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EXISTENCE OF VERTICAL ENDS OF MEAN CURVATURE 1/2 IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We prove the existence of graphs over exterior domains of $\mathbb{H}^2 \times \{0\}$, of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ and weak growth equal to the embedded rotational examples.

1. Introduction

In this paper, we prove the existence of vertical graphs over exterior domains of $\mathbb{H}^2 \times \{0\}$, with constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ (Theorem 3.4), provided that the boundary curve satisfies some geometric conditions. An easy example of boundary curve could be a hyperbolic ellipse with small eccentricity (Example 3.5).

As far as we know, the graphs thus obtained are the first examples of embedded vertical ends (a vertical end is a topological compact disk minus one interior point, with no asymptotic point at finite height) that are not rotational. Furthermore, they have weak growth (see Definition 3.3) equal to the embedded rotational examples.

Such examples suggest a new discussion in the theory of constant mean curvature $H = \frac{1}{2}$ ends in $\mathbb{H}^2 \times \mathbb{R}$, inspired by that of minimal ends in $\mathbb{R}^3$ (see [H-K], [SC], [R]). In analogy with the minimal case, we ask:

Question 1. Is every vertical end, that is also a graph of mean curvature $H = \frac{1}{2}$, uniformly asymptotic to the end of a rotational graph?

Question 2. Let M be a surface of mean curvature $H = \frac{1}{2}$, properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, with two vertical ends that are graphs. Is M a rotational surface?

We can prove that the ends of our graphs are trapped between two rotational surfaces, which gives a contribution to the answer to the first question.

It is worth noticing that there are many examples of entire graphs of mean curvature $H = \frac{1}{2}$ with a nonvertical end; see for instance [SE]. A significant example is given by the graph of the following function $f : \mathbb{H}^2 \times \{0\} \longrightarrow \mathbb{R}$ (in the halfplane model for $\mathbb{H}^2$):

$$f(x, y) = \frac{\sqrt{x^2 + y^2}}{y}, \quad y > 0.$$

The graph of f has mean curvature $H = \frac{1}{2}$ and its asymptotic boundary contains two vertical half straight lines going from 1 to $+\infty$ (see [SE], equation (31), figure (12)).

We prove our existence result by solving a certain Dirichlet problem for the mean curvature equation $H = \frac{1}{2}$, over some exterior domains in $\mathbb{H}^2 \times \{0\}$. The proof is

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obtained by the Perron Process. The geometric condition satisfied by the boundary guarantees that the rotational surfaces of mean curvature $H = \frac{1}{2}$ can be used as barriers at the boundary points.

The third author and E. Toubiana proved an existence theorem analogous to Theorem 3.4 for the minimal case in $\mathbb{R}^3$ [SE-T3] and in $\mathbb{H}^n \times \mathbb{R}$ [SE-T4]. They used catenoids as barriers. In [N-SE], the second and the third authors studied surfaces in $\mathbb{H}^2 \times \mathbb{R}$ of mean curvature $H = \frac{1}{2}$: they proved a vertical halfspace theorem.

This paper is organized as follows. In Section 2, we describe the geometry of rotational ends of mean curvature $H = \frac{1}{2}$ and we prove the Convex Hull Lemma. In Section 3 we prove the existence theorem.

2. Rotational surfaces with $H \leqslant \frac{1}{2}$

The third author and E. Toubiana found explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ in [SE-T2]. A careful description of the geometry of these surfaces is contained in the Appendix of [N-SE-S-T].

In this section, we recall some properties of rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$, and we describe their asymptotic behavior. Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function defined on a subset Ω of $\mathbb{H}^2 \times \{0\}$. The *vertical graph* of u is the subset of $\mathbb{H}^2 \times \mathbb{R}$ given by $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$. The vertical graph of a function $u : \mathbb{H}^2 \times \{0\} \longrightarrow \mathbb{R}$ has constant mean curvature H, with respect to the upward unit normal vector field $N = -\frac{\nabla_{\mathbb{H}} u}{W_u} + \frac{1}{W_u}\frac{\partial}{\partial t}$, if and only if u satisfies the following partial differential equation:

$$\operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 2H, \tag{2.1}$$

where $\operatorname{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, $|\cdot|_{\mathbb{H}}$ being the norm in $\mathbb{H}^2 \times \{0\}$.

We take the disk model for $\mathbb{H}^2$ with Euclidean coordinates (x, y). If we set $F = \left(\frac{1-x^2-y^2}{2}\right)^2$ and develop equation (2.1), we obtain

$$u_{xx}(1+Fu_y^2) - 2Fu_x u_y u_{xy} + u_{yy}(1+Fu_x^2) + \sqrt{F}(u_x^2+u_y^2)(xu_x+yu_y) - \frac{2HW_u^3}{F} = 0. \tag{2.2}$$

Denote by ρ the hyperbolic distance from the origin in $\mathbb{H}^2 \times \{0\}$. Then, the function whose graph is a rotational surface with constant mean curvature H satisfies (cf. Lemma 11 in [SE-T2])

$$\frac{1}{\sinh \rho}\left(\frac{\sinh \rho u_\rho}{\sqrt{1+u_\rho^2}}\right) = 2H.$$

The family of first integrals of the previous equation is the following (cf. formula (21) in [SE-T2], with $l = 0$, $\alpha = -d$ or formula (9) in [N-SE-S-T]):

$$u_\alpha^H(\rho) = \int_{r_\alpha^H}^{\rho} \frac{-\alpha + 2H\cosh r}{\sqrt{\sinh^2 r - (-\alpha + 2H\cosh r)^2}} dr, \tag{2.3}$$

where α is a real parameter and $r_\alpha^H = \operatorname{arccosh}\left(\frac{-2\alpha H + \sqrt{1-4H^2+\alpha^2}}{1-4H^2}\right)$, for $0 \leqslant H < \frac{1}{2}$ and $r_\alpha^{\frac{1}{2}} = |\ln \alpha|$.

