

ROHLIN PROPERTIES FOR \mathbb{Z}^d ACTIONS ON THE CANTOR SET

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ABSTRACT. We study the space $\mathcal{H}(d)$ of continuous \mathbb{Z}^d -actions on the Cantor set, particularly questions related to density of isomorphism classes. For $d = 1$, Kechris and Rosendal showed that there is a residual conjugacy class. We show, in contrast, that for $d \geq 2$ every conjugacy class in $\mathcal{H}(d)$ is meager, and that while there are actions with dense conjugacy class and the effective actions are dense, no effective action has dense conjugacy class. Thus, the action by the group homeomorphisms on the space of actions is topologically transitive but one cannot construct a transitive point. Finally, we show that in the spaces of transitive and minimal actions the effective actions are nowhere dense, and in particular there are minimal actions that are not approximable by minimal shifts of finite type.

1. INTRODUCTION

This work concerns the space of continuous actions of \mathbb{Z}^d on the Cantor set, particularly the existence and nature of actions with dense conjugacy class and obstructions to the approximation of certain actions by others (see below for precise definitions). Questions of this sort are classical in ergodic theory, going back to the theorem of Halmos that, in the space of automorphisms of a Lebesgue space, every aperiodic automorphism has a dense conjugacy class [12].¹ This means, roughly speaking, that when viewed at a finite resolution, it is impossible to distinguish between aperiodic isomorphism types. The same is true for probability-preserving actions of amenable groups. Similar questions arise in the smooth category, where the situation is quite different: rigidity phenomena appear and one finds open sets of actions within which only one isomorphism type is represented. Thus in the smooth category one sees separation of dynamical types.

The situation in the topological category is somewhere between these two and has received attention only relatively recently. For actions on the Cantor set generated by a single homeomorphism, the situation is now quite well understood [9, 4, 17, 1, 13], and we shall discuss it further below. The aim of the present paper is to initiate the study of higher-rank actions on the Cantor set, which are of interest both from the point of view of abstract topological dynamics and also as the topological systems underlying a large number of lattice models in statistical mechanics and probability. As we shall see, the higher-rank case differs significantly from actions of \mathbb{Z} .

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¹This result is often attributed to Rohlin, who gave a simpler proof. See [9].

Let us first recall some basic definitions. Given a countable group G , a topological G -system consists of a pair (X, φ) , where X is a compact metric space and $\varphi : G \rightarrow \text{homeo}(X)$, $g \mapsto \varphi^g$, is a homomorphism of G into the group of homeomorphisms of X . A *subsystem* is a closed, non-empty subset $X' \subseteq X$ invariant under the action, to which we may restrict the action and obtain a dynamical system. Two G -systems (X, φ) and (Y, ψ) are *isomorphic* if there is a homeomorphism $\pi : X \rightarrow Y$ intertwining the action, that is, satisfying $\pi\varphi^g = \psi^g\pi$ for all $g \in G$. If π is onto and continuous (but not necessarily invertible), then (Y, ψ) is a *factor* of (X, φ) .

We now specialize to actions of \mathbb{Z}^d on the Cantor set. Denote the Cantor set by K and let

$$\mathcal{H} = \text{homeo}(K)$$

be the group of homeomorphisms of K , which is Polish when endowed with the topology of uniform convergence. The space of actions of \mathbb{Z}^d by homeomorphisms on K is

$$\mathcal{H}(d) = \text{hom}(\mathbb{Z}^d, \mathcal{H}).$$

We identify each action $\varphi \in \mathcal{H}(d)$ with the dynamical system (K, φ) . The space $\mathcal{H}(d)$ inherits a Polish topology as a closed subset of the countable product $\mathcal{H}^{\mathbb{Z}^d}$. Equivalently, this is the topology in which $\varphi_n \rightarrow \varphi$ if and only if $\varphi_n^u \rightarrow \varphi^u$ uniformly, for each $u \in \mathbb{Z}^d$ (or, equivalently, for $u = \pm e_i$, where e_i are the standard generators of \mathbb{Z}^d).

The group \mathcal{H} acts on $\mathcal{H}(d)$ by *conjugation*, that is, $\pi \in \mathcal{H}$ maps the action $\varphi = (\varphi^u)_{u \in \mathbb{Z}^d}$ to $\pi\varphi = (\pi\varphi^u\pi^{-1})_{u \in \mathbb{Z}^d}$, and the orbits of \mathcal{H} in $\mathcal{H}(d)$ are called *conjugacy classes*.² When φ, ψ are isomorphic actions on K , it follows that the homeomorphism $\pi \in \mathcal{H}$ which realizes the isomorphism conjugates φ and ψ . Thus the conjugacy class of an action $\varphi \in \mathcal{H}(d)$ is precisely the set of actions isomorphic to it.

The group \mathbb{Z}^d is said to have the *weak topological Rohlin property* (WTRP) if $\mathcal{H}(d)$ has a dense conjugacy class; it has the *strong topological Rohlin property* (STRP) if $\mathcal{H}(d)$ has a residual conjugacy class. This terminology has evolved recently in connection with questions about the largeness of conjugacy classes in topological groups. See [10] for a recent survey and extensive bibliography.

Over the past decade there has been significant progress in the understanding of the space $\mathcal{H}(1)$. Glasner and Weiss [9] showed that \mathbb{Z} has the WTRP, that is, there exist actions $\varphi \in \mathcal{H}(1)$ whose conjugacy class is dense (although it is not true that this is so for every aperiodic φ , as in the ergodic setting). More recently this result has been subsumed by a remarkable theorem of Kechris and Rosendal [17], who proved that \mathbb{Z} has the STRP: there is a residual conjugacy class; thus, generically, there is only one \mathbb{Z} -action on K . This action was described explicitly by Akin, Glasner and Weiss [1], and it turns out to be rather degenerate from a dynamical point of view: for example, it has no dense orbits. We also note that in the Polish subspace of $\mathcal{H}(1)$ consisting of transitive actions (i.e., those which have a dense orbit) there is also a residual conjugacy class [13].

²One can identify \mathcal{H} and $\mathcal{H}(1)$ by associating to $\varphi \in \mathcal{H}$ the action $(\varphi^n)_{n \in \mathbb{Z}} \in \mathcal{H}(1)$ generated by it. With this identification the conjugation action on $\mathcal{H}(1)$ is just the usual conjugation in the group \mathcal{H} , and conjugacy classes have the usual group-theoretic meaning. We shall maintain the distinction between \mathcal{H} and $\mathcal{H}(1)$ for consistency with the higher-rank case.

Let us now turn to \mathbb{Z}^d . A first difference between the case $d = 1$ and $d \geq 2$ is the following:

Theorem 1.1. *For $d \geq 2$, \mathbb{Z}^d does not have the strong topological Rohlin property, i.e., every action $\varphi \in \mathcal{H}(d)$ has a meager conjugacy class.*

We already noted that \mathbb{Z} has the WTRP [9]. Using the separability of $\mathcal{H}(d)$ it is not hard to show that this is true for any d :

Proposition 1.2. *\mathbb{Z}^d has the weak topological Rohlin property, i.e., there exist actions $\varphi \in \mathcal{H}(d)$ with dense conjugacy class.*

However, there is an interesting twist. In $\mathcal{H}(1)$ one can give explicit examples of systems with dense conjugacy class. In contrast, in $\mathcal{H}(d)$, $d \geq 2$, such actions exist, but it is formally impossible to construct one explicitly.

