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A SUPPORT THEOREM FOR A GAUSSIAN RADON TRANSFORM IN INFINITE DIMENSIONS

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Abstract. We prove that in infinite dimensions, if a bounded, suitably continuous, function has zero Gaussian integral over all hyperplanes outside a closed bounded convex set, then the function is zero outside this set. This is an infinite-dimensional form of the well-known Helgason support theorem for Radon transforms in finite dimensions.

1. Introduction

The Radon transform [15] associates to a function f on the finite-dimensional space $\mathbb{R}^n$ the function Rf on the set of all hyperplanes in $\mathbb{R}^n$ whose value on any hyperplane P is the integral of f over P:

$$Rf(P) = \int_P f(x)\,dx, \tag{1.1}$$

the integration here being with respect to Lebesgue measure on P. This transform does not generalize directly to infinite dimensions because there is no useful notion of Lebesgue measure in infinite dimensions. However, there is a well-developed theory of Gaussian measures in infinite dimensions, and so it is natural to extend the Radon transform to infinite dimensions using Gaussian measure:

$$Gf(P) = \int f\,d\mu_P, \tag{1.2}$$

where μ_P is Gaussian measure for any hyperplane P in a Hilbert space H_0. (By 'hyperplane' we shall always mean a translate of a closed linear subspace of codimension one.) This transform was developed in [14], but we shall present a self-contained account in section 2.

A central feature of the classical Radon transform R is the Helgason support theorem (Helgason [10]): if f is a rapidly decreasing continuous function and $Rf(P)$ is 0 on every hyperplane P lying outside a compact convex set K, then f is 0 off K. In this paper we prove an infinite-dimensional version of this support theorem.

A support theorem, even in the finite-dimensional case, works for a class of suitably regular functions, such as continuous functions of rapid decrease. In the infinite-dimensional setting it is first of all necessary to choose a framework for

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the Gaussian measures with respect to which the transform is defined (Bogachev [3] is an extensive account of the general theory of Gaussian measures on infinite-dimensional spaces). There are two standard frameworks: (i) nuclear spaces and their duals [5, 6, 13]; (ii) Abstract Wiener Spaces [8, 12]. For our purposes, we restrict to a more elementary setting, using only Hilbert spaces, as this highlights the essential ideas involved in proving our support theorem. Becnel [2] proves the result in the white noise analysis framework for a class of functions called Hida test functions.

The finite-dimensional Radon transform is of central significance in tomography (albeit here only the three-dimensional theory matters), and there is a vast body of results of purely mathematical interest, including one of the great early results which specifies the range of the space of Schwartz functions. The authors freely acknowldege that the theory in infinite dimensions is in its infancy and at this stage can only express their hope that larger developments, techniques, and possible applications lie in the future. The motivation for our investigation of the infinite-dimensional theory arose in a stochastic context. Consider a random functional F, of suitable regularity, of a Brownian motion $t \mapsto B_t$. One may wish to recover information about F from the conditional expectation values $\mathbb{E}[F \mid \int_0^\infty f(t)\, dB_t = c]$, with f running over a suitable collection of functions and c over real numbers. Such a problem is essentially a problem concerning the Gauss-Radon transform in the setting of Gaussian measure over the Hilbert space $L^2([0,\infty))$. We shall not pursue this or other applications in the present paper where we develop the theory and central result in an abstract setting.

2. The Gauss-Radon transform

The Gauss-Radon transform Gf of a function f for a real separable Hilbert space H_0 associates to each hyperplane P in H_0 the integral of f with respect to the Gaussian measure μ_P for the hyperplane P. Recall that by 'hyperplane' we mean a translate of a closed linear subspace of codimension one. In this section we will spell out the details of this, including a precise specification of the measure μ_P and the space on which f is defined.

Throughout this paper H_0 is a separable real Hilbert space. The inner-product on H_0 will be denoted $\langle\cdot,\cdot\rangle_0$ or simply $\langle\cdot,\cdot\rangle$, and the corresponding norm by $\|\cdot\|_0$.