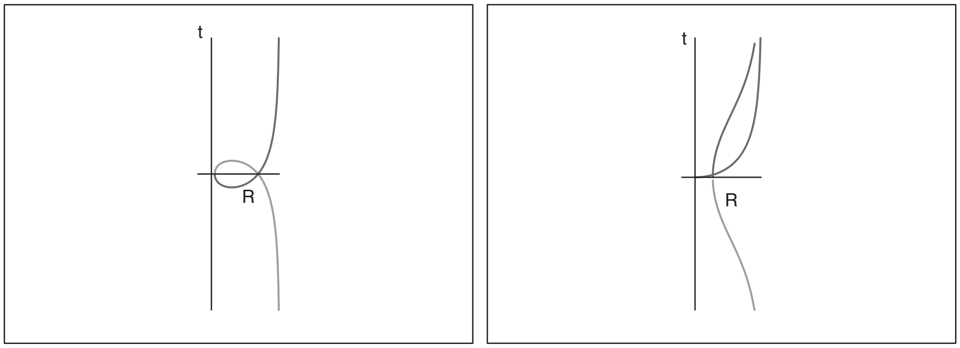

FIGURE 1. $H = \frac{1}{2}$: the profile curve in the immersed and embedded case ($R = \tanh \rho/2$).

The function u_α^H is defined up to an additive constant that corresponds to a vertical translation of the rotational surface. When $\alpha = 2H$, the function u_{2H}^H is defined on $\mathbb{H}^2 \times \{0\}$, and its graph is an entire rotational surface, denoted by S^H. For any $\alpha \neq 2H$, the graph of u_α^H is defined outside the disk D_α^H of radius r_α^H and it is vertical along the boundary of D_α^H. We choose the integration constant such that $u_\alpha^H(r_\alpha^H) = 0$. Denote by $\mathcal{H}_\alpha^H$ the annulus obtained by the union of the graph of u_α^H with its symmetry with respect to the slice $t = 0$. As it is shown in Lemma 5.2 in [N-SE-S-T], for $\alpha > 1$ the surfaces $\mathcal{H}_\alpha^H$ are immersed, while they are embedded for $\alpha \leqslant 1$.

We are especially interested in the asymptotic behavior of the rotational surfaces with mean curvature $H = \frac{1}{2}$ (see Figure 1).

For simplicity, we denote by u_α, D_α, r_α, $\mathcal{H}_\alpha$, S, the previous $u_\alpha^{\frac{1}{2}}$, $D_\alpha^{\frac{1}{2}}$, $r_\alpha^{\frac{1}{2}}$, $\mathcal{H}_\alpha^{\frac{1}{2}}$, $S^{\frac{1}{2}}$, respectively.

For any $\alpha \neq 1$, $\mathcal{H}_\alpha$ is a rotational annulus symmetric with respect to the plane $t = 0$. Replacing $H = \frac{1}{2}$ in formula (2.3) one has

$$u_\alpha(\rho) = \int_{|\ln \alpha|}^{\rho} \frac{-\alpha + \cosh r}{\sqrt{2\alpha \cosh r - 1 - \alpha^2}} dr. \tag{2.4}$$

As we said before, the radius of the disk D_α is $r_\alpha = |\ln \alpha|$ and the function u_α is vertical along the boundary of D_α. Furthermore, r_α is always greater than or equal to zero; it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \longrightarrow 0$. In this case, the unique simply connected example is the graph of the entire function $u_1(\rho) = 2\cosh \frac{\rho}{2}$.

A straightforward computation shows that the integrand function in (2.4) is equivalent to $\frac{1}{2\sqrt{\alpha}} e^{\frac{r}{2}} - c_\alpha e^{-\frac{r}{2}}$, for $r \longrightarrow \infty$, where c_α is a constant depending only on α. Then, by integrating, one has that $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}} + 2c_\alpha e^{-\frac{\rho}{2}} + k$, for $\rho \longrightarrow \infty$, where k is the integration constant.

By a more careful computation, one can prove that the asymptotic behavior of the function u_α is the following:

$$u_\alpha(\rho) = \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}} + \frac{3\alpha^2 - 1}{2\alpha^{\frac{3}{2}}} e^{-\frac{\rho}{2}} + k + O\left(e^{-\frac{3\rho}{2}}\right), \quad \rho \longrightarrow \infty. \tag{2.5}$$

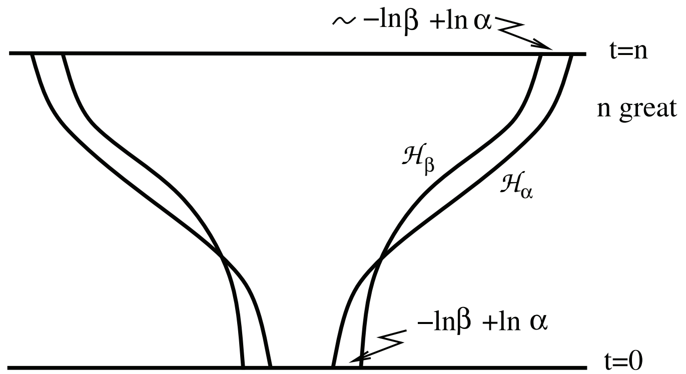

FIGURE 2. Distance between $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$.

