitrary field $\mathbf{k}$ of characteristic zero. Note that $\mathbb{Q} \subset \mathbf{k}$.

Definition 1. Let $\mathbf{k}$ be a field of characteristic zero. The $\mathbf{k}$-minimal model of a space X is $(\wedge V_X \otimes \mathbf{k}, d)$. We say that X and Y have the same $\mathbf{k}$-homotopy type if and only if the $\mathbf{k}$-minimal models $(\wedge V_X \otimes \mathbf{k}, d)$ and $(\wedge V_Y \otimes \mathbf{k}, d)$ are isomorphic.

Note that if $\mathbf{k}_1 \subset \mathbf{k}_2$, then the fact that X and Y have the same $\mathbf{k}_1$-homotopy type implies that X and Y have the same $\mathbf{k}_2$-homotopy type.

Recall that a nilmanifold is a quotient $N = G/\Gamma$ of a nilpotent connected Lie group by a discrete cocompact subgroup (i.e., the resulting quotient is compact). The minimal model of N is precisely the Chevalley-Eilenberg complex $(\wedge \mathfrak{g}^*, d)$ of the nilpotent Lie algebra $\mathfrak{g}$ of G (see [12]). Here, $\mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{Q})$ is assumed to be concentrated in degree 1, and the differential $d : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ reflects the Lie bracket via the pairing

$$dx(X, Y) = -x([X, Y]), \quad x \in \mathfrak{g}^*, \ X, Y \in \mathfrak{g}.$$

Indeed, consider a basis $\{X_i\}$ of $\mathfrak{g}$, such that

$$[X_j, X_k] = \sum_{i<j,k} a^i_{jk} \, X_i \, . \tag{2}$$

Let $\{x_i\}$ be the dual basis for $\mathfrak{g}^*$, so that $a^i_{jk} = x_i([X_j, X_k])$. Then the differential is expressed as

$$dx_i = - \sum_{j,k>i} a^i_{jk} \, x_j x_k \, . \tag{3}$$

Mal'cev proved that the existence of a basis $\{X_i\}$ of $\mathfrak{g}$ with *rational* structure constants a^i_{jk} in (2) is equivalent to the existence of a cocompact $\Gamma \subset G$. The minimal model of the nilmanifold $N = G/\Gamma$ is

$$(\wedge(x_1, \dots, x_n), d),$$

where $V = \langle x_1, \dots, x_n \rangle = \bigoplus_{i=1}^n \mathbb{Q} x_i$ is the vector space generated by $x_1, \dots, x_n$ over $\mathbb{Q}$, with $|x_i| = 1$ for every $i = 1, \dots, n$ and dx_i is defined according to (3).

We prove the following:

Theorem 2. *Let* $\mathbf{k}$ *be a field of characteristic zero. The number of minimal models of* 6*-dimensional nilmanifolds, up to* $\mathbf{k}$*-homotopy type, is* $26 + 4s$*, where* s *denotes the cardinality of* $\mathbb{Q}^*/((\mathbf{k}^*)^2 \cap \mathbb{Q}^*)$*. In particular:*

- *There are* 30 *complex homotopy types of* 6*-dimensional nilmanifolds.*
- *There are* 34 *real homotopy types of* 6*-dimensional nilmanifolds.*
- *There are infinitely many rational homotopy types of* 6*-dimensional nilmanifolds.*

One of the consequences is the existence of pairs of nilmanifolds M_1, M_2 which have the same real homotopy type, but for which there is no map $f : M_1 \to M_2$ inducing an isomorphism in the real minimal models.

Theorem 2 is a consequence of the following classification of all minimal algebras generated in degree 1 by a vector space of dimension less than or equal to 6, in which we also give an explicit representative of each isomorphism class. (From now on, by the dimension of a minimal algebra $(\wedge V, d)$ we mean the dimension of V.)

Theorem 3. *Let* $\mathbf{k}$ *be any field of any characteristic* $\mathrm{char}(\mathbf{k}) \neq 2$*. There are* $26+4r$ *isomorphism classes of* 6*-dimensional minimal algebras generated in degree* 1 *over* $\mathbf{k}$*, where* r *is the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$*.*

As the Chevalley-Eilenberg complex, defined as above over a nilpotent Lie algebra, gives a one-to-one correspondence between these objects and minimal algebras generated in degree 1, we obtain the following

Corollary 4. *There are* $26 + 4r$ *isomorphism classes of* 6*-dimensional nilpotent Lie algebras over* $\mathbf{k}$*, where* r *is the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$*. In particular:*

- *There are* 30 *isomorphism classes of* 6*-dimensional nilpotent real Lie algebras.*
- *There are* 34 *isomorphism classes of* 6*-dimensional nilpotent complex Lie algebras.*
- *For finite fields* $\mathbf{k} = \mathbb{F}_{p^n}$*, with* $p \neq 2$*, the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$ *is* $r = 2$*. So there are* 34 *isomorphism classes of* 6*-dimensional nilpotent Lie algebras defined over* $\mathbb{F}_{p^n}$*,* $p \neq 2$*.*

This result is already known in the literature (see for instance [1] or [5]), but we obtain it from a new perspective: our starting point is the classification of minimal models.

Note that the classification of real homotopy types of 6-dimensional nilmanifolds already appears in the literature (see for instance [4] and [10]).

We finish the paper by determining which 6-dimensional nilmanifolds admit a symplectic structure. In particular, there are 27 real homotopy types of 6-dimensional nilmanifolds admitting symplectic forms. This already appears in [14], but we have decided to include it here for completeness, and to write down explicit symplectic forms in the cases where the nilmanifold does admit them.

2. Preliminaries

Let $\mathbf{k}$ be a field of characteristic different from 2. Let $V = \langle x_1, \dots x_n \rangle = \bigoplus_{i=1}^n \mathbf{k} x_i$ be a finite-dimensional vector space over $\mathbf{k}$ with $\dim V \geq 2$. We want to analyse minimal algebras of the type

$$(\wedge(x_1,\dots,x_n),d),$$

where $|x_i|=1$, for every $i=1,\dots,n$, and dx_i is defined according to (3), with $a_{ij}^k\in\mathbf{k}$. Write $(\wedge V,d)$ with $V=V^1$ (i.e., $\wedge V$ is generated as an algebra by elements of degree 1). Set

$$\begin{aligned} W_1 &= \ker(d)\cap V,\\ W_k &= d^{-1}(\wedge^2 W_{k-1}),\ \text{for } k\geq 2\,. \end{aligned}$$

This is a filtration of V intrinsically defined. We see that $W_k\subset W_{k+1}$, for $k\geq 1$, as follows. First notice that $W_1\subset W_2$ since $W_1=d^{-1}(0)$. By induction, suppose that $W_{k-1}\subset W_k$. Then we have

$$d(W_k)=d(d^{-1}(\wedge^2 W_{k-1}))\subset\wedge^2 W_{k-1}\subset\wedge^2 W_k\,.$$

This proves that $W_k\subset W_{k+1}$, as required.

Now define

$$\begin{aligned} F_1 &= W_1,\\ F_k &= W_k/W_{k-1}\ \text{for } k\geq 2\,. \end{aligned}$$

Then, in a noncanonical way, one has $V=\bigoplus F_i$. The numbers $f_k=\dim(F_k)$ are invariants of V. Notice that $f_k=0$ eventually. Under the splitting $W_k=W_{k-1}\oplus F_k$, the differential decomposes as[1]

$$d: W_{k+1}\longrightarrow\wedge^2 W_k=\wedge^2 W_{k-1}\oplus(W_{k-1}\otimes F_k)\oplus\wedge^2 F_k.$$

If we project to the second and third summands, we have

$$d: W_{k+1}\longrightarrow\frac{\wedge^2 W_k}{\wedge^2 W_{k-1}}=(W_{k-1}\otimes F_k)\oplus\wedge^2 F_k,$$

which vanishes on W_k and hence induces a map

$$\bar{d}: F_{k+1}\longrightarrow(W_{k-1}\otimes F_k)\oplus\wedge^2 F_k=((F_1\oplus\cdots\oplus F_{k-1})\otimes F_k)\oplus\wedge^2 F_k\,. \tag{4}$$

This map is injective because $W_k=d^{-1}(\wedge^2 W_{k-1})$. Notice that the map (4) is not canonical since it depends on the choice of the splitting.

The differential d also determines a well-defined map (independent of choice of splitting)

$$\hat{d}: F_{k+1}\to H^2(\wedge(F_1\oplus\cdots\oplus F_k),d)\,,$$

which is also injective.

By considering $\bar{d}: F_2\to\wedge^2 F_1$, we see that $f_1\geq 2$. Moreover, if $f_1=2$, then $f_2=1$ and $\bar{d}: F_2\to\wedge^2 F_1$ is an isomorphism.

We shall make extensive use of the following (easy) result.

Lemma 5. *Let W be a $\mathbf{k}$-vector space of dimension k, where $\mathbf{k}$ is a field of characteristic different from 2. Given any element $\varphi\in\wedge^2 W$, there is a (not unique) basis $x_1,\dots,x_k$ of W such that $\varphi=x_1\wedge x_2+\cdots+x_{2r-1}\wedge x_{2r}$, for some $r\geq 0$, $2r\leq k$.*

The $2r$-dimensional space $\langle x_1,\dots,x_{2r}\rangle\subset W$ is well defined (independent of the basis).

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[1] We use the notation $W_{k-1}\otimes F_k$ instead of $W_{k-1}\cdot F_k$, tacitly using the natural isomorphism $W_{k-1}\cdot F_k\cong W_{k-1}\otimes F_k$. We prefer this notation, as the other one could lead to some apparent incoherences throughout the paper.

Proof. Interpret φ as an antisymmetric bilinear map $W^* \times W^* \to \mathbb{Q}$. Let $2r$ be its rank, and consider a basis $e_1, \dots, e_k$ of W^* such that $\varphi(e_{2i-1}, e_{2i}) = 1$, $1 \leq i \leq r$, and the other pairings are zero. Then the dual basis $x_1, \dots, x_k$ does the job. □

3. Classification in low dimensions

As we said in the introduction, a minimal algebra $(\wedge V, d)$ is of dimension k if $\dim V = k$. We start with the classification of minimal algebras over $\mathbf{k}$ of dimensions 2, 3 and 4.

Dimension 2. It should be $f_1 = 2$, so there is just one possibility:

$$(\wedge(x_1, x_2), dx_1 = dx_2 = 0)\,.$$

The corresponding Lie algebra is abelian.

For $\mathbf{k} = \mathbb{Q}$, where we are classifying 2-dimensional nilmanifolds, the corresponding nilmanifold is the 2-torus.