From well-known methods of Gaussian measure theory in infinite dimensions there is a measurable space $(\Omega, \mathcal{F})$ and a linear map $x \mapsto \hat{x}$ taking vectors x in some dense subspace H_1 of H_0 to measurable functions on Ω, such that for any closed subspace $F \subset H_0$ and any $u \in F$, there is a probability measure $\mu_{u+F^\perp}$ on $(\Omega, \mathcal{F})$ satisfying

$$\int e^{i\hat{x}}\, d\mu_{u+F^\perp} = e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{F^\perp}\|_0^2}, \tag{2.1}$$

where $x_{F^\perp}$ denotes the orthogonal projection of x onto $F^\perp$. See, for instance, Kuo [12, 13] and Gelfand et al. [6] for Gaussian measures in infinite dimensions and Bogachev [3, Theorem 3.10.2] for Gaussian measures in infinite-dimensionals spaces and on hyperplanes of such spaces.

There are several possible choices for the space $(\Omega, \mathcal{F})$. In this paper we will stay within the Hilbert-space setting (which goes back, in a more general context, to Sazonov [17] and Gross [9]) and make the following choice. Fix, once and for all, a Hilbert-Schmidt operator $T : H_0 \to H_0$ with $0 \le T \le I$, and let

$$H_1 = T^{1/2}(H_0), \tag{2.2}$$

with Hilbert space structure given by the inner-product

$$\langle x, y\rangle_1 = \langle T^{-1/2}x, T^{-1/2}y\rangle_0. \tag{2.3}$$

Identifying H_0 with its dual H_0' in the usual way produces Hilbert-Schmidt inclusions

$$H_1 \subset H_0 \simeq H_0' \subset H_{-1} \stackrel{\text{def}}{=} H_1', \tag{2.4}$$

where H_{-1} is the dual space to the Hilbert space H_1. Note that $\|\cdot\|_0 \le \|\cdot\|_1$ on H_1 and $\|\cdot\|_{-1} \le \|\cdot\|_0$ on H_0. The measurable space $(\Omega, \mathcal{F})$ is then just H_{-1} with its Borel σ-algebra. For $x \in H_1$ we take $\hat{x}$ on H_{-1} to be the evaluation map $x' \mapsto \langle x', x\rangle$.

The characteristic function of $\mu_{u+F^\perp}$ provided by (2.1) implies that, with respect to the probability measure $\mu_{u+F^\perp}$, the random variable $\hat{x}$ has Gaussian distribution with mean $\langle x, u\rangle_0$ and variance $\|x_{F^\perp}\|_0^2$. Hence,

$$\|\hat{x}\|^2_{L^2(\mu_{u+F^\perp})} = |\langle u, x\rangle_0|^2 + \|x_{F^\perp}\|_0^2 \le (\|u\|_0^2 + 1)\|x\|_0^2. \tag{2.5}$$

Thus, $x \mapsto \hat{x}$ is continuous as a map $H_{-1} \to L^2(H_{-1}, \mu_{u+F^\perp})$ and so extends to a continuous linear map

$$H_0 \to L^2(\mu_{u+F^\perp}) : x \mapsto \hat{x},$$

with $\hat{x}$ satisfying (2.1), i.e. $\hat{x}$ is a Gaussian variable with mean $\langle x, u\rangle_0$ and variance $\|x_{F^\perp}\|_0^2$, and hence also the bound (2.5). Note, however, that for x in H_0 outside H_{-1}, the definition of $\hat{x}$ depends on $\mu_{u+F^\perp}$ and hence on the affine subspace $u+F^\perp$.

The measure $\mu_{u+F^\perp}$ is indeed concentrated 'on' the affine subspace $u + F^\perp$, in the sense that $\hat{x}$, for $x \in F$, is equal to the constant $\langle x, u\rangle_0$ almost surely with respect to the measure $\mu_{u+F^\perp}$.

Definition 2.1. If f is a bounded Borel measurable function on H_{-1}, then its *Gauss-Radon transform* Gf is the function which associates to each hyperplane P in H_0 the value

$$Gf(P) = \int_{H_{-1}} f\, d\mu_P. \tag{2.6}$$

To stress the role of H_0, we may sometimes write G_{H_0} instead of G.

3. Some geometric and limiting results

We work within the framework of infinite-dimensional Hilbert spaces $H_1 \subset H_0 \subset H_{-1}$ described before and denote the closed ball of radius R in H_{-1} by $D_{-1}(R)$:

$$D_{-1}(R) = \{x' \in H_{-1} : \|x'\|_{-1} \le R\}. \tag{3.1}$$

Proposition 3.1. *Let F be a finite-dimensional subspace of the separable real Hilbert space H_0, and let U be a bounded subset of F. Then for any $\epsilon > 0$, there is an $R \in (0, \infty)$ such that*

$$\mu_{u+F^\perp}[D_{-1}(R)] > 1 - \epsilon, \tag{3.2}$$

for all $u \in U$.