To make this precise, we must explain what we mean by an “explicit construction”. An action $\varphi \in \mathcal{H}(d)$ is said to be *strongly effective* if there is an algorithmic procedure³ for deciding, given a finite set $F \subseteq \mathbb{Z}^d$ and a family $\{C_u\}_{u \in F}$ of closed and open subsets of K , whether $\bigcap_{u \in F} \varphi^u(C_u) = \emptyset$. The action is *effective* if the emptiness of this intersection can be semi-decided, in the sense that if it is empty the algorithm must detect this, but otherwise it need not return a decision. This notion and others related to it are discussed in Section 3.1.

For $d = 1$ the Kechris-Rosendal system can be realized as an effective (and even strongly effective) action, and one can also explicitly construct other actions with dense conjugacy class, as in [9]. In contrast,

Theorem 1.3. *For $d \geq 2$ there are no effective actions with dense conjugacy class.*

Recall that a point in a topological dynamical system is called *transitive* if its orbit is dense, and the system is (*topologically*) *transitive* if it has a transitive point. One can now restate Theorem 1.3 as follows: the action of \mathcal{H} on $\mathcal{H}(d)$ by conjugation is topologically transitive, but it is formally impossible to construct a transitive point.

In spite of the above, the effective systems are dense in $\mathcal{H}(d)$ (see Proposition 4.2 below). This means that in a certain sense the entire space $\mathcal{H}(d)$ is accessible to us. It turns out that this is not the case for some other interesting spaces. Let $\mathcal{T}(d) \subseteq \mathcal{H}(d)$ denote the set of transitive actions, and let $\mathcal{M}(d) \subseteq \mathcal{T}(d)$ denote the set of minimal actions; recall that a dynamical system is *minimal* if it has no non-trivial subsystems, or, equivalently, every orbit is dense. Both these spaces are G_δ sets. Minimal and transitive actions have been studied extensively as analogues of ergodic actions in the measure-preserving category, and minimal systems admit a rich structure theory [2].

In dimension $d = 1$ it is possible to effectively construct dense families of minimal actions $\mathcal{T}(1)$ and $\mathcal{M}(1)$. Indeed, in [13] we showed that the universal odometer, i.e., the unique (up to isomorphism) transitive subsystem of the product of all finite cycles, has a residual conjugacy class in both, and one can construct a dense family of conjugates explicitly. The following theorem shows that in higher dimensions the situation is quite different.

³An algorithm is, formally, a Turing machine; informally it is a computer program without restrictions on memory or run time. Note that the word “effective” here is used to indicate that a problem can be decided by formal computation but does not make any guarantees about the efficiency of the computation.

Theorem 1.4. *Let $d \geq 2$. Then, within the spaces $\mathcal{T}(d)$ and $\mathcal{M}(d)$, the effective actions are nowhere dense.*

Note that a system may be conjugate to an effective action without being effective itself, so the statement above is stronger than the statements that the effective actions are not dense.

An important class of \mathbb{Z}^d actions, which will also be central to our proofs, are subshifts (also called *symbolic systems*), and specifically shifts of finite type (SFTs). To define these, we recall some definitions from symbolic dynamics. Let Σ be a finite alphabet, $|\Sigma| \geq 2$, and let $\Sigma^{\mathbb{Z}^d}$ be the set of Σ -colorings of \mathbb{Z}^d endowed with the product topology under which it is a Cantor set. The shift action is the action $\sigma = (\sigma^u)_{u \in \mathbb{Z}^d}$ on $\Sigma^{\mathbb{Z}^d}$ given by translation: $(\sigma^u x)_v = x_{u+v}$ for $x \in \Sigma^{\mathbb{Z}^d}$ and $u, v \in \mathbb{Z}^d$. A *subshift* $X \subseteq \Sigma^{\mathbb{Z}^d}$ is a subsystem of $(\Sigma^{\mathbb{Z}^d}, \sigma)$, that is, a closed, non-empty, shift-invariant set which we identify with the dynamical system (X, σ) . Thus, for example, if X, Y are subshifts, we write $X \cong Y$ instead of $(X, \sigma) \cong (Y, \sigma)$. We note, for later use, that a shift action may be defined in the same way on the space $K^{\mathbb{Z}^d}$, and we denote it also by σ . A closed, shift-invariant subset $X \subseteq K^{\mathbb{Z}^d}$ can be identified with a dynamical system in the same way as a subshift.

A shift of finite type (SFT) is a subshift X consisting of the points in $\Sigma^{\mathbb{Z}^d}$ whose orbit avoids a given closed and open set C , that is, $X = \Sigma^{\mathbb{Z}^d} \setminus \bigcup_{u \in \mathbb{Z}^d} \sigma^u(C)$ (this is equivalent to the more customary definition, requiring that the orbits avoid a finite collection of cylinder sets). An action $\varphi \in \mathcal{H}(d)$ is called a shift of finite type if it is isomorphic to the action of the shift σ on a shift of finite type $X \subseteq \Sigma^{\mathbb{Z}^d}$.

Shifts of finite type are effective (see Section 3.1), and it follows from the theorem above that:

Corollary 1.5. *For $d \geq 2$, the minimal (resp. transitive) \mathbb{Z}^d -SFTs are nowhere dense in the space of minimal (resp. transitive) \mathbb{Z}^d -actions.*

This result is somewhat unexpected: in dimension $d \geq 2$, SFTs display a wealth of dynamics, including minimal dynamics, and this has led to the impression that they can represent quite general systems [15, 18]. Theorem 1.5 shows that this is far from the case.

Shifts of finite type will play a central role in our proofs. We rely on the following rudimentary (and elementary) stability property possessed by SFTs, which explains their special role in $\mathcal{H}(d)$:

Theorem 1.6. *Let Y be a \mathbb{Z}^d -shift of finite type and $\varphi \in \mathcal{H}(d)$ an action on K which factors into Y . Then there is a neighborhood U of φ so that every action $\psi \in U$ factors into Y .*

Thus, controlling the subsystems of Y gives one control over the dynamics of actions in a neighborhood of φ . We achieve the required control by combining the recent constructive methods from [16, 15] with the recursive invariants introduced by Simpson in [21].

We conclude this introduction with the remark that, rather than work in the space of actions on K , one may alternatively formulate all the results above in the space of closed subsystems of $K^{\mathbb{Z}^d}$ or of $Q^{\mathbb{Z}^d}$, where Q is the Hilbert cube (the topology on subsystems is that induced by the Hausdorff metric). This model was studied in [13], and the methods there can be used to translate the present results to that setting.

The rest of this paper is organized as follows. Section 2 contains some notation. In Section 3 we review some results about effective systems and Medvedev degrees, which provide invariants for them. In Section 4 we discuss SFTs and the WTRP and prove Theorems 1.6 and 1.3, Proposition 1.2, and the part of Theorem 1.4 relating to $\mathcal{M}(d)$. Section 5 is mostly devoted to proving Theorem 1.1 and also completes the proof of Theorem 1.4 for the space $\mathcal{T}(d)$. In Section 6 we discuss some open problems.

2. PRELIMINARIES

This section contains additional notation and background.

Let Σ be a finite set and $\Sigma^{\mathbb{Z}^d}$ the full shift over Σ , on which we have defined the shift action $\sigma = (\sigma^u)_{u \in \mathbb{Z}^d}$. Let $F \subseteq \mathbb{Z}^d$ be a finite set and $a \in \Sigma^F$. One may think of a as a coloring of F by colors from Σ , and we refer to a as a pattern. The cylinder set $[a]$ is defined by

$$[a] = \{x \in \Sigma^{\mathbb{Z}^d} : x|_F = a\}.$$

In particular, for $i \in \Sigma$, we write

$$[i] = \{X \in \Sigma^{\mathbb{Z}^d} : x_0 = i\}.$$

Cylinder sets are open and closed and form a basis for the topology of $\Sigma^{\mathbb{Z}^d}$. We shall call an open and closed set *clopen* for short.