Dimension 3. Now there are two possibilities:

- $f_1 = 3$. Then the minimal algebra is $(\wedge(x_1, x_2, x_3), dx_1 = dx_2 = dx_3 = 0)$. The corresponding Lie algebra is abelian. In the case $\mathbf{k} = \mathbb{Q}$, the associated nilmanifold is the 3-torus.
- $f_1 = 2$ and $f_2 = 1$. Then $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism. We choose a generator $x_3 \in F_2$ such that $dx_3 = x_1x_2 \in \wedge^2 F_1$. The minimal algebra is $(\wedge(x_1, x_2, x_3), dx_1 = dx_2 = 0, dx_3 = x_1x_2)$. The corresponding Lie algebra is the Heisenberg Lie algebra. And for $\mathbf{k} = \mathbb{Q}$, the associated nilmanifold is known as the Heisenberg nilmanifold (see [13]).

We summarize the classification in the following table:

(f_i)	dx_1	dx_2	dx_3	$\mathfrak{g}$
(3)	0	0	0	A_3
$(2,1)$	0	0	x_1x_2	L_3

In the last column we have the corresponding Lie algebra: the abelian one, A_3, and the Lie algebra of the Heisenberg group, which we denote by L_3.

Dimension 4. The minimal algebra is of the form $(\wedge(x_1, x_2, x_3, x_4), d)$. We have to consider the following cases:

- $f_1 = 4$. Then the four elements x_i have zero differential. The corresponding Lie algebra is abelian.
- $f_1 = 3$, $f_2 = 1$. As the map $\bar{d} : F_2 \to \wedge^2 F_1$ is injective, there is a nonzero element in the image $\varphi_4 \in \wedge^2 F_1$. Using Lemma 5, we can choose a basis x_1, x_2, x_3 for F_1 such that $\varphi_4 = x_1x_2$. Then choose $x_4 \in F_2$ such that $dx_4 = \varphi_4 = x_1x_2$. Obviously, $dx_1 = dx_2 = dx_3 = 0$.
- $f_1 = 2$, $f_2 = 1$, $f_3 = 1$. In this case, we have a basis for $F_1 \oplus F_2$ such that $dx_1 = 0, dx_2 = 0$ and $dx_3 = x_1x_2$. The map

$$\bar{d} : F_3 \to F_1 \otimes F_2$$

is injective, hence the image determines a line $\ell \subset F_1$ such that $\bar{d}(F_3) = \ell \otimes F_2$. As $d(F_1 \oplus F_2) = \wedge^2 F_1$, we can choose $F_3 \subset W_3$ such that $d(F_3) = \ell \otimes F_2$. We choose the basis as follows: let $x_1 \in F_1$ be a vector spanning ℓ; x_2 another vector so that x_1, x_2 is a basis of F_1; let $x_3 \in F_2$ so that $dx_3 = x_1x_2$; finally choose x_4 such that $dx_4 = x_1x_3$.

The results are collected in the following table:

(f_i)	dx_1	dx_2	dx_3	dx_4	$\mathfrak{g}$
(4)	0	0	0	0	A_4
$(3,1)$	0	0	0	x_1x_2	$L_3 \oplus A_1$
$(2,1,1)$	0	0	x_1x_2	x_1x_3	L_4

The n-dimensional abelian Lie algebra is A_n; L_4 denotes the (unique) irreducible 4-dimensional nilpotent Lie algebra.

4. Classification in dimension 5

The minimal algebra is of the form $(\wedge(x_1,x_2,x_3,x_4,x_5),d)$. The possibilities for the numbers f_k are the following: $(f_1)=(5)$, $(f_1,f_2)=(4,1)$, $(f_1,f_2)=(3,2)$, $(f_1,f_2,f_3)=(3,1,1)$, $(f_1,f_2,f_3)=(2,1,2)$, $(f_1,f_2,f_3,f_4)=(2,1,1,1)$ (noting that $f_1\geq 2$ and that $f_1=2 \implies f_2=1$). We study all these possibilities in detail:

Case (5). All the elements have zero differential.

Case (4, 1). Then F_1 is a 4-dimensional vector space. Now the image of $\bar{d}: F_2 \to \wedge^2 F_1$ defines a line generated by some nonzero element $\varphi_5 \in \wedge^2 F_1$. By Lemma 5, we have two cases, according to the rank of φ_5 (by the *rank* of φ_5, we mean henceforth its rank as a bivector):

(1) There is a basis $F_1=\langle x_1,x_2,x_3,x_4\rangle$ such that $dx_5=\varphi_5=x_1x_2$.
(2) There is a basis $F_1=\langle x_1,x_2,x_3,x_4\rangle$ such that $dx_5=\varphi_5=x_1x_2+x_3x_4$.

Case (3, 2). Now F_1 is a 3-dimensional vector space, and $\bar{d}: F_2 \hookrightarrow \wedge^2 F_1$. By Lemma 5, every nonzero element $\varphi \in \wedge^2 F_1$ is of the form $\varphi = x_1x_2$ for a suitable basis x_1,x_2,x_3 of F_1, and determines a well-defined plane $\pi=\langle x_1,x_2\rangle \subset F_1$.

Now $F_2 \subset \wedge^2 F_1$ is a 2-dimensional vector space. Consider two linearly independent elements of F_2, which give two different planes in F_1, and let x_1 be a vector spanning their intersection. Now take a vector x_2 completing a basis for the first plane and a vector x_3 completing a basis for the second plane. Then we get the differentials $dx_4=x_1x_2$, $dx_5=x_1x_3$.

Case (3, 1, 1). F_1 is 3-dimensional, and the image of $\bar{d}: F_2 \hookrightarrow \wedge^2 F_1$ determines a plane $\pi \subset F_1$. Now

$$\bar{d}: F_3 \hookrightarrow F_1 \otimes F_2$$

determines a line $\ell \subset F_1$ (such that $\bar{d}(F_3)=\ell\otimes F_2$). We easily compute

(5)
$$H^2(\wedge(F_1\oplus F_2),d)=\frac{\ker(d:\wedge^2(F_1\oplus F_2)\to \wedge^3(F_1\oplus F_2))}{\operatorname{im}(d: F_1\oplus F_2 \to \wedge^2(F_1\oplus F_2))}=(\wedge^2 F_1/d(F_2))\oplus(\pi\otimes F_2)\,.$$

(The map $d: F_1\otimes F_2 \hookrightarrow F_1\otimes\wedge^2F_1 \to \wedge^3 F_1$ sends $v\otimes F_2 \mapsto 0$ if and only if $v\in\pi$). Hence $\ell\subset\pi$. We can arrange a basis x_1,x_2,x_3,x_4,x_5 with $\ell=\langle x_1\rangle$, $\pi=\langle x_1,x_2\rangle$, $F_1=\langle x_1,x_2,x_3\rangle$, so that $\varphi_4=dx_4=x_1x_2$, $\varphi_5=dx_5=x_1x_4+v$, where $v\in\wedge^2F_1$. Recall that F_2, F_3 are not well defined (only $W_1\subset W_2\subset W_3$ is a well-defined filtration.) In particular, this means that φ_4 is well defined, but φ_5 is only well defined up to $\varphi_5 \mapsto \varphi_5 + \mu\varphi_4$. But then $\varphi_5^2 \in \wedge^4 W_2$ is well defined, so we can distinguish cases according to the rank (as a bilinear form) of $\varphi_5 \in \wedge^2(F_1 \oplus F_2)$:

(1) φ_5 is of rank 2. This determines a plane $\pi' \subset W_2 = F_1 \oplus F_2$. The intersection of π' with F_1 is the line ℓ. Take an element $x_4 \in \pi'$ not in the line, and declare $F_2 \subset W_2$ to be the span of x_4. Therefore $dx_5 = x_1x_4$.

(2) φ_5 is of rank 4. The vector v is well defined in $\wedge^2 F_1/d(F_2)$. Thus $v = ax_1x_3+bx_2x_3$ with $b \neq 0$. We do the change of variables $x_4' = x_4+ax_3$, $x_3' = bx_3$. Then $x_1, x_2, x_3', x_4', x_5$ is a basis with $dx_4' = x_1x_2$, $dx_5' = x_1x_4' + x_2x_3'$.

Case $(2,1,2)$. Now F_1 is 2-dimensional; then $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism and $\bar{d} : F_3 \to F_1 \otimes F_2$ is an isomorphism. Therefore there is a basis x_1, x_2, x_3, x_4, x_5 such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, and $dx_5 = x_2x_3$.

Case $(2,1,1,1)$. Now $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism and the image of $\bar{d} : F_3 \to F_1 \otimes F_2$ produces a line $\ell \subset F_1$. Write $\ell = \langle x_1 \rangle$, $F_1 = \langle x_1, x_2 \rangle$, $F_2 = \langle x_3 \rangle$, and $F_3 = \langle x_4 \rangle$ so that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$.

For studying F_4, compute

$$H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3). \tag{6}$$

(Clearly, $d(F_1 \otimes F_2) = 0$, $d : F_1 \otimes F_3 \to \wedge^2 F_1 \otimes F_2$ has kernel equal to $\ell \otimes F_3$, and $d : F_2 \otimes F_3 \to \wedge^2 F_1 \otimes F_3$ is injective, so $\ker d = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus (\ell \otimes F_3)$; on the other hand $\operatorname{im} d = \wedge^2 F_1 \oplus (\ell \otimes F_2)$.) Recall that the element φ_5 generating $d(F_4)$ should have nonzero projection to $\ell \otimes F_3$. Also, φ_5 can be understood as a bivector in $W_3 = F_1 \oplus F_2 \oplus F_3$. This is well defined up to the addition of elements in $d(W_3) = \wedge^2 F_1 \oplus (\ell \otimes F_2)$; so $\varphi_5^2 \in \wedge^2 W_3$ is well defined, and hence we can talk about the rank of φ_5. We have two cases:

(1) φ_5 is of rank 2. This determines a plane $\pi' \subset W_3$, which intersects $F_1 \oplus F_2$ in a line. Let v span this line and x_4 be another generator of π'. Write $\varphi_5 = vx_4$. It must be $\langle v \rangle = \ell$, so $v = x_1$. Then $dx_3 = x_1x_2$, $dx_4 = x_1x_3$ and $dx_5 = x_1x_4$.

(2) φ_5 is of rank 4. Then the projection of φ_5 to the first summand in (6) must be nonzero. So there is a choice of basis so that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, and $dx_5 = x_1x_4 + x_2x_3$.