Now, it is very natural to give the following definition.

Definition 2.1. We define $\frac{1}{\sqrt{\alpha}}$ to be the *growth* of the surface $\mathcal{H}_\alpha$.

As it is shown in Lemma 5.2 in [N-SE-S-T], the growth of any immersed rotational surface is smaller than the growth of the simply connected surface S and the growth of any embedded rotational surface is greater than the growth of S.

Now, we describe how two rotational embedded surfaces $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ intersect for $0 < \beta < \alpha < 1$. In this case, both ends of $\mathcal{H}_\beta$ are contained in the mean convex side of $\mathcal{H}_\alpha$. Furthermore, if one restricts to the halfspace $t \geqslant 0$, then $\mathcal{H}_\alpha \cap \mathcal{H}_\beta \cap \{t \geqslant 0\}$ is a horizontal circle for every $0 < \alpha \neq \beta < 1$. For every $\alpha \in (0, 1)$, $S \cap \mathcal{H}_\alpha$ is a circle as well, and, as $\alpha \longrightarrow 1$, the circle $S \cap \mathcal{H}_\alpha$ approaches the origin, while the upper end of $\mathcal{H}_\alpha$ approaches the end of S.

In order to simplify the notation, we set $\mathcal{H}_1 = S$.

Definition 2.2. Let $0 < \beta < \alpha \leqslant 1$. We define the *horizontal distance* between $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ as the distance between $\mathcal{H}_\alpha \cap \{t = 0\}$ and $\mathcal{H}_\beta \cap \{t = 0\}$, i.e. the positive number $r_\beta - r_\alpha = -\ln \frac{\beta}{\alpha}$.

The next lemma guarantees that, for n great, the distance between $\mathcal{H}_\alpha \cap \{t = n\}$ and $\mathcal{H}_\beta \cap \{t = n\}$ is almost the same as the distance at height zero, i.e. the horizontal distance. This result will be crucial in the barrier argument in the proof of Theorem 3.4.

Lemma 2.3. *Let $0 < \beta < \alpha \leqslant 1$. Denote by R_α the radius of $\mathcal{H}_\alpha \cap \{t = n\}$ and by R_β the radius of $\mathcal{H}_\beta \cap \{t = n\}$. Then $R_\alpha \simeq 2\ln n + \ln \alpha$ and $R_\beta \simeq 2\ln n + \ln \beta$, for n great enough, that is, $R_\alpha - R_\beta \simeq -\ln \frac{\beta}{\alpha}$.*

Proof. It is a straightforward computation using the asymptotic behavior in (2.5) (see Figure 2). ☐

It is clear that any surface obtained from $\mathcal{H}_\alpha$ either by a vertical translation or by a horizontal hyperbolic translation has growth $\frac{1}{\sqrt{\alpha}}$. The effect of a vertical translation on formula (2.5) is obviously an additive constant. The image of $\mathcal{H}_\alpha$ by a horizontal hyperbolic translation intersects any slice in a circle. All such circles have hyperbolic center on the same vertical geodesic, that is, the image of the t-axis. Then, the translated surface has an asymptotic expansion as in (2.5), where ρ is the distance from the vertical geodesic image of the t-axis.

In the following, we will refer to formula (2.5) for any surface obtained from $\mathcal{H}_\alpha$ either by a vertical translation or by a horizontal hyperbolic translation.

Remark 2.4. The asymptotic behavior of rotational surfaces with mean curvature $H < \frac{1}{2}$ is quite different from the $H = \frac{1}{2}$ case. In fact, the parameter α does not appear in the leading term of the asymptotic expansion. Hence, for any fixed $H < \frac{1}{2}$, all the $\mathcal{H}_\alpha^H$ behave analogously at infinity. Furthermore, if $H < \frac{1}{2}$, the leading term has linear growth (see [SE-T2], [N-SE-S-T]).

We close this section by proving a very interesting consequence of the existence of the rotational simply connected surfaces. We say that a surface S is *congruent* to S^H if there is either a vertical or a horizontal translation, or a symmetry with respect to a horizontal slice, say T, such that $T(S) = S^H$. Let K be a compact set in $\mathbb{H}^2 \times \mathbb{R}$. For any $H \in (0, \frac{1}{2}]$, we define $\mathcal{F}_K^H$ as follows:

$$\mathcal{F}_K^H = \{B \subset \mathbb{H}^2 \times \mathbb{R} \mid K \subset B,\ \partial B \text{ is congruent to } S^H\}.$$

Lemma 2.5 (Convex Hull Lemma)**.** *(a) Let M be a compact surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in (0, \frac{1}{2}]$. Then M is contained in the convex hull of the family $\mathcal{F}_{\partial M}^H$.*

(b) Let M be a compact surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with prescribed mean curvature function $H : M \longrightarrow (0, \frac{1}{2}]$. Then M is contained in the convex hull of the family $\mathcal{F}_{\partial M}^{\frac{1}{2}}$.

