TORUS MANIFOLDS WITH NON-ABELIAN SYMMETRIES

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Abstract. Let $G$ be a connected compact non-abelian Lie group and $T$ be a maximal torus of $G$. A torus manifold with $G$-action is defined to be a smooth connected closed oriented manifold of dimension $2\dim T$ with an almost effective action of $G$ such that $M^G \neq \emptyset$. We show that if there is a torus manifold $M$ with $G$-action, then the action of a finite covering group of $G$ factors through $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^{l_0}$. The action of $\tilde{G}$ on $M$ restricts to an action of $\tilde{G}' = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod U(l_i) \times T^{l_0}$ which has the same orbits as the $\tilde{G}$-action.

We define invariants of torus manifolds with $G$-action which determine their $\tilde{G}'$-equivariant diffeomorphism type. We call these invariants admissible 5-tuples. A simply connected torus manifold with $G$-action is determined by its admissible 5-tuple up to a $\tilde{G}'$-equivariant diffeomorphism. Furthermore, we prove that all admissible 5-tuples may be realised by torus manifolds with $\tilde{G}''$-action, where $\tilde{G}''$ is a finite covering group of $\tilde{G}'$.

1. Introduction

A $2n$-dimensional smooth connected closed oriented manifold $M$ with an almost effective action of an $n$-dimensional torus $T$ is called a torus manifold if $M^T \neq \emptyset$. If each point of $M$ has an invariant open neighborhood which is weakly equivariantly diffeomorphic to an open subset of the standard action of $T$ on $\mathbb{C}^n$, then the orbit space $M/T$ is an $n$-dimensional manifold with corners [15 pp. 720-721]. In this case $M$ is said to be quasitoric if $M/T$ is face preserving homeomorphic to a simple polytope $P$. In that case there are strong relations between the topology of $M$ and the combinatorics of $P$ [6, 5].

In this article we study torus manifolds for which the $T$-action may be extended by an action of a connected compact non-abelian Lie group $G$. To state our results, we introduce a bit more notation which we use to describe the structure of torus manifolds.

A closed, connected submanifold $M_i$ of codimension two of a torus manifold $M$, which is pointwise fixed by a one-dimensional subtorus $\lambda(M_i)$ of $T$ and which contains a $T$-fixed point, is called a characteristic submanifold of $M$.

All characteristic submanifolds $M_i$ are orientable, and an orientation of $M_i$ determines a complex structure on the normal bundle $N(M_i,M)$ of $M_i$.
We denote the set of unoriented characteristic submanifolds of \( M \) by \( \mathcal{F} \). If \( M \) is quasitoric, the characteristic submanifolds of \( M \) are given by the preimages of the facets of \( P \). In this case we identify \( \mathcal{F} \) with the set of facets of \( P \).

Let \( G \) be a connected compact non-abelian Lie group. We call a smooth connected closed oriented \( G \)-manifold \( M \) a torus manifold with \( G \)-action if \( G \) acts almost effectively on \( M \), \( \dim M = 2 \operatorname{rank} G \) and \( M^T \neq \emptyset \) for a maximal torus \( T \) of \( G \). That means that \( M \) with the action of \( T \) is a torus manifold. Because all maximal tori of \( G \) are conjugated, \( M \) together with the action of any other maximal torus \( T' \) is also a torus manifold. Moreover, for all choices of a maximal torus of \( G \), we get up to weakly equivariant diffeomorphism the same torus manifold. The \( G \)-action on \( M \) induces an action of the Weyl group \( W(G) \) on \( \mathcal{F} \) and the \( T \)-equivariant cohomology of \( M \). Results of Masuda \[14\] and Davis-Januszkiewicz \[6\] make a comparison of these actions possible. From this comparison we get a description of the action on \( \mathcal{F} \) and the isomorphism type of \( W(G) \). Namely, there is a partition of \( \mathcal{F} = \mathcal{F}_0 \sqcup \cdots \sqcup \mathcal{F}_L \) and a finite covering group \( \tilde{G} = \prod_{j=1}^L G_j \times T^{b_j} \) of \( G \) such that each \( G_{j_0} \) is non-abelian and \( W(G_{j_0}) \) acts transitively on \( \mathcal{F}_{j_0} \) and trivially on \( \mathcal{F}_j, j \neq j_0 \), and the orientation of each \( M_i \in \mathcal{F}_j, j \neq j_0 \), is preserved by \( W(G_{j_0}) \) (see section \[2\]).

We call such \( G \) the elementary factors of \( \tilde{G} \).

By looking at the orbits of the \( T \)-fixed points, we find that we may assume without loss of generality that all elementary factors are isomorphic to \( SU(l_1+1) \), \( SO(2l_1) \) or \( SO(2l_1+1) \) (see section \[3\]). If \( M \) is quasitoric, then all elementary factors are isomorphic to \( SU(l_1+1) \).

Now assume \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SO(2l_1) \) elementary. Then the restriction of the action of \( G_1 \) to \( U(l_1) \) restricts the same orbits as the \( G_1 \)-action (see section \[6\]). The following theorem shows that the classification of simply connected torus manifolds with \( \tilde{G} \)-action reduces to the classification of torus manifolds with \( U(l_1) \times G_2 \)-action.

**Theorem 1.1** (Theorem \[6.3\]). Let \( M, M' \) be two simply connected torus manifolds with \( \tilde{G} \)-action, \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SO(2l_1) \) elementary. Then \( M \) and \( M' \) are \( \tilde{G} \)-equivariantly diffeomorphic if and only if they are \( U(l_1) \times G_2 \)-equivariantly diffeomorphic.

By applying a blow up construction along the fixed points of an elementary factor of \( \tilde{G} \) isomorphic to \( SU(l_1+1) \) or \( SO(2l_1+1) \), we get a fiber bundle over a complex or real projective space with some torus manifold as the fiber.

This construction may be reversed, and we call the inverse construction a blow down. With this notation we get:

**Theorem 1.2** (Corollaries \[5.0\] \[5.14\] \[7.2\] Theorem \[7.5\]). Let \( \tilde{G} = G_1 \times G_2 \) and let \( M \) be a torus manifold with \( G \)-action such that \( G_1 \) is elementary and \( l_2 = \operatorname{rank} G_2 \). Then:

- If \( G_1 = SU(l_1+1) \) and \( \# \mathcal{F}_{l_1} = 2 \) in the case \( l_1 = 1 \), then \( M \) is the blow down of a fiber bundle \( \bar{M} \) over \( \mathbb{C}P^{l_1} \) with the fiber being some \( 2l_2 \)-dimensional torus manifold with \( G_2 \)-action along an invariant submanifold of codimension two. Here the \( G_1 \)-action on \( \bar{M} \) covers the standard action of \( SU(l_1+1) \) on \( \mathbb{C}P^{l_1} \).
- If \( G_1 = SO(2l_1+1) \) and \( \# \mathcal{F}_{l_1} = 1 \) in the case \( l_1 = 1 \), then \( M \) is a blow down of a fiber bundle \( \bar{M} \) over \( \mathbb{R}P^{2l_1} \) with the fiber being some \( 2l_2 \)-dimensional
torus manifold with $G_2$-action along an invariant submanifold of codimension one or a Cartesian product of a $2l_1$-dimensional sphere and a $2l_2$-dimensional torus manifold with $G_2$-action. In the first case the $G_1$-action on $\tilde{M}$ covers the standard action of $SO(2l_1 + 1)$ on $\mathbb{R}P^{2l_1}$. In the second case $G_1$ acts in the usual way on $S^{2l_1}$.

If all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i + 1)$ or $SU(l_i + 1)$, then we may iterate this construction. By this iteration we get a complete classification of torus manifolds with $\tilde{G}$-action up to a $\tilde{G}$-equivariant diffeomorphism in terms of admissible 5-tuples (Theorem 3.6). For general $G$ we have $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times SO(2l_i) \times T^{l_i}$. We may restrict the action of $\tilde{G}$ to $\prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod U(l_i) \times T^{l_i}$. Therefore we get invariants for torus manifolds with $G$-action from the above classification. With Theorem 1.1 we see that these invariants determine the $G$-equivariant diffeomorphism type of simply connected torus manifolds with $G$-action.

At the end we apply our classification to get more explicit results in special cases. These are:

For the special case $G_2 = \{1\}$ we get:

**Corollary 1.3** (Corollary 3.6). Assume that $G$ is elementary and $M$ is a torus manifold with $G$-action. Then $M$ is equivariantly diffeomorphic to $S^{2l}$ or $\mathbb{C}P^l$ if $G = SO(2l + 1), SO(2l)$ or $G = SU(l + 1)$, respectively.

We recover certain results of Kuroki [13, 11, 12] who gave a classification of torus manifolds with $G$-action and $\dim M/G \leq 1$ (see Corollaries 8.10 and 8.11).

For quasitoric manifolds we have the following result.

**Theorem 1.4** (Corollary 8.9). If $G$ is semi-simple and $M$ is a quasitoric manifold with $G$-action, then

$$\tilde{G} = \prod_{i=1}^k SU(l_i + 1)$$

and $M$ is equivariantly diffeomorphic to a product of complex projective spaces.

Furthermore, we give an explicit classification of simply connected torus manifolds with $G$-action such that $\tilde{G}$ is semi-simple and has two simple factors.

**Theorem 1.5** (Corollaries 3.6, 8.12, 8.14). Let $\tilde{G} = G_1 \times G_2$ with $G_i$ simple and $M$ be a simply connected torus manifold with $G$-action. Then $M$ is one of the following:

$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}$, $\mathbb{C}P^{l_1} \times S^{2l_2}$, $\#_i(S^{2l_1} \times S^{2l_2})$, $S^{2l_1 + 2l_2}$.

The $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

The paper is organized as follows. In section 2 we investigate the action of the Weyl group of $G$ on $\mathcal{S}$ and $H^*_T(M)$. In section 3 we determine the orbit types of the $T$-fixed points in $M$ and the isomorphism types of the elementary factors of $G$. In section 4 the basic properties of the blow up construction are established. In section 5 actions with an elementary factor $G_1 = SU(l_1 + 1)$ are studied. In section 6 we give an argument which reduces the classification problem for actions with an elementary factor $G_1 = SO(2l_1)$ to that with an elementary factor $SU(l_1)$. In section 7 we classify torus manifolds with $G$-action with elementary factor $G_1 = SO(2l_1 + 1)$. In section 8 we iterate the classification results of the previous
sections and illustrate them with some applications. There are two appendices with preliminary facts on Lie groups and torus manifolds.

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2. The action of the Weyl group on \( \mathfrak{g} \)

Let \( G \) be a compact connected Lie group of rank \( n \) and \( T \) be a maximal torus of \( G \). Moreover, let \( M \) be a torus manifold with \( G \)-action. That means that \( G \) acts almost effectively on the \( 2n \)-dimensional smooth closed connected oriented manifold \( M \) such that \( M^T \neq \emptyset \). We call a closed connected submanifold \( M_i \) of codimension two of \( M \), which is pointwise fixed by a one-dimensional subtorus \( \lambda(M_i) \) of \( T \) and which contains a \( T \)-fixed point, a characteristic submanifold of \( M \). If \( g \) is an element of the normalizer \( N_G T \) of \( T \) in \( G \), then, for every characteristic submanifold \( M_i \), \( gM_i \) is also a characteristic submanifold. Therefore there are actions of \( N_G T \) and the Weyl group of \( G \) on \( \mathfrak{g} \).

In this section we describe this action of the Weyl group of \( G \) on \( \mathfrak{g} \). At first we recall the definition of the equivariant cohomology of a \( G \)-space \( X \). Let \( EG \to BG \) be a universal principal \( G \)-bundle. Then \( EG \) is a contractible free right \( G \)-space. If \( T \) is a maximal torus of \( G \), then we may identify \( ET = EG \) and \( BT = EG/T \). The Borel construction \( X_G \) of \( X \) is the orbit space of the right action \( ((e, x), g) \mapsto (eg, g^{-1}x) \) on \( EG \times X \). The equivariant cohomology \( H^*_G(X) \) of \( X \) is defined as the cohomology of \( X_G \).

In this section we take all cohomology groups with coefficients in \( \mathbb{Q} \).

The \( G \)-action on \( EG \times X \) induces a right action of the normalizer of \( T \) on \( X_T \). Therefore it induces a left action of the Weyl group of \( G \) on the \( T \)-equivariant cohomology of \( X \).

Now let \( X = M \) be a torus manifold with \( G \)-action. Denote the characteristic submanifolds of \( M \) by \( M_i \), \( i = 1, \ldots, m \). Then, for any \( g \in N_G T \), \( M_{g(i)} = gM_i \) is also a characteristic submanifold which depends only on the class \( [g] \in W(G) = N_G T/T \). Therefore we get an action of the Weyl group of \( G \) on \( \mathfrak{g} \). Notice that \( M_i \in \mathfrak{g} \) is a fixed point of the \( W(G) \)-action on \( \mathfrak{g} \) if and only if it is invariant under the action of \( N_G T \) on \( M \).

A choice of an orientation for each characteristic submanifold of \( M \) together with an orientation for \( M \) is called an omniorientation of \( M \). If we fix an omniorientation for \( M \), then the \( T \)-equivariant Poincaré dual \( \tau_i \) of \( M_i \) is well defined.