Summary of results. We gather all the results in the following table; the first three columns display the nonzero differentials. The fourth one gives the corresponding Lie algebras, and the last one refers to the list contained in [1]:

(f_i)	dx_3	dx_4	dx_5	$\mathfrak{g}$	[1]
(5,0)	0	0	0	A_5	–
(4,1)	0	0	x_1x_2	$L_3 \oplus A_2$	–
	0	0	$x_1x_2 + x_3x_4$	$L_{5,1}$	$\mathcal{N}_{5,6}$
(3,2)	0	x_1x_2	x_1x_3	$L_{5,2}$	$\mathcal{N}_{5,5}$
(3,1,1)	0	x_1x_2	x_1x_4	$L_4 \oplus A_1$	–
	0	x_1x_2	$x_1x_4 + x_2x_3$	$L_{5,3}$	$\mathcal{N}_{5,4}$
(2,1,2)	x_1x_2	x_1x_3	x_2x_3	$L_{5,5}$	$\mathcal{N}_{5,3}$
(2,1,1,1)	x_1x_2	x_1x_3	x_1x_4	$L_{5,4}$	$\mathcal{N}_{5,2}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{5,6}$	$\mathcal{N}_{5,1}$

As before, $L_{5,k}$ denotes the nonsplit 5-dimensional nilpotent Lie algebras.

Recall that this classification works over any field $\mathbf{k}$. In the case $\mathbf{k} = \mathbb{Q}$, this means in particular that there are nine nilpotent Lie algebras of dimension 5 over $\mathbb{Q}$ and, as a consequence, nine rational homotopy types of 5-dimensional nilmanifolds.

5. Classification in dimension 6

Now we move to study minimal algebras of the form $(\wedge(x_1, x_2, x_3, x_4, x_5, x_6), d)$, where $|x_i| = 1$. The numbers $\{f_k\}$ can be the following: $(f_1) = (6)$, $(f_1, f_2) = (5, 1)$, $(f_1, f_2) = (4, 2)$, $(f_1, f_2, f_3) = (4, 1, 1)$, $(f_1, f_2) = (3, 3)$, $(f_1, f_2, f_3) = (3, 2, 1)$, $(f_1, f_2, f_3) = (3, 1, 2)$, $(f_1, f_2, f_3, f_4) = (3, 1, 1, 1)$, $(f_1, f_2, f_3, f_4) = (2, 1, 2, 1)$, $(f_1, f_2, f_3, f_4) = (2, 1, 1, 2)$ and $(f_1, f_2, f_3, f_4) = (2, 1, 1, 1, 1)$.

The case $(2, 1, 3)$ does not appear due to the injectivity of the differential $\bar{d} : F_3 \to W_1 \otimes F_2$. Also the case $(2, 1, 1, 2)$ does not show up, as we will see at the end of this section. Now we consider all the cases in detail.

Case (6). In this case we have $F_1 = V$, $d(F_1) = 0$. This corresponds to the abelian Lie algebra.

Case (5, 1). Here F_1 is a 5-dimensional vector space and F_2 is 1-dimensional, $F_2 = \langle x_6 \rangle$; $\bar{d}(F_2) \subset \wedge^2 F_1$. Let $\varphi_6 = dx_6 \in \wedge^2 F_1$ be a generator of $d(F_2)$. By Lemma 5, we have the following cases:

(1) $\operatorname{rank}(\varphi_6) = 2$. Then there exists a basis of F_1 such that $dx_6 = x_1 x_2$.
(2) $\operatorname{rank}(\varphi_6) = 4$. Then there exists a basis of F_1 such that $dx_6 = x_1 x_2 + x_3 x_4$.

Case (4, 2). Here F_1 is a 4-dimensional vector space and $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$. This defines a projective line ℓ in $\mathbb{P}(\wedge^2 F_1) = \mathbb{P}^5$.

The skew-symmetric matrices of dimension 4 with rank ≤ 2 are given as the zero locus of the single quadratic homogeneous equation

$$a_1 a_6 - a_2 a_5 + a_3 a_4 = 0\,,$$

where

$$A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}$$

is a skew-symmetric matrix. This defines a smooth quadric $\mathcal{Q}$ in $\mathbb{P}^5$.

Now we have to look at the intersection of ℓ with $\mathcal{Q}$. It is here that the field of definition matters.

(1) $\ell \cap \mathcal{Q} = \{p_1, p_2\}$, two different points. Choose $\varphi_5, \varphi_6 \in \wedge^2 F_1$ so that they correspond to the points $p_1, p_2 \in \mathbb{P}(\wedge^2 F_1)$. Accordingly, choose x_5, x_6 generators of F_2 so that $\varphi_5 = dx_5$, $\varphi_6 = dx_6$. Note that both are bivectors of F_1 of rank 2, but the elements $a\varphi_5 + b\varphi_6$, $ab \neq 0$ are of rank 4. By Lemma 5, a rank 2 element determines a plane in F_1. The two planes correspond to φ_5, φ_6 intersect transversally (otherwise, we are in case (2) below). Thus we can choose a basis x_1, x_2, x_3, x_4 for F_1 so that $dx_5 = x_1 x_2$ and $dx_6 = x_3 x_4$. Note that the elements $ax_1x_2 + bx_3x_4$ are of rank 4 when $ab \neq 0$.
(2) $\ell \subset \mathcal{Q}$. We choose a basis x_5, x_6 so that both $\varphi_5 = dx_5$, $\varphi_6 = dx_6$ have rank 2. All linear combinations $a\,dx_5 + b\,dx_6$ are also of rank 2. The planes determined by φ_5, φ_6 do not intersect transversally (otherwise we are in case (1) above), so they intersect in a line. Then we can choose a basis

x_1, x_2, x_3, x_4 for F_1 so that $dx_5 = x_1x_2$ and $dx_6 = x_1x_3$, the line being $\langle x_1 \rangle$. Note that all elements $a\varphi_5 + b\varphi_6 = x_1(ax_2 + bx_3)$ are of rank 2.

(3) $\ell \cap \mathcal{Q} = \{p\}$. This means that ℓ is tangent to $\mathcal{Q}$. Let $\varphi_5 \in \wedge^2 F_1$ correspond to p. This is of rank 2, so it determines a plane $\pi \subset F_1$. The plane π is described by some equations $e_3 = e_4 = 0$, where $e_3, e_4 \in F_1^*$. Now consider $\varphi_6 \in \wedge^2 F_1$ giving another point $q \in \ell$. So φ_6 is of rank 4 (see Lemma 5). If $\varphi_6(e_3, e_4) = 1$, then choose e_1, e_2 so that $\varphi_6 = x_1x_2 + x_3x_4$, but then $\varphi_5 = \lambda x_1x_2$, with $\lambda \neq 0$, and $\varphi_6 - \lambda\varphi_5$ is also of rank 2, which is a contradiction.

Therefore $\varphi_6(e_3, e_4) = 0$, and so $\langle e_3, e_4 \rangle$ is Lagrangian in (F_1^*, φ_6). We can complete the basis to e_1, e_2, e_3, e_4 so that $dx_6 = \varphi_6 = x_1x_3 + x_2x_4$. Normalize φ_5 so that $dx_5 = \varphi_5 = x_1x_2$. All forms $dx_6 + a\,dx_5$ are of rank 4.

(4) $\ell \cap \mathcal{Q} = \emptyset$. This means that ℓ and $\mathcal{Q}$ intersect in two points with coordinates in the algebraic closure of $\mathbf{k}$. As this intersection is invariant by the Galois group, there must be a quadratic extension $\mathbf{k}' \supset \mathbf{k}$ where the coordinates of the two points lie; the two points are conjugate by the Galois automorphism of $\mathbf{k}'|\mathbf{k}$. Therefore, there is an element $a \in \mathbf{k}^*$ such that $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, a is not a square in $\mathbf{k}$, and the differentials

$$dx_5 = x_1x_2, \qquad dx_6 = x_3x_4,$$

satisfy that the planes $\pi_1 = \langle x_1, x_2 \rangle$ and $\pi_2 = \langle x_3, x_4 \rangle$ are conjugate under the Galois map $\sqrt{a} \mapsto -\sqrt{a}$. Write:

$$\begin{aligned} x_1 &= y_1 + \sqrt{a}y_2, \\ x_2 &= y_3 + \sqrt{a}y_4, \\ x_3 &= y_1 - \sqrt{a}y_2, \\ x_4 &= y_3 - \sqrt{a}y_4, \\ x_5 &= y_5 + \sqrt{a}y_6, \\ x_6 &= y_5 - \sqrt{a}y_6, \end{aligned}$$

where $y_1, \ldots, y_6$ are defined over $\mathbf{k}$. Then $dy_5 = y_1y_3 + ay_2y_4$, $dy_6 = y_1y_4 + y_2y_3$.

This is the "canonical" model. Two of these minimal algebras are not isomorphic over $\mathbf{k}$ for different quadratic field extensions, since the equivalence would be given by a $\mathbf{k}$-isomorphism, therefore commuting with the action of the Galois group.

The quadratic field extensions are parametrized by elements $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. Note that for $a = 1$, we recover case (1), where $dy_5 + dy_6 = (y_1 + y_2)(y_3 + y_4)$ and $dy_5 - dy_6 = (y_1 + y_2)(y_3 - y_4)$ are of rank 2.

Remark 6. If $\mathbf{k} = \mathbb{C}$ (or any algebraically closed field), then case (4) above does not appear.

For $\mathbf{k} = \mathbb{R}$, we have that $\mathbb{R}^*/(\mathbb{R}^*)^2 - \{1\} = \{-1\}$, and there is only one minimal algebra in this case, given by $dy_5 = y_1y_3 - y_2y_4$, $dy_6 = y_1y_4 + y_2y_3$.

The case $\mathbf{k} = \mathbb{Q}$ is very relevant, as it corresponds to the classification of rational homotopy types of nilmanifolds. Note that in this case the classes in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ are parametrized bijectively by elements $\pm p_1 p_2 \cdots p_k$, where p_i are different primes, and $k \geq 0$. In particular, if a is a square in $\mathbb{Q}$, then we fall again in (1) above.

Remark 7. Note that we get examples of distinct rational homotopy types of nilmanifolds which have the same real homotopy type. Also, we get nilmanifolds with different real homotopy types but the same complex homotopy type.