Proof. (a) Up to vertical translation, there exists a copy of S^H with the end on the top and containing M in its convex side. By abuse of notation, we denote by S^H any surface obtained by S^H by a hyperbolic isometry. Now, move S^H by a translation along some horizontal geodesic. If the first contact point p between S^H and M is an interior point of M, then the two surfaces are tangent at p and they have the same mean curvature vector at p. This is a contradiction by the maximum principle. Hence, one can move S^H horizontally until it touches ∂M. One can do the same for any horizontal geodesic, and one can move S^H vertically as well. Furthermore, one can start with a surface with the end on the bottom and proceed in the same manner. The result follows.

(b) The proof is analogous to the proof of (a). $\square$

The Convex Hull Lemma gives horizontal and vertical distance estimates in many geometric situations; for example, in the proof of existence of vertical graphs in $\mathbb{H}^2 \times \mathbb{R}$ with mean curvature $H \leqslant \frac{1}{2}$ and in the proof of uniqueness of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with mean curvature $H \leqslant \frac{1}{2}$ and boundary in two parallel horizontal planes [N-SE-S-T].

3. Existence of complete graphs on exterior domains with $H = \frac{1}{2}$

Our main goal is to construct examples of $H = \frac{1}{2}$ vertical graphs in $\mathbb{H}^2 \times \mathbb{R}$, over an exterior domain in $\mathbb{H}^2$. The ends of our graphs are trapped between two rotational surfaces. This fact suggests that they are asymptotically rotational.

Let us start by defining the geometric condition on the boundary curve. Let $r > 0$ and denote by C_r the hyperbolic circle centered at the origin of radius r. We recall that the hyperbolic curvature of C_r is $\coth r$.

Definition 3.1. Let $\Gamma \subset \mathbb{H}^2 \times \{0\}$ be a C^2 Jordan curve and let $0 < b < c$ be two real numbers. We say that Γ satisfies the *circles condition* (b, c) if the following hold:

- Γ is contained in the annulus whose boundary is $C_c \cup C_b$.
- The curvature of Γ at any point is in the interval $(\coth c, \coth b)$.

Notice that if Γ satisfies the circles condition (b, c), then for any $p \in \Gamma$ there exists a hyperbolic translation of C_b (respectively of C_c) with length $< c - b$ such that C_b (respectively C_c) is tangent to Γ at p.

Before stating the theorem, we need two further definitions.

Definition 3.2. Let D be a bounded domain and let E be the graph of a C^2 function u defined on the exterior domain $\mathbb{H}^2 \times \{0\} \setminus D$. If $u_{|\partial D}$ is bounded and either $u(p) \longrightarrow +\infty$ or $u(p) \longrightarrow -\infty$, as p approaches the asymptotic boundary of $\mathbb{H}^2 \times \{0\}$, we call E a *vertical graph end*.

If E is a vertical graph end, then $\partial_\infty E$ is disjoint from $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$. The ends of the rotational surfaces described in Section 2 are vertical graph ends.

Equation (2.5) describes the asymptotic behavior of a rotational vertical graph end of growth $\frac{1}{\sqrt{\alpha}}$. Now, we give the notion of growth for a general vertical graph end of mean curvature $H = \frac{1}{2}$.

Definition 3.3. Let $E = graph(u)$ be a vertical graph end of constant mean curvature $H = \frac{1}{2}$.

- We say that E has *growth* $\frac{1}{\sqrt{\alpha}}$ if the asymptotic behavior of u is the same as u_α in (2.5).
- We say that E has *weak growth* $\frac{1}{\sqrt{\alpha}}$ if there exists a constant K such that
$$|u - u_\alpha| < K.$$

Recall that the end of the simply connected surface S has growth one, while each end of an annulus $\mathcal{H}_\alpha$ has growth $\frac{1}{\sqrt{\alpha}}$ ($\alpha \leqslant 1$ if $\mathcal{H}_\alpha$ is embedded and $\alpha > 1$ if $\mathcal{H}_\alpha$ is nonembedded).

Now, we can make Question 1, stated in the Introduction, more precise.

Question 1′. Is every vertical graph end of mean curvature $H = \frac{1}{2}$ asymptotic to a rotational graph end? If a vertical graph end of mean curvature $H = \frac{1}{2}$ has weak growth $\frac{1}{\sqrt{\alpha}}$, does it have growth $\frac{1}{\sqrt{\alpha}}$?

Let us state our main result.

Theorem 3.4. *Let c be the radius of the circle $S \cap \{t = 1\}$. Let Γ be a simple C^2 closed curve contained in the disk of $\mathbb{H}^2 \times \{1\}$ bounded by $S \cap \{t = 1\}$ and let D be the compact domain bounded by the projection of Γ in $\mathbb{H}^2 \times \{0\}$. Assume that, for some b such that $2c < 3b$, Γ satisfies the circles condition (b, c). Then, for any α such that $e^{c-2b} < \alpha < e^{-c+b}$, there exists a complete vertical graph on $\Omega = \mathbb{H}^2 \times \{0\} \setminus D$ with boundary Γ, mean curvature $H = \frac{1}{2}$ and weak growth $\frac{1}{\sqrt{\alpha}}$.*

Example 3.5. Any small C^2 deformation of a hyperbolic circle of radius r gives a domain satisfying Definition 3.1. Then, nontrivial examples of curves satisfying the hypotheses of Theorem 3.4 are hyperbolic ellipses with small eccentricity and any small C^2 deformation of them.

Let us give some details about the hyperbolic ellipse. The main reference for this computation is the book by the third author and E. Toubiana [SE-T5]. More properties about the hyperbolic ellipse can be found in the thesis of P. Castillon [C].