It is the image of the Thom class of \( N(M_i, M)_T \) under the natural map

\[
\psi: H^2(N(M_i, M)_T, N(M_i, M)_T - (M_i)_T) \to H^2(T, M_T - (M_i)_T) \to H^2_T(M).
\]

Because of the uniqueness of the Thom class [17, p. 110] and because \( \psi \) commutes with the action of \( W(G) \), we have

\[
(2.1) \quad \tau_{g(i)} = \pm g^*\tau_i.
\]

Here the minus sign occurs if and only if \( g|_{M_i}: M_i \to M_{g(i)} \) is orientation reversing. We say that the class \( [g] \in W(G) \) acts orientation preserving at \( M_i \) if this map is orientation preserving. If \( [g] \) acts orientation preserving at all characteristic submanifolds, then we say that \( [g] \) preserves the omniorientation of \( M \).
Let $S = H^>(BT)$ and $\hat{H}^2_*(M) = H^2_*(M)/S$-torsion. Because $MT \neq \emptyset$, there is an injection $H^2(BT) \hookrightarrow H^2_*(M)$ and
\begin{equation}
H^2(BT) \cap S\text{-torsion} = \{0\}.
\end{equation}

By [14, pp. 240-241], the $\tau_i$ are linearly independent in $\hat{H}^2_*(M)$. By Lemma 3.2 of [14, p. 246], they form a basis of $\hat{H}^2_*(M)$.

The Lie algebra $LG$ of $G$ may be endowed with a Euclidean inner product which is invariant for the adjoint representation. This allows us to identify the Weyl group $W(G)$ of $G$ with a group of orthogonal transformations on the Lie algebra $LT$ of $T$. It is generated by reflections in the walls of the Weyl chambers of $G$ [4, pp. 192-193]. In the following we say that an element of $W(G)$ is a reflection if and only if it is a reflection in a wall of a Weyl chamber of $G$. An element $w \in W(G)$ is a reflection if and only if it acts as a reflection on $H^2(BT)$.

Here we say that $A \in \text{Gl}(L)$ acts as a reflection on the $\mathbb{Q}$-vector space $L$ if there is a decomposition $L = L_+ \oplus L_-$ with $\dim_{\mathbb{Q}} L_- = 1$ and $A|_{L_\pm} = \pm \text{Id}$. Notice that $A \in \text{Gl}(L)$ acts as a reflection on $L$ if and only if $\text{ord} A = 2$ and $\text{trace}(A, L) = \dim_{\mathbb{Q}} L - 2$.

**Lemma 2.1.** Let $w \in W(G)$ be a reflection. Then there are the following possibilities for the action of $w$ on $\mathfrak{g}$:

1. $w$ fixes all except exactly two elements of $\mathfrak{g}$. It acts orientation preserving at all characteristic submanifolds.
2. $w$ fixes all except exactly two elements of $\mathfrak{g}$. Denote the elements of $\mathfrak{g}$ which are not fixed by $w$ by $M_1, M_2$. The action of $w$ is orientation preserving at all characteristic submanifolds of $M$ except $M_1, M_2$. It is orientation reversing at $M_1, M_2$.
3. $w$ fixes all elements of $\mathfrak{g}$. It acts orientation reversing at exactly one characteristic submanifold of $M$.

**Proof.** Using the arguments given before Lemma 2.1 we have the following commutative diagram of $W(G)$-representations with exact rows and columns:

\[
\begin{array}{cccccc}
S\text{-torsion} & \hookrightarrow & H^2(BT) & \xrightarrow{\phi} & H^2_*(M) & \xrightarrow{\hat{\phi}} & H^2(M) \\
& \downarrow & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{H}^2_*(M) & & & & 0.
\end{array}
\]

Here $\hat{\phi}$ denotes the natural map $H^2_*(M) \rightarrow H^2(M)$.

Because $G$ is connected, the $W(G)$-action on $H^2(M)$ is trivial. By (2.2) the $S$-torsion in $H^2_*(M)$ injects into $H^2(M)$. Therefore $W(G)$ acts trivially on the $S$-torsion in $H^2_*(M)$.  

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Because \(w\) is a reflection, we have \(\text{trace}(w, H^2(BT)) = \dim Q H^2(BT) - 2\). From the exact row in the diagram we get
\[
\text{trace}(w, H^2_T(M)) = \text{trace}(w, H^2(BT)) + \text{trace}(w, \text{im}\phi) = \dim_Q H^2(BT) - 2 + \dim_Q \text{im}\phi = \dim_Q H^2_T(M) - 2.
\]

Similarly we get
\[
\text{trace}(w, \hat{H}^2_T(M)) = \text{trace}(w, H^2_T(M)) - \text{trace}(w, S\text{-torsion in } H^2_T(M)) = \dim_Q \hat{H}^2_T(M) - 2.
\]

Now the statement follows from (2.1) because the \(\tau_i\) form a basis of \(\hat{H}^2_T(M)\).

Lemma 2.2. An element \(w \in W(G)\) acts as a reflection on \(\hat{H}^2_T(M)\) if and only if it is a reflection.

Proof. Because, by (2.2), \(H^2(BT)\) injects into \(\hat{H}^2_T(M)\), \(W(G)\) acts effectively on \(\hat{H}^2_T(M)\). Therefore we may identify \(W(G)\) with a subgroup of \(\text{Gl}(\hat{H}^2_T(M))\).

If \(w \in W(G)\), then, as in the proof of Lemma 2.1 we see that
\[
\dim_Q H^2(BT) - \text{trace}(w, H^2(BT)) = \dim_Q \hat{H}^2_T(M) - \text{trace}(w, \hat{H}^2_T(M)).
\]

Therefore, by the remark before Lemma 2.1 an element of \(W(G)\) of order two is a reflection if and only if it acts as a reflection on \(\hat{H}^2_T(M)\). 

Let \(\mathcal{F}_0\) be the set of characteristic submanifolds which are fixed by the \(W(G)\)-action on \(\mathfrak{F}\) and at which \(W(G)\) acts orientation preserving. Furthermore let \(\mathcal{F}_i, i = 1, \ldots, k\), be the other orbits of the \(W(G)\)-action on \(\mathfrak{F}\) and \(V_i\) the subspace of \(\hat{H}^2_T(M)\) spanned by the \(\tau_j\) with \(M_j \in \mathcal{F}_i\). Then \(W(G)\) acts trivially on \(V_i\). For \(i > 0\), let \(W_i\) be the subgroup of \(W(G)\) which is generated by the reflections which act non-trivially on \(V_i\). Then, by Lemma 2.1 \(W_i\) acts trivially on \(V_j, j \neq i\).

By (2.2), \(H^2(BT)\) injects into \(\hat{H}^2_T(M)\). Therefore \(W(G)\) acts effectively on \(\hat{H}^2_T(M)\). This fact implies that the subgroups \(W_i, i = 1, \ldots, k\), of \(W(G)\) pairwise commute and \((W_1, \ldots, W_i) \cap W_{i+1} = \{1\}\) for all \(i = 1, \ldots, k-1\). Here \((W_1, \ldots, W_i)\) denotes the subgroup of \(W(G)\) which is generated by \(W_1, \ldots, W_i\). Hence, we have an injective group homomorphism \(\prod W_i \to W(G), (w_1, \ldots, w_k) \mapsto w_1 \ldots w_k\).

Lemma 2.3. The group homomorphism \(\prod W_i \to W(G), (w_1, \ldots, w_k) \mapsto w_1 \ldots w_k\) is an isomorphism.

Proof. Because \(W(G)\) is generated by reflections and each reflection is contained in a \(W_i\), the above homomorphism is surjective. As noted before, it is injective. Therefore it is an isomorphism.

Lemma 2.4. For each pair \(M_{j_1}, M_{j_2} \in \mathcal{F}_i, i > 0, \) with \(M_{j_1} \neq M_{j_2}\), there is a reflection \(w \in W_i\) with \(w(M_{j_1}) = M_{j_2}\).

Proof. Because \(\mathcal{F}_i\) is an orbit of the \(W(G)\)-action on \(\mathfrak{F}\) and \(W(G)\) is generated by reflections, there is an \(M'_{j_1} \in \mathcal{F}_i\) with \(M'_{j_1} \neq M_{j_2}\) and a reflection \(w \in W_i\) with \(w(M'_{j_1}) = M_{j_2}\).
Because $W_i$ is generated by reflections and acts transitively on $\mathcal{F}_i$, the natural map $W_i \to S(\mathcal{F}_i)$ to the permutation group $S(\mathcal{F}_i)$ of $\mathcal{F}_i$ is a surjection by Lemma 2.1 and Lemma 3.10 of [1, p. 51]. Therefore there is a $w' \in W_i$ with

$$w'(M_{j_1}) = M'_{j_1}, \quad w'(M_{j_2}) = M_{j_2}.$$  

Now $w'^{-1}ww' \in W_i$ is a reflection with the required properties. \qed

It follows from Lemma 2.1 that for each pair $M_{j_1}, M_{j_2} \in \mathcal{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there are at most two reflections which map $M_{j_1}$ to $M_{j_2}$.

If $M_{j_1}, M_{j_2} \in \mathcal{F}_i$ is another pair with $M_{j_1} \neq M_{j_2}$, then one sees as in the proof of Lemma 2.1 that there is a $w' \in W_i$ with

$$w'(M_{j_1}) = M_{j_1}, \quad w'(M_{j_2}) = M_{j_2}.$$  

Therefore there is a bijection

$$\{w \in W_i; \text{ w reflection}, w(M_{j_1}) = M_{j_2}\} \to \{w \in W_i; \text{ w reflection}, w(M_{j_2}) = M_{j_2}\},$$  

$$w \mapsto w'^{-1}ww'.$$

In particular, the number of reflections which map $M_{j_1}$ to $M_{j_2}$ does not depend on the choice of $M_{j_1}, M_{j_2} \in \mathcal{F}_i$.

**Lemma 2.5.** Assume $\#\mathcal{F}_i > 1$ and $i > 0$. If for each pair $M_{j_1}, M_{j_2} \in \mathcal{F}_i$ with $M_{j_1} \neq M_{j_2}$ there is exactly one reflection in $W_i$, which maps $M_{j_1}$ to $M_{j_2}$, then $W_i$ is isomorphic to $S(\mathcal{F}_i) \cong W(SU(l_i + 1))$ with $l_i + 1 = \#\mathcal{F}_i$.

**Proof.** First we show that there is no reflection of the third type as described in Lemma 2.1 in $W_i$. Assume that $w' \in W_i$ is a reflection of the third type. Then let $M_1 \in \mathcal{F}_i$ be the characteristic submanifold at which $w'$ acts orientation reversing. Furthermore, let $M_1 \neq M_2 \in \mathcal{F}_i$.

Then by Lemma 2.1 there is a reflection $w \in W_i$ such that $wM_1 = M_2$. Hence, $ww'$ is a reflection with $ww'M_1 = M_2$. Because $w$ and $ww'$ have a different orientation behaviour at $M_1$, we have $w \neq ww'$, contradicting our assumption.

To prove the lemma, it is sufficient to show that the kernel of the natural map $W_i \to S(\mathcal{F}_i)$ is trivial. Let $w$ be an element of this kernel. Then for each $\tau_j \in V_i$ we have

$$w\tau_j = \pm \tau_j.$$  

If we have $w\tau_j = \tau_j$ for all $\tau_j \in V_i$, then $w = \text{Id}$.

Now assume that $w\tau_{j_0} = -\tau_{j_0}$ for a $\tau_{j_0} \in V_i$. Then there are reflections $w_1, \ldots, w_n \in W_i$, $n \geq 2$, with $-\tau_{j_0} = w\tau_{j_0} = w_1 \ldots w_n\tau_{j_0}$. After removing some of the $w_i$, we may assume that

$$w_i \ldots w_n\tau_{j_0} \neq \pm \tau_{j_0} \quad \text{for all } i = 2, \ldots, n,$$

$$w_{i+1} \ldots w_n\tau_{j_0} \neq \pm w_i \ldots w_n\tau_{j_0} \quad \text{for all } i = 2, \ldots, n.$$  

Therefore, by Lemma 2.1 we have $w_i\tau_{j_0} = \tau_{j_0}$ for $2 \leq i < n$. This equation together with $w\tau_{j_0} = -\tau_{j_0}$ implies

$$w_n \ldots w_2w_1w_2 \ldots w_n\tau_{j_0} = -w_n\tau_{j_0}.$$  

Therefore $w_n \ldots w_2w_1w_2 \ldots w_nM_{j_0} = w_nM_{j_0}$.

But $w_n \ldots w_2w_1w_2 \ldots w_n$ is a reflection. Therefore, by assumption, we have

$$w_n \ldots w_2w_1w_2 \ldots w_n = w_n.$$
and
\[ w_n \tau_{j_0} = w_n w_{n-1} \ldots w_2 w_1 w_2 \ldots w_n \tau_{j_0} = -w_n \tau_{j_0}. \]

Because \( w_n \tau_{j_0} \neq 0 \), this is impossible. Hence, our assumption that \( w \tau_{j_0} = -\tau_{j_0} \) is false.

Therefore the kernel is trivial. \( \square \)

To get the isomorphism type of \( W_i \) in the case where there is a pair \( M_{j_1}, M_{j_2} \in \bar{F}_i, i > 0 \), with \( M_{j_1} \neq M_{j_2} \) and exactly two reflections in \( W_i \) which map \( M_{j_1} \) to \( M_{j_2} \), we first give a description of the Weyl groups of some Lie groups.