Proof. Fix an orthonormal basis $v_1, v_2, \ldots$ of the Hilbert space H_1. The complement $D_{-1}(R)^c$ is given by

$$D_{-1}(R)^c = \left[\frac{1}{R^2}\sum_{n=1}^{\infty} \hat{v}_n^2 > 1\right] \subset H_{-1}. \tag{3.3}$$

So, as in Chebyshev's inequality,

$$\begin{aligned}
\mu_{u+F^\perp}[D_{-1}(R)^c] &\leq \int_{H_{-1}} \frac{1}{R^2}\sum_{n=1}^{\infty} \hat{v}_n^2 \, d\mu_{u+F^\perp} \\
&= \frac{1}{R^2}\sum_{n=1}^{\infty} \|\hat{v}_n\|^2_{L^2(\mu_{u+F^\perp})} \\
&\leq \frac{1}{R^2}\sum_{n=1}^{\infty} (\|u\|_0^2 + 1)\|v_n\|_0^2 \qquad \text{(by (2.5))} \\
&\leq \frac{1}{R^2}\left(\sum_{n=1}^{\infty} \|v_n\|^2_{-1}\right)(\sup_{u\in U} \|u\|_0^2 + 1),
\end{aligned} \tag{3.4}$$

which is $< \epsilon$ when R is large enough, since U is bounded and $\sum_{n=1}^{\infty} \|v_n\|^2_{-1} < \infty$. □

Here is a convenient way to look at integrals over affine subspaces:

Proposition 3.2. *If F is a closed subspace of the real separable Hilbert space H_0 and $u \in F$, then*

$$\int f(x')\, d\mu_{u+F^\perp}(x') = \int f(x'+u)\, d\mu_{F^\perp}(x') \tag{3.5}$$

whenever f is a measurable function on H_{-1}, and the equality here holds in the sense that if either side is defined, so is the other, and the integrals are then equal.

This result is checked for $f = e^{i\hat{x}}$, with $x \in H_{-1}$, and then extends to general f by routine arguments. Observe that F sits inside H_0, which we are viewing as a subspace of H_1'. Thus $u \in F$ is also a function on H_1, mapping any $x \in H_1$ to $\langle u, x\rangle_0$.

Proposition 3.3. *Suppose that $f : H_{-1} \to \mathbb{R}$ is a bounded function on the Hilbert space H_{-1} which is either uniformly continuous in the Hilbert $\|\cdot\|_{-1}$-norm topology or sequentially continuous in the weak dual topology on H_1'. Then for every finite-dimensional subspace F of H_0, the function $u \mapsto \int f\, d\mu_{u+F^\perp}$ is continuous and bounded in $u \in F$.*

The version with weak sequential continuity will not be needed for our main result, Theorem 4.1, but is useful for the analogous result for nuclear spaces.

Proof. Let U be a bounded neighborhood of u in F. Let $\epsilon > 0$. Then, by Proposition 3.1, there is an $R \in (0, \infty)$ such that

$$\mu_{y+F^\perp}[D_{-1}(R)] > 1 - \epsilon$$

for all $y \in U$, where $D_{-1}(R)$ is the closed R-ball in H_{-1}. Then, for any $v \in U$, we have

(3.6)
$$\left|\int f\,d\mu_{u+F^\perp} - \int f\,d\mu_{v+F^\perp}\right| \le \int_{D_{-1}(R)} |f(x'+u) - f(x'+v)|\,d\mu_{F^\perp}(x') + 2\epsilon\|f\|_{\sup} \quad \text{(by Proposition 3.2)}$$
$$\le \sup_{x'\in D_{-1}(R)} |f(x'+u) - f(x'+v)| + 2\epsilon\|f\|_{\sup}.$$