Consider the halfspace model for the hyperbolic plane. Denote by L the vertical geodesic through the origin. We will write the equation of a hyperbolic ellipse, E, with focuses on L, at points i and $ie^{2\varepsilon a}$, where $\varepsilon \in [0,1]$ is the eccentricity of the ellipse. Notice that $2\varepsilon a$ is the distance between the two focuses and $2a$ is the hyperbolic length of the major axis. Denote by $d_{\mathbb{H}}$ the distance in $\mathbb{H}^2$. Then $z \in \mathbb{H}^2$ is a point of the ellipse if and only if

$$d_{\mathbb{H}}(z, ie^{2\varepsilon a}) + d_{\mathbb{H}}(z, i) = 2a. \tag{3.1}$$

Consider the *polar coordinates* (r, θ) defined as follows. For any $p \in \mathbb{H}^2$, $p \neq i$, let $r = d_{\mathbb{H}}(p, i)$ and θ be the oriented angle between the vertical geodesic L and the unique geodesic through p and i.

In order to write equation (3.1) in polar coordinates we must introduce a new system of coordinates (u, v) (cf. exercise 2.6.1 in [SE-T5]) that are completely determined by the following formulas (cf. formulas (2.10), (2.12) and (2.13) in [SE-T5]):

$$\begin{aligned} \tanh u &= \tanh r \cos\theta, \\ \sinh v &= \sinh r \sin\theta, \\ \cosh v &= \frac{\cosh r}{\cosh u}. \end{aligned} \tag{3.2}$$

In such coordinates, a generic point $z \in \mathbb{H}^2$ is $z = e^u \tanh v + \frac{e^u}{\cosh v} i$. Hence, by the formula given in the exercise 2.5.2 in [SE-T5], one has

$$\cosh d_{\mathbb{H}}(z, ie^{2\varepsilon a}) = \cosh v \cosh(u - 2\varepsilon a). \tag{3.3}$$

Equation (3.1) is equivalent to

$$\cosh d_{\mathbb{H}}(z, ie^{2\varepsilon a}) = \cosh(2a - r). \tag{3.4}$$

By using (3.2) and (3.3), equation (3.4) writes as

$$\tanh r = \frac{\tanh a \left(1 - \left(\dfrac{\sinh \varepsilon a}{\sinh a}\right)^2\right)}{1 - \dfrac{\sinh 2\varepsilon a}{\sinh 2a}\cos\theta}. \tag{3.5}$$

The curvature of the ellipse can be computed by using the last formula of exercise 2.6.2 in [SE-T5], and one has

$$\kappa(\theta) = \frac{\sinh a \cosh a}{\sinh^2 \varepsilon a - \sinh^2 a}\left(\frac{\sinh^2 a \cosh a \cosh 2\varepsilon a + \sinh \varepsilon a F}{\sinh^2 a \cosh a \cosh 2\varepsilon a + \sinh \varepsilon a G}\right)^{\frac{3}{2}}, \tag{3.6}$$

where

$$\begin{aligned} F &= \cosh^2 \varepsilon a \sinh \varepsilon a \cos^2\theta - \cosh \varepsilon a \sinh 2a \cos\theta - \sinh^3 \varepsilon a, \\ G &= -\cosh \varepsilon a \sinh 2a \cos\theta + \sinh \varepsilon a. \end{aligned} \tag{3.7}$$

Denote by M the geodesic through the point $ie^{\varepsilon a}$, orthogonal to L. The ellipse E is invariant by reflections with respect to L and M. Furthermore, the points $V_1,\ \bar{V}_1 \in E \cap M$ and $V_2,\ \bar{V}_2 \in E \cap L$ are the only critical points of the curvature κ.

The polar coordinates for $V_1,\ \bar{V}_1,\ V_2,\ \bar{V}_2$ are the following:

$$V_1 = \left(a, \arccos \frac{\tanh \varepsilon a}{\tanh a}\right),\ \bar{V}_1 = \left(a, 2\pi - \arccos \frac{\tanh \varepsilon a}{\tanh a}\right), \tag{3.8}$$

$$V_2 = (a(1+\varepsilon), 0),\ \bar{V}_2 = (a(1-\varepsilon), \pi). \tag{3.9}$$

Also, by formula (3.6),

(3.10)

$$\kappa_1 := \kappa\left(\arccos \frac{\tanh \varepsilon a}{\tanh a}\right) = \kappa\left(2\pi - \arccos \frac{\tanh \varepsilon a}{\tanh a}\right) = \frac{\sqrt{\cosh 2a - \cosh 2\varepsilon a}}{\sqrt{2}\sinh a \tanh a},$$

$$\kappa_2 := \kappa(0) = \kappa(\pi) = \frac{2\sinh^2 a}{(\cosh 2a - \cosh 2\varepsilon a)\tanh a}. \tag{3.11}$$

Denote $A = \frac{\sqrt{\cosh 2a - \cosh 2\varepsilon a}}{\sqrt{2}\sinh a}$; then $\kappa_1 = \frac{A}{\tanh a}$, $\kappa_2 = \frac{1}{A^2 \tanh a}$. Notice that $A \leqslant 1$ and equality holds if and only if $\varepsilon = 0$, i.e. for the circle.

Since $\kappa_1 = A^3 \kappa_2$, then $\kappa_2 \geqslant \kappa_1$ and equality holds for $A = 1$, i.e. for the circle.