Case $(4,1,1)$. Now F_1 is 4-dimensional, and $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$ determines an element $\varphi_5 \in \wedge^2 F_1$. Clearly, $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2)$. The differential $d : F_1 \otimes F_2 \to \wedge^3 F_1$ is given as wedge by φ_5. So if φ_5 is of rank 4, then this map is an isomorphism and

$$\ker(d : \wedge^2(F_1 \oplus F_2) \to \wedge^3(F_1 \oplus F_2)) = \wedge^2 F_1.$$

So there cannot be an injective map $\bar{d} : F_3 \to F_1 \otimes F_2$. This shows that φ_5 must be of rank 2, and therefore it determines a plane $\pi \subset F_1$. Now the closed elements are given as $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. The differential $\bar{d} : F_3 \to \pi \otimes F_2$ determines a line $\ell \subset \pi$. Let x_1 be a generator for ℓ, and $\pi = \langle x_1, x_2 \rangle$. Then there is a basis x_1, x_2, x_3, x_4 such that $dx_5 = x_1 x_2$ and $dx_6 = x_1 x_5 + \varphi'$, where $\varphi' \in \wedge^2 F_1$. We are allowed to change x_5 by $x_5' = x_5 + v$ with $v \in F_1$. This has the effect of changing dx_6 by adding $x_1 v$. This means that we may assume that φ' does not contain x_1, so $\varphi' \in \wedge^2(F_1/\ell)$. Actually, wedging $\varphi_6 = dx_6 \in \wedge^2 F_1 \oplus (\pi \otimes F_2)$ by x_1, we get an element $\varphi_6\, x_1 \in \wedge^3 F_1$ which is the image of φ' under the map $\wedge^2(F_1/\ell) \stackrel{x_1}{\hookrightarrow} \wedge^3 F_1$. It is then easy to see that φ' is well defined (independent of the choices of F_2, F_3).

We have the following cases:

(1) $\varphi' = 0$. So $dx_6 = x_1 x_5$.
(2) φ' is nonzero, so it is of rank 2. Therefore it determines a plane π' in F_1/ℓ. If this is transversal to the line π/ℓ, then $\varphi' = x_3 x_4$, and we have that $dx_6 = x_1 x_5 + x_3 x_4$.
(3) If π' contains π/ℓ, then $\varphi' = x_2 x_3$, and we have $dx_6 = x_1 x_5 + x_2 x_3$.

Case $(3,3)$. This case is very easy, since F_1 is 3-dimensional, and $\bar{d} : F_2 \to \wedge^2 F_1$ must be an isomorphism. So there exists a basis such that $dx_4 = x_1 x_2$, $dx_5 = x_1 x_3$, and $dx_6 = x_2 x_3$.

Case $(3,2,1)$. We have a 3-dimensional space F_1. Then there is a 2-dimensional space F_2 with a map $\bar{d} : F_2 \to \wedge^2 F_1$. Note that any element in F_2 determines a plane in F_1. Intersecting those planes, we get a line $\ell \subset F_1$. Then the differential gives an isomorphism $h : F_2 \stackrel{\cong}{\to} F_1/\ell$ (defined up to a nonzero scalar). Choosing $\ell = \langle x_1 \rangle$, we take basis such that $h(x_4) = x_2$ and $h(x_5) = x_3$. So

$$dx_4 = x_1 x_2, \quad dx_5 = x_1 x_3\,.$$

Let us compute the closed elements in $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus \wedge^2 F_2$. Clearly, $d : \wedge^2 F_2 \hookrightarrow \wedge^2 F_1 \otimes F_2$. Also the map $d : F_1 \otimes F_2 \cong F_1 \otimes (F_1/\ell) \to \wedge^3 F_1$ is the map $(u, v) \mapsto u \wedge v \wedge x_1$. As $\operatorname{im} d = d(F_2)$, we have that

$$H^2(\wedge(F_1 \oplus F_2), d) = \wedge^2(F_1/\ell) \oplus \ker(F_1 \otimes F_2 \to \wedge^3 F_1),$$

and F_3 determines an element φ_6 in that space. Let π_4, π_5 be the planes in F_1 corresponding to dx_4, dx_5. There are vectors $v_2 \in \pi_4$, $v_3 \in \pi_5$ and $\lambda \in \mathbf{k}$ so that $\varphi_6 = \lambda x_2 x_3 + v_2 x_4 + v_3 x_5$. We have the following cases:

(1) Suppose that $\varphi_6^2 x_1 \neq 0$ (this condition is well defined, independently of the choices of F_2, F_3). This is an element in $\wedge^3 F_1 \otimes \wedge^2 F_2 \cong x_1 \otimes \wedge^2(F_1/\ell) \otimes \wedge^2 F_2 \cong (\wedge^2 F_2)^2$. Taking an isomorphism $\wedge^2 F_2 \cong \mathbf{k}$, we have that the class of $\varphi_6^2 x_1 \in (\wedge^2 F_2)^2 \cong \mathbf{k}$ gives a well-defined element in $\mathbf{k}^*/(\mathbf{k}^*)^2$.

The condition $\varphi_6^2 x_1 \neq 0$ translates into v_2, v_3, x_1 being linearly independent. So we can arrange $x_2 = a_2 v_2$, $x_3 = a_3 v_3$, with $a_2, a_3 \neq 0$. Normalizing x_6, we can assume $a_2 = 1$. So $dx_4 = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = \lambda x_2 x_3 + x_2 x_4 + a x_3 x_5$. Note that the class defined by $\varphi_6^2 x_1$ is $-2a \in \mathbf{k}^*/(\mathbf{k}^*)^2$. (If we change the basis $x_3' = \mu x_3$, $x_5' = \mu x_5$, we obtain $dx_6 = x_2 x_4 + a\mu^{-2} x_3' x_5'$. We see again that $-2a$ is defined in $\mathbf{k}^*/(\mathbf{k}^*)^2$.)

Changing the basis as $x_4' = x_4 + \lambda x_3$, we get $dx_4' = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = x_2 x_4' + a x_3 x_5$.

(2) Now suppose $\varphi_6^2 x_1 = 0$, $\varphi_6 x_1 \notin \wedge^3 F_1$, and $\varphi_6^2 \notin \wedge^3 F_1 \otimes F_2$ (again these conditions are independent of the choices of F_2, F_3). Then $v_2 v_3 x_1 = 0$ and $v_2 v_3 \neq 0$. We can choose the coordinates x_2, x_3 (and x_4, x_5 accordingly through h) so that $v_2 = x_2$, $v_3 = x_1$. Therefore $\varphi_6 = \lambda x_2 x_3 + x_2 x_4 + x_1 x_5$. Now the change of variable $x_4' = x_4 + \lambda x_3$ gives the form $dx_4' = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = x_2 x_4' + x_1 x_5$.

(3) Suppose that $\varphi_6^2 \in \wedge^3 F_1 \otimes F_2$ and $\varphi_6 x_1 \notin \wedge^3 F_1$. Then $v_2 v_3 = 0$ but x_1 is linearly independent with $\langle v_2, v_3 \rangle$. Choose coordinates so that $v_2 = x_2$ and $v_3 = 0$. So $\varphi_6 = \lambda x_2 x_3 + x_2 x_4$. The change of variable $x_4' = x_4 + \lambda x_3$ gives the form $dx_4' = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = x_2 x_4'$.

(4) Suppose that $\varphi_6 x_1 \in \wedge^3 F_1$, $\varphi_6^2 \neq 0$. So that we can choose $v_2 = x_1$, $v_3 = 0$. We have $dx_4 = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = \lambda x_2 x_3 + x_1 x_4$, where $\lambda \neq 0$. Now take $x_3' = \lambda x_3$ and $x_5' = \lambda x_5$. So $dx_4 = x_1 x_2$, $dx_5' = x_1 x_3'$, $dx_6 = x_2 x_3' + x_1 x_4$

(5) Finally, we have $\varphi_6 x_1 \in \wedge^3 F_1$, $\varphi_6^2 = 0$, and this gives the minimal algebra $dx_4 = x_1 x_2$, $dx_5 = x_1 x_3$, $dx_6 = x_1 x_4$.

Case $(3,1,2)$. We have a 3-dimensional vector space F_1. Then $\bar{d} : F_2 \to \wedge^2 F_1$ determines a well-defined plane $\pi \subset F_1$. Looking at $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2)$, we see that the closed elements are $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. The differential is defined by

$$\hat{d} : F_3 \to H^2(\wedge(F_1 \oplus F_2), d) = (\wedge^2 F_1 / d(F_2)) \oplus (\pi \otimes F_2)\,, \tag{7}$$

where the projection $\bar{d} : F_3 \to \pi \otimes F_2$ is injective, hence an isomorphism. So we identify $F_3 \cong \pi \otimes F_2$. Let x_1, x_2 be a basis for π, and x_5, x_6 the corresponding basis of F_3 through the above isomorphism. So $dx_4 = x_1 x_2$, $dx_5 = x_1 x_4 + v_5$, $dx_6 = x_2 x_4 + v_6$, where $v_5, v_6 \in \wedge^2 F_1 / d(F_2)$.

The map (7) together with $\bar{d}^{-1} : \pi \otimes F_2 \to F_3$ gives a map $\phi : \pi \otimes F_2 \to (\wedge^2 F_1 / d(F_2))$. It is easy to see that the pairing $F_1 \otimes \wedge^2 F_1 \to \wedge^3 F_1$ induces a nondegenerate pairing $\pi \otimes (\wedge^2 F_1 / d(F_2)) \to \wedge^3 F_1$, and hence an isomorphism $(\wedge^2 F_1 / d(F_2)) \cong \pi^* \otimes \wedge^3 F_1$. Hence $\phi : \pi \otimes F_2 \to \pi^* \otimes \wedge^3 F_1$, and using that $\pi^* \cong \pi \otimes \wedge^2 \pi^*$, we finally get a map

$$\phi : \pi \to \pi \otimes (\wedge^2 \pi^* \otimes \wedge^3 F_1 \otimes F_2^*).$$

This gives an endomorphism of π defined up to a constant.

Now let us see the indeterminacy of ϕ. With the change of variables $x_4' = x_4+\mu x_3+\nu x_2+\eta x_1$, we get $dx_5 = x_1x_4'+v_5'$, $dx_6 = x_2x_4'+v_6'$, where $v_5' = v_5-\mu x_1x_3$, $v_6' = v_6 - \mu x_2x_3$. Therefore the corresponding map $\phi' = \phi - \mu\,\mathrm{Id}$. So ϕ is defined up to addition of a multiple of the identity.

We get the following classification:

(1) Suppose that ϕ is zero (or a scalar multiple of the identity). Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4$.

(2) Suppose that ϕ is diagonalizable. Adding a multiple of the identity, we can assume that one of the eigenvalues is zero and the other is not. Let x_2 generate the image and x_1 be in the kernel. Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4 + x_2x_3$.