Let \( L \) be an \( l \)-dimensional \( \mathbb{Q} \)-vector space with basis \( e_1, \ldots, e_l \). For \( 1 \leq i < j \leq l \) let \( f_{ij}, g_i \in \text{Gl}(L) \) such that
\[
\begin{align*}
   f_{ij} &+ e_k = \begin{cases} 
   e_i & \text{if } k = j, \\
   e_j & \text{if } k = i, \\
   e_k & \text{else,}
   \end{cases} \\
   f_{ij} &- e_k = \begin{cases} 
   -e_i & \text{if } k = j, \\
   -e_j & \text{if } k = i, \\
   e_k & \text{else,}
   \end{cases} \\
   g_i e_k &= \begin{cases} 
   -e_i & \text{if } k = i, \\
   e_k & \text{else.}
   \end{cases}
\end{align*}
\]

Then we have the following isomorphisms of groups \cite{[3] pp. 171-172}:
\[
\begin{align*}
   W(SU(l-1)) &\cong S(l) \cong \langle f_{ij}; 1 \leq i < j \leq l \rangle, \\
   W(SO(2l)) &\cong \langle f_{ij}; 1 \leq i < j \leq l \rangle, \\
   W(SO(2l+1)) &\cong W(Sp(l)) \cong \langle f_{ij}, g_1; 1 \leq i < j \leq l \rangle.
\end{align*}
\]

From this description and Lemma \ref{2.1} we get:

**Lemma 2.6.** If for each pair \( M_{j_1}, M_{j_2} \in \bar{F}_i, i > 0 \), with \( M_{j_1} \neq M_{j_2} \) there are exactly two reflections in \( W_i \) which map \( M_{j_1} \) to \( M_{j_2} \), then with \( l_i = \# \bar{F}_i \) we have:

1. \( W_i \cong W(SO(2l_i)) \) if there is no reflection of the third type as described in Lemma \ref{2.1} in \( W_i \).
2. \( W_i \cong W(SO(2l_i + 1)) \cong W(Sp(l_i)) \) if there is a reflection of the third type in \( W_i \).

By \cite{[3] p. 233}, \( G \) has a finite covering group \( \bar{G} \) such that \( \bar{G} = \prod G_i \times T^{10} \), where the \( G_i \) are simple and simply connected compact Lie groups. The Weyl group of \( G \) is given by \( W(G) = \prod W(G_i) \).

We call two reflections \( w, w' \in W(G) \) equivalent if there are reflections \( w_1, \ldots, w_k \in W(G) \) such that
\[
\begin{align*}
   w &= w_1, \\
   w' &= w_k, \\
   [w_i, w_{i+1}] &\neq 1.
\end{align*}
\]

Here \( [w_i, w_{i+1}] \) denotes the commutator of \( w_i \) and \( w_{i+1} \). Because the Dynkin diagram of a simple Lie group is connected, each \( W(G_i) \) is generated by equivalent reflections. Therefore each \( W(G_i) \) is contained in a \( W_j \). Therefore we get \( W_i = \prod_{j \in J_i} W(G_j) \). Using Lemmas \ref{2.4} and \ref{2.6} we deduce:
\[
\begin{align*}
   W_i &= \begin{cases} 
   W(G_j) & \text{for some } j \text{ if } W_i \ncong W(SO(4)), \\
   W(G_{j_1}) \times W(G_{j_2}) & \text{with } G_{j_1} \cong G_{j_2} \cong SU(2) \text{ if } W_i \cong W(SO(4)).
   \end{cases}
\end{align*}
\]
Therefore we may write $\tilde{G} = \prod_i G_i \times T^{i_0}$ with $W_i = W(G_i)$ and $G_i$ simple and simply connected or $G_i = \text{Spin}(4)$. In the following we will call these $G_i$ the elementary factors of $\tilde{G}$.

We summarize the above discussion in the following lemma.

**Lemma 2.7.** Let $M$ be a torus manifold with $G$-action and $\tilde{G}$ as above. Then all $G_i$ are non-exceptional, i.e. $G_i = SU(l_i + 1), \text{Spin}(2l_i), \text{Spin}(2l_i + 1), \text{Sp}(l_i)$.

The Weyl group of an elementary factor $G_i$ of $\tilde{G}$ acts transitively on $\tilde{F}_i$ and trivially on $\tilde{F}_j$, $j \neq i$.

For a given isomorphism type of $G_i$, there are at most two possible values of $\#\tilde{F}_i$. The possible values of $\#\tilde{F}_i$ are listed in the following table:

<table>
<thead>
<tr>
<th>$G_i$</th>
<th>$#\tilde{F}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2) = \text{Spin}(3) = \text{Sp}(1)$</td>
<td>1, 2</td>
</tr>
<tr>
<td>$\text{Spin}(4)$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Spin}(5) = \text{Sp}(2)$</td>
<td>2</td>
</tr>
<tr>
<td>$SU(4) = \text{Spin}(6)$</td>
<td>3, 4</td>
</tr>
<tr>
<td>$SU(l_i + 1), l_i \neq 1, 3$</td>
<td>$l_i + 1$</td>
</tr>
<tr>
<td>$\text{Spin}(2l_i + 1), l_i &gt; 2$</td>
<td>$l_i$</td>
</tr>
<tr>
<td>$\text{Spin}(2l_i), l_i &gt; 3$</td>
<td>$l_i$</td>
</tr>
<tr>
<td>$\text{Sp}(l_i), l_i &gt; 2$</td>
<td>$l_i$</td>
</tr>
</tbody>
</table>

If we restrict our attention to quasitoric manifolds with $G$-action, then we get a much shorter list of possible isomorphism types of the elementary factors. In fact, if $M$ is a quasitoric manifold with $G$-action, then, as shown in the next lemma, all elementary factors of $G$ are isomorphic to $SU(l_i + 1)$ for some $l_i \geq 1$.

**Lemma 2.8.** Let $M$ be a quasitoric manifold with $G$-action. Then there is a covering group $\tilde{G}$ of $G$ with $\tilde{G} = \prod_{i=1}^{k_1} SU(l_i + 1) \times T^{i_0}$.

**Proof.** First we show for $i > 0$:

\begin{equation}
W_i \cong S(\tilde{F}_i).
\end{equation}

To do so, it is sufficient to prove that there is an omniorientation on $M$ which is preserved by the action of $W(G)$. This is true if for every characteristic submanifold $M_i$ and $g \in N_G T$ such that $g M_i = M_i$, $g$ preserves the orientation of $M_i$. Since $G$ is connected, $g$ preserves the orientation of $M$ and acts trivially on $H^2(M)$.

Because each vertex of the orbit polytope $P$ of $M$ is the intersection of exactly $n$ facets of $P$, every fixed point of the $T$-action on $M$ is the transverse intersection of exactly $n$ characteristic submanifolds. Thus, the Poincaré dual $PD(M_i) \in H^2(M)$ of $M_i$ is non-zero because $M_i \cap M^T \neq \emptyset$. Therefore $g$ preserves the orientation of $M_i$ since otherwise

\begin{equation}
PD(M_i) = \frac{1}{2} \left( PD(M_i) + PD(M_i) \right) = \frac{1}{2} \left( PD(M_i) + g^* PD(M_i) \right) \quad (g \text{ acts trivially on } H^2(M))
\end{equation}

\begin{equation}
= \frac{1}{2} \left( PD(M_i) - PD(M_i) \right) \quad (g \text{ reverses the orientation of } M_i)
\end{equation}

= 0.
Lemma 2.10. Let $SU$ be some groups having a Weyl group isomorphic to some symmetric group are isomorphic to some $SU(l_i+1)$. Therefore all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i+1)$.

From this the statement follows.

Remark 2.9. In [15] Masuda and Panov show that the cohomology with coefficients in $Z$ of a torus manifold $M$ is generated by its degree-two part if and only if the torus action on $M$ is locally standard and the orbit space $M/T$ is a homology polytope. That means that all faces of $M/T$ are acyclic and all intersections of facets of $M/T$ are connected. In particular, each $T$-fixed point is the transverse intersection of $n$ characteristic submanifolds. Therefore the above lemma also holds in this case.

For a characteristic submanifold $M_i$ of $M$, let $\lambda(M_i)$ denote the one-dimensional subtorus of $T$ which fixes $M_i$ pointwise. The normalizer $N_G T$ of $T$ in $G$ acts by conjugation on the set of one-dimensional subtori of $T$. The following lemma shows that $\lambda : S \rightarrow \{\text{one-dimensional subtori of } T\}$ is $N_G T$-equivariant.

Lemma 2.10. Let $M$ be a torus manifold with $G$-action, $g \in N_G T$ and $M_i \subset M$ be a characteristic submanifold. Then we have:

1. $\lambda(gM_i) = g\lambda(M_i)g^{-1}$.
2. If $gM_i = M_i$, then $g$ acts orientation preserving on $M_i$ if and only if $\lambda(M_i) \rightarrow \lambda(M_i) \quad t \rightarrow gtg^{-1}$ is orientation preserving.

Proof. First we prove (1). Let $x \in M_i$ be a generic point. Then the identity component $T^0_x$ of the stabilizer of $x$ in $T$ is given by $T^0_x = \lambda(M_i)$. Therefore we have

$$\lambda(gM_i) = T^0_{gx} = gT^0_xg^{-1} = g\lambda(M_i)g^{-1}.$$ 

Now we shall prove (2). An orientation of $M_i$ induces a complex structure on $N(M_i, M)$. We fix an isomorphism $\rho : \lambda(M_i) \rightarrow S^1$ such that the action of $t \in \lambda(M_i)$ on $N(M_i, M)$ is given by multiplication with $\rho(t)^m$, $m > 0$. The differential $Dg : N(M_i, M) \rightarrow N(M_i, M)$ is orientation preserving if and only if it is complex linear. Otherwise it is complex anti-linear. Therefore for $v \in N(M_i, M)$ we have

$$\rho(gtg^{-1})^m v = (Dg)(Dt)(Dg)^{-1}v = (Dg)\rho(t)^m(Dg)^{-1}v = \rho(t^{\pm m})(Dg)(Dg)^{-1}v = \rho(t^{\pm 1})^m v.$$ 

This equation implies that $\rho(gtg^{-1}t^{\pm 1}) \in \mathbb{Z}/m\mathbb{Z}$. Because $\lambda(M_i)$ is connected and $\mathbb{Z}/m\mathbb{Z}$ is discrete, $gtg^{-1} = t^{\pm 1}$ follows, where the plus sign arises if and only if $g$ acts orientation preserving on $M_i$. □

3. G-action on $M$

In this section we consider torus manifolds with $G$-action such that $\tilde{G}$ has only one elementary factor $G_1$, i.e. $\tilde{G} = G_1 \times T^0$. There are two cases:

1. There is a $T$-fixed point which is not fixed by $G_1$.
2. There is a $G$-fixed point.

We first discuss the case where there is a $T$-fixed point which is not fixed by $G_1$. 
Lemma 3.1. Let \( \tilde{G} = G_1 \times T^{l_0} \) with \( G_1 \) elementary, rank \( G_1 = l_1 \) and \( M \) a torus manifold with \( G \)-action of dimension \( 2n = 2(l_0 + l_1) \). If there is an \( x \in M^T \) which is not fixed by the action of \( G_1 \), then

1. \( G_1 = SU(l_1 + 1) \) or \( G_1 = Spin(2l_1 + 1) \), and the stabilizer of \( x \) in \( G_1 \) is conjugated to \( S(U(l_1) \times U(1)) \) or \( Spin(2l_1) \), respectively.

2. The \( G_1 \)-orbit of \( x \) equals the component of \( M^{T_0} \) which contains \( x \).

Moreover, if \( G_1 = SU(4) \), one has \#\( \mathcal{F}_1 \) = 4.

Proof. The \( G_1 \)-orbit of \( x \) is contained in the component \( N \) of \( M^{T_0} \) containing \( x \). Therefore we have

\[
\text{codim} G_{1x} = \dim G_1 / G_{1x} = \dim G_1 x \leq \dim N \leq 2l_1.
\]

Furthermore the stabilizer \( G_{1x} \) of \( x \) has maximal rank \( l_1 \). In particular, its identity component \( G_{1x}^0 \) is a closed connected maximal rank subgroup.