Now we claim uniform continuity: *there is a $\delta > 0$ such that if $v \in F$ with $\|v-u\|_0 < \delta$, then* $\sup_{x'\in D_{-1}(R)} |f(x'+u) - f(x'+v)|$ *is* $< \epsilon$. Of course, this is given if f is uniformly continuous in the strong topology (note that $v - u \in F$, and, of course, there the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are equivalent on F). Now suppose f is sequentially continuous in the weak topology, and assume that the uniform continuity claim does not hold. Then there is a sequence of points $v_n \in F$ converging to u and a sequence of points $x'_n \in D_{-1}(R)$ such that $|f(x'_n+u) - f(x'_n+v_n)|$ is $\ge \epsilon$. Now, by the Banach-Alaoglu theorem and separability, $D_{-1}(R)$ is compact and metrizable in the weak topology of $H_{-1} = H'_1$, and so we may assume that $x'_n \to x'$ weakly for some $x' \in D_{-1}(R)$. Hence also $x'_n + v_n \to x' + u$, and so

$$f(x'_n+u) - f(x'_n+v_n) \to f(x+u) - f(x+u) = 0,$$

contradicting the assumption made.

Thus, indeed, for all v in some neighborhood of u in F we have

$$\sup_{x'\in D_{-1}(R)} |f(x'+u) - f(x'+v)| < \epsilon.$$

Then we have for such v, from (3.6),

$$\left|\int f\,d\mu_{u+F^\perp} - \int f\,d\mu_{v+F^\perp}\right| \le (1 + 2\|f\|_{\sup})\epsilon, \tag{3.7}$$

which shows that $\int f\,d\mu_{u+F^\perp}$ depends continuously on $u \in F$. ☐

Next we have a key limiting result:

Proposition 3.4. *Suppose that f is a bounded Borel function on H_{-1}, continuous at $u \in H_0 \subset H_{-1}$ relative to the norm $\|\cdot\|_{-1}$. Then*

$$\lim_{u\in F\to H_0} \int f\,d\mu_{u+F^\perp} = f(u),$$

in the sense that if $F_1 \subset F_2 \subset \cdots$ is a sequence of finite-dimensional subspaces, with $u \in F_1$, such that $\bigcup_{n\ge1} F_n$ is dense in H_0, then

$$\lim_{n\to\infty} \int f\,d\mu_{u+F_n^\perp} = f(u). \tag{3.8}$$

Proof. Let $\epsilon > 0$. Then there is a closed ball $D_{-1}(R)$ in the Hilbert space H_{-1}, centered at 0 and having radius some $R \in (0,\infty)$, such that

$$\sup_{x'\in D_{-1}(R)} |f(u+x') - f(u)| < \epsilon. \tag{3.9}$$

Let $v_1, v_2, \ldots$ be an orthonormal basis of the Hilbert space H_1. For any finite-dimensional subspace F in H_0, we have

(3.10)
$$\begin{aligned}\mu_{F^\perp}\big(D_{-1}(R)^c\big) = \mu_{F^\perp}\left[\sum_{m=1}^{\infty} \hat{v}_m^2 > R^2\right] &\leq \frac{1}{R^2}\sum_{m=1}^{\infty}\int \hat{v}_m^2\, d\mu_{F^\perp} \\ &= \frac{1}{R^2}\sum_{m=1}^{\infty} \|(v_m)_{F^\perp}\|_0^2 \qquad \text{(by (2.5))}.\end{aligned}$$

Note that this series is dominated termwise by the convergent series $\sum_{m=1}^{\infty}\|v_m\|_0^2$.

Now, to apply this to the subspaces F_n, observe first that the projection $x_{F_n^\perp} = x - x_{F_n}$ converges to 0. Hence, by dominated convergence,

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\|(v_m)_{F_n^\perp}\|_0^2 = 0.$$

Consequently,

$$\lim_{n\to\infty}\mu_{F_n^\perp}\big(D_{-1}(R)^c\big) = 0. \tag{3.11}$$

Now

(3.12)
$$\begin{aligned}\left|\int f\, d\mu_{u+F_n^\perp} - f(u)\right| &\leq \int |f(u+x') - f(u)|\, d\mu_{F_n^\perp}(x') \\ &\leq \sup_{x'\in D_{-1}(R)} |f(u+x') - f(u)| + 2\|f\|_{\sup}\mu_{F_n^\perp}\big(D_{-1}(R)^c\big).\end{aligned}$$

By the choice of R, the first term on the right is $< \epsilon$, as noted in (3.9). Next, with this R, the second term is $< \epsilon$ when n is large enough. Hence we have the limiting result (3.8). □

Subspaces X and Y, and any of their translates, of a Hilbert space are said to be *perpendicular* if neither is a subspace of the other and they can be split into mutually orthogonal subspaces

$$X = (X\cap Y) + (X\cap Y^\perp) \qquad \text{and} \qquad Y = (X\cap Y) + (X^\perp \cap Y). \tag{3.13}$$

This means that a vector in X (or Y) which is orthogonal to $X\cap Y$ is in fact orthogonal to Y (or X).