(3) Suppose that ϕ is not diagonalizable. Adding a multiple of the identity, we can assume that the eigenvalues are zero. Let x_1 generate the image, so that x_1 is in the kernel. Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4 + x_1x_3$.

(4) Finally, ϕ can be nondiagonalizable if $\mathbf{k}$ is not algebraically closed. To diagonalize ϕ, we need a quadratic extension of $\mathbf{k}$. Let $a \in \mathbf{k}^*$ so that ϕ diagonalizes over $\mathbf{k}' = \mathbf{k}(\sqrt{a})$. If we arrange ϕ to have zero trace (by adding a multiple of the identity), then the minimum polynomial of ϕ is $T^2 - a$. So we can choose a basis such that $\phi(x_1) = x_2$, $\phi(x_2) = ax_1$. Thus $dx_4 = x_1x_2$, $dx_5 = x_1x_4+x_2x_3$, $dx_6 = x_2x_4+ax_1x_3$. The minimal algebras are parametrized by $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. (The value $a = 1$ recovers case (2).)

Case $(3,1,1,1)$. Now F_1 is of dimension 3. We have a 1-dimensional space given as the image of $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$, which determines a plane $\pi \subset F_1$. The closed elements in $\wedge^2(F_1 \oplus F_2)$ are $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. Therefore, $\varphi_5 = dx_5$ determines a line $\ell \subset \pi$. But it also determines an element in $\wedge^2 F_1$, up to $d(F_2)$ and up to $\ell \wedge F_1$, i.e., in $\wedge^2(F_1/\ell)$. Then

(1) $dx_4 = x_1x_2$, $dx_5 = x_1x_4$. Now we compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$ to be $\wedge^2 F_1 \oplus (\pi \otimes F_2) \oplus (\ell \otimes F_3)$. The element $\varphi_6 = dx_6$ has nonzero last component in $\ell \otimes F_3$. It is well defined up to $\ell \wedge F_1$ and up to $\ell \otimes F_2$. There are several cases:
   - (a) $\varphi_6 \in \ell \otimes F_3$. Then $dx_6 = x_1x_5$.
   - (b) $\varphi_6 \in (\pi \otimes F_2) \oplus (\ell \otimes F_3)$. Then $dx_6 = x_2x_4 + x_1x_5$.
   - (c) $\varphi_6 \in \wedge^2 F_1 \oplus (\ell \otimes F_3)$, then $dx_6 = x_2x_3 + x_1x_5$.
   - (d) φ_6 has nonzero components in all summands. Then $dx_6 = \lambda x_2x_3 + x_2x_4 + x_1x_5$. We can arrange $\lambda = 1$ by choosing $x_3' = \lambda x_3$.

   (We can check that these cases are not equivalent: the first one is characterised by $\varphi_6 x_1 = 0$; the second one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 = 0$; the third one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 \neq 0$, $\varphi_6\varphi_4 = 0$; the last one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 \neq 0$, $\varphi_6\varphi_4 \neq 0$.)

(2) $dx_4 = x_1x_2$, $dx_5 = x_1x_4 + x_2x_3$. Then the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$ are those in
$$\wedge^2 F_1 \oplus (\pi \otimes F_2) \oplus \langle x_1x_5 + x_4x_3\rangle.$$
So $\varphi_6 = ax_1x_3 + bx_2x_3 + cx_1x_4 + dx_2x_4 + x_1x_5 + x_4x_3$. The change of variables $x_6' = x_6 - bx_5$ arranges $b = 0$. Then the change of variables $x_3' = -dx_2 + x_3$ and $x_5' = ax_3 + x_5$ arranges $a = 0$ and $d = 0$. Thus $\varphi_6 = cx_1x_4 + x_1x_5 + x_4x_3$. Finally $x_3' = -\frac{c}{2}x_1 + x_3$, $x_5' = \frac{c}{2}x_4 + x_5$ arranges $c = 0$. Hence $\varphi_6 = x_1x_5 - x_3x_4$.

Case $(2,1,2,1)$. Now we have a 2-dimensional space F_1, and an isomorphism $\bar{d}: F_2 \to \wedge^2 F_1$. Also $\bar{d}: F_3 \to \wedge^2(F_1 \oplus F_2)/\wedge^2 F_1 = F_1 \otimes F_2$ is an isomorphism. Then there is a basis for $F_1 \oplus F_2 \oplus F_3$ such that

$$dx_3 = x_1x_2, \ dx_4 = x_1x_3, \quad \text{and } dx_5 = x_2x_3 .$$

Let us compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$. First, $d: F_2 \otimes F_3 \to \wedge^2 F_1 \otimes F_3$ is an isomorphism; second $d: \wedge^2 F_3 \hookrightarrow F_1 \otimes F_2 \otimes F_3$ is an injection; finally, $d: F_1 \otimes F_3 \cong F_1 \otimes F_1 \otimes F_2 \to \wedge^2 F_1 \otimes F_2$. So the kernel of d is isomorphic to $\wedge^2(F_1 \oplus F_2) \oplus (s^2 F_1 \otimes F_2)$. Then

$$\varphi_6 \in H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = s^2 F_1 \subset F_1 \otimes F_1 \cong F_1 \otimes F_3$$

determines a nonzero quadratic form on F_1 up to multiplication by scalar, call it A. (Here we use the natural identification $F_3 \cong F_1$, $x_4 \mapsto x_1$, $x_5 \mapsto x_2$, defined up to scalar.)

We have the following cases:

(1) If $\operatorname{rank}(A) = 1$, then A has nonzero kernel. We get a basis such that $dx_6 = x_1x_4$.

(2) If $\operatorname{rank}(A) = 2$, then $\det(A) \neq 0$. This determines a 2×2-matrix A defined up to conjugation $A \mapsto M^T AM$ and up to $A \mapsto \lambda A$. Note that the class of the determinant $a = \det(A) \in \mathbf{k}^*/(\mathbf{k}^*)^2$ is well defined. Take a basis diagonalizing A. We can arrange that $A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. So $dx_6 = x_1x_4 + ax_2x_5$. (Note that for $a = 0$ we recover case (1).)

Case $(2,1,1,2)$. Now F_1 is 2-dimensional, and $\bar{d}: F_2 \to \wedge^2 F_1$ is an isomorphism. F_3 is 1-dimensional and $\bar{d}: F_3 \to \wedge^2(F_1 \oplus F_2)/\wedge^2 F_1 = F_1 \otimes F_2$. Therefore there exists a line $\ell \subset F_1$ such that $d(F_3) = \ell \otimes F_2$.

We compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3) = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus (F_1 \otimes F_3) \oplus (F_2 \otimes F_3)$. As $d: F_1 \otimes F_3 \to \wedge^2 F_1 \otimes F_2$ has kernel $\ell \otimes F_3$ and $d: F_2 \otimes F_3 \hookrightarrow \wedge^2 F_1 \otimes F_3$, we have that

$$H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3) .$$

As $\bar{d}: F_4 \to \wedge^2(F_1 \oplus F_2 \oplus F_3)/\wedge^2 (F_1 \oplus F_2)$ is injective, and $\dim(\ell \otimes F_3) = 1$, it cannot be that $f_4 = 2$.

Case $(2,1,1,1,1)$. We work as in the previous case. Now $\bar{d}: F_4 \to ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3)$ produces an isomorphism $F_4 \cong \ell \otimes F_3$ and hence a map

$$\phi : \ell \otimes F_3 \to (F_1/\ell) \otimes F_2.$$

Note that this map is well defined, independent of the choice of F_3 satisfying $W_2 \oplus F_3 = W_3$. We have the following cases:

(1) Suppose that $\phi = 0$. So there is a basis such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, $dx_5 = x_1x_4$, where we have chosen $\ell = \langle x_1 \rangle$, $F_1 = \langle x_1, x_2 \rangle$. We can easily compute

$$H^2(\wedge(x_1, x_2, x_3, x_4, x_5), d) = \langle x_1x_5, x_2x_3, x_2x_5 - x_3x_4 \rangle .$$

Then

$$\varphi_6 = dx_6 = ax_1x_5 + bx_2x_3 + c(x_2x_5 - x_3x_4) . \tag{8}$$

Table 1. Classification of minimal algebras over $\mathbf{k}$

(f_i)	dx_3	dx_4	dx_5	dx_6	$\mathfrak{g}$
(6,0)	0	0	0	0	A_6
(5,1)	0	0	0	x_1x_2	$L_3 \oplus A_3$
	0	0	0	$x_1x_2 + x_3x_4$	$L_{5,1} \oplus A_1$
(4,2)	0	0	x_1x_2	x_1x_3	$L_{5,2} \oplus A_1$
	0	0	x_1x_2	x_3x_4	$L_3 \oplus L_3$
	0	0	x_1x_2	$x_1x_3 + x_2x_4$	$L_{6,1}$
	0	0	$x_1x_3 + ax_2x_4$	$x_1x_4 + x_2x_3$	$L^a_{6,2}, a \in \Lambda - \{1\}$
(4,1,1)	0	0	x_1x_2	x_1x_5	$L_4 \oplus A_2$
	0	0	x_1x_2	$x_1x_5 + x_3x_4$	$L_{6,3}$
	0	0	x_1x_2	$x_1x_5 + x_2x_3$	$L_{5,3} \oplus A_1$
(3,3)	0	x_1x_2	x_1x_3	x_2x_3	$L_{6,4}$
(3,2,1)	0	x_1x_2	x_1x_3	x_1x_4	$L_{6,5}$
	0	x_1x_2	x_1x_3	x_2x_4	$L_{6,6}$
	0	x_1x_2	x_1x_3	$x_1x_5 + x_2x_4$	$L_{6,7}$
	0	x_1x_2	x_1x_3	$x_2x_4 + ax_3x_5$	$L^a_{6,8}, a \in \Lambda$
	0	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{6,9}$
(3,1,2)	0	x_1x_2	x_1x_4	x_2x_4	$L_{5,5} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_2x_3 + x_2x_4$	$L_{6,10}$
	0	x_1x_2	x_1x_4	$x_1x_3 + x_2x_4$	$L_{6,11}$
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_3 + ax_2x_4$	$L^a_{6,12}, a \in \Lambda - \{1\}$
(3,1,1,1)	0	x_1x_2	x_1x_4	x_1x_5	$L_{5,4} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,13}$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_4$	$L_{5,6} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3 + x_2x_4$	$L_{6,14}$
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_5 - x_3x_4$	$L_{6,15}$
(2,1,2,1)	x_1x_2	x_1x_3	x_2x_3	x_1x_4	$L_{6,16}$
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 + ax_2x_5$	$L^a_{6,17}, a \in \Lambda$
(2,1,1,1,1)	x_1x_2	x_1x_3	x_1x_4	x_1x_5	$L_{6,18}$
	x_1x_2	x_1x_3	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,19}$
	x_1x_2	x_1x_3	x_1x_4	$x_2x_5 - x_3x_4$	$L_{6,20}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_1x_5 + x_2x_4$	$L_{6,21}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_2x_5 - x_3x_4$	$L_{6,22}$

We have

(a) If $\varphi_6 x_1 = 0$, then $b = c = 0$. We can choose generators so that $dx_6 = x_1x_5$.