Next we use the theory of Lie groups to determine the isomorphism types of \( G_1 \) and \( G_{1x} \). At first we consider the case \( G_1 \neq Spin(4) \). From the classification of closed connected maximal rank subgroups of a compact Lie group given in [2] p. 219 we get the following connected maximal rank subgroups \( H \) of maximal dimension:

<table>
<thead>
<tr>
<th>( G_1 )</th>
<th>( H )</th>
<th>( \text{codim} H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SU(2) = Spin(3) = Sp(1) )</td>
<td>( S(U(1) \times U(1)) )</td>
<td>2</td>
</tr>
<tr>
<td>( Spin(5) = Sp(2) )</td>
<td>( Spin(4) )</td>
<td>4</td>
</tr>
<tr>
<td>( SU(4) = Spin(6) )</td>
<td>( S(U(3) \times U(1)) )</td>
<td>6</td>
</tr>
<tr>
<td>( SU(l_1 + 1), l_1 \neq 1, 3 )</td>
<td>( S(U(l_1) \times U(1)) )</td>
<td>2l_1</td>
</tr>
<tr>
<td>( Spin(2l_1 + 1), l_1 &gt; 2 )</td>
<td>( Spin(2l_1) )</td>
<td>2l_1</td>
</tr>
<tr>
<td>( Spin(2l_1), l_1 &gt; 3 )</td>
<td>( Spin(2l_1 - 2) \times Spin(2) )</td>
<td>4l_1 - 4</td>
</tr>
<tr>
<td>( Sp(l_1), l_1 &gt; 2 )</td>
<td>( Sp(l_1 - 1) \times Sp(1) )</td>
<td>4l_1 - 4</td>
</tr>
</tbody>
</table>

Because \( H \) is unique up to conjugation and

\[
\text{codim} H \leq \text{codim} G_{1x}^0 = \text{codim} G_{1x} \leq 2l_1,
\]

we see \( G_1 = SU(l_1 + 1) \) or \( G_1 = Spin(2l_1 + 1) \). Moreover, \( G_{1x} \) is conjugated to a subgroup of \( G_1 \) which contains \( S(U(l_1) \times U(1)) \) or \( Spin(2l_1) \), respectively.

If \( l_1 > 1 \), then \( S(U(l_1) \times U(1)) \) is a maximal subgroup of \( SU(l_1 + 1) \) by Lemma A.1. Therefore, if \( G_1 = SU(l_1 + 1) \) and \( l_1 > 1 \), then \( G_{1x} \) is conjugated to \( S(U(l_1) \times U(1)) \). Because \( \text{codim} S(U(l_1) \times U(1)) = 2l_1 \geq \dim N \geq \text{codim} G_{1x} \), we have \( G_{1x} = N \) in this case.

If \( G_1 = Spin(2l_1 + 1) \), \( l_1 \geq 1 \), then by Lemma A.3 there are two proper subgroups of \( G_1 \) which contain \( Spin(2l_1) \), \( Spin(2l_1) \) and its normalizer \( H_0 \). Because of dimension reasons we have \( N = G_{1x} \). Because \( \text{Spin}(2l_1 + 1) / H_0 \) is not orientable and \( M^{T_0} \) is orientable, \( G_{1x} = Spin(2l_1) \) follows. The case \( G_1 = SU(2) \) is included in the discussion in this paragraph because \( SU(2) = Spin(3) \).

Now we prove the last statement of the lemma. If \( G_1 = SU(4) \), then \( G_{1x} \) is \( G_1 \)-equivariantly diffeomorphic to \( \mathbb{C}P^3 \) by the above discussion. Because \( \mathbb{C}P^3 \) has four characteristic submanifolds with pairwise non-trivial intersections, by Lemmas B.2 and B.3 there are four characteristic submanifolds \( M_1, \ldots, M_4 \) which intersect...
transversely with $G_1x = N$. Because $G_1x$ is a component of $M^{T^0}$ we have by Lemma B.1 that $\lambda(M_i) \not\in T^0$. Therefore $\lambda(M_i)$ is not fixed pointwise by the action of $W(G_1)$ on $T$. Here $W(G_1)$ acts on $T$ by conjugation. Now it follows with Lemma B.10 that $M_1, \ldots, M_4$ belong to $F_1$.

Now we turn to the case $G_1 = \Spin(4) = SU(2) \times SU(2)$.

Then there are the following proper closed connected maximal rank subgroups $H$ of $G_1$ of codimension at most 4:

$SU(2) \times S(U(1) \times U(1)),\ S(U(1) \times U(1)) \times SU(2),\ S(U(1) \times U(1)) \times S(U(1) \times U(1))$.

The last has codimension four in $G_1$. The others have codimension two in $G_1$.

At first assume that $G_1x$ has dimension four. Then we have

$G_{1x}^0 = S(U(1) \times U(1)) \times S(U(1) \times U(1))$.

There are five proper subgroups of $\Spin(4)$ which contain $S(U(1) \times U(1)) \times S(U(1) \times U(1))$ as a maximal connected subgroup, namely:

$H_1' = S(U(1) \times U(1)) \times S(U(1) \times U(1))$,
$H_2' = N_{SU(2)} S(U(1) \times U(1)) \times S(U(1) \times U(1))$,
$H_3' = S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1))$,
$H_4' = N_{SU(2)} S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1))$,
$H_5' = \{(g_1, g_2) \in N_{SU(2)} S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1));
\quad g_1 \in S(U(1) \times U(1)) \Leftrightarrow g_2 \in S(U(1) \times U(1))\}$.

Therefore $G_1x$ is $G_1$-equivariantly diffeomorphic to one of the following spaces:

$\Spin(4)/H_1' = S^2 \times S^2$,
$\Spin(4)/H_2' = S^2 \times Z_2^2 S^2$ = orientable double cover of $\mathbb{R}P^2 \times \mathbb{R}P^2$,
$\Spin(4)/H_3' = \mathbb{R}P^2 \times S^2$,
$\Spin(4)/H_4' = S^2 \times \mathbb{R}P^2$,
$\Spin(4)/H_5' = \mathbb{R}P^2 \times \mathbb{R}P^2$.

Since $G_1x = M^{T^0}$ is orientable, the latter three do not occur.

For $N = G_1x = S^2 \times S^2, S^2 \times Z_2 S^2$, let $N^{(1)}$ be the union of the $T$-orbits in $N$ of dimension less than or equal to one. Then $W(G_1) = Z_2 \times Z_2$ acts on the orbit space $N^{(1)}/T$. This space is given by one of the following graphs:

\begin{align*}
(S^2 \times S^2)^{(1)}/T & \quad (S^2 \times Z_2 S^2)^{(1)}/T
\end{align*}

Here the edges correspond to orbits of dimension one and the vertices to the fixed points. The arrows indicate the action of the generators $w_1, w_2 \in W(G_1)$ on this space. Let $M_1, M_2$ be the two characteristic submanifolds of $M$ which intersect
transversely with $N$ in $x$. Because $N$ is a component of $M T^{l_0}$, $\lambda(M_i)$, $i = 1, 2$, is not a subgroup of $T^{l_0}$ by Lemma 3.1. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 2.10, this fact implies $M_1, M_2 \in \mathfrak{F}_1$. Therefore there is a $w \in W(G_1)$ with $w(M_1) = M_2$. But from the pictures above we see that $M_1$ and $M_2$ are not in the same $W(G_1)$-orbits. Therefore the case $\dim G_1 x = 4$ does not occur.

Now assume that $G_1 x$ has dimension two. Then we may assume without loss of generality that $G_1^0 = SU(2) \times S(U(1) \times U(1))$. Therefore $G_1 x \subset M^{SU(2) \times SU(1)}$. Because $G_1 x \subset M T^{l_0}$, $G_1 x$ is a component of $M^{SU(1) \times U(1)} \times T^{l_0}$ in this case. Therefore, by Lemmas 3.1 and 3.2, there are characteristic submanifolds $M_2, \ldots, M_{l_0+2}$ of $M$ such that $G_1 x$ is a component of $\bigcap_{i=2}^{l_0+2} M_i$. Furthermore, we may assume that $\lambda(M_2) \not\subset T^{l_0}$. Therefore, by Lemma 2.10, we have $M_2 \in \mathfrak{F}_1$.

But there is also a characteristic submanifold $M_1$ of $M$ which intersects $G_1 x$ transversely in $x$. With the Lemmas 3.1 and 2.10 we see $M_1 \in \mathfrak{F}_1$.

Therefore there is a $w \in W(G_1)$ with $w(M_2) = M_1$. But this is impossible because $M_2 \supset G_1 x \not\subset M_1$.

Therefore $G_1 \neq \text{Spin}(4)$ and the lemma is proved. □

Remark 3.2. If, in the situation of Lemma 3.1, $T \cap G_1$ is the standard maximal torus of $G_1$, then it follows by Proposition 2 of [8, p. 325] that $G_1 x$ is conjugated to the groups given in Lemma 3.1 by an element of the normalizer of the maximal torus.

Lemma 3.3. In the situation of the previous lemma $x$ is contained in the intersection of exactly $l_1$ characteristic submanifolds belonging to $\mathfrak{F}_1$.

Proof. Because $N = G_1 x$ has dimension $2l_1$, $x$ is contained in exactly $l_1$ characteristic submanifolds of $N$. By Lemmas 3.2 and 3.3 we know that they are components of intersections of characteristic submanifolds $M_1, \ldots, M_{l_1}$ of $M$ with $N$.

Because $G_1 x$ is a component of $M T^{l_0}$, $\lambda(M_i)$ is not a subgroup of $T^{l_0}$ for $i = 1, \ldots, l_1$ by Lemmas 3.1 and 3.2. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 2.10, this implies that $M_i$ belongs to $\mathfrak{F}_1$.

By Lemmas 3.3 and 3.1 $G_1 x$ is the intersection of $l_0$ characteristic submanifolds $M_{l_1+1}, \ldots, M_n$ of $M$. We show that these manifolds do not belong to $\mathfrak{F}_1$. Assume that there is an $i \geq l_1 + 1$ such that $M_i$ belongs to $\mathfrak{F}_1$. Because $W(G_1)$ acts transitively on $\mathfrak{F}_1$, there is a $w \in W(G_1)$ with $w(M_i) = M_j$, $j \leq l_1$. But this is impossible because $M_i \supset G_1 x \not\subset M_j$. □

Now we turn to the case where there is a $T$-fixed point which is fixed by $G_1$.

Lemma 3.4. Let $\tilde{G} = G_1 \times T^{l_0}$ with $G_1$ elementary, rank $G_1 = l_1$ and $M$ a torus manifold with $G$-action of dimension $2n = 2(l_0 + l_1)$. If there is a $T$-fixed point $x \in M^T$ which is fixed by $G_1$, then $G_1 = SU(l_1 + 1)$ or $G_1 = \text{Spin}(2l_1)$.

Moreover, if $G_1 \neq \text{Spin}(8)$ one has

(3.1) $T_x M = V_1 \oplus V_2 \otimes_C W_1$ if $G_1 = SU(l_1 + 1)$ and $\# \mathfrak{F}_1 = 4$ in the case $l_1 = 3$,

(3.2) $T_x M = V_3 \oplus W_2$ if $G_1 = \text{Spin}(2l_1)$ and $\# \mathfrak{F}_1 = 3$ in the case $l_1 = 3$,

where $W_1$ is the standard complex representation of $SU(l_1+1)$ or its dual, $W_2$ is the standard real representation of $SO(2l_1)$ and the $V_i$ are complex $T^{l_0}$-representations.
In the case $G_1 = \text{Spin}(8)$, one may change the action of $G_1$ on $M$ by an automorphism of $G_1$, which is independent of $x$, to reach the situation described in (3.2).

Furthermore, we have $x \in \bigcap_{M_i \in \mathfrak{S}_1} M_i$. If $l_1 = 1$, then we have $\#\mathfrak{S}_1 = 2$.

**Proof.** Let $M_1, \ldots, M_n$ be the characteristic submanifolds of $M$ which intersect in $x$. Then the weight spaces of the $G$-representation $T_x M$ are given by

$$N_x(M_1, M), \ldots, N_x(M_n, M).$$

For $g \in N_G T$ we have $M_i = gM_i$ if and only if $N_x(M_i, M) = gN_x(M_j, M)$. Because $G_1$ acts non-trivially on $T_x M$, there is at least one $M_i$, $i \in \{1, \ldots, n\}$, such that $M_i \in \mathfrak{S}_1$.

In the following a weight space of $T_x M$ together with a choice of an orientation for this weight space is called an oriented weight space of $T_x M$. The action of $G_1$ on $T_x M$ induces an action of $W(G_1)$ on the set of oriented weight spaces of $T_x M$.

Because $W(G_1)$ acts transitively on $\mathfrak{S}_1$ and $x$ is a $G$-fixed point, we have

$$\frac{1}{2} \#\{\text{oriented weight spaces of } T_x M \text{ which are not fixed by } W(G_1)\} = \#\mathfrak{S}_1$$

and $x \in \bigcap_{M_i \in \mathfrak{S}_1} M_i$.