Proposition 3.5. *Consider a hyperplane P in the infinite-dimensional Hilbert space H_0 and a finite-dimensional subspace $F \neq \{0\}$ of H_0. Then P and F are perpendicular if and only if P can be expressed as*

$$P = u + u_0^\perp,$$

for some non-zero vector $u_0 \in F$ and u is a multiple of u_0. Moreover, $P\cap F$ is a hyperplane within the finite-dimensional space F.

We omit the proof, which is elementary.

The following relates the Gauss-Radon transform in infinite dimensions to that for finite-dimensional subspaces by a disintegration process.

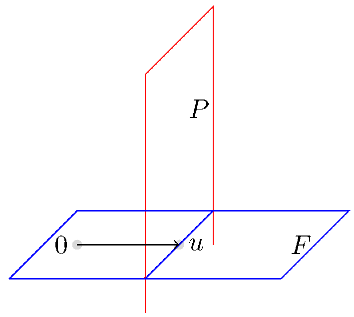


FIGURE 1. A hyperplane P intersecting a subspace F

Proposition 3.6. *Let P be a hyperplane in H_0, and let $F \neq 0$ be a finite-dimensional subspace of H_0 which is perpendicular to P. Then for any bounded Borel function f on H_{-1} we have*

$$Gf(P) = G_F(f^*)(P \cap F), \tag{3.14}$$

where, on the right, $G_F(f^*)$ is the Gauss-Radon transform, within the subspace F, of the function f^* on F given by

$$f^*(y) = \int_{H_{-1}} f \, d\mu_{y+F^\perp}.$$

Part of the conclusion here is that f^* is a measurable function on F.

Proof. Let u_0 be a non-zero vector in F orthogonal to the hyperplane P, and let u be the point on $P \cap F$ closest to the origin. Then (see Figure 1, with u_0 along u):

$$P \cap F = u + (u_0^\perp \cap F).$$

It will suffice to prove the result for a function of the form $\phi = e^{i\hat{x}}$, where $x \in H_1$. For such ϕ, we have

$$\begin{aligned} G_F(\phi^*)(P \cap F) &= \int_F \left(\int_{\mathcal{H}'} e^{i\hat{x}} \, d\mu_{y+F^\perp} \right) d\mu_{P\cap F}(y) \\ &= \int_F e^{i\langle x,y\rangle_0 - \frac{1}{2}\|x_{F^\perp}\|_0^2} \, d\mu_{u+(u_0^\perp \cap F)}(y) \\ &= e^{-\frac{1}{2}\|x_{F^\perp}\|_0^2} \int e^{i\langle x_F, y\rangle_0} \, d\mu_{u+(u_0^\perp\cap F)}(y) \\ &= e^{-\frac{1}{2}\|x_{F^\perp}\|_0^2 + i\langle x_F, u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp \cap F}\|_0^2} \\ &= e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp}\|^2}, \end{aligned} \tag{3.15}$$

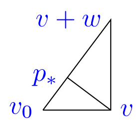
where, in the last step, we used the fact that the component of $x_{u_0^\perp}$ orthogonal to $u_0^\perp \cap F$ is simply $x_{F^\perp}$. This is because $u_0^\perp$ and F are perpendicular so that a vector in $u_0^\perp$ which is orthogonal to $u_0^\perp \cap F$ is just a vector in $u_0^\perp$ orthogonal to F. Comparing with $G\phi(P) = e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp}\|_0^2}$, we see that the result is proved for $\phi = e^{i\hat{x}}$. The general case now follows by routine arguments. $\square$

Lastly we have a geometric observation:

Proposition 3.7. *Let K_0 be a closed convex subset of the Hilbert space H_0, and let v be a point outside K_0. Then there is a finite-dimensional subspace $F_0 \subset H_0$ containing v, and a sequence of finite-dimensional subspaces $F_n \subset H_0$ with $F_0 \subset F_1 \subset \cdots$ and $\bigcup_{n\geq 0} F_n$ dense in H_0 such that $v + F_n^\perp$ is disjoint from K_0 for each $n \in \{1, 2, 3, ...\}$. Moreover, v lies outside the orthogonal projection $\mathrm{pr}_{F_n}(K_0)$ of K_0 onto F_n:*

$$v \notin \mathrm{pr}_{F_n}(K_0) \qquad \text{for all } n \in \{1, 2, 3, ...\}. \tag{3.16}$$