(b) If $\varphi_6 x_1 \neq 0$ and $\varphi_6 x_1 x_2 = 0$, then $c = 0$ and $a, b \neq 0$. We can arrange $a = 1$ by normalizing x_6 and then do the change of variables $x_2' = bx_2$ $x_3' = bx_3$, $x_4' = bx_4$, $x_5' = bx_5$, $x_6' = bx_6$. This produces an equation such as (8) with $b = 1$. Hence $dx_6 = x_1x_5 + x_2x_3$.

(c) If $\varphi_6 x_1 x_2 \neq 0$, then $c \neq 0$. We can arrange $c = 1$ by normalizing x_6. Now put $x_2' = x_2 + ax_1$ to arrange $a = 0$. Finally take $x_5' = x_5 + bx_3$, $x_4' = x_4 + bx_2$ to be able to put $b = 0$. So $dx_6 = x_2x_5 - x_3x_4$.

(2) Suppose that $\phi \neq 0$. Then there is a basis for $F_1 \oplus F_2 \oplus F_3 \oplus F_4$ such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, $dx_5 = x_1x_4 + x_2x_3$. We can easily compute

$$H^2(\wedge(x_1, x_2, x_3, x_4, x_5), d) = \langle x_1x_4, x_1x_5 + x_2x_4, x_2x_5 - x_3x_4\rangle .$$

Then

$$\varphi_6 = dx_6 = ax_1x_4 + b(x_1x_5 + x_2x_4) + c(x_2x_5 - x_3x_4) .$$

We have

(a) If $\varphi_6x_1x_2 = 0$, then $c = 0$. We can suppose $b = 1$ and put $x'_2 = x_2 + \frac{a}{2}x_1$, $x'_5 = x_5 + \frac{a}{2}x_4$ to arrange $a = 0$. So $dx_6 = x_1x_5 + x_2x_4$.

(b) If $\varphi_6x_1x_2 \neq 0$, then we can suppose $c = 1$. Put $x'_2 = bx_1 + x_2$ and $x'_5 = bx_4 + x_5$ to eliminate b. Finally, do the change of variables $x'_4 = x_4 - \frac{a}{2}x_2$, $x'_5 = x_5 - \frac{a}{2}x_3$ and $x'_6 = -ax_5 + x_6$ to arrange $a = 0$. Hence $dx_6 = x_2x_5 - x_3x_4$.

Classification of minimal algebras over k. Let $\mathbf{k}$ be any field of characteristic different from 2. The above work can be summarized in Table 1.

The first four columns display the nonzero differentials, and the fifth one is a labelling of the corresponding Lie algebra. Denote $\Lambda = \mathbf{k}^*/(\mathbf{k}^*)^2$. There are four families which are indexed by a parameter a: $L^a_{6,2}$ and $L^a_{6,12}$, which are indexed by $a \in \Lambda - \{1\}$; $L^a_{6,8}$ and $L^a_{6,17}$, which are indexed by $a \in \Lambda$. Thus, if we denote by r the cardinality of Λ, we obtain $28 + 2(r - 1) + 2r = 26 + 4r$ minimal algebras.

If $\mathbf{k}$ is algebraically closed (e.g., $\mathbf{k} = \mathbb{C}$), then there are 30 minimal models over $\mathbf{k}$. We can assume $a = 1$ in lines $L^a_{6,8}$ and $L^a_{6,17}$, while lines $L^a_{6,2}$ and $L^a_{6,12}$ disappear (actually, they are equivalent to lines $L_3 \oplus L_3$ and L_{10}, respectively).

Notice that when we set $a = 0$, the minimal algebra $L^a_{6,2}$ reduces to $L_{6,1}$; the minimal algebra $L^a_{6,8}$ reduces to $L_{6,6}$; the minimal algebra $L^a_{6,12}$ reduces to $L_{6,9}$; and the minimal algebra $L^a_{6,17}$ reduces to $L_{6,16}$.

Finally, recall that this classification yields the classification of nilpotent Lie algebras of dimension 6 over $\mathbf{k}$.

6. k-HOMOTOPY TYPES OF 6-DIMENSIONAL NILMANIFOLDS

In the case $\mathbf{k} = \mathbb{Q}$, the classification in Table 1 gives all rational homotopy types of 6-dimensional nilmanifolds. Note that $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is indexed by rational numbers up to squares, hence by $a = \pm p_1p_2 \cdots p_k$, where p_i are different primes, and $k \geq 0$.

Let us explicitly give the classification of real homotopy types of 6-dimensional nilmanifolds. Note that $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{\pm 1\}$. Therefore there are 34 real homotopy types, and we have Table 2.

Notice that all these minimal algebras do actually correspond to nilmanifolds, since they are defined over $\mathbb{Q}$.

The fifth column is a labeling of the nilpotent Lie algebra corresponding to the associated minimal algebra; for instance, when we write $L_{5,1} \oplus A_1$ we mean that the 6-dimensional nilpotent Lie algebra splits as the sum of a 5-dimensional nilpotent Lie algebra with an abelian Lie algebra of dimension 1. In geometric terms, the corresponding 6-dimensional nilmanifold is the product of the corresponding 5-dimensional nilmanifold with S^1.

The sixth column refers to the list contained in [1]. In [1], the problem of classifying 6-dimensional nilmanifolds is treated in a different way. Cerezo classifies 6-dimensional nilpotent Lie algebras over $\mathbb{R}$. Let us explain how we derived the

Table 2. Real homotopy types of 6-dimensional nilmanifolds

(f_i)	dx_3	dx_4	dx_5	dx_6	$\mathfrak{g}$	[1]	b_1	b_2	b_3	$\sum_i b_i$
(6,0)	0	0	0	0	A_6	–	6	15	20	64
(5,1)	0	0	0	x_1x_2	$L_3 \oplus A_3$	–	5	11	14	48
	0	0	0	$x_1x_2 + x_3x_4$	$L_{5,1} \oplus A_1$	–	5	9	10	40
(4,2)	0	0	x_1x_2	x_1x_3	$L_{5,2} \oplus A_1$	–	4	9	12	40
	0	0	x_1x_2	x_3x_4	$L_3 \oplus L_3$	–	4	8	10	36
	0	0	x_1x_2	$x_1x_3 + x_2x_4$	$L_{6,1}$	$\mathcal{N}_{6,24}$	4	8	10	36
	0	0	$x_1x_3 - x_2x_4$	$x_1x_4 + x_2x_3$	$L_{6,2}$	$\mathcal{N}_{6,23}$	4	8	10	36
(4,1,1)	0	0	x_1x_2	x_1x_5	$L_4 \oplus A_2$	–	4	7	8	32
	0	0	x_1x_2	$x_1x_5 + x_3x_4$	$L_{6,3}$	$\mathcal{N}_{6,22}$	4	6	6	28
	0	0	x_1x_2	$x_1x_5 + x_2x_3$	$L_{5,3} \oplus A_1$	–	4	7	8	32
(3,3)	0	x_1x_2	x_1x_3	x_2x_3	$L_{6,4}$	$\mathcal{N}_{6,21}$	3	8	12	36
(3,2,1)	0	x_1x_2	x_1x_3	x_1x_4	$L_{6,5}$	$\mathcal{N}_{6,20}$	3	6	8	28
	0	x_1x_2	x_1x_3	x_2x_4	$L_{6,6}$	$\mathcal{N}_{6,18}$	3	6	8	28
	0	x_1x_2	x_1x_3	$x_1x_5 + x_2x_4$	$L_{6,7}$	$\mathcal{N}_{6,17}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_2x_4 + x_3x_5$	$L_{6,8}^+$	$\mathcal{N}_{6,15}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_2x_4 - x_3x_5$	$L_{6,8}^-$	$\mathcal{N}_{6,16}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{6,9}$	$\mathcal{N}_{6,19}$	3	6	8	28
(3,1,2)	0	x_1x_2	x_1x_4	x_2x_4	$L_{5,5} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_2x_3 + x_2x_4$	$L_{6,10}$	$\mathcal{N}_{6,12}$	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_3 + x_2x_4$	$L_{6,11}$	$\mathcal{N}_{6,13}$	3	5	6	24
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_3 - x_2x_4$	$L_{6,12}$	$\mathcal{N}_{6,14}$	3	5	6	24
(3,1,1,1)	0	x_1x_2	x_1x_4	x_1x_5	$L_{5,4} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,13}$	$\mathcal{N}_{6,11}$	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_4$	$L_{5,6} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3 + x_2x_4$	$L_{6,14}$	$\mathcal{N}_{6,10}$	3	5	6	24
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_5 - x_3x_4$	$L_{6,15}$	$\mathcal{N}_{6,9}$	3	4	4	20
(2,1,2,1)	x_1x_2	x_1x_3	x_2x_3	x_1x_4	$L_{6,16}$	$\mathcal{N}_{6,8}$	2	4	6	20
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 + x_2x_5$	$L_{6,17}^+$	$\mathcal{N}_{6,6}$	2	4	6	20
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 - x_2x_5$	$L_{6,17}^-$	$\mathcal{N}_{6,7}$	2	4	6	20
(2,1,1,1,1)	x_1x_2	x_1x_3	x_1x_4	x_1x_5	$L_{6,18}$	$\mathcal{N}_{6,5}$	2	3	4	16
	x_1x_2	x_1x_3	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,19}$	$\mathcal{N}_{6,4}$	2	3	4	16
	x_1x_2	x_1x_3	x_1x_4	$x_2x_5 - x_3x_4$	$L_{6,20}$	$\mathcal{N}_{6,2}$	2	2	2	12
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_1x_5 + x_2x_4$	$L_{6,21}$	$\mathcal{N}_{6,3}$	2	3	4	16
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_2x_5 - x_3x_4$	$L_{6,22}$	$\mathcal{N}_{6,1}$	2	2	2	12

correspondence between our list and his. Consider, for example, the nilmanifold with real minimal model associated to the Lie algebra $L_{6,14}$. The 6-dimensional nilpotent Lie algebra $\mathcal{N}_{6,10}$ considered by Cerezo has generators $\langle X_1, \ldots, X_6 \rangle$ and commutators

$$[X_1, X_2] = X_4,\ [X_1, X_4] = X_5,\ [X_1, X_5] = X_6,\ [X_2, X_3] = X_6, \quad \text{and} \quad [X_2, X_4] = X_6.$$

Using the correspondence between nilpotent Lie algebras and minimal algebras, according to formula (3), we associate the Lie algebra $\mathcal{N}_{6,10}$ to the nilmanifold $L_{6,14}$. To check the other correspondences, it might be necessary to switch variables.