For the $G$-representation $T_x M$ we have

$$T_x M = N_x(M_{T^{l_0}}, M) \oplus T_x M_{T^{l_0}}.$$

If $l_0 = 0$, then we have $N_x(M_{T^{l_0}}, M) = \{0\}$. Otherwise the action of $T^{l_0}$ induces a complex structure on $N_x(M_{T^{l_0}}, M)$. By [4, p. 68] and [4, p. 82], we have

$$N_x(M_{T^{l_0}}, M) = \bigoplus_i V_i \otimes \mathbb{C} W_i,$$

where the $V_i$ are one-dimensional complex $T^{l_0}$-representations and the $W_i$ are irreducible complex $G_1$-representations. Since $T^{l_0}$ acts almost effectively on $M$, there are at least $n - l_1$ summands in this decomposition. Therefore we get

$$\dim \mathbb{C} W_i = \dim \mathbb{C} N_x(M_{T^{l_0}}, M) - \sum_{j \neq i} \dim \mathbb{C} V_j \otimes \mathbb{C} W_j \leq n - (n - l_1 - 1) = l_1 + 1.$$

Furthermore,

$$\dim \mathbb{R} T_x M_{T^{l_0}} \leq 2(n - l_0) = 2l_1.$$

If there is a $W_{i_0}$ with $\dim \mathbb{C} W_{i_0} = l_1 + 1$, then from equation (3.5) we get, for all other $W_i$,

$$\dim \mathbb{C} W_i = \dim \mathbb{C} N_x(M_{T^{l_0}}, M) - \dim \mathbb{C} V_{i_0} \otimes \mathbb{C} W_{i_0} - \sum_{j \neq i, i_0} \dim \mathbb{C} V_j \otimes \mathbb{C} W_j \leq 1.$$

So they are one-dimensional. Therefore they are trivial. Furthermore, we have

$$\dim \mathbb{C} N_x(M_{T^{l_0}}, M) = \sum_i \dim \mathbb{C} V_i \otimes \mathbb{C} W_i \geq n$$

because there are at least $n - l_1$ summands in the decomposition (3.5). Therefore $T_x M_{T^{l_0}}$ is zero-dimensional in this case.
If \( \dim_R T_x M^{T^{i_0}} = 2l_1 \), then we have
\[
\dim_C W_i = \dim_C N_x(M^{T^{i_0}}, M) - \sum_{j \neq i} \dim_C V_j \otimes_C W_j \leq 1.
\]
Therefore all \( W_i \) are one-dimensional, so they are trivial in this case.

There are the following lower bounds \( d_R, d_C \) for the dimension of real and complex non-trivial irreducible representations of \( G_1 \) \[19\] pp. 53-54:

<table>
<thead>
<tr>
<th>( G_1 )</th>
<th>( d_R )</th>
<th>( d_C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SU(2) = \text{Spin}(3) = Sp(1) )</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( \text{Spin}(4) )</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( \text{Spin}(5) = Sp(2) )</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>( SU(4) = \text{Spin}(6) )</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>( SU(l_1 + 1), l_1 \neq 1, 3 )</td>
<td>( 2l_1 + 2 )</td>
<td>( l_1 + 1 )</td>
</tr>
<tr>
<td>( \text{Spin}(2l_1 + 1), l_1 &gt; 2 )</td>
<td>( 2l_1 + 1 )</td>
<td>( 2l_1 + 1 )</td>
</tr>
<tr>
<td>( \text{Sp}(l_1), l_1 &gt; 2 )</td>
<td>( 2l_1 )</td>
<td>( 2l_1 )</td>
</tr>
</tbody>
</table>

In \[19\] pp. 53-54] the dominant weights of the \( G_1 \)-representations realising these bounds are also given. They are important in the discussion below.

Because \( G_1 \) acts non-trivially on \( T_x M \), one of the \( W_i \)'s or \( T_x M^{T^{i_0}} \) is a non-trivial \( G_1 \)-representation. Therefore we have \( d_R \leq 2l_1 \) or \( d_C \leq l_1 + 1 \) by \( \text{(3.6)} \) and \( \text{(3.7)} \). Therefore \( G_1 \neq Sp(l_1), l_1 > 1 \), and \( G_1 \neq \text{Spin}(2l_1 + 1), l_1 > 1 \).

If \( G_1 = \text{Spin}(2l_1), l_1 > 3 \), then all \( W_i \) are trivial because
\[
\dim_C W_i \leq l_1 + 1 < 2l_1 = d_C.
\]
Moreover, \( T_x M^{T^{i_0}} \) has dimension \( 2l_1 \). Therefore it is the standard real \( SO(2l_1) \)-representation if \( l_1 > 4 \). If \( l_1 = 4 \), then there are three eight-dimensional real representations of \( \text{Spin}(8) \), namely the standard real \( SO(8) \)-representation and the two half spinor representations. They have three different kernels. Notice that the kernel of the \( G_1 \)-representation \( T_x M^{T^{i_0}} \) is equal to the kernel of the \( G_1 \)-action on \( M \). Therefore, if one of them is isomorphic to \( T_x M^{T^{i_0}} \), then it is isomorphic to \( T_y M^{T^{i_0}} \) for all \( y \in M^T \). So we may -- after changing the action of \( \text{Spin}(8) \) on \( M \) by an automorphism -- assume that \( T_x M^{T^{i_0}} \) is the standard real \( SO(8) \)-representation.

If \( G_1 = SU(l_1 + 1), l_1 \neq 1, 3 \), then only one \( W_i \) is non-trivial and \( T_x M^{T^{i_0}} \) has dimension zero. The non-trivial \( W_i \) is the standard representation of \( SU(l_1 + 1) \) or its dual depending on the complex structure of \( N_x(M^{T^{i_0}}, M) \).

If \( G_1 = SU(4) \), then there is one real representation of dimension 6 and two complex representations of dimension 4. If the first representation occurs in the decomposition of \( T_x M \), then, by \( \text{(5.3)} \), we have \( \#\mathfrak{F}_1 = 3 \). If one of the others occurs, then \( \#\mathfrak{F}_1 = 4 \).

If \( G_1 = SU(2) \), then there is one non-trivial \( W_i \) of dimension 2. Therefore, by \( \text{(3.3)} \), one has \( \#\mathfrak{F}_1 = 2 \).

If \( G_1 = \text{Spin}(4) \), then \( T_x M \) is an almost faithful representation. Because all almost faithful complex representations of \( \text{Spin}(4) \) have at least dimension four, there is no \( W_i \) of dimension three.

If there is one \( W_{i_0} \) of dimension two, then we see as in \( \text{(3.8)} \) that all other \( W_i \) and \( T_x M^{T^{i_0}} \) have dimension less than or equal to two. Because there is no non-trivial two-dimensional real \( \text{Spin}(4) \)-representation, there is another \( W_i \) of dimension two.
Therefore there are eight oriented weight spaces of $T_x M$ which are not fixed by the action on $W(G_1)$. But this contradicts \( \text{(3)} \) because $\# \mathfrak{h}_1 = 2$.

Therefore all $W_i$ are one-dimensional. Hence, they are trivial. $T_x M^{T_{\alpha}}$ has to be the standard four-dimensional real representation of $\text{Spin}(4)$. \( \square \)

With the Lemmas \( \text{3.1} \) and \( \text{3.4} \) we see that there is no elementary factor of $\tilde{G}$, which is isomorphic to $\text{Sp}(l)_{1}$ for $l_1 > 2$.

Now let $G_1 = \text{Spin}(2l)$. If $l = 3$, we assume $\# \mathfrak{h}_1 = 3$. Then, by looking at the $G_1$-representation $T_x M$, one sees with Lemma \( \text{3.4} \) that the $G_1$-action factors through $\text{SO}(2l)$.

Now let $G_1 = \text{Spin}(2l + 1), l > 1$. Then, by Lemma \( \text{3.1} \) we have $G_{1x} \cong \text{Spin}(2l)$. Because the $G_{1x}$-action on $N_x(G_{1x}, M)$ is trivial by Lemma \( \text{3.4} \) the $G_1$-action factors through $\text{SO}(2l + 1)$.

In the case $G_1 = \text{Spin}(3)$ and $\# \mathfrak{h}_1 = 1$ we have $G_{1x} = S^2$. The characteristic submanifold $M_1 \in \mathfrak{f}_1$ intersects $G_{1x}$ transversely in $x$. Because $\# \mathfrak{h}_1 = 1$, $\lambda(M_1)$ is invariant under the action of $W(G_1)$ on the maximal torus of $G$. Because, by Lemma \( \text{2.10} \) the non-trivial element of $W(G_1)$ reverses the orientation of $\lambda(M_1)$, it is a maximal torus of $G_1$. Therefore the center of $G_1$ acts trivially on $M$. Hence, the $G_1$-action on $M$ factors through $\text{SO}(3)$.

If in the case $G_1 = \text{Spin}(3)$ and $\# \mathfrak{h}_1 = 2$ the principal orbit type of the $G_1$-action is given by Spin(3)/Spin(2), then the $G_1$-action factors through $\text{SO}(3)$.

Therefore in the following we may replace an elementary factor $G_i$ of $\tilde{G}$ isomorphic to Spin($l$), which satisfies the above conditions, by $\text{SO}(l)$.

\textbf{Convention 3.5.} If we say that an elementary factor $G_i$ is isomorphic to $\text{SU}(2)$ or $\text{SU}(4)$, then we mean that $\# \mathfrak{h}_i = 2$ or $\# \mathfrak{h}_i = 4$, respectively. Conversely, if we say that $G_i$ is isomorphic to $\text{SO}(3)$ we mean that $\# \mathfrak{h}_i = 1$ or $\# \mathfrak{h}_i = 2$ and the $\text{SO}(3)$-action has principal orbit type $SO(3)/SO(2)$. If we say $G_i = \text{SO}(6)$, then we mean $\# \mathfrak{h}_i = 3$.

\textbf{Corollary 3.6.} Assume that $G$ is elementary. Then $M$ is equivariantly diffeomorphic to $\mathbb{C}P^{l_1}$ or $M = S^{2l_1}$ if $\tilde{G} = \text{SU}(l_1 + 1)$ or $\tilde{G} = \text{SO}(2l_1 + 1), \text{SO}(2l_1)$, respectively.

\textbf{Proof.} If $G$ is elementary, we may assume that $G = \tilde{G} = \text{SO}(2l_1), \text{SO}(2l_1 + 1), \text{SU}(l_1 + 1)$ and $\dim M = 2l_1$.

If $G = \text{SO}(2l_1)$, then, by Lemmas \( \text{3.1} \) and \( \text{3.4} \) the principal orbit type of the $\text{SO}(2l_1)$-action is given by $\text{SO}(2l_1)/\text{SO}(2l_1 - 1)$, which has codimension one in $M$.

The group $SO(2l_1 - 1) \times O(1)$ is the only proper subgroup of $SO(2l_1)$ which contains $SO(2l_1 - 1)$ properly. Because $SO(2l_1)/SO(2l_1 - 1) \times O(1) = \mathbb{R}P^{2l_1 - 1}$ is orientable, all orbits of the $\text{SO}(2l_1)$-action are of types $SO(2l_1)/SO(2l_1 - 1)$ or $SO(2l_1)/SO(2l_1)$ by [3, p. 185].

By [3] pp. 206-207, we have

$$M = D_1^{2l_1} \cup_\phi D_2^{2l_1},$$

where $SO(2l_1)$ acts on the disks $D_i^{2l_1}$ in the usual way and

$$\phi : S^{2l_1 - 1} = SO(2l_1)/SO(2l_1 - 1) \to S^{2l_1 - 1} = SO(2l_1)/SO(2l_1 - 1)$$

is given by $gSO(2l_1 - 1) \mapsto gnSO(2l_1 - 1)$, where $n \in N_{SO(2l_1)}SO(2l_1 - 1) = S(O(2l_1 - 1) \times O(1))$.

Therefore $\phi = \pm \text{Id}_{S^{2l_1 - 1}}$ and $M = S^{2l_1}$.
If $G = SO(2l_1 + 1)$, then

$$M = SO(2l_1 + 1)/SO(2l_1) = S^{2l_1}$$

follows directly from Lemmas 3.1 and 3.4.

Now assume $G = SU(l_1 + 1)$. Because $\dim M = 2l_1$, the intersection of $l_1 + 1$ pairwise distinct characteristic submanifolds of $M$ is empty. By Lemma 3.4 no $T$-fixed point is fixed by $G$. Therefore from Lemma 3.1 we get

$$M = SU(l_1 + 1)/S(U(l_1) \times U(1)) = \mathbb{C}P^{l_1}.$$

□

Remark 3.7. Another proof of this statement follows from the classification given in section 8.

4. blowing up

In this section we describe blow ups of torus manifolds with $G$-action. They are used in the following sections to construct from a torus manifold $M$ with $G$-action another torus manifold $\tilde{M}$ with $G$-action, such that an elementary factor of the covering group $\tilde{G}$ of $G$ has no fixed point in $\tilde{M}$.

References for this construction are [7, pp. 602-611] and [16, pp. 269-270].

As before we write $\tilde{G} = \prod_{i=1}^k G_i \times T^o$ with $G_i$ elementary and $T^o$ a torus.

We will see in sections 5 and 7 that there are the following two cases:

(1) A component $N$ of $M^{G_1}$ has odd codimension in $M$.

(2) A component $N$ of $M^{G_1}$ has even codimension in $M$, and there is a $g \in Z(\tilde{G})$ such that $g$ acts trivially on $N$ and $g^2$ acts as $-\text{Id}$ on $N(N,M)$.

In the second case the action of $g$ on $N(N,M)$ induces a $G$-invariant complex structure. We equip $N(N,M)$ with this structure. Let $E = N(N,M) \oplus \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ in the first case and $\mathbb{K} = \mathbb{C}$ in the second case.

In the following we call case (1) the real case and case (2) the complex case.

Lemma 4.1. The projectivication $P_\mathbb{K}(E)$ is orientable.

Proof. Because $M$ is orientable the total space of the normal bundle of $N$ in $M$ is orientable. Therefore

$$E = N(N,M) \oplus \mathbb{K} = N(N,M) \times \mathbb{K}$$

and the associated sphere bundle $S(E)$ are orientable.