Proof. Since K_0 is closed and convex in the Hilbert space H_0, there is a unique point $v_0 \in K_0$ closest to v. Then the hyperplane through v orthogonal to the vector $u_0 = v - v_0$, i.e. the hyperplane $v + u_0^\perp$, does not contain any point of K_0. For, otherwise, there would be some $w \in u_0^\perp$ with $v + w$ in K_0, and then in the right angled triangle



formed by the points v_0, v, and $v+w$ (which has a right angle at the point v) there would be a point p_* on the hypotenuse, joining v_0 and $v + w$, and hence lying in the convex set K, which would be closer to v than is v_0.

Let F_0 be the subspace of H_0 spanned by the vectors v_0 and u_0; note that $v \in F_0$. Now choose an orthonormal basis $u_1, u_2, ...$ of the closed subspace $F_0^\perp$, and let

$$F_n = F_0 + \text{linear span of } u_1, ..., u_n. \tag{3.17}$$

This gives an increasing sequence of finite-dimensional subspaces whose union contains F_0 as well as all the vectors u_n, and hence is dense in H_0. Next observe that

$$v + F_n^\perp \subset v + F_0^\perp \subset v + u_0^\perp.$$

As noted before, $v + u_0^\perp$ is disjoint from K_0, and so $v + F_n^\perp$ is disjoint from K_0.

Since the hyperplane $v + u_0^\perp$ is precisely the set of points in H_0 whose inner-product with u_0 equals $\langle u_0, v\rangle$, it follows that no point in K_0 has an inner-product with u_0 equal to $\langle u_0, v\rangle$. In particular, the orthogonal projection of K_0 on F_n cannot contain v, for if a point p in K_0 projected orthogonally onto F_n produced v, then its inner-product $\langle u_0, p\rangle$ with u_0 would be the same as $\langle u_0, v\rangle$. This proves (3.16). □

4. The support theorem

We continue with the framework set up in the preceding sections, and now turn to our main result. Thus, H_0 is a separable, infinite-dimensional real Hilbert space, which is a Hilbert-Schmidt completion of a Hilbert space $H_1 \subset H_0$, and H_{-1} is the dual H_1'.

Theorem 4.1. *Suppose that $f : H_{-1} \to \mathbb{R}$ is a bounded function on the Hilbert space H_{-1} which is uniformly continuous in the strong (Hilbert) topology on H_{-1}. Let K_0 be a closed, bounded, convex subset of H_0. If the Gauss-Radon transform*

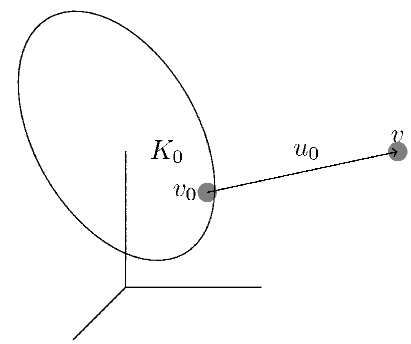


FIGURE 2. A compact convex set K_0 and a point v outside it

of f is 0 *on hyperplanes which do not intersect K_0, then f is* 0 *on the complement of K_0 in H_0.*

Proof. We work with a point $v \in H_0$ outside K_0.

In Proposition 3.7 we constructed a sequence of finite-dimensional subspaces $F_n \subset H_0$ containing v, with $F_1 \subset F_2 \subset \cdots$ such that $\bigcup_{n\geq 1} F_n$ is dense in H_0 and $v + F_n^{\perp}$ is disjoint from K_0 for every positive integer n. Recall briefly how this was done. First we chose a point $u \in K_0$ closest to v, we set

$$u_0 = v - u,$$

then we chose $u_1, u_2, \ldots$ an orthonormal basis of $u_0^{\perp}$, set F_0 to be the linear span of v and u_0, and took F_n to be the linear span of $v, u_0, u_1, \ldots, u_n$. We showed that the hyperplane

$$v + u_0^{\perp}$$

is disjoint from K_0 and that v is outside the orthogonal projection $\mathrm{pr}_{F_n}(K_0)$ of K_0 onto F_n.