The last columns contain the Betti numbers of the nilmanifolds and the total dimension of the cohomology. The computation of the Betti numbers has been perfomed using the following facts.

- Thanks to Poincaré duality, we have $b_0 = b_6$, $b_1 = b_5$, and $b_2 = b_4$, where $b_i = \dim H^i(N)$.
- Nilmanifolds are parallelizable, and parallelizable manifolds have Euler characteristic zero, so

$$\sum_{i=0}^{n} (-1)^i \, b_i = 0\,. \tag{9}$$

- To compute b_3, we use Poincaré duality and (9). We obtain

$$b_3 = 2(b_0 - b_1 + b_2). \tag{10}$$

- $b_0 = 1$ and $b_1 = f_1$.

Thus it is enough to compute b_2 to obtain the whole information. As an example, we compute the Betti numbers of the nilmanifold $N = L_{6,12}$. We have $b_0 = b_6 = 1$ and $b_1 = b_5 = f_1 = 3$. The computation of b_2 goes as follows: a basis for $\ker d \cap \wedge^2 V$ is given by

$$\langle x_1x_2, x_1x_3, x_1x_4, x_1x_5 + x_2x_6, x_1x_6 - x_2x_5, x_2x_3, x_2x_4, x_3x_4 + x_2x_6 \rangle\,,$$

and $\ker d \cap \wedge^2 V$ is 8-dimensional. On the other hand, $\dim(\operatorname{im} d \cap \wedge^2 V) = n - f_1 = 3$. Thus $b_2 = \dim H^2(N) = 8 - 3 = 5 = b_4$. This gives, according to (10), $b_3 = 6$ and $\sum_i b_i = 24$.

Note that $\min \dim H^*(N) = 12$. This agrees with [9, proposition 3.3].

We end up with the proof of Theorem 2.

Proof of Theorem 2. If $(\wedge V, d)$ is a minimal model of a nilmanifold, then it is defined over $\mathbb{Q}$. So it is a minimal algebra in Table 1, with the condition that $a \in \mathbb{Q}^*$ if we are dealing with any of the four cases with parameter. (This element a is an invariant of the minimal algebra.)

Now, two nilmanifolds with minimal models $(\wedge V_1, d)$, $(\wedge V_2, d)$ are of the same $\mathbf{k}$-homotopy type if $(\wedge V_1 \otimes \mathbf{k}, d)$ and $(\wedge V_2 \otimes \mathbf{k}, d)$ are isomorphic (over $\mathbf{k}$). Then, first they should be in the same line in Table 1; second, if they correspond to a parameter case, with respective parameters $a_1, a_2 \in \mathbb{Q}^*$, then the $\mathbf{k}$-minimal models are isomorphic if and only if there exists $\lambda \in \mathbf{k}^*$ with $a_1 = \lambda^2 a_2$. Therefore a_1, a_2 define the same class in $\mathbb{Q}^*/((\mathbf{k}^*)^2 \cap \mathbb{Q}^*)$. □

Remark 8. A consequence of Theorem 2 is the following:

(1) There are nilmanifolds which have the same real homotopy type but different rational homotopy type.
(2) There are nilmanifolds which have the same complex homotopy type but different real homotopy type.
(3) There are nilmanifolds M_1, M_2 for which the CDGAs $(\Omega^*(M_1), d)$ and $(\Omega^*(M_2), d)$ are joined by chains of quasi-isomorphisms (i.e., they have the same *real* minimal model), but for which there is no $f : M_1 \to M_2$ inducing a quasi-isomorphism $f^* : (\Omega^*(M_2), d) \to (\Omega^*(M_1), d)$. Just consider M_1, M_2 not of the same rational homotopy type. If there was such f, then there is a map on the rational minimal models $f^* : (\wedge V_2, d) \to (\wedge V_1, d)$ such that $f^*_{\mathbb{R}} : (\wedge V_2 \otimes \mathbb{R}, d) \to (\wedge V_1 \otimes \mathbb{R}, d)$ is an isomorphism. Hence f^* is an isomorphism itself, and M_1, M_2 would be of the same rational homotopy type.

Remark 9. The fact that there exist nilpotent Lie algebras that are isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$ was already noticed by Lehmann in [8]. He gave a particular example of two nilpotent 6-dimensional Lie algebras that are isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$.

7. Symplectic nilmanifolds

In this section we study which of the above rational homotopy types of nilmanifolds admit a symplectic structure. The subject is important because symplectic nilmanifolds which are not a torus supply a large source of examples of symplectic nonKähler manifolds (see for instance [13]).

In the 2-dimensional case we have only the torus $\mathbb{T}^2$ which carries the symplectic area form $\omega = x_1x_2$.

The three 4-dimensional examples are symplectic. We recall them:

(1) $dx_i = 0$ for $i = 1, 2, 3, 4$. Here a symplectic form is given, for instance, by $\omega = x_1x_2 + x_3x_4$.
(2) $dx_i = 0$ for $i = 1, 2, 3$ and $dx_4 = x_1x_2$. Here we can take, for example, $\omega = x_1x_3 + x_2x_4$.
(3) $dx_i = 0$ for $i = 1, 2$, $dx_3 = x_1x_2$ and $dx_4 = x_1x_3$. Take $\omega = x_1x_4 + x_2x_3$.

In the 6-dimensional case our approach is based on the following simple remark: if there is a symplectic form, then there is an invariant symplectic form. Let $\omega \in \wedge^2(x_1, \ldots, x_6)$. We can assume that it has rational coefficients, i.e.,

$$\omega = \sum_{i<j} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Q}. \tag{11}$$

In order for it to be a symplectic form, ω must be closed ($d\omega = 0$) and nondegenerate ($\omega^3 \neq 0$). The second condition implies that ω must be of the form

$$\omega = a_{i_1i_2} x_{i_1} x_{i_2} + a_{i_3i_4} x_{i_3} x_{i_4} + a_{i_5i_6} x_{i_5} x_{i_6} + \omega' , \tag{12}$$

where $i_1, \ldots, i_6$ is a permutation of $1, 2, 3, 4, 5, 6$. If this is not possible, then there is no symplectic form ω and hence no symplectic structure on the associated nilmanifold. We list the symplectic 6-dimensional nilmanifolds in Table 3. In the first column we mention the Lie algebra of Table 2 associated to the rational homotopy type of the nilmanifold. In the second column either we produce an explicit symplectic form for the type, or we say that there do not exist symplectic structures on it.

As an example of computations, we show that the nilmanifold $L_{5,5} \oplus A_1$ is not symplectic and also how we constructed one possible symplectic form on $L_{6,9}$. The minimal model of $L_{5,5} \oplus A_1$ is $(\wedge V, d)$ with

$$dx_4 = x_1x_2, \; dx_5 = x_1x_4, \; \text{ and } dx_6 = x_2x_4.$$

It is easy to see that the space of closed elements of degree 2 is generated by

$$x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5 + x_1x_6, x_2x_6 ,$$

so ω is a linear combination of these terms. But now, according to (12), the subindices $5, 6$ do not go together, and 5 goes either with 1 or 2, whereas 6 goes either with 1 or 2. This implies that $3, 4$ should form a pair, which is impossible.

To show that some nilmanifold admits some symplectic structure is much easier as it is enough to find a symplectic form. If we take $L_{6,9}$ we have the minimal

TABLE 3. Symplectic 6-dimensional nilmanifolds

Type	**Symplectic form**	**Type**	**Symplectic form**
A_6	$x_1x_2 + x_3x_4 + x_5x_6$	$L_{5,5} \oplus A_1$	Not symplectic
$L_3 \oplus A_3$	$x_1x_6 + x_2x_3 + x_4x_5$	$L_{6,10}$	$x_1x_6 + x_2x_5 - x_3x_4$
$L_{5,1} \oplus A_1$	Not symplectic	$L_{6,11}$	$x_1x_5 + x_2x_6 + x_3x_4$
$L_{5,2} \oplus A_1$	$x_1x_5 + x_2x_4 + x_3x_6$	$L_{6,12}$	$x_1x_6 + 2x_2x_5 + x_3x_4$
$L_3 \oplus L_3$	$x_1x_5 + x_3x_6 + x_2x_4$	$L_{5,4} \oplus A_1$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{6,1}$	$x_1x_3 + x_2x_6 + x_3x_5$	$L_{6,13}$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{6,2}$	$x_1x_6 + x_2x_5 + x_3x_4$	$L_{5,6} \oplus A_1$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_4 \oplus A_2$	$x_1x_6 + x_2x_5 + x_3x_4$	$L_{6,14}$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{5,3} \oplus W$	$x_1x_6 + x_2x_4 - x_3x_5$	$L_{6,15}$	$x_1x_4 + x_2x_6 + x_3x_5$
$L_{6,3}$	Not symplectic	$L_{6,16}$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,4}$	$x_1x_4 + x_2x_6 + x_3x_5$	$L^+_{6,17}$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,5}$	$x_1x_6 + x_2x_4 + x_3x_5$	$L^-_{6,17}$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,6}$	$x_1x_4 + x_2x_6 + x_3x_5$	$L_{6,18}$	$x_1x_6 + x_2x_5 - x_3x_4$
$L_{6,7}$	Not symplectic	$L_{6,19}$	$x_1x_6 + x_2x_4 + x_2x_5 - x_3x_4$
$L^+_{6,8}$	Not symplectic	$L_{6,20}$	Not symplectic
$L^-_{6,8}$	Not symplectic	$L_{6,21}$	$2x_1x_6 + x_2x_5 + x_3x_4$
$L_{6,9}$	$x_1x_6 + 2x_2x_5 + x_3x_4$	$L_{6,22}$	Not symplectic

model $(\wedge V, d)$ with the following differentials:

$$dx_4 = x_1x_2,\ dx_5 = x_1x_3, \text{ and } dx_6 = x_1x_4 + x_2x_3.$$

Now $d(x_1x_6) = d(x_3x_4) = -x_1x_2x_3$ and $d(x_2x_5) = x_1x_2x_3$. Therefore,

$$\omega = x_1x_6 + 2x_2x_5 + x_3x_4$$

is closed, and we easily see that $\omega^3 = 12\, x_1x_2x_3x_4x_5x_6 \neq 0$. Thus ω is symplectic.