Let $Z_\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$ if $\mathbb{K} = \mathbb{R}$ and $Z_\mathbb{K} = S^1$ if $\mathbb{K} = \mathbb{C}$. Then $Z_\mathbb{K}$ acts on $E$ and $S(E)$ by multiplication on the fibers. Now $P_\mathbb{K}(E)$ is given by $S(E)/Z_\mathbb{K}$. If $\mathbb{K} = \mathbb{C}$, then $Z_\mathbb{K}$ is connected. Therefore it acts orientation preserving on $S(E)$.

If $\mathbb{K} = \mathbb{R}$, then $\dim E$ is even. Therefore the restriction of the $Z_\mathbb{K}$-action to a fiber of $E$ is orientation preserving. Hence, it preserves the orientation of $S(E)$.

Because the action of $Z_\mathbb{K}$ is orientation preserving on $S(E)$, $P_\mathbb{K}(E)$ is orientable.

□

Choose a $G$-invariant Riemannian metric on $N(N,M)$ and a $G$-equivariant closed tubular neighborhood $B$ around $N$. Then one may identify

$$B = \{z_0 \in N(N,M); |z_0| \leq 1\} = \{(z_0 : 1) \in P_\mathbb{K}(E); |z_0| \leq 1\}.$$

By gluing the complements of the interior of $B$ in $M$ and $P_\mathbb{K}(E)$ along the boundary of $B$, we get a new torus manifold with $G$-action $\tilde{M}$, the blow up of
extends to a differentiable map along $M$.

$M$ is oriented in such a way that the induced orientation on $M - \tilde{B}$ coincides with the orientation induced from $M$. This forces the inclusion of $P_K(E) - \tilde{B}$ to be orientation reversing. Because $G_1$ is elementary there is no one-dimensional $G_1$-invariant subbundle of $N(N, M)$. Therefore we have $\#\pi_0(M^{G_1}) = \#\pi_0(M^{G_1}) - 1$.

So by iterating this process over all components of $M^{G_1}$ one ends up at a torus manifold $\tilde{M}'$ with $G$-action without $G_1$-fixed points. In the following we will call $\tilde{M}'$ the blow up of $M$ along $M^{G_1}$.

**Lemma 4.2.** There is a $G$-equivariant map $F : \tilde{M} \to M$ which maps the exceptional submanifold $M_0 = P_K(N(N, M) \oplus \{0\})$ to $N$ and is the identity on $M - B$. Moreover, $F$ restricts to a diffeomorphism $\tilde{M} - M_0 \to M - N$. Its restriction to $M_0$ is the bundle projection $P_K(N(N, M) \oplus \{0\}) \to N$.

**Proof.** The $G$-equivariant map

$$f : P_K(E) - \tilde{B} \to B \quad (z_0 : z_1) \mapsto (z_0 \bar{z}_1 : |z_0|^2) \quad (z_0 \in N(N, M), z_1 \in \mathbb{K})$$

is the identity on $\partial B$. Therefore it may be extended to a continuous map $h : \tilde{M} \to M$ which is the identity outside of $P_K(E) - \tilde{B}$.

Because $f|_{P_K(E) - \tilde{B} - M_0} : P_K(E) - \tilde{B} - M_0 \to B - N$ is a diffeomorphism there is a $G$-equivariant diffeomorphism $F' : \tilde{M} - M_0 \to M - N$ which is the identity outside $P_K(E) - \tilde{B} - M_0$ and coincides with $f$ near $M_0$ by [10, pp. 24-25]. Therefore $F'$ extends to a differentiable map $F : \tilde{M} \to M$ such that $F|_{M_0} = f|_{M_0}$ is the bundle projection. \hfill \Box

**Lemma 4.3.** Let $H$ be a closed subgroup of $G$. Then there is a bijection

$$\{\text{components of } M^H \nsubseteq N\} \to \{\text{components of } \tilde{M}^H \nsubseteq M_0\}$$

such that

$$N' \mapsto \tilde{N}' = \left(P_K(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B}\right) \cup_{\partial B \cap N'} \left(N' - \tilde{B}\right)$$

and its inverse is given by

$$F(N'') \leftrightarrow N'',$

where $N'$ is a component of $M^H$ and $N''$ is one of $\tilde{M}^H$. Here $F(N'')$ is the image of $N''$ under the map $F$ defined in Lemma 4.2. For a component $N'$ of $M^H$, we call $\tilde{N}'$ the proper transform of $N'$.

**Proof.** At first we calculate the fixed point set of the $H$-action on $\tilde{M}$:

$$M^H = \left(P_K(E) - \tilde{B}\right) \cup_{\partial B} \left(M - \tilde{B}\right)^H = \left(P_K(E) - \tilde{B}\right)^H \cup_{\partial B^H} \left(M - \tilde{B}\right)^H.$$

Because $H$ is compact, there are pairwise distinct $i$-dimensional non-trivial irreducible $H$-representations $V_{ij}$ and $H$-vector bundles $E_{ij}$ over $N^H$ such that

$$N(N, M)|_{N^H} = N(N, M)|_{N^H}^H \oplus \bigoplus_{i,j} E_{ij},$$
and the $H$-representation on each fiber of $E_{i,j}$ is isomorphic to $\mathbb{K}^{d_{i,j}} \otimes \mathbb{K} V_{i,j}$, where $\mathbb{K}^{d_{i,j}}$ denotes the trivial $H$-representation of dimension $d_{i,j}$.

Now the $H$-fixed points in $P_K(E)$ are given by

$$P_K(E)^H = P_K(N(N, M) \oplus \mathbb{K})^H_{N,M} = P_K(N(N, M) \oplus \mathbb{K}) \prod_j P_K(E_{i,j} \oplus \{0\}).$$

Because $N(N, M)^H_{N,M} = N(N^H, M^H)$ we get

$$\tilde{M}^H = \left(\left(P_K(N(N^H, M^H) \oplus \mathbb{K}) - \tilde{B}^H\right) \cup_{\partial B^H} \left(\tilde{M} - \tilde{B}\right)^H\right) \prod_j P_K(E_{i,j} \oplus \{0\})$$

$$= \prod_{N' \subset M^H} \tilde{N}' \prod_j P_K(E_{i,j} \oplus \{0\}),$$

where $N'$ runs through the connected components of $M^H$ which are not contained in $N$. Thus the statement follows.

By replacing $H$ in Lemma 4.3 by a one-dimensional subtorus of $T$, we get:

**Corollary 4.4.** There is a bijection between the characteristic submanifolds of $M$ and the characteristic submanifolds of $\tilde{M}$, which are not contained in $M_0$.

**Proof.** The only thing that is to be proved here is that for a characteristic submanifold $M_i$ of $M$, $\tilde{M}_i^T$ is non-empty. If $(M_i - N)^T \neq \emptyset$, then this is clear.

If $p \in (M_i \cap N)^T$, then $P_K(N(M_i \cap N, M_i) \oplus \{0\})_p$ is a $T$-invariant submanifold of $\tilde{M}_i$ which is diffeomorphic to $CP^k$ or $RP^{2k}$. Therefore it contains a $T$-fixed point.

This bijection is compatible with the action of the Weyl group of $G$ on the sets of characteristic submanifolds of $\tilde{M}$ and $M$.

In the real case the exceptional submanifold $M_0$ has codimension one in $\tilde{M}$ and is $G$-invariant. Because there is no $S^1$-representation of real dimension one, $M_0$ does not contain a characteristic submanifold of $\tilde{M}$ in this case.

In the complex case $M_0$ is $G$-invariant and may be a characteristic submanifold of $\tilde{M}$.

Therefore there is a bijection between the non-trivial orbits of the $W(G)$-actions on the sets of characteristic submanifolds of $M$ and $\tilde{M}$. Hence we get the same elementary factors for the $G$-actions on $\tilde{M}$ and $M$.

**Corollary 4.5.** Let $H$ be a closed subgroup of $G$ and $N'$ be a component of $M^H$ such that $N \cap N'$ has codimension one in the real case or two in the complex case in $N'$. Then $F$ induces an $(N_G H)^0$-equivariant diffeomorphism of $\tilde{N}'$ and $N'$.

**Proof.** Because of the dimension assumption the $(N_G H)^0$-equivariant map

$$f|_{P_K(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B} \cap N'} : P_K(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B} \cap N' \to B \cap N'$$
from the proof of Lemma 4.2 is a diffeomorphism. Because the restriction of $F$ to $\tilde{M} - M_0$ is a $G$-equivariant diffeomorphism, the restriction $F|_{\tilde{N}' - M_0} : \tilde{N}' - M_0 \to N' - N$ is an $(N_GH)^0$-equivariant diffeomorphism. Therefore $F|_{\tilde{N}'} : \tilde{N}' \to N'$ is a diffeomorphism. 

Lemma 4.6. In the complex case $\tilde{E} = N(N,M)^* \oplus \mathbb{C}$, where $N(N,M)^*$ is the normal bundle of $N$ in $M$ equipped with the dual complex structure. Then there is a $G$-equivariant diffeomorphism

$$\tilde{M} \to P_C(\tilde{E}) - \tilde{B} \cup_{\partial B} M - \tilde{B}.$$ 

This means that the diffeomorphism type of $\tilde{M}$ does not change if we replace the complex structure on $N(N,M)$ by its dual.

Proof. We have $P_C(E) = E/\sim$ and $P_C(\tilde{E}) = E/\sim'$, where

$$(z_0, z_1) \sim (z'_0, z'_1) \iff \exists t \in \mathbb{C}^* \ (tz_0, tz_1) = (z'_0, z'_1),$$

$$(z_0, z_1) \sim' (z'_0, z'_1) \iff \exists t \in \mathbb{C}^* \ (tz_0, \bar{t}z_1) = (z'_0, z'_1).$$

Therefore

$$E \to E \quad (z_0, z_1) \mapsto (z_0, \bar{z}_1)$$

induces a $G$-equivariant diffeomorphism $P_C(E) - \tilde{B} \to P_C(\tilde{E}) - \tilde{B}$ which is the identity on $\partial B$. By [10, pp. 24-25] the result follows. 

Lemma 4.7. If in the complex case $G_1 = SU(l_1 + 1)$ and codim $N = 2l_1 + 2$ or in the real case $G_1 = SO(2l_1 + 1)$ and codim $N = 2l_1 + 1$, then $F : \tilde{M} \to M$ induces a homeomorphism $\tilde{F} : \tilde{M}/G_1 \to M/G_1$.

Proof. Because $F|_{\tilde{M} - M_0} : \tilde{M} - M_0 \to M - N$ is an equivariant diffeomorphism and $\tilde{M}/G_1, M/G_1$ are compact Hausdorff spaces, the only thing that has to be checked is that

$$F|_{P_k(N(N,M))} : P_k(N(N,M)) \to N$$

induces a homeomorphism of the orbit spaces. But this map is just the bundle map $P_k(N(N,M)) \to N$.

If $G_1 = SU(l_1 + 1)$, then, because of dimension reasons [19, pp. 53-54], the $G_1$-representation on the fibers of $N(N,M)$ is the standard representation of $G_1$ or its dual. If $G_1 = SO(2l_1 + 1)$, then, by [19, pp. 53-54], the $G_1$-representation on the fibers of $N(N,M)$ is the standard representation of $G_1$.

Thus, in both cases the $G_1$-action on the fibers of $P_k(N(N,M)) \to N$ is transitive. Therefore the statement follows. 

Remark 4.8. All statements proved above also hold for non-connected groups of the form $G \times K$, where $K$ is a finite group and $G$ is connected if we replace $N$ by a $K$-invariant union of components of $M^{G_1}$.

Now we want to reverse the construction of a blow up. Let $A$ be a closed $G$-manifold and $E \to A$ be a $G$-vector bundle such that $G_1$ acts trivially on $A$. If $E$ is even-dimensional, we assume that there is a $g \in Z(G)$ such that $g$ acts trivially on $A$ and $g^2$ acts on $E$ as $-\text{Id}$. In this case we equip $E$ with the complex structure induced by the action of $g$. 

Assume that $\tilde{M}$ is a $G$-manifold and there is a $G$-equivariant embedding of $P_k(E) \hookrightarrow M$ such that the normal bundle of $P_k(E)$ is isomorphic to the tautological bundle over $P_k(E)$.

Then one may identify a closed $G$-equivariant tubular neighborhood $B^c$ of $P_k(E)$ in $\tilde{M}$ with

$$B^c = \{(z_0 : 1) \in P_k(E \oplus K) : |z_0| \geq 1\} \cup \{(z_0 : 0) \in P_k(E \oplus K)\}.$$ 

By gluing the complements of the interior of $B^c$ in $\tilde{M}$ and $P_k(E \oplus K)$, we get a $G$-manifold $M$ such that $A$ is $G$-equivariantly diffeomorphic to a union of components of $M^{G_1}$.

We call $M$ the blow down of $\tilde{M}$ along $P_k(E)$.

It is easy to see that the $G$-equivariant diffeomorphism type of $M$ does not depend on the choices of a metric on $E$ and the tubular neighborhood of $P_k(E)$ in $\tilde{M}$ if $G_1$ acts transitively on the fibers of $P_k(E) \to A$.

It is also easy to see that the blow up and blow down constructions are inverse to each other.

5. The case $G_1 = SU(l_1 + 1)$

In this section we discuss actions of groups which have a covering group of the form $G_1 \times G_2$, where $G_1 = SU(l_1 + 1)$ is elementary and $G_2$ acts effectively on $M$. It turns out that the blow up of $M$ along $M^{G_1}$ is a fiber bundle over $CP^{l_1}$. This fact leads to our first classification result.