Let $f^*_{F_n}$ be the function on F given by

$$f^*_{F_n}(x) = \int f \, d\mu_{x+F_n^{\perp}}.$$

We will show that this is 0 whenever x lies outside $\mathrm{pr}_{F_n}(K_0)$. In particular, it will follow that $f^*_{F_n}(v)$ is 0 for all $n \in \{1, 2, 3, \ldots\}$, and so, by the limiting result of Proposition 3.4,

$$f(v) = \lim_{n\to\infty} f^*_{F_n}(v) = 0. \tag{4.1}$$

We drop the subscript n in the following, and work with a finite-dimensional space $F \subset H_0$ for which

$$v \in F, \text{ but } v \notin \mathrm{pr}_F(K_0). \tag{4.2}$$

Let P' be a hyperplane within the subspace F. Then

$$P' = P \cap F,$$

where P is the hyperplane in H_0 given by

$$P = P' + F^{\perp},$$

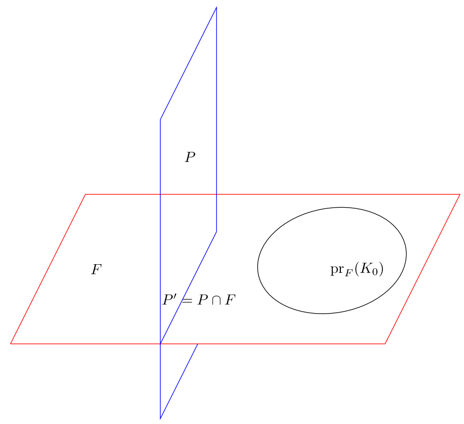


FIGURE 3. The projection of a compact set K_0 on F, disjoint from a hyperplane $P' = P \cap F$ in P

which is perpendicular to F. If P' is disjoint from $\mathrm{pr}_F(K_0)$, then P is disjoint from K_0 because any point in $P \cap K_0$ projects by pr_F to a point which is both P' and $\mathrm{pr}_F(K_0)$. Now recall the disintegration formula (3.14) says

$$Gf(P) = G_F(f_F^*)(P'). \tag{4.3}$$

By hypothesis, this is 0 if P is disjoint from K_0, i.e. if P' is disjoint from $\mathrm{pr}_F(K_0)$. Now observe that $\mathrm{pr}_F(K_0)$ is a convex, compact subset of F (compactness follows because K_0, being convex, closed and bounded is weakly compact, and hence any finite-dimensional projection is compact). Note that by Proposition 3.3, the function f_F is continuous on F and is, of course, also bounded. It then follows by the finite-dimensional Radon transform support theorem (applied to f_F times a Gaussian density) that f_F^* is 0 outside $\mathrm{pr}_F(K_0)$. In particular, $f_F^*(v)$ is 0, which is what was needed to complete the proof. □

4.1. **Support theorem in nuclear spaces.** As noted in the introduction, there are two frameworks for analysis in infinite dimensions used widely: (i) Abstract Wiener Spaces [8, 12], and (ii) Nuclear spaces. The latter form a useful setting for both quantum field theory (for instance [7, 16]) and stochastic analysis (for instance [4, 11]). Our results have been framed entirely in terms of Hilbert spaces, but they extend readily to the nuclear space setting. Suppose H_0 is a real separable Hilbert space, $T : H_0 \to H_0$ is a Hilbert-Schmidt operator, with $0 \leq T \leq I$, $H_p = T^{p/2}(H_0)$ for $p \in \{0, 1, 2, ..\}$ and $\mathcal{H} = \bigcap_{p\in\{0,1,2,..\}} H_p$, equipped with the topology generated by the norms given by $\|x\|_p = \|T^{-p/2}x\|_0$. The dual $\mathcal{H}'$ of the 'nuclear' space $\mathcal{H}$ has the weak topology as well as a strong topology [5, 6], and the hyperplane measures $\mu_{u+F^\perp}$ may be realized on $\mathcal{H}' \supset H_{-1} = H_1'$. In the space $\mathcal{H}'$, a strongly continuous function is strongly sequentially continuous. This is equivalent to weak sequential continuity, and so Proposition 3.3 is applicable. Then the support theorem says that if f is a bounded, strongly continuous function on $\mathcal{H}'$ whose Gauss-Radon transform vanishes on hyperplanes lying outside a convex, closed and bounded set $K_0 \subset H_0$, then f is 0 off K_0.

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