As κ_1 is the minimum of the curvatures of the ellipse and κ_2 is the maximum of the curvatures of the ellipse, one takes b such that $\tanh(b) = \frac{1}{\kappa_2}$ and c such that $\tanh(c) = \frac{1}{\kappa_1}$.

Then, the condition of Theorem 3.4, that is, $2c < 3b$, is equivalent to the following inequality:

$$\left(\frac{A + \tanh a}{A - \tanh a}\right)^2 < \left(\frac{1 + A^2 \tanh a}{1 - A^2 \tanh a}\right)^3. \tag{3.12}$$

If $\varepsilon = 0$, i.e. $A = 1$, the previous inequality is verified for any a; hence it is verified for any ε sufficiently small.

In our proof, we strongly use the geometry of the family of embedded, rotational surfaces of mean curvature $H = \frac{1}{2}$. In fact, we obtain a priori height estimates using them as geometric barriers. Furthermore, the vertical graph is obtained as a limit of vertical graphs defined on compact subsets of the domain, and the geometric barriers guarantee the right boundary values for the limit vertical graph.

The proof of Theorem 3.4 is obtained by the Perron method, which we describe below. First, we need some preliminaries.

In the following, Ω will be an open domain of $\mathbb{H}^2 \times \{0\}$.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function. By equation (2.1), the graph of u has mean curvature $H = \frac{1}{2}$ with respect to the upward normal vector field if and only if u satisfies

$$\operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 1. \tag{3.13}$$

Let $f : \partial\Omega \to [0, +\infty[$ be a continuous function. We consider the Dirichlet problem

$$P(\Omega, f) \begin{cases} \mathcal{D}(u) = \operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) - 1 = 0 \ \text{ in } \ \Omega, \ \ u \in C^2(\Omega) \cap C^0(\bar{\Omega}), \\ u|_{\partial\Omega} = f. \end{cases}$$

For our proof we use a Compactness Theorem for constant mean curvature graphs in $\mathbb{H}^2\times\mathbb{R}$, and we need to guarantee the solvability of the Dirichlet problem $P(U,f)$, where $U\subset\Omega$ is a small closed disk. The Compactness Theorem for constant mean curvature graphs in $\mathbb{H}^2\times\mathbb{R}$ yields that any bounded sequence of solutions of equation (3.13) on a domain Ω in $\mathbb{H}^2$ admits a subsequence that converges uniformly in the C^2 topology, on any compact subset of Ω, to a solution of equation (3.13). The result follows from the purely interior gradient bound at any point $p\in\Omega$ (see [SI] and Theorem 1.1 in [SP]) and from a standard argument using Schauder theory and the Ascoli-Arzelà Theorem.

The existence of the solution of the Dirichlet problem in a small disk U is stated in [SP] (Theorem 1.4). We give a sketch of the proof here, for completeness. Our proof is analogous to that given in [SE-T4] for minimal hypersurfaces in $\mathbb{H}^n\times\mathbb{R}$.

By an approximation argument (see the proof of Theorem 16.8 in [G-T]), we can assume that the boundary data f is $C^{2,\alpha}$.

By classical elliptic theory (see chapter 11 in [G-T]), in order to prove existence of $C^{2,\alpha}$ solutions of $P(U,f)$, it is enough to obtain a priori global C^1 estimates. By Theorem 3.1 in [SP], a priori height estimates and a priori boundary gradient estimates yield a priori global C^1 estimates. Using the Convex Hull Lemma, we get a priori height estimates. So we are left with the a priori boundary gradient estimates. Notice that the eigenvalues of the symmetric matrix of the coefficients of the terms of the second order of equation (2.2) are 1 and $1+F|\nabla u|^2$. A straightforward computation yields that, if the radius of the disk U is small enough, equation (2.2) satisfies the structure conditions (14.33) in [G-T]. Then, one can apply Corollary 14.5 [G-T] to obtain the desired a priori boundary gradient estimates for the solution of $P(U,f)$. Notice that the Maximum Principle guarantees that the solution of $P(U,f)$ is unique.

Let $u:\overline{\Omega}\to\mathbb{R}$ be a continuous function and $U\subset\Omega$ be a small closed disk. Let $\tilde{u}_U$ be the unique extension of $u_{|\partial U}$ as a graph of mean curvature $H=\frac{1}{2}$ over U, continuous up to ∂U.

Definition 3.6. Let u and Ω be as above. We define the continuous function $M_U(u)$ on $\bar{\Omega}$ by

$$M_U(u)=\begin{cases} u(x), & \text{if } x\in\bar{\Omega}\setminus U,\\ \tilde{u}_U(x), & \text{if } x\in U.\end{cases}$$

We say that $u\in C^0(\bar{\Omega})$ is a *subsolution* (resp. *supersolution*) of the problem $P(\Omega,f)$ if

(1) For any small closed disk $U\subset\Omega$, we have $u\leqslant M_U(u)$ (resp. $u\geqslant M_U(u)$),
(2) $u|_{\partial\Omega}\leqslant f$ (resp. $u|_{\partial\Omega}\geqslant f$).