Recall that an SFT is a subshift $X \subseteq \Sigma^{\mathbb{Z}^d}$ of the form $X = \bigcap_{u \in \mathbb{Z}^d} \sigma^u(C)$, where $C \subseteq \Sigma^{\mathbb{Z}^d}$ is a closed and open set. C can be written as the disjoint union of finitely many cylinder sets, $C = \bigcup_{i=1}^n [a^{(i)}]$. We say that $x \in \Sigma^{\mathbb{Z}^d}$ contains the pattern a if $\sigma^u x \in [a]$ for some $u \in \mathbb{Z}^d$; thus an SFT X may be described equivalently as the set of configurations $x \in \Sigma^{\mathbb{Z}^d}$ which do not contain patterns from a fixed, finite list of patterns $a^{(1)}, \dots, a^{(n)}$. We note that the property of being an SFT is an isomorphism invariant, that is, if two subshifts are isomorphic and one is an SFT, then so is the other (though of course they are defined by different sets of forbidden patterns).

Given a partition $\alpha = \{A_1, \dots, A_n\}$ of K into clopen sets and an action $\varphi \in \mathcal{H}(d)$, we write $c_{\alpha, \varphi} : K \rightarrow \{1, \dots, n\}^{\mathbb{Z}^d}$ for the coding map that assigns to $x \in K$ its α -itinerary, i.e., the configuration $(x_u)_{u \in \mathbb{Z}^d} \in \{1, \dots, n\}^{\mathbb{Z}^d}$ with $x_u = i$ if and only if $\varphi^u x \in A_i$. We denote the image of the map $c_{\alpha, \varphi}$ by $\widehat{c}_\alpha(\varphi)$. Thus \widehat{c}_α maps actions to subshifts of $\{1, \dots, n\}^{\mathbb{Z}^d}$, and $c_{\alpha, \varphi}$ is a factor map from the system (K, φ) onto the system $(\widehat{c}_\alpha(\varphi), \sigma)$. If $c_{\alpha, \varphi}$ is an isomorphism of (K, φ) and $(\widehat{c}_\alpha(\varphi), \sigma)$, then we say that α is a *generating partition* for φ .

There is a converse to this: if $\varphi \in \mathcal{H}(d)$ is an action, $Y \subseteq \Sigma^{\mathbb{Z}^d}$ is a subshift, and $\pi : K \rightarrow Y$ is a factor map from (K, φ) onto (Y, σ) , then $\pi = c_{\alpha, \varphi}$, where α is the partition $\alpha = \{\pi^{-1}([i]) : i \in \Sigma\}$.

It will be convenient to introduce compatible metric on the spaces K, \mathcal{H} and $\mathcal{H}(d)$. We denote all of these by $d(\cdot, \cdot)$. Which is intended will be clear from the context.

Finally, recall that a metric space is perfect if it contains no isolated points, and the Cantor set is characterized up to homeomorphism as the unique compact, totally disconnected perfect metrizable space. In particular, any non-empty clopen subset of K is homeomorphic to K .

3. EFFECTIVE DYNAMICS AND MEDVEDEV DEGREES

In this section we review the notions of effectiveness for actions and subshifts, and define Medvedev degrees, which provide computation-theoretic invariants for effective actions.

3.1. Effective subshifts. We begin with some definitions from the theory of computation; for more information see [20].

A sequence (a_n) of integers is *recursive* (R) if there is an algorithm A (formally a Turing machine) that, upon input $n \in \mathbb{N}$, outputs a_n . A set of integers is *recursively enumerable* (RE) if it is the set of elements of some recursive sequence. Equivalently, $E \subseteq \mathbb{N}$ is RE if there is an algorithm which semi-decides membership to E in the sense that, given $k \in E$, the algorithm returns an affirmative answer in finite time, but given $k \notin E$ it either detects this in finite time or runs forever.

By identifying the integers with other sets, we can speak of recursive sequences of other elements. For example, since $\mathbb{N} \cong \mathbb{N}^2$ (and the bijection can be made effective), we can speak of recursive sequences of pairs of integers; and in the same way of sequences of finite sequences of integers.

We shall assume from here on that the Cantor set is parametrized in an explicit way. We shall use several such parametrizations, representing K as $\{0, 1\}^{\mathbb{N}}$, $\{1, 2, \dots, k\}^{\mathbb{Z}^d}$ and $(\{0, 1\}^{\mathbb{N}})^{\mathbb{Z}^d} = K^{\mathbb{Z}^d}$. All three may be identified by explicit homeomorphisms in such a way that a family of cylinder sets in one is R or RE if and only if the corresponding family of cylinder sets in the other parametrizations are also R or RE, respectively. Note that the basic operations on cylinder sets are computable, that is, it is algorithmically decidable whether two finite unions of cylinder sets are equal, disjoint, or one is contained in the other, etc.

Definition 3.1. A subset $X \subseteq K$ is *effective* if its complement is the union of a recursive sequence of cylinder sets.

Effective sets are, by definition, closed, and have been extensively studied in the recursion theory literature; see, e.g., [20].

A closed, shift-invariant subset $X \subseteq \{1, 2, \dots, k\}^{\mathbb{Z}^d}$ or $X \subseteq K^{\mathbb{Z}^d}$ is effective if it is effectively closed as a set. The following is obvious from the definition:

Proposition 3.2. *SFTs are effective.*

Effectiveness is not an isomorphism invariant for subsystems of $K^{\mathbb{Z}^d}$. Indeed, there are countably many effective subsets of $K^{\mathbb{Z}^d}$, since each is defined by some algorithm, and there are countably many algorithms but there are uncountably many ways to embed a given system as a subsystem of $K^{\mathbb{Z}^d}$. However, effectiveness is preserved under symbolic factors (i.e., factors onto subshifts).

Proposition 3.3. *A symbolic factor of an effective system is effective.*

Proof. See [15, Proposition 3.3]. □

This implies that effectiveness is an invariant of symbolic systems, that is, if $X \subseteq \Sigma^{\mathbb{Z}^d}$ and $Y \subseteq \Delta^{\mathbb{Z}^d}$ for finite sets Σ, Δ , and if X, Y are isomorphic, then one is effective if and only if the other is. This can be shown rather simply from the well-known fact that isomorphisms of subshifts are given by computable maps (so-called sliding block codes).

Since we are working in the space of actions rather than the space of subshifts, we also have the following definition, which was outlined in the introduction:

Definition 3.4. Let $\varphi \in \mathcal{H}(d)$. A finite sequence of pairs $\{(C_i, u_i)\}_{i=1}^n$, where C_i is a cylinder set and $u_i \in \mathbb{Z}^d$, is φ -disjoint if

$$\bigcap_{i=1}^n \varphi^{u_i}(C_i) = \emptyset.$$

φ is *effective* if the set of φ -disjoint sequences is RE, or in other words, if there is an algorithm that can recognize a φ -disjoint sequence in finite time (but may or may not identify non-disjoint ones).