APPENDIX

This appendix is devoted to the study of the minimal model of commutative differential graded algebras defined over fields of characteristic $p \neq 2$. Let $\mathbf{k}$ be a field of arbitrary characteristic $p \neq 2$.

Theorem 10. *Any CDGA (A, d) has a Sullivan model: there exist a minimal algebra $(\wedge V, d)$ (in the sense of the definition given in the introduction) and a quasi-isomorphism $(\wedge V, d) \to (A, d)$.*

Proof. The proof of the existence is the same as in the case of characteristic zero, given in ([3, chapter 14]). □

Now we want to study the issue of uniqueness of the minimal model. It is not known in general whether it is necessarily true that if $(\wedge V, d) \to (A, d)$ and $(\wedge W, d) \to (A, d)$ are two minimal models, then $(\wedge W, d) \cong (\wedge V, d)$. This is known in characteristic zero ([15]), but it is an open question in positive characteristic $p \neq 2$ (see [7]).

Here we give a positive answer for the case of CDGAs with a minimal model generated in degree 1. However, some of the results which follow are valid in full generality.

Lemma 11. *Let $(\wedge V, d)$ be a minimal algebra, and let (A, d) and (B, d) be two CDGAs. Suppose that $f : (\wedge V, d) \to (A, d)$ is a CDGA morphism and that $\pi : (B, d) \to (A, d)$ is a surjective quasi-isomorphism. Then f can be lifted to a CDGA map $g : (\wedge V, d) \to (B, d)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} & & (B,d) \\ & \overset{g}{\nearrow} & \downarrow \pi \\ (\wedge V, d) & \xrightarrow{f} & (A,d). \end{array}$$

Moreover, if f is a quasi-isomorphism, then so is g.

Proof. We work inductively. By minimality, there is an increasing filtration $\{V_\mu\}$ of V such that d maps V_μ to $\wedge(V_{<\mu})$ (V_μ is the span of those generators x_τ with $\tau \leq \mu$). Suppose that g has been constructed on $V_{<\mu}$ and consider $x = x_\mu$. Since $dx \in \wedge(V_{<\mu})$, $g(dx)$ is well defined. We want to solve

$$\begin{cases} g(dx) = dy, \\ f(x) = \pi(y), \end{cases} \tag{13}$$

so that we can set $g(x) = y$.

There is some $b \in B$ such that $\pi(b) = f(x)$. Then $\pi(g(dx)) = f(dx) = d(f(x)) = d(\pi(b)) = \pi(db)$, so $c = g(dx) - db \in \ker \pi$. We compute $dc = d(g(dx)) = 0$, so c is closed. But $[c] \in H^*(B) \cong H^*(A)$ and $\pi(c) = 0$, so $[c] = 0$, i.e., there is some $e \in B$ such that $c = de$. Now $d\pi(e) = \pi(c) = 0$, so $\pi(e)$ is closed and $[\pi(e)] \in H^*(A) \cong H^*(B)$. Hence there is some closed $\beta \in B$ and $\alpha \in A$ such that $\pi(e) = \pi(\beta) + d\alpha$. Using the surjectivity of π again, $\alpha = \pi(\psi)$, for some $\psi \in B$. So $\pi(e) = \pi(\beta + d\psi)$. Now take $y = b + e - \beta - d\psi$. Clearly, $\pi(y) = \pi(b) = f(x)$ and $dy = db + de = g(dx)$.

Now suppose that f is a quasi-isomorphism and denote f_* and π_* the maps induced by f and π, respectively, at cohomology level. One has $f = \pi \circ g$, hence $f_* = \pi_* \circ g_*$; thus $g_* = \pi_*^{-1} \circ f_*$ is also an isomorphism. □

Now we particularise to minimal algebras generated in degree 1. In this case, we do not need surjectivity to prove a lifting property.

Theorem 12. *Let $(\wedge V, d)$ be a minimal algebra generated in degree 1 (i.e., $V = V^1$), and let (A, d) and (B, d) be two CDGAs. Suppose that $A^0 = \mathbf{k}$. If $f : (\wedge V, d) \to (A, d)$ is a CDGA morphism and $\psi : (B, d) \to (A, d)$ is a quasi-isomorphism, then there exists a CDGA map $g : (\wedge V, d) \to (B, d)$ such that $\psi \circ g = f$. Moreover, if f is a quasi-isomorphism, then so is g.*

Proof. We work as in the proof of Lemma 11. Consider generators $\{x_\tau\}$ of $V = V^1$. Assume that g has been defined for $V_{<\mu}$, and let $x = x_\mu$. Since $dx \in \wedge^2(V_{<\mu})$, $g(dx)$ is well defined. As before, we want to solve (13).

Now $d(g(dx)) = g(dd(x)) = 0$, so $[g(dx)] \in H^2(B,d)$. But $\psi_*[g(dx)] = = [f(dx)] = [d(f(x))] = 0$, so $[g(dx)] = 0$. Therefore, there exists $\xi \in B^1$ such that $g(dx) = d\xi$. Now $d(\psi(\xi)) = \psi(g(dx)) = f(dx) = d(f(x))$, so $\psi(\xi) - f(x) \in A^1$ is closed. As $A^0 = \mathbf{k}$, we have that $H^1(A,d) = Z^1(A,d) = \ker(d : A^1 \to A^2)$. Clearly, the quasi-isomorphism $\psi : (B,d) \to (A,d)$ gives a surjective map $Z^1(B,d) \to Z^1(A,d)$. Therefore, there exists $b \in Z^1(B,d) \subset B^1$ such that $\psi(\xi) - f(x) = \psi(b)$. Take $y = \xi - b$, to solve (13). □

Lemma 13. *Suppose $\varphi : (\wedge V, d) \to (\wedge W, d)$ is a quasi-isomorphism between minimal algebras. Then φ is an isomorphism.*

Proof. We can assume inductively that $\wedge(V^{<n}) \cong \wedge(W^{<n})$. We first show that $\varphi : \wedge(V^{\le n}) \to \wedge(W^{\le n})$ is injective. It is enough to see that the composition $\bar\varphi : V^n \to (\wedge W^{\le n})^n \to W^n$ is injective. Suppose $v \in V^n$ satisfies $\bar\varphi(v) = 0$. Then there exists $v' \in \wedge(W^{<n}) \cong \wedge(V^{<n})$ such that $\varphi(v) = \varphi(v')$. Then $\varphi(v'') = 0$, where $v'' = v - v'$. Then

$$0 = d(\varphi(v'')) = \varphi(dv'').$$

Thus $dv'' = 0$. Since φ is a quasi-isomorphism and $\varphi_*[v''] = 0$, we have that $v'' = d(v''')$ for some $v''' \in (\wedge V)^{n-1}$; but this is impossible since $\wedge V$ is a minimal algebra.

Now we prove the surjectivity of $\varphi : \wedge(V^{\le n}) \to \wedge(W^{\le n})$. First note that the minimality condition means the existence of an increasing filtration V_i^n such that $d(V_i^n) \subset \wedge(V^{<n} \oplus V_{i-1}^n)$ (and an analogous filtration W_i^n for W^n). We assume by induction that $\wedge(V^{<n} \oplus V_{i-1}^n) \cong \wedge(W^{<n} \oplus W_{i-1}^n)$. Consider

$$\mathcal{V}_i = V_i^n \oplus \wedge(V^{<n} \oplus V_{i-1}^n).$$

These are differential vector subspaces. Write $\mathcal{V}_i \hookrightarrow \wedge V \to C$, where C is the cokernel. Then C has only terms of degree $\ge n$. Moreover if we take the filtration with V_i^n maximal, i.e., $\mathcal{V}_i = d^{-1}(\wedge(V^{<n} \oplus V_{<i}^n))$, then $H^n(C) = 0$. This implies that $H^{\le n}(\mathcal{V}_i) \cong H^{\le n}(\wedge V)$ and $H^{n+1}(\mathcal{V}_i) \hookrightarrow H^{n+1}(\wedge V)$.

We define analogously $\mathcal{W}_i = W_i^n \oplus \wedge(W^{<n} \oplus W_{i-1}^n)$. Clearly, $\varphi : \mathcal{V}_i \to \mathcal{W}_i$. We have an exact sequence $0 \to \mathcal{V}_i \to \mathcal{W}_i \to Q \to 0$, where $Q = W_i^n/V_i^n$ is the cokernel. Again, Q does not have terms of degree $< n$. Also d on Q^n is zero, so $H^n(Q) = Q^n$. Note that the isomorphism $H^*(\wedge V) \cong H^*(\wedge W)$ implies that $H^{\le n}(\mathcal{V}_i) \cong H^{\le n}(\mathcal{W}_i)$ and $H^{n+1}(\mathcal{V}_i) \hookrightarrow H^{n+1}(\mathcal{W}_i)$. The long exact sequence in cohomology gives $H^n(Q) = Q^n = 0$, and hence $\mathcal{V}_i \cong \mathcal{W}_i$, which completes the induction. □

This gives us the uniqueness of the minimal model for the CDGAs that we are interested in.

Theorem 14. *Let (A,d) be a CDGA, defined over a field $\mathbf{k}$ of characteristic $p \ne 2$, such that $A^0 = \mathbf{k}$. Suppose that its minimal model $\varphi : (\wedge V, d) \to (A,d)$ satisfies that $(\wedge V, d)$ is a minimal algebra generated in degree 1. If $(\wedge W, d) \to (A,d)$ is another minimal model for (A,d), then $(\wedge W, d) \cong (\wedge V, d)$.*

Proof. By Theorem 12, there exists a quasi-isomorphism $g : (\wedge V, d) \to (\wedge W, d)$. By Lemma 13, g is an isomorphism. □

We have the following refinement.

Corollary 15. *Consider the category of CDGAs* (A, d) *with* $A^0 = \mathbf{k}$ *and whose minimal model is generated in degree* 1*. If two of such CDGAs* (A, d) *and* (B, d) *are quasi-isomorphic, then they have the same minimal model.*

Proof. Without loss of generality, we may assume that there is a quasi-isomorphism $\psi : (B, d) \to (A, d)$. If $\varphi : (\wedge V, d) \to (A, d)$ is a minimal model for (A, d), then there exists a quasi-isomorphism $g : (\wedge V, d) \to (B, d)$. Any other minimal model of (B, d) is isomorphic to $(\wedge V, d)$ by Theorem 14. □

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