The assumption on $G_2$ is no restriction on $G$, because one may replace any covering group $\tilde{G}$ by the quotient $\tilde{G}/H$ where $H$ is a finite subgroup of $G_2$ acting trivially on $M$. Following Convention 3.3 we also assume $\#\tilde{F}_1 = 2$ or $\#\tilde{F}_1 = 4$ in the cases $G_1 = SU(2)$ or $G_1 = SU(4)$, respectively. Furthermore, we assume after conjugating $T$ with some element of $G_1$ that $T_1 = T \cap G_1$ is the standard maximal torus of $G_1$.

5.1. The $G_1$-action on $M$. We have the following lemma:

**Lemma 5.1.** Let $M$ be a torus manifold with $G$-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$ elementary. Then the $W(S(U(l_1) \times U(1)))$-action on $\tilde{F}_1$ has an orbit $\tilde{F}'_1$ with $l_1$ elements and there is a component $N_1$ of $\bigcap_{M_i \in \tilde{F}_1} M_i$ which contains a $T$-fixed point.

**Proof.** We know that $W(SU(l_1 + 1)) = S_{l_1 + 1} = S(\tilde{F}_1)$ and $W(S(U(l_1) \times U(1))) = S_{l_1} \subset S_{l_1 + 1}$. Therefore the first statement follows. Let $x \in M^T$. Then, by Lemmas 3.3 and 3.4 $x$ is contained in the intersection of $l_1$ characteristic submanifolds of $M$ belonging to $\tilde{F}_1$. Because $W(G_1) = S(\tilde{F}_1)$ there is a $g \in N_{G_1}T_1$ such that $gx \in \bigcap_{M_i \in \tilde{F}_1} M_i$. Therefore the second statement follows. \hfill \Box

**Remark 5.2.** We will see in Lemma 5.10 that $\bigcap_{M_i \in \tilde{F}_1} M_i$ is connected.

**Lemma 5.3.** Let $M$ be a torus manifold with $G$-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$ elementary. Furthermore, let $N_1$ be as in Lemma 5.1. Then there is a group homomorphism $\psi_1 : S(U(l_1) \times U(1)) \to Z(G_2)$ such that, with

$$H_0 = SU(l_1 + 1) \times \text{im } \psi_1,$$

$$H_1 = S(U(l_1) \times U(1)) \times \text{im } \psi_1,$$

$$H_2 = \{(g, \psi_1(g)) \in H_1 : g \in S(U(l_1) \times U(1))\},$$
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(1) \( \text{im} \psi_1 \) is the projection of \( \lambda(M_i) \) to \( G_2 \), for all \( M_i \in \mathfrak{g}_1 \).

(2) \( N_1 \) is a component of \( M^{T_2} \).

(3) \( N_1 \) is invariant under the action of \( G_2 \).

(4) \( M = G_1 N_1 = H_0 N_1 \).

Proof. Denote by \( T_2 \) the maximal torus \( T \cap G_2 \) of \( G_2 \). Let \( x \in N_l^T \). If \( x \in M^{SU(l_1+1)} \), then we have, by Lemma 3.4, the \( SU(l_1 + 1) \times T_2 \)-representation

\[
T_x M = W \otimes_C V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i,
\]

where \( W \) is the standard complex representation of \( SU(l_1 + 1) \) or its dual and the \( V_i \) are one-dimensional complex representations of \( T_2 \). Because \( G_2 \) acts effectively on \( M \) the weights of the \( V_i \) form a basis of the integral lattice in \( LT_2 \). From the description of the weight spaces of \( T_x M \) given in the proof of Lemma 3.3, we get that \( T_x N_1 \) is \( S(U(l_1) \times U(1)) \)-invariant and that there is a one-dimensional complex representation \( W_1 \) of \( S(U(l_1) \times U(1)) \) such that

\[
T_x N_1 = W_1 \otimes_C V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i.
\]

Now assume that \( x \) is not fixed by \( SU(l_1 + 1) \). Because, by Lemma 3.1, \( G_1 x \subset M^{T_2} \) is \( G_1 \)-equivariantly diffeomorphic to \( CP^1 \), we see by the definition of \( N_1 \) that \( G_1 x = S(U(l_1) \times U(1)) \).

At the point \( x \), we get a representation of \( S(U(l_1) \times U(1)) \times T_2 \) of the form

\[
T_x M = T_x N_1 \oplus T_x G_1 x.
\]

Since \( T_2 \) acts effectively on \( M \) and trivially on \( G_1 x \), there is a decomposition

\[
T_x N_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_C W_i,
\]

where the \( W_i \) are one-dimensional complex \( S(U(l_1) \times U(1)) \)-representations and the \( V_i \) are one-dimensional complex \( T_2 \)-representations whose weights form a basis of the integral lattice in \( LT_2 \).

Therefore, in both cases, there is a homomorphism \( \psi_1 : S(U(l_1) \times U(1)) \rightarrow S^1 \rightarrow T_2 \) such that, for all \( g \in S(U(l_1) \times U(1)) \), \( (g, \psi_1(g)) \) acts trivially on \( T_x N_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_C W_i \).

Hence the component of the identity of the isotropy subgroup of the torus \( T \) for generic points in \( N_1 \) is given by

\[
H_3 = \{(t, \psi_1(t)) \in T_1 \times T_2 \}.
\]

With Lemma 3.1 we see that

\[
H_3 = \langle \lambda(M_i); M_i \in \mathfrak{g}_1, M_i \supset N_1 \rangle.
\]

Because the Weyl group of \( G_2 \) acts trivially and orientation preserving on \( \mathfrak{g}_1 \), \( \lambda(M_i), M_i \in \mathfrak{g}_1 \), is pointwise fixed by the action of \( W(G_2) \) on \( T \) by Lemma 2.10.

It follows from (5.2) that \( H_3 \) is pointwise fixed by the action of \( W(G_2) \) on \( T \). Here \( W(G_2) \) acts on \( T \) by conjugation. Therefore the image of \( \psi_1 \) is contained in the center of \( G_2 \). Furthermore \( \text{im} \psi_1 \) is the projection of \( \lambda(M_i) \), \( M_i \in \mathfrak{g}_1 \), to \( T_2 \).

Because \( H_3 \) commutes with \( G_2 \) it follows that \( N_1 \) is \( G_2 \)-invariant. So we have proved the first and the third statement.
Now we turn to the second and fourth parts. 
Because $T_y N_1 = (T_y M)^{H_2} = (T_y M)^{H_2}$, $N_1$ is a component of $M^{H_2}$. Because, by Lemma A.2, $H_1$ is the only proper closed connected subgroup of $H_0$ which contains $H_2$ properly, for $y \in N_1$ there are the following possibilities:

- $H^0_{0y} = H_0$,
- $H^0_{0y} = H_1$ and $\dim H_0 y = 2l_1$,
- $H^0_{0y} = H_2$ and $\dim H_0 y = 2l_1 + 1$,

where $H^0_{0y}$ is the identity component of the stabilizer of $y$ in $H_0$. If $g \in H_0$ such that $gy \in N_1$, then we have $H^0_{0gy} = g H^0_{0y} g^{-1} \in \{ H_0, H_1, H_2 \}$. Therefore

$$g \in N_{H_0} H^0_{0y} = \begin{cases} H_0 \quad &\text{if } y \in M^{H_0}, \\ H_1 \quad &\text{if } y \notin M^{H_0} \text{ and } l_1 > 1, \\ N_{G_1 T_1} \times \im \psi_1 \quad &\text{if } H^0_{0y} = H_1 \text{ and } l_1 = 1, \\ T_1 \times \im \psi_1 \quad &\text{if } H^0_{0y} = H_2, l_1 = 1 \text{ and } \im \psi_1 \neq \{1\}. \end{cases}$$

Now let $y \in N_1$ such that $H^0_{0y} \neq H_0$. Because $N_1$ is a component of $M^{H_2}$ and $H_0 y$ is $H_2$ invariant, $N_1 \cap H_0 y$ is an union of some components of $(H_0 y)^{H_2}$. Therefore $N_1 \cap H_0 y$ is a submanifold of $M$. Moreover,

$$T_y N_1 \cap T_y H_0 y = (T_y M)^{H_2} \cap T_y H_0 y = (T_y H_0 y)^{H_2} = T_y (N_1 \cap H_0 y).$$

Hence,

$$\dim T_y N_1 \cap T_y H_0 y = \dim N_1 \cap H_0 y = \dim H_1 y$$

follows. Therefore $N_1$ intersects $H_0 y$ transversely in $y$. It follows, by Lemma A.3, that $G N_1 - N_1^{H_0} = H_0 N_1 - N_1^{H_0}$ is an open subset of $M$.

Because $M$ is connected and $\codim M^{H_0} \geq 4$, $M - M^{H_0}$ is connected. Since $(M - M^{H_0}) \cap H_0 N_1 = H_0 N_1 - N_1^{H_0}$ is closed in $M - M^{H_0}$, we have $M - M^{H_0} = H_0 N_1 - N_1^{H_0}$. Hence

$$M = (M - M^{H_0}) \amalg M^{H_0} = \left( H_0 N_1 - N_1^{H_0} \right) \amalg M^{H_0}$$

$$= \left( H_0 N_1 - N_1^{H_0} \right) \amalg \left( M^{H_0} \cap N_1 \right) \amalg \left( M^{H_0} - N_1^{H_0} \right)$$

$$= H_0 N_1 \amalg \left( M^{H_0} - N_1^{H_0} \right).$$

Because $N_1$ is a component of $M^{H_2}$, $N_1^{H_0}$ is a union of components of $M^{H_0}$. Therefore $M^{H_0} - N_1^{H_0}$ is closed in $M$. Because $H_0 N_1$ is closed in $M$ it follows that $M = G N_1 = H_0 N_1 = G_1 N_1$. \(\square\)

The following lemma guarantees together with Lemma A.3 that if $l_1 > 1$, then the homomorphism $\psi_1$ is independent of all choices made in its construction, namely the choice of $N_1$ and of $x \in N_1^{T_1}$.

**Lemma 5.4.** In the situation of Lemma A.3 let $T' = T_2$ or $T' = \im \psi_1$. Then the principal orbit type of the $G_1 \times T'$-action on $M$ is given by $(G_1 \times T')/H_2$. 

Proof. Let $H \subset G_1 \times T'$ be a principal isotropy subgroup. Then, by Lemma 5.3 we may assume $H \supset H_2$. Consider the projection
\[ \pi_1 : G_1 \times T' \to G_1 \]
on the first factor.

At first we show that the restriction of $\pi_1$ to $H$ is injective. Because $(G_1 \times T')_x \cap T' = T'_x$ for all $x \in M$ and the $T'$-action on $M$ is effective, there is an $x \in M$ such that
\[ (G_1 \times T')_x \cap T' = \{1\}. \]
Furthermore, there is a $g \in G_1 \times T'$ such that $(G_1 \times T')_x \supset gHg^{-1}$.
Because $T'$ is contained in the center of $G_1 \times T'$, we get
\[ gHg^{-1} \cap T' = \{1\}, \]
\[ H \cap g^{-1}T'g = \{1\}, \]
\[ H \cap T' = \{1\}. \]
Therefore the restriction of $\pi_1$ to $H$ is injective.

Furthermore, $\pi_1(H) \supset \pi_1(H_2) = S(U(l_1) \times U(1))$. Therefore, by Lemma A.1 we have
\[ \pi_1(H) = \begin{cases} SU(l_1 + 1), S(U(l_1) \times U(1)) & \text{if } l_1 > 1, \\ SU(l_1 + 1), S(U(l_1) \times U(1)), N_{G_1}T_1 & \text{if } l_1 = 1. \end{cases} \]
There is a left inverse $\phi : \pi_1(H) \to H \hookrightarrow G_1 \times T' \to \pi_1|_H$. Therefore there is a group homomorphism $\psi' : \pi_1(H) \to T'$ such that
\[ H = \phi(\pi_1(H)) = \{(g, \psi'(g)) \in G_1 \times T'; g \in \pi_1(H)\}. \]
Because $H_2$ is a subgroup of $H$, we see that $\psi'|_{S(U(l_1) \times U(1))} = \psi_1$.
At first we discuss the cases $\pi_1(H) = SU(l_1 + 1)$ and $\pi_1(H) = S(U(l_1) \times U(1))$.
Because $T'$ is abelian we have in these cases
\[ H = \phi(\pi_1(H)) = \begin{cases} G_1 & \text{if } \pi_1(H) = SU(l_1 + 1), \\ H_2 & \text{if } \pi_1(H) = S(U(l_1) \times U(1)). \end{cases} \]
The first case does not occur because $G_1$ acts non-trivially on $M$.
Now we discuss the case $l_1 = 1$ and $\pi_1(H) = N_{G_1}T_1$. Because for $t \in T_1$ and $g \in N_{G_1}T_1 - T_1$ we have
\[ \psi(t)^{-1} = \psi'(gtg^{-1}) = \psi'(g)\psi'(t)\psi'(g)^{-1} = \psi'(t), \]
it follows that $\psi_1$ is trivial in this case.