These definitions are related as follows. Given an action $\varphi \in \mathcal{H}(d)$ and $x \in K$, let $\pi_\varphi(x) \in K^{\mathbb{Z}^d}$ be the point $(\pi_\varphi(x))_u = \varphi^u x$. Writing $X_\varphi = \pi_\varphi(K)$, one verifies that $\pi_\varphi : K \rightarrow X_\varphi$ is an isomorphism of (K, φ) and (X_φ, σ) . One can then show that φ is effective in the sense of Definition 3.4 if and only if X_φ is effective in the sense of Definition 3.1.⁴

Another natural way to define computability of an action $\varphi \in \mathcal{H}(d)$ is to require that each of the functions φ^{e_i} which generate the action is computable in the sense of Braverman and Cook [6], i.e., approximable to arbitrarily good precision. This notion implies effectiveness in the sense of Definition 3.4. Other definitions for effectiveness for sets, functions, and dynamical systems have received some attention recently; see [11, 5, 7].

3.2. Medvedev degrees. Given $X \subseteq \{0, 1\}^{\mathbb{N}}$, a function $f : X \rightarrow \{0, 1\}^{\mathbb{N}}$ is *computable* if there is an algorithm A such that, when given as input a point $x \in X$ and an integer k , outputs the k -th component of $f(x)$.⁵ Note that x is an infinite sequence of zeros and ones, but the algorithm will perform finitely many operations before halting, so it will only read a finite number of these bits. If $x' \in \{0, 1\}^{\mathbb{N}}$ differs from x at coordinates which were not read by the algorithm when run on x, k , then running it on x', k will produce exactly the same computation and give the same result as it did for x, k . It follows that a computable function is continuous on X .

An effective subset $Y \subseteq \{0, 1\}^{\mathbb{N}}$ is *reducible* to an effective subset $X \subseteq \{0, 1\}^{\mathbb{N}}$ if there exists a computable function $f : X \rightarrow Y$ (not necessarily onto). We denote this relation by $X \succeq Y$ and note that it is a partial order; we write $X \succ Y$ if $X \succeq Y$ but $Y \not\succeq X$.

One should interpret the relation $X \succeq Y$ as follows. Suppose we want to show that Y is not empty by producing in some manner a point $y \in Y$. If $X \succeq Y$ and if we can produce a point $x \in X$, then we can, by applying the computable function f , obtain the point $y = f(x)$. Thus X is at least as complicated as Y , in the sense that demonstrating that $X \neq \emptyset$ is at least as hard as demonstrating that $Y \neq \emptyset$. Notice that, contrary to one's expectation,

$$X \subseteq Y \implies X \succeq Y$$

because the inclusion map is computable. Also, if $y \in Y$ is computable as a function $y : \mathbb{N} \rightarrow \{0, 1\}$ (i.e., if there is an algorithm that given k computes the k -th

⁴Note that if $Y \subseteq K^{\mathbb{Z}^d}$ is a closed and shift invariant Cantor set which is effective, it can certainly happen that Y is not of the form $Y = X_\psi$ for any action ψ on K .

⁵The infinite sequence x is, technically, an oracle for the computation.

coordinate of y), then $X \succeq Y$ for all X , because there is a computable function $X \rightarrow Y$, i.e., the constant map y .

We say that X, Y are Medvedev equivalent if $X \succeq Y$ and $Y \succeq X$. The equivalence class of X is denoted $m(X)$ and called the *Medvedev degree* of X , and the partial order \succeq on effective sets induces a partial order on degrees, denoted in the same way. By the above, there is a minimal Medvedev degree consisting of all effective sets containing computable points; we denote the minimal degree by 0 . There are infinitely many distinct Medvedev degrees forming a distributive lattice whose structure is still rather mysterious. For more information about degrees, see [20].

Effective subsets of $K^{\mathbb{Z}^d}$ and Medvedev degrees for such sets are defined similarly.

Lemma 3.5. *If $X \subseteq \{0, 1\}^{\mathbb{N}}$ is effective and has an isolated point, then $m(X) = 0$.*

Proof. First, we claim that if a singleton $\{x\} \subseteq \{0, 1\}^{\mathbb{N}}$ is an effective set, then x is computable (as a function $\mathbb{N} \rightarrow \{0, 1\}$). Indeed, let $(C_n)_{n=1}^{\infty}$ be a recursive sequence of cylinder sets such that $\{x\} = \{0, 1\}^{\mathbb{N}} \setminus \bigcup_{n=1}^{\infty} C_n$. Given n , one can compute x_n as follows: Enumerate all pairs (a, N) with $a \in \{0, 1\}^n$, and find the first pair such that $\{0, 1\}^{\mathbb{N}} \setminus \bigcup_{i=1}^N C_i \subseteq [a]$. Such a pair exists because $[x_1 \cdots x_n] \cup \bigcup_{i=1}^{\infty} C_i = \{0, 1\}^{\mathbb{N}}$, so by compactness the same will be true for some finite union. Clearly, the a that we found is $a = x_1 \cdots x_n$, so a_n is the desired digit.

Now, if $x \in X$ were an isolated point, then there would be a cylinder set E such that $X \cap E = \{x\}$. The complement of E can be written as the union of finitely many cylinder sets. Therefore the complement of $\{x\}$ can be written as the union of a recursive sequence of cylinder sets. Therefore $\{x\}$ is effective, so x is computable, and $m(X) = 0$. \square

Medvedev degrees were introduced into the study of SFTs by S. Simpson [21], who proved the following:

Theorem 3.6 (Simpson, [21]). *For every $d \geq 2$, every Medvedev degree occurs as a d -dimensional shift of finite type. Furthermore, if $X \rightarrow Y$ is a factor map between SFTs, then $m(X) \succeq m(Y)$. Thus $X \mapsto m(X)$ is a functor from the category of SFTs and factor maps onto the category of Medvedev degrees.*

We shall mainly use the second part regarding monotonicity of degrees under factor maps. This part follows from the observation that factor maps between SFTs (and symbolic systems in general) are computable, since they arise from a clopen partition α and a map of the form $c_{\alpha, \varphi}$ (or, in the language of symbolic dynamics, they are given by sliding block codes).

A natural question, which is still far from understood, concerns the relation of the Medvedev degree of an effective subshift and its dynamics. One such connection is the following, which will be central to our argument.

Proposition 3.7. *If $\emptyset \neq X \subseteq K^{\mathbb{Z}^d}$ is effective, closed, shift-invariant, and minimal under the shift, then $m(X) = 0$.*

Proof. The proof follows that of [15, Proposition 9.4], where the claim was established for SFTs; we provide it here for completeness. First, since X is a minimal dynamical system, every non-empty open set $U \subseteq X$ satisfies $\bigcup_{u \in F} \sigma^u(U) = X$ for some finite set $F \subseteq \mathbb{Z}^d$. Indeed, $Y = X \setminus \bigcup_{u \in \mathbb{Z}^d} \sigma^u(U)$ is closed, invariant,

and $Y \neq X$, so, since X is minimal, $Y = \emptyset$, and the existence of F follows by compactness.

Since X is effective there is a recursive sequence C_1, C_2, \dots of cylinder sets in $K^{\mathbb{Z}^d}$ such that $X = K^{\mathbb{Z}^d} \setminus \bigcup_{n=1}^{\infty} C_n$. We must show that we can compute a point in X , or, equivalently, that there is a computable, decreasing sequence $B_1 \subseteq B_2 \subseteq \dots \subseteq K^{\mathbb{Z}^d}$ of clopen sets of diameter decreasing to 0 such that $X \cap B_n \neq \emptyset$ for all n . For this it suffices to show that there is an algorithm which, given a clopen set B with $X \cap B \neq \emptyset$, computes a clopen set $B' \subseteq B$ such that $X \cap B' \neq \emptyset$, and B' has half the diameter of B . For this, it suffices to prove that there is an algorithm which, given a clopen $B \subseteq K^{\mathbb{Z}^d}$, decides whether $X \cap B = \emptyset$.

The algorithm operates as follows. Given B , for each n , check whether either of the following holds:

- (1) $B \subseteq \bigcup_{i=1}^n C_i$.
- (2) $(\bigcup_{i=1}^n C_i) \cup (\bigcup_{\|u\| \leq n} \sigma^u(B)) = K^{\mathbb{Z}^d}$.

Clearly, if (1) holds, then $X \cap B = \emptyset$, and if (2) holds, then $X \cap B \neq \emptyset$ (note that we are assuming $X \neq \emptyset$). Each of these conditions can be checked algorithmically so it suffices to show that eventually one of them will occur. Indeed, if $X \cap B = \emptyset$, then $B \subseteq \bigcup_{i=1}^{\infty} C_i$, and since B is compact, there will be an n satisfying (1); on the other hand, if $X \cap B \neq \emptyset$, then $X \cap B$ is open and non-empty in X , and by minimality there is a finite set $F \subseteq \mathbb{Z}^d$ such that $X \subseteq \bigcup_{u \in F} \sigma^u(B)$, so (2) will hold for large enough n . □

4. SHIFTS OF FINITE TYPE AND THE WTRP

In this section we establish basic properties of SFTs in $\mathcal{H}(d)$ and prove Theorem 1.6. We then prove the WTRP (Proposition 1.2) and Theorem 1.3 about non-effectiveness of actions with dense conjugacy class. Finally, we prove the part of Theorem 1.4 relating to minimal actions.

4.1. Shifts of finite type in $\mathcal{H}(d)$. We say that an action $\psi \in \mathcal{H}(d)$ is an SFT if it is isomorphic to one. This means, in particular, that there is a clopen partition α of K such that the subshift $\widehat{c}_\alpha(\psi)$ is an SFT.

The following proposition is a reformulation and proof of Theorem 1.6.

Proposition 4.1. *Suppose $\alpha = \{A_1, \dots, A_n\}$ is a clopen partition of K and $\varphi \in \mathcal{H}(d)$ is mapped via \widehat{c}_α into a shift of finite type $X \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$ (i.e., $\widehat{c}_\alpha(\varphi) \subseteq X$). Then there is a neighborhood of φ in $\mathcal{H}(d)$ whose members are mapped via the factor map \widehat{c}_α to subsystems of X .*

Proof. Suppose $X \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$ is specified by disallowed patterns $b_1, \dots, b_k \in \{1, \dots, n\}^F$ for a finite $F \subseteq \mathbb{Z}^d$. Since A_i are clopen, it follows that whenever ψ is an action close enough to φ , then $\varphi^u(x), \psi^u(x)$ belong to the same set A_i for every $u \in F$ and $x \in K$. Thus, since $c_{\alpha, \varphi}(x)|_F \neq b_i$ for $i = 1, \dots, k$ and $x \in K$, similarly $c_{\alpha, \psi}(x)|_F \neq b_i$. Hence $\widehat{c}_\alpha(\psi) \subseteq X$. □

Proposition 4.2. *The shifts of finite type are dense in $\mathcal{H}(d)$.*

Proof. The proof is, essentially, to show that the “Markov approximations” of an action converge to it.

Let $\varphi \in \mathcal{H}(d)$, $\varepsilon > 0$, and choose a clopen partition $\alpha = \{A_1, \dots, A_n\}$ of K whose atoms are of diameter $< \varepsilon$. We must construct an SFT $\psi \in \mathcal{H}(d)$ such that

$\varphi^u(x), \psi^u(x)$ lie in the same atom of α for all $x \in K$ and all $u = \pm e_1, \dots, \pm e_d$, where e_i are the standard generators of \mathbb{Z}^d .

Let $Y = \widehat{c}_\alpha(\varphi)$, let $F = \{0, \pm e_1, \dots, \pm e_d\}$, let L denote the set of patterns on F appearing in Y , that is, $L = \{y|_F : y \in Y\}$. Let $\beta = \{B_a : a \in L\}$ denote the clopen partition of K corresponding to the patterns in L , that is,

$$B_a = c_{\alpha, \varphi}^{-1}([a]).$$

By definition, if $x \in B_a$, then $\varphi^u x \in A_{a(u)}$ for $u \in F$.

Let L' denote the patterns on F which do not belong to L , that is, $L' = \{1, \dots, n\}^F \setminus L$. Let X be the SFT defined by excluding all patterns from L' . Then $Y \subseteq X$, so X is not empty, and we have $L = \{x|_F : x \in X\}$.

Let $\gamma = \{C_a : a \in L\}$ denote the clopen partition of X according to the cylinder sets from L , that is,

$$C_a = X \cap [a].$$

Suppose first that X is perfect, i.e., has no isolated points. Then each C_a is homeomorphic to the Cantor set, as are the sets B_a , so we can choose a homeomorphism $\pi : K \rightarrow X$ such that $\pi(B_a) = C_a$ for $a \in L$. Let

$$\psi = \pi^{-1} \sigma \pi,$$

where σ is the shift on X . Then $\psi \in \mathcal{H}(d)$ is isomorphic to X . Now, if $x \in B_a$, then $\varphi^u x \in A_{a(u)}$ for $u \in F$. On the other hand $\pi x \in C_a$, so $(\sigma^u \pi x)_0 = a(u)$. It follows that $\psi^u x = \pi^{-1} \sigma^u \pi x \in A_{a(u)}$, as desired.

Finally, if X is not perfect we replace X by $X' = X \times \{0, 1\}^{\mathbb{Z}^d}$, which is an SFT as well, and replace γ with $\gamma' = \{C_a \times \{0, 1\}^{\mathbb{Z}^d} : a \in L\}$. Now the atoms of γ' are Cantor sets, and we proceed as before. \square

The proof of Theorem 1.3 relies on the following:

Theorem 4.3. *For $d \geq 2$, no effective \mathbb{Z}^d -action factors into every perfect SFT.*

The proof of this result is essentially identical to the proof given in [14], where it was shown that if there were an effective system that factors onto every SFT, then it could be used as part of an algorithm that decides whether a given finite set of patterns defines an empty SFT, and this is undecidable by Berger’s theorem [3, 19]. Two modifications to the proof are needed to deduce the version above. First, an inspection of the proof in [14] shows that it does not use the fact that the factor map is onto; thus the same proof works with the present hypothesis that the map is into. Second, to prove the version above, we must show that, given the rules of an SFT which is either empty or perfect, it is undecidable whether it is empty. Indeed, if we could decide this, then we could decide whether an arbitrary SFT were empty, since an SFT X is empty if and only if the SFT $X \times \{0, 1\}^{\mathbb{Z}^d}$ is empty, and the latter is either empty or perfect. Finally, the notion of effectiveness used in [14], and its relation to the one we are using, is discussed in Section 3.