Remark 3.7. We list below some classical facts about subsolutions and supersolutions:

(1) If $u\in C^2(\Omega)$, the first condition in Definition 3.6 is equivalent to $\mathcal{D}(u)\geqslant 0$ for a subsolution or $\mathcal{D}(u)\leqslant 0$ for a supersolution. Then, solutions of $\mathcal{D}(u)=0$ are both sub and supersolutions.
(2) The supremum (resp. infimum) of subsolutions (resp. supersolutions) is again a subsolution (resp. a supersolution).
(3) Let Ω be a bounded domain. Suppose that u is a subsolution and that v is a supersolution of $P(\Omega,f)$; then $u\leqslant v$ in Ω.

Definition 3.8. Let Ω be a bounded domain. We say that $p \in \partial\Omega$ admits a barrier for the problem $P(\Omega, f)$ if there exist a supersolution ϕ and a subsolution φ both in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi(p) = \varphi(p) = f(p)$.

Then we have the following proposition.

Proposition 3.9 (Perron Process). *Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. Suppose that the problem $P(\Omega, f)$ has a supersolution ϕ. Set*

$$\mathcal{S}_\phi = \{v|\ v \textit{ is a subsolution of } P(\Omega, f) \textit{ with } v \leqslant \phi\}.$$

Then one has:

(1) *If $\mathcal{S}_\phi \neq \emptyset$, then the function $u(x) = \sup_{v \in \mathcal{S}_\phi} v(x)$ is defined for any $x \in \Omega$, is C^2 on Ω and satisfies equation* (3.13).
(2) *Suppose that Ω is bounded and that $p \in \partial\Omega$ admits a barrier for the problem $P(\Omega, f)$. Then the solution u extends continuously at p setting $u(p) = f(p)$.*

The proof of Proposition 3.9 is analogous to that of Theorem 3.4 in [SE-T1] and Theorem 4.1 in [SE-T4], where the authors proved the Perron Process for minimal surfaces in $\mathbb{H}^3$ and $\mathbb{H}^n \times \mathbb{R}$, respectively.

Now we are ready to prove the main theorem.

Proof of Theorem 3.4. We use the Perron technique to construct a sequence of surfaces with mean curvature $H = \frac{1}{2}$, each of them being the graph of a function v_n defined on an annulus Ω_n, whose interior boundary is $\partial\Omega$ and whose exterior boundary is a circle γ_n, to be defined below. The annuli Ω_n exhaust Ω, and the function v_n takes values 1 on $\partial\Omega$ and n on γ_n. Then, we let n go to infinity and we prove that the sequence $\{v_n\}$ converges to a solution u of

$$\begin{cases} \operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 1 \ \text{ in } \ \mathbb{H}^2 \times \{0\} \setminus \overline{\Omega}, \\ \\ u = 1 \ \text{ on } \ \partial\Omega \end{cases} \tag{3.14}$$

with the desired weak growth.

We start by constructing the circle γ_n. Consider the surface $\mathcal{H}_\beta$ with $\beta = e^{-b}$. By Lemma 2.3, for any $\alpha > e^{c-2b}$, the surface $\mathcal{H}_\alpha$ has the following property: the distance between $\mathcal{H}_\beta \cap \{t = n\}$ and $\mathcal{H}_\alpha \cap \{t = n\}$ is almost $-\ln\beta + \ln\alpha > c - b$, for n great enough. Denote by $\bar{\mathcal{H}}_\beta$ (resp. $\bar{\mathcal{H}}_\alpha$) the vertical upward translation of length one, of the surface $\mathcal{H}_\beta \cap \{t \geqslant 0\}$ (resp. $\mathcal{H}_\alpha \cap \{t \geqslant 0\}$). Define $\Gamma_n = \bar{\mathcal{H}}_\alpha \cap \{t = n\}$ and let γ_n be the projection of Γ_n on the plane $t = 0$. Notice that this choice of α allows us to translate horizontally $\bar{\mathcal{H}}_\beta$ by a distance $c - b$, without touching Γ_n with $\bar{\mathcal{H}}_\beta \cap \{t = n\}$.

Let $f_n : \partial\Omega_n \longrightarrow \mathbb{R}$ be defined by

$$\begin{cases} f_n(p) = 1 \ \text{ for } \ p \in \partial\Omega, \\ \\ f_n(p) = n \ \text{ on } \ p \in \gamma_n. \end{cases}$$

Let $\bar{u}_\alpha$ be the function whose graph is $\bar{\mathcal{H}}_\alpha$, which is clearly a supersolution of the problem $P(\Omega_n, f_n)$.

We consider the following function:

$$v_n(x) = \sup\{v(x)|\ v \text{ is a subsolution of } P(\Omega_n, f_n) \text{ with } \ v \leqslant \bar{u}_\alpha\}. \tag{3.15}$$

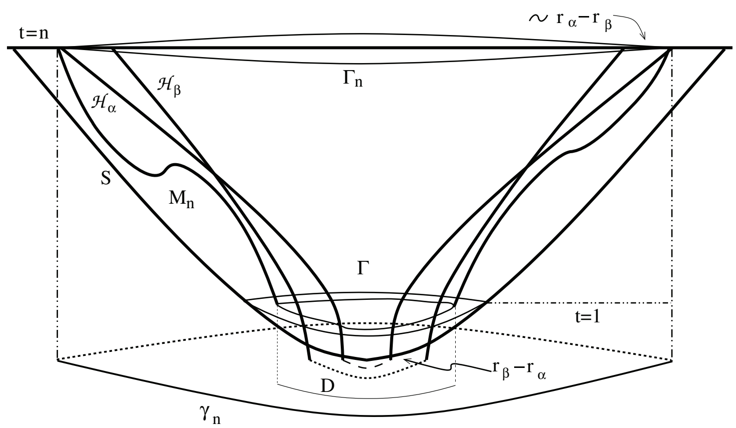

FIGURE 3. Proof of Theorem 3.4

As the function u_1, whose graph gives S, is a subsolution of $P(\Omega_n, f_n)$, $u_1 \leqslant \bar{u}_\alpha$, the set in (3.15) is not empty (see Figure 3). By (1) of Proposition 3.9, we conclude that v_n is well defined and satisfies

$$\operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} v_n}{W_{v_n}}\right) = 1 \quad \text{in} \quad \Omega_n.$$