Let $x \in M^T$. Then it follows by the definition of $\psi_1$ in the proof of Lemma 5.3 that $x$ is not a fixed point of $G_1$. By Lemma 5.1 we know that
\[ G_{1x} = S(U(l_1) \times U(1)) = T_1. \]
Therefore $(G_1 \times T')_x = T_1 \times T'$ is abelian. But $H$ is non-abelian if $\pi_1(H) = N_{G_1}T_1$. This is a contradiction because $H$ is conjugated to a subgroup of $(G_1 \times T')_x$. \hfill \Box

If $l_1 = 1$, we have $\#T_1 = 2$ and $W(S(U(l_1) \times U(1))) = \{1\}$. Therefore there are two choices for $N_1$. Denote them by $M_1$ and $M_2$.

Lemma 5.5. In the situation described above let $\psi_1$ be the homomorphism constructed for $M_i$, $i = 1, 2$. Then we have $\psi_1 = \psi_2^{-1}$. 

Corollary 5.6. If in the situation of Lemma 5.3, the \( G_1 \)-action on \( M \) has no fixed point, then \( M \) is the total space of a \( G \)-equivariant fiber bundle over \( \mathbb{C}P^1 \) with fiber some torus manifold. More precisely \( M = H_0 \times H_1 \). Proof. \( H_0 \times H_1 \) is defined to be the space \( H_0 \times N_1 / \sim_1 \), where \((g_1, y_1) \sim_1 (g_2, y_2) \) if \( \exists h \in H_1 \) such that \( g_1 h^{-1} = g_2 \) and \( h y_1 = y_2 \). By Lemma 5.3, we have that \( M = H_0 N_1 = (H_0 \times N_1) / \sim_2 \), where \((g_1, y_1) \sim_2 (g_2, y_2) \) if \( g_1 y_1 = g_2 y_2 \). We show that the two equivalence relations \( \sim_1, \sim_2 \) are equal.

For \((g_1, y_1), (g_2, y_2) \in H_0 \times N_1 \) we have

\[
(g_1, y_1) \sim_1 (g_2, y_2) \iff \exists h \in H_1 \text{ s.t. } g_1 h^{-1} = g_2 \text{ and } h y_1 = y_2
\]

\[
(g_1, y_1) \sim_2 (g_2, y_2) \iff g_1 y_1 = g_2 y_2.
\]

For the last equivalence, we have to show the implication from the second to the third line. If \( l_1 > 1 \), then \( N_1 \mid_{H_0} H_{0 y_1}^0 \) is equal to \( H_1 \) because \( y_1 \) is not a \( H_0 \)-fixed point. So we have \( h \in H_1 \).

If \( l_1 = 1 \), then \( N_1 \) is a characteristic submanifold of \( M \) belonging to \( S_1 \). If \( H_{0 y_1}^0 = H_2 \) we are done because \( N_{H_2, H_{0 y_1}^0} = H_1 \).

Now assume that \( H_{0 y_1}^0 = H_1 \) and there is an \( h \in N_{G_1} T_1 \times \text{im } \psi_1 - T_1 \times \text{im } \psi_1 \) such that \( y_2 = h y_1 \in N_1 \). Then \( y_2 \in N_1 \cap N_2 \subset M T_1 \times \text{im } \psi_1 \), where \( N_2 \) is the other characteristic submanifold of \( M \) belonging to \( S_1 \).

As shown in the proof of Lemma 5.3, \( N_1 \) intersects \( H_0 y_2 \) transversely in \( y_2 \). Therefore one has

\[
T_{y_2} N_1 \oplus T_{y_2} H_0 y_2 = T_{y_2} M = T_{y_2} N_2 \oplus T_{y_2} H_0 y_2
\]

as \( T_1 \times \text{im } \psi_1 \)-representations. This implies

\[
T_{y_2} N_1 = T_{y_2} N_2
\]

as \( T_1 \times \text{im } \psi_1 \)-representations. Therefore \( T_1 \times \text{im } \psi_1 \) acts trivially on both \( N_1 \) and \( N_2 \). Therefore we have \( \text{im } \psi_1 = \{1\} \) and \( \lambda(N_1) = \lambda(N_2) = T_1 \). Hence, we get a contradiction because the intersection of \( N_1 \) and \( N_2 \) is non-empty. \( \square \)

Corollary 5.7. In the situation of Lemma 5.3 we have \( M^{G_1} = M^{H_0} = \bigcap_{M_i \in S_i} M_i \).
Proof: At first let \( l_1 > 1 \). By Lemma 5.3 we know \( M^{H_0} \subseteq M^{G_1} \subseteq N_1 \). Therefore \( M^{G_1} \subseteq \bigcap_{g \in N_{G_1}, T_1} gN_1 = \bigcap_{M_i \in \mathfrak{S}_1} M_i \). There is a \( g \in N_{G_1}, T_1 - T_1 \) with \( gH_2g^{-1} \not\subseteq H_1 \). Thus, the subgroup \( \langle H_2, gH_2g^{-1} \rangle \) of \( H_0 \), which is generated by \( H_2 \) and \( gH_2g^{-1} \), contains \( H_2 \) as a proper subgroup. Therefore \( \langle H_2, gH_2g^{-1} \rangle = H_0 \) follows by Lemma A.2 Because \( H_2 \) acts trivially on \( N_1 \), this equation implies
\[
\bigcap_{g \in N_{G_1}, T_1} gN_1 = \bigcap_{M_i \in \mathfrak{S}_1} M_i.
\]

Now let \( l_1 = 1 \). Then \( \mathfrak{S}_1 \) contains two characteristic submanifolds \( M_1 \) and \( M_2 \). As in the first case one can show that \( M^{H_0} \subseteq M^{G_1} \subseteq M_1 \cap M_2 \).

So \( M^{H_0} \supseteq M_1 \cap M_2 \) remains to be shown. Assume that there is a \( y \in M_1 \cap M_2 - M^{H_0} \). Then we also have \( y \in M^{H_1} \). Now the above assumption leads to a contradiction as in the proof of Corollary 5.6. \( \square \)

**Corollary 5.8.** If in the situation of Lemma 5.3 \( \psi_1 \) is trivial, then \( M^{G_1} \) is empty. Otherwise the normal bundle of \( M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{S}_1} M_i \) possesses a \( G \)-invariant complex structure. It is induced by the action of some element \( g \in \text{im} \psi_1 \). Furthermore, it is unique up to conjugation.

Proof. If \( \psi_1 \) is trivial, then \( \langle \lambda(M_i); M_i \in \mathfrak{S}_1 \rangle \) is contained in the \( l_1 \)-dimensional maximal torus of \( G_1 \) by Lemma 5.3. By Corollary 5.7 and Lemma B.1 it follows that \( M^{H_0} \) is empty.

If \( \psi_1 \) is non-trivial, then for \( y \in M^{H_0} \) we have
\[
\text{N}_y(M^{H_0}, M) = V_C \oplus V_R,
\]
where \( \text{im} \psi_1 \) acts non-trivially on the \( H_0 \)-representation \( V_C \) and trivially on the \( H_0 \)-representation \( V_R \). Clearly \( V_C \) has at least real dimension two, and the action of \( \text{im} \psi_1 \) induces an \( H_0 \)-invariant complex structure on \( V_C \). Because \( M^{H_0} \) has codimension \( 2l_1 + 2 \) by Corollary 5.7 and Lemma B.1 the dimension of \( V_R \) is at most \( 2l_1 \). So it follows from [19 pp. 53-54] that \( V_R \) is trivial if \( l_1 \neq 3 \).

If \( l_1 = 3 \), we have \( SU(4) = \text{Spin}(6) \), and there are two possibilities:

1. \( V_R \) is trivial.
2. \( V_R \) is the standard representation of \( SO(6) \) and \( V_C \) a one-dimensional complex representation of \( \text{im} \psi_1 \).

Because the principal orbits are dense in \( M \), it follows with the slice theorem that the principal orbit types of the \( H_0 \)-actions on \( \text{N}_y(M^{H_0}, M) \) and \( M \) are equal. Therefore in the second case the principal orbit type of the \( H_0 \)-action on \( M \) is given by \( \text{Spin}(6) \times S^1/\text{Spin}(5) \times \{1\} \). Therefore we see with Lemma 5.4 that the second case does not occur.

Because of dimension reasons we get
\[
\text{N}_y(M^{H_0}, M) = V_C = W \otimes_C V,
\]
where \( W \) is the standard complex representation of \( SU(l_1 + 1) \) or its dual and \( V \) is a complex one-dimensional \( \text{im} \psi_1 \)-representation. Because \( \text{im} \psi_1 \subset Z(G) \), we see that \( \text{N}(M^{H_0}, M) \) has a \( G \)-invariant complex structure, which is induced by the action of some \( g \in \text{im} \psi_1 \).
Next we prove the uniqueness of this complex structure. Assume that there is another \( g' \in Z(G) \cap G_y \) whose action induces a complex structure on \( N_y(M^{H_0}, M) \). Then \( g' \) induces a – with respect to the complex structure induced by \( g \) – complex linear \( H_0 \)-equivariant map

\[
J : N_y(M^{H_0}, M) \to N_y(M^{H_0}, M)
\]

with \( J^2 + \text{Id} = 0 \). Because \( N_y(M^{H_0}, M) \) is an irreducible \( H_0 \)-representation, it follows by Schur’s Lemma that \( J \) is multiplication with \( \pm i \). Therefore \( g' \) induces up to conjugation the same complex structure as \( g \).

**Corollary 5.9.** If in the situation of Lemma 5.5 \( M^{G_1} = M^{H_0} \neq \emptyset \), then \( \ker \psi_1 = SU(l_1) \).

**Proof.** Let \( y \in M^{H_0} \). Then by the proof of Corollary 5.8 we have

\[
N_y(M^{H_0}, M) = W \otimes_{\mathbb{C}} V,
\]

where \( W \) is the standard complex \( SU(l_1 + 1) \)-representation or its dual and \( V \) is a one-dimensional complex im \( \psi_1 \)-representation. Furthermore, im \( \psi_1 \) acts effectively on \( M \).

Because the principal orbits are dense in \( M \), it follows by the slice theorem that the principal orbit types of the \( H_0 \)-actions on \( N_y(M^{H_0}, M) \) and \( M \) are equal. Therefore a principal isotropy subgroup of the \( H_0 \)-action on \( M \) is given by

\[
H = \left\{ (g, g_{l+1}^{\pm 1}) \in H_1; g = \begin{pmatrix} A & 0 \\ 0 & g_{l+1} \end{pmatrix} \in S(U(l_1) \times U(1)) \text{ with } A \in U(l_1) \right\}.
\]

Now the statement follows by the uniqueness of the principal orbit type and Lemmas 5.3 and 5.8.

**Lemma 5.10.** In the situation of Lemma 5.5, the intersection \( \bigcap_{M_i \in \mathcal{B}_1} M_i = N_1 \) is connected.

**Proof.** Let \( \tilde{M} \) be the blow up of \( M \) along \( M^{G_1} \) and \( \tilde{N}_1 \) be the proper transform of \( N_1 \) in \( M \). By Corollary 5.8 we have \( \tilde{M} = H_0 \times H_1 \tilde{N}_1 \), which is a fiber bundle over \( CP^{l_1} \). The characteristic submanifolds of \( \tilde{M} \), which are permuted by \( W(G_1) \), are given by the preimages of the characteristic submanifolds of \( CP^{l_1} \) under the bundle map. By Corollary 4.4 and the discussion following this corollary, they are also given by the proper transforms \( \tilde{M}_i \) of the characteristic submanifolds \( M_i \) in \( \mathcal{B}_1 \) of \( M \). Because \( l_1 \) characteristic submanifolds of \( CP^{l_1} \) intersect in a single point, we see that \( \bigcap_{M_i \in \mathcal{B}_1} \tilde{M}_i = \tilde{N}_1 \). Therefore this intersection is connected. Because \( \bigcap_{M_i \in \mathcal{B}_1} \tilde{M}_i \) is mapped by \( F \) to \( \bigcap_{M_i \in \mathcal{B}_1} M_i \), we see that \( \bigcap_{M_i \in \mathcal{B}_1} M_i = N_1 \) is connected.

5.2. **Blowing up along** \( M^{G_1} \). By blowing up a torus manifold \( M \) with \( G \)-action along \( M^{G_1} \), one gets a torus manifold \( \tilde{M} \) without \( G_1 \)-fixed points.

Denote by \( \tilde{N}_1 \) the proper transform of \( N_1 \) as defined in Lemma 5.1. Then by Corollary 5.7 there is an \( (H_1, G_2) \)-equivariant diffeomorphism \( F : \tilde{N}_1 \to N_1 \).

As in section 4 we denote by \( M_0 = P_C(N(M^{G_1}, M) \oplus \{0\}) \) the exceptional submanifold of \( \tilde{M} \). Because \( M_0 \cap \tilde{N}_1 \) is mapped by this diffeomorphism to \( M^{G_1} = M^{H_0} = N_1^{H_0} \), \( H_1 \) acts trivially on \( M_0 \cap \tilde{N}_1 \). By Corollary 5.6 we know that \( \tilde{M} \) is diffeomorphic to \( H_0 \times H_1 \tilde{N}_1 = H_0 \times H_1 N_1 \).
A natural question arising here is: When is a torus manifold of this form a blow up of another torus manifold with G-action?

We claim that this is the case if and only if $N_1$ has a codimension two submanifold, which is fixed by the $