Proof of Theorem 1.3. If $\varphi \in \mathcal{H}(d)$ has a dense conjugacy class, then, by Theorem 1.6, it would factor into every $\psi \in \mathcal{H}(d)$ that is conjugate to an SFT, that is, into every perfect SFT. Thus, by Theorem 4.3, φ is not effective. \square

4.2. Weak topological Rohlin property.

Proof of Proposition 1.2. We must show that there is a dense conjugacy class in $\mathcal{H}(d)$. The argument is similar to that given in [8] for the measure-preserving category and works for any countable group. Since $\mathcal{H}(d)$ is separable, we may choose a dense sequence $\varphi_1, \varphi_2, \dots$ in it, and let $\psi = \varphi_1 \times \varphi_2 \times \dots$ be the product action on $K^{\mathbb{N}}$, i.e.,

$$\psi(x_1, x_2 \dots) = (\varphi_1(x_1), \varphi_2(x_2), \dots).$$

It suffices to show that every φ_i is the limit of actions isomorphic to ψ . Fix i , parametrize K as $K = \{0, 1\}^{\mathbb{N}}$, and fix an integer $n \in \mathbb{N}$. By uniform continuity of $\varphi_i^{e_1}, \dots, \varphi_i^{e_d}$ there is a $k(n)$ such that, for each $j = 1, \dots, d$, the first n coordinates of $\varphi_i^{e_j}(x)$ depend only on the first $k(n)$ coordinates of x . Choose a partition I_1, I_2, \dots of \mathbb{N} into infinite sets with $\{1, \dots, k(n)\} \subseteq I_i$. Let $\pi_m : K \rightarrow \{0, 1\}^{I_m}$ denote the restriction map, and define $\varphi \in \mathcal{H}(d)$ so that $\pi_m \varphi = \varphi_m \pi_m$, that is, φ acts on $\{0, 1\}^{I_m}$ as φ_m acts on $K = \{0, 1\}^{\mathbb{N}}$. Clearly, $\varphi \cong \psi$, and for each $j = 1, \dots, d$ the first n coordinates of $\varphi^{e_j}(x)$ and $\varphi_i^{e_j}(x)$ agree for all $x \in K$. Thus by choosing n large we can make φ arbitrarily close to φ_i . This completes the proof of Proposition 1.2. \square

For the proof of Theorem 1.4 we also require the following:

Proposition 4.4. *Each of the spaces $\mathcal{T}(d)$ and $\mathcal{M}(d)$ contains a dense conjugacy class.*

Proof. The proof is similar to the one above. Consider $\mathcal{T}(d)$ for example. Choose a dense countable sequence of actions $\varphi_1, \varphi_2, \dots \in \mathcal{T}(d)$ and form $\varphi = \varphi_1 \times \varphi_2 \times \dots$ acting on $K^{\mathbb{N}}$. Although φ may not act transitively on $K^{\mathbb{N}}$, we may choose a transitive point $x_i \in K$ for φ_i and form the subset $X \subseteq K^{\mathbb{N}}$, which is the closure of the orbit of $x = (x_1, x_2, \dots)$ under φ . Then X projects to K on each coordinate (it is a *joining* of the φ_i 's). It is now possible to embed $(K^{\mathbb{N}}, \varphi)$ densely in $\mathcal{T}(d)$ in a manner similar to the proof above; we omit the details. A similar construction works for $\mathcal{M}(d)$, by choosing a dense set of representatives $\varphi_i \in \mathcal{M}(d)$ and a minimal subsystem of the product action. \square

4.3. Effective systems are nowhere dense in $\mathcal{M}(d)$. We can now prove the part of Theorem 1.4 regarding the space $\mathcal{M}(d)$. By Theorem 3.6 there is an SFT Y with non-minimal Medvedev degree. Let $Y_0 \subseteq Y$ be a minimal subsystem of Y (such a subsystem always exists).

We first claim that Y_0 is homeomorphic to K . Indeed, if Y_0 had an isolated point y , then the orbit E of y would be open and invariant in Y_0 and hence $Y_0 \setminus E$ would be closed and invariant, and by minimality must be empty. Thus $E = Y_0$ is compact and discrete, so it must be finite, so y is a periodic configuration and is trivially computable. This contradicts $m(Y) \succ 0$, so Y_0 is perfect, proving the claim.

We can therefore choose an action $\varphi \in \mathcal{H}(d)$ isomorphic to (Y_0, σ) . By Theorem 1.6 there is an open neighborhood $U \subseteq \mathcal{H}(d)$ of ψ so that every $\psi \in U$ factors into Y . We claim that U does not contain any minimal SFTs. Indeed, if $\psi \in U$ were a minimal SFT, then every symbolic factor of (K, ψ) is effective and minimal, and hence has minimal Medvedev degree. But this would imply that Y contains an effective subshift with minimal degree, and so Y itself has minimal degree, contrary to assumption.

We have shown that the systems conjugate to effective minimal systems do not meet the open set U . The fact that they are nowhere dense in $\mathcal{M}(d)$ follows from the fact that, by Proposition 4.4, the open set of all conjugates of U is dense in $\mathcal{M}(d)$.

The proof of Theorem 1.4 for $\mathcal{T}(d)$ is similar, except that Proposition 3.7 is not available. The proof is given in Section 5.2, where we construct an SFT Y with some additional special properties.

5. THE STRONG ROHLIN PROPERTY

The main goal of this section is the proof of Theorem 1.1. The idea, roughly speaking, is as follows. Let Y be an SFT and $\varphi \in \mathcal{H}(d)$ isomorphic to Y via $c_{\alpha, \varphi}$ for a generating partition α , and let U be an open neighborhood of φ in which every action factors into Y via \widehat{c}_α (see Proposition 1.6). Now fix an action $\theta \in \mathcal{H}(d)$; we wish to show that the conjugacy class of θ is meager. The Polish group \mathcal{H} acts on the Polish space $\mathcal{H}(d)$ continuously and with a dense orbit (Proposition 4.1), and therefore, by [10], every orbit of the action (i.e., every conjugacy class) is either meager or co-meager. Thus it suffices to show the isomorphism class of θ is meager in U .

Observe that there are at most countably many subsystems Y_1, Y_2, \dots of Y which are factors of θ , because each such factor arises from a clopen partition of K , and there are only countably many of these. For each i let

$$V_i = \{\psi \in U : \widehat{c}_\alpha(\psi) \neq Y_i\}.$$

The map $\psi \mapsto \widehat{c}_\alpha(\psi)$ is continuous (see Section 5.1) so these sets are open, and if we could show that they are dense in U we would be done, since then $V = \bigcap V_i$ would be a residual set in U , and if $\theta' \in V$, then $\widehat{c}_\alpha(\theta') \neq Y_i$ for all i , so $\theta' \not\cong \theta$.

Suppose now that Y had a very rich supply of subsystems, rich enough that any subsystem can be perturbed by an arbitrarily small amount. We would like to argue that, if $\psi \in U$ and $\widehat{c}_\alpha(\psi) = Y_i$, then we can perturb Y_i slightly in Y to obtain $Y' \neq Y_i$, and then lift the perturbed subshift to a perturbation ψ' of ψ with $\widehat{c}_\alpha(\psi') = Y' \neq Y_i$. Clearly, this would establish that V_i is dense in U .

It is not obvious, however, that such a perturbation can be made. For example, by Theorem 1.6, if ψ is an SFT, then the perturbed system Y' will necessarily be a subsystem of Y_i , and if Y_i is minimal, then its only subsystem is Y_i itself. Even if Y_i is not minimal, it might not be possible to perturb it by an arbitrarily small amount; this depends on the subsystem structure of Y .

We overcome these obstacles by making a careful choice of Y . This is done in the following sections, and on the way proving Theorem 1.4. We complete the proof outlined above in Section 5.4.

5.1. The space of subshifts. Recall that if X is a metrizable space with metric d , then the Hausdorff distance between compact subsets $\emptyset \neq A, B \subseteq X$ is defined by the condition that $d(A, B) < \varepsilon$ if and only if for each $a \in A$ there is a $b \in B$ with $d(a, b) < \varepsilon$, and the same with the roles of A, B reversed. The topology induced by the Hausdorff metric is independent of the metric we began with, is compact if X is, and is totally disconnected if X is.

Recall that if $\alpha = \{A_1, \dots, A_k\}$ is a clopen partition on K , then \widehat{c}_α is the map associating to $\varphi \in \mathcal{H}(d)$ the subshift obtained by coding itineraries according to α ; see Section 2.

Lemma 5.1. *Let $\alpha = \{A_1, \dots, A_k\}$ be a clopen partition on K . Then \widehat{c}_α is continuous as a map from $\mathcal{H}(d)$ to the space of subshifts of $\{1, \dots, k\}^{\mathbb{Z}^d}$.*

The proof is by direct verification and we omit it.

Lemma 5.2. *Let $W \subseteq \{1, \dots, k\}^{\mathbb{Z}^d}$. If W is an SFT, then the subsystems of W which are SFTs are dense among the subsystems of W ; and similarly if W is effective, then its effective subsystems are dense.*

Proof. We prove the SFT case, the effective case being similar. Let $W = \Sigma^{\mathbb{Z}^d} \setminus \bigcup_{u \in \mathbb{Z}^d} \sigma^u(C)$ for some clopen set $C \subseteq \Sigma^{\mathbb{Z}^d}$, and let $W' \subseteq W$ be a subsystem. $\Sigma^{\mathbb{Z}^d} \setminus W'$ is open and contains C , so we can write $W' = \Sigma^{\mathbb{Z}^d} \setminus \bigcup C_n$, where $C \subseteq C_1 \subseteq C_2 \cdots$ are clopen sets. Let $W_n = \Sigma^{\mathbb{Z}^d} \setminus \bigcup_{u \in \mathbb{Z}^d} \sigma^u(C_n)$. Then W_n are SFTs and $W' = \bigcap W_n$, so $W_n \rightarrow W$. \square

5.2. Construction of the SFT Y . In this section we construct an SFT Y whose subsystems are easily perturbed. More precisely, we construct an SFT Y and a symbolic factor Z of Y such that Z is the union of its minimal subsystems, these subsystems are not isolated, and Z has non-trivial Medvedev degree.

Let $\Omega \subseteq \{0, 1\}^{\mathbb{N}}$ be an effective, closed set of non-trivial Medvedev degree. To each $\omega \in \Omega$, we assign the point $z_\omega \in \{0, 1\}^{\mathbb{Z}}$ defined as follows. Select the coordinates of z_ω that form the arithmetic progression of period 2 passing through 0, and assign to them the symbol $\omega(1)$. Next, choose the arithmetic progression of period 4 passing through the smallest coordinate in absolute value not yet assigned, and to these coordinates assign the symbol $\omega(2)$. At the k -th step, assign the symbol $\omega(k)$ the coordinates belonging to the arithmetic progression of period 2^k that passes through the unassigned coordinate of least absolute value.

Let Z_ω denote the orbit closure of z_ω . One can show that it is possible to recover ω from any $z \in Z_\omega$, and furthermore the function $\bigcup Z_\omega \mapsto \omega$ is computable. Thus the Z_ω are pairwise disjoint, and one may verify that $Z' = \bigcup_{\omega \in \Omega} Z_\omega$ is closed and effective. The map $\omega \mapsto z_\omega$ is a computable function $\Omega \rightarrow Z'$, and we have already noted that there is a computable function $Z' \rightarrow \Omega$. Thus Z', Ω are Medvedev equivalent.

To obtain an SFT Y from Z' , we rely on the construction in Section 6 of [16].⁶ Recall that a sofic shift is, by definition, a symbolic factor of an SFT.

Theorem 5.3. *For $d \geq 2$ there is a \mathbb{Z}^d sofic shift Z such that $(Z, \sigma^{e_1}) \cong Z'$ and $\sigma^{e_j}, j = 2, 3, \dots, d$ act as the identity on Z .*

Let Y be an SFT factoring onto Z , and let $\rho : Y \rightarrow Z$ be the factor map. Then $m(Y) \succeq m(Z) = m(\Omega) \succ 0$.

We can now complete the proof of Theorem 1.4 for the space $\mathcal{T}(d)$ of transitive actions in $\mathcal{H}(d)$. Proceed as in the proof of the theorem for $\mathcal{M}(d)$ as presented in Section 4.3: select a minimal (and hence transitive) subshift $Y_0 \subseteq Y$, an action $\varphi \in \mathcal{T}(d)$ conjugate to Y_0 , and a neighborhood U of φ such that every $\psi \in U$ factors into Y . If $\psi \in U$ were an effective transitive system, then it factors into a transitive, effective subshift $Y' \subseteq Y$. This in turn factors to a transitive, effective subshift $Z' \subseteq Z$. But, since Z is the union of its minimal subsystems, it follows that Z' must be minimal. This is a contradiction since, as a minimal effective

⁶In [16] the construction is performed in dimension two, but from this it follows for any dimension.

subshift, Z' must have minimal Medvedev degree, contradicting $m(Y) \succ 0$. Thus the conjugates of effective transitive actions do not intersect U ; by Proposition 4.4 it follows that they are nowhere dense in $\mathcal{T}(d)$.

5.3. Subsystems of Z . Let $\rho : Y \rightarrow Z$ be the factor and systems constructed in the previous section. Compare the following to Lemma 3.5.

Lemma 5.4. *If $X \subseteq Z$ is an effective subsystem, then in the space of subsystems of X no minimal subsystem is isolated in the Hausdorff metric.*

Proof. Suppose $X' \subseteq X$ were an isolated minimal system. By Lemma 5.2 the effective subsystems of X are dense, so X' is an effective minimal system and therefore $m(X') = 0$ by Proposition 3.7. Since $X' \subseteq X \subseteq Z$, we have $m(Z) \preceq m(X) \preceq m(X')$, so $m(Z) = 0$, a contradiction. \square

Lemma 5.5. *If $X \subseteq Z$ is an effective subsystem, then every minimal subsystem of X is the accumulation point of minimal subsystems of X .*

Proof. Let $X_0 \subseteq X$ be a minimal subsystem. By the previous lemma there are subsystems of X arbitrarily close to X_0 ; we must show that this is also true for minimal subsystems. Let $\varepsilon > 0$ and $x_0 \in X_0$, and choose a finite $F \subseteq \mathbb{Z}^d$ so that $\{\sigma^u x_0\}_{u \in F}$ is ε -dense in X_0 . Let X_1 be a system ε -close to X_0 in the Hausdorff metric and close enough that there is a point $x_1 \in X_1$ such that $d(\sigma^u x_0, \sigma^u x_1) < \varepsilon$ for $u \in F$. The orbit closure X'_1 of x_1 is minimal since Z (and hence X) is the union of its minimal subsystems. The proof will be completed by showing that X'_1 is within distance 2ε of X_0 . To see this, note that if $x \in X_0$, then $d(x, \sigma^u x_0) < \varepsilon$ for some $u \in F$, hence $d(x, T^u x_1) < 2\varepsilon$; and on the other hand if $x' \in X'_1$, then $x' \in X_1$, so, since $d(X_0, X_1) < \varepsilon$, there is an $x \in X$ with $d(x, x') < \varepsilon$. This implies $d(X_0, X'_1) < 2\varepsilon$, as required. \square

Proposition 5.6. *Let $\rho : Y \rightarrow Z$ as above, W an SFT, and $\pi : W \rightarrow Y$ a shift-commuting map onto a subsystem of Y . Then for every $\varepsilon > 0$, there is an SFT $W_0 \subseteq W$ such that $d(W_0, W) < \varepsilon$ in the Hausdorff metric and $\pi(W_0) \neq \pi(W)$.*

Proof. Consider the diagram

$$\begin{array}{ccc} & & W \\ & & \pi \downarrow \\ Y & \supseteq & X = \pi(W) \\ \rho \downarrow & & \rho \downarrow \\ Z & \supseteq & X' = \rho(X). \end{array}$$

X' is a symbolic factor of an SFT, so it is effective. Let C_1, \dots, C_n be a partition of W into cylinder sets of diameter $< \varepsilon$. Since Z is the disjoint union of its minimal subsystems, so is X' . By Lemma 5.5, none of the minimal subsystems of X' is isolated, and since X is totally disconnected so is the space of minimal subsystems. We can therefore partition X' into clopen, pairwise disjoint invariant subsystems X'_1, \dots, X'_{n+1} . For each C_i , $i = 1, \dots, n$, there is at least one X'_j such that $C_i \cap (\rho\pi)^{-1}(X'_j) \neq \emptyset$, so without loss of generality, $j = i$. Thus $W'_0 = \bigcup_{i=1}^n (\rho\pi)^{-1}(X'_i)$ satisfies most of the desired properties: $\pi(W) \neq \pi(W'_0)$ because $\rho\pi(W'_0) \cap X_{n+1} = \emptyset$, and $d(W, W'_0) < \varepsilon$ because $W_0 \subseteq W$ and $W_0 \cap C_i \neq \emptyset$ for all i . Of course, W_0 is not an SFT, but the subsystems that are SFTs are dense among the subsystems of W by Lemma 5.2, and any SFT W_0 close enough to W_0 has the desired properties. \square

It remains to translate this approximation lemma to the space $\mathcal{H}(d)$.

Corollary 5.7. *Let Y, W and $\pi : W \rightarrow Y$ be as in the previous lemma, and let $\varphi \in \mathcal{H}(d)$ be conjugate to W by coding with respect to a partition $\alpha = \{A_1, \dots, A_n\}$ of K . Then for every $\varepsilon > 0$, there is an SFT ψ with $d(\varphi, \psi) < \varepsilon$, and ψ factors via $c_{\alpha, \psi}$ to an SFT $W_0 \subseteq W$ with $\pi(W_0) \neq \pi(W)$.*

Proof. Since α generates for φ , there is an $r > 0$ so that the atoms of the partition

$$\beta = \bigvee_{\|u\| \leq r} \sigma^u \alpha = \left\{ \bigcap_{\|u\| \leq r} \sigma^u(A_u) : A_u \in \alpha \right\}$$

are of diameter $< \eta$ for a parameter η we shall specify later. By the previous lemma, we may choose a subshift $W_0 \subseteq W$ so that $K_0 = c_{\alpha, \varphi}^{-1}(W_0)$ intersects each atom of β . We define $\psi_0 = \varphi|_{K_0} : K_0 \rightarrow K_0$; notice that $(K_0, \psi_0) \cong (W_0, \sigma)$.

Since W_0 is an SFT it is effective, and since $W_0 \subseteq W$ and $m(W) \neq 0$, we have $m(W_0) \neq 0$, so by Lemma 3.5, W_0 has no isolated points. Therefore $K_0 \cap B$ is homeomorphic to a Cantor set for each atom $B \in \beta$, and we may choose a homeomorphism $\tau : K \rightarrow K_0$ satisfying $\tau(B) = K_0 \cap B$ for $B \in \beta$. It follows that $d(x, \tau(x)) < \eta$ for $x \in K$, so if η is small enough, the action $\psi = \tau^{-1}\psi_0\tau$ will satisfy $d(\varphi, \psi) < \varepsilon$. Finally, the β -itineraries of a point $x \in K$ are the same for the actions ψ and ψ_0 since $\tau(B) = K_0 \cap B$. Thus $c_{\beta, \psi}$ is a factor map $(K, \psi) \rightarrow W_0$, and the lemma follows. \square

5.4. The strong topological Rohlin property. We now have all the parts we need to prove our main theorem.

Proof of Theorem 1.1. We proceed as described in the introduction to Section 5. Let $\varphi \in \mathcal{H}(d)$ be conjugate, via a partition α , to the SFT Y constructed in Section 5.2. Choose a neighborhood U of φ such that $\widehat{c}_\alpha(\psi) \subseteq Y$ for $\psi \in U$ (Proposition 4.1).

Fix $\theta \in \mathcal{H}(d)$, whose isomorphism class we wish to show is meager. Let Y_1, Y_2, \dots be an enumeration of all the subsystems of Y that are factors of θ , and let

$$V_i = \{\psi \in U : \widehat{c}_\alpha(\psi) \neq Y_i\},$$

which is an open subset of U . We will be done once we show that the V_i 's are dense in U , since then $V = \bigcap V_i$ is residual in U , and if $\psi \in V$, then $\widehat{c}_\alpha(\psi) \neq Y_i$ for all i , implying that $\psi \not\cong \theta$.

By Proposition 4.2 the SFTs are dense in U , so it suffices to show that each V_i is dense among the SFTs. Fix i , and let $\psi \in U$ be an SFT. We must show that there are actions ψ' arbitrarily close to ψ with $\widehat{c}_\alpha(\psi') \neq Y_i$. If $\widehat{c}_\alpha(\psi) \neq Y_i$, then there is nothing to do. If $\widehat{c}_\alpha(\psi) = Y_i$, then, for every $\varepsilon > 0$, we can apply Corollary 5.7 to get an SFT action ψ' within distance ε of ψ , so that $\widehat{c}_\alpha(\psi') \neq \widehat{c}_\alpha(\psi) = Y_i$.

Finally, it is a general fact that, for a topologically transitive action of a Polish group on a Polish space, every orbit is either meager or co-meager [10]. We know that \mathcal{H} acts transitively on $\mathcal{H}(d)$ by Proposition 1.2, and we have shown that the conjugacy class of θ is not residual in U and therefore not residual; so it is meager. \square

6. TWO PROBLEMS

The picture emerging from these results is that the space of \mathbb{Z}^d actions on K is quite complicated. The closure of the space of effective systems may be better behaved, and we conclude with a couple of questions about this space.

Recall that an action $\varphi \in \mathcal{H}(d)$ is *strongly irreducible* if there is an $R > 0$ such that, for every pair of open sets $\emptyset \neq A, B \subseteq K$, we have $\varphi^u A \cap B \neq \emptyset$ for every $u \in \mathbb{Z}^d$ with $\|u\| \geq R$. The class of SFTs with this property has been widely studied in thermodynamics as the class with the best hope of developing something of a thermodynamic formalism, and in symbolic dynamics as a fairly manageable class where embedding and factoring relations may be well behaved (note that the factor of a strongly irreducible system is itself strongly irreducible).

Problem 6.1. Can every strongly irreducible action be approximated by a strongly irreducible SFT?

In dimension 1 the answer is affirmative. Note that strongly irreducible SFTs, like minimal SFTs, have Medvedev degree 0 [16, Corollary 3.5]. Thus a negative answer would follow if we could construct an SFT of non-trivial degree having some strongly irreducible subsystem (which of course will not be effective).

With regard to the space of minimal systems, we have shown that within $\mathcal{M}(d)$ the (relative) closure of the effective systems, and thus of the minimal SFTs, has empty (relative) interior. It is still open if these closures are the same. In other words,

Problem 6.2. Can every minimal effective system be approximated by a minimal SFT?

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