We claim that, for each $p \in \partial\Omega$ and for each sufficiently large n, we can construct a barrier at p, i.e. a subsolution h_p and a supersolution H_p of $P(\Omega_n, f_n)$ such that $h_p(p) = H_p(p) = 1$.

In fact, in view of Lemma 2.3 and inequality $\alpha < e^{-c+b}$, we can move S horizontally until the curve $S \cap \{t = 1\}$ is tangent to Γ at the point $(p, 1)$ without touching Γ_n. Thus, we obtain the subsolution h_p of $P(\Omega_n, f_n)$. As we have already pointed out, we can translate $\bar{\mathcal{H}}_\beta$ horizontally until it touches Γ at $(p, 1)$ without touching Γ_n. The graph thus obtained gives the supersolution H_p of $P(\Omega_n, f_n)$. Then, by (2) in Proposition 3.9, v_n extends continuously at p setting $v_n(p) = 1$.

In order to prove that v_n extends continuously to any point of $p \in \gamma_n$, one constructs a supersolution and a subsolution, holding n at the point p. The supersolution is given by the horizontal plane $t = n$. For the subsolution, we make a suitable downward translation of S and then we translate it horizontally until it touches Γ_n at (p, n). This is possible because of the asymptotic behavior of S. In fact, translate down S by a constant $C(n)$. As S is the graph of $u_1(\rho) = 2\cosh(\rho/2)$, the translated surface $\tilde{S}$ is given by the graph of $2\cosh(\rho/2) - C(n)$. Let $\rho(s)$ be the radius of the circle $\tilde{S} \cap \{t = s\}$. The function $\rho(s)$ is strictly increasing in $s \in [1, n]$. Furthermore, by Lemma 2.3, for n large enough,

$$\rho(n) - \rho(1) \sim 2 \ln \frac{1 + \frac{n}{C(n)}}{1 + \frac{1}{C(n)}}.$$

If we choose $C(n)$ such that $\displaystyle\lim_{n \longrightarrow \infty} \frac{n}{C(n)} = 0$, then $\displaystyle\lim_{n \longrightarrow \infty} \rho(n) - \rho(1) = 0$. That is, $\tilde{S} \cap \{1 \leqslant t \leqslant n\}$ is almost a vertical cylinder.

Then, by (2) in Proposition 3.9, v_n extends continuously at any $p \in \gamma_n$, setting $v_n(p) = n$.

For n sufficiently large, the sequence $\{v_n\}$ is uniformly bounded above by H_p and below by h_p, for any $p \in \partial\Omega$. Such C^0 estimates for $\{v_n\}$ independent of n, combined with interior gradient estimates (see [SI] and [SP]), yield $C^{2,\alpha}$ estimates for $\{v_n\}$ on a compact subset of Ω. Then, the Ascoli-Arzelà Theorem guarantees that a subsequence of $\{v_n\}$ converges uniformly on compact set, in the C^2 topology, to a solution u of the equation $\mathcal{D}(u) = 0$ in Ω.

By construction, for any $q \in \Omega$,

$$h_p(q) \leqslant u(q) \leqslant H_p(q). \tag{3.16}$$

Now, letting q go to $p \in \partial\Omega$ in (3.16), one has $u(p) = 1$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

It remains to show the growth property for the solution u. Let Γ_α be the intersection of $\bar{\mathcal{H}}_\alpha$ with the cylinder $\Gamma \times \mathbb{R}$. We translate $\bar{\mathcal{H}}_\alpha$ until Γ_α lies below Γ. Then, for each n, the translation of $\bar{\mathcal{H}}_\alpha$ gives a subsolution of $P(\Omega_n, f_n)$, being, therefore, below the graph of v_n and a fortiori below the graph of u. □

Remark 3.10. Notice that, in the proof of Theorem 3.4, we prove the existence of graphs of mean curvature $H = \frac{1}{2}$, over annuli.

The proof of the following result is analogous to the proof of Theorem 3.4.

Theorem 3.11. *Let c be the radius of the circle $\mathcal{H}_\delta \cap \{t = 1\}$, $\delta \in (0, 1)$. Let Γ be a simple C^2 closed curve contained in the disk of $\mathbb{H}^2 \times \{1\}$ bounded by $\mathcal{H}_\delta \cap \{t = 1\}$ and let D be the compact domain bounded by the projection of Γ in $\mathbb{H}^2 \times \{0\}$. Assume that, for some b such that $c - \frac{\ln\delta}{2} < \frac{3b}{2}$, Γ satisfies the circles condition (b, c). Then, for any α such that $e^{c-2b} < \alpha < \delta e^{-c+b}$, there exists a complete vertical graph on $\Omega = \mathbb{H}^2 \times \{0\} \setminus D$ with boundary Γ, mean curvature $H = \frac{1}{2}$ and weak growth $\frac{1}{\sqrt{\alpha}}$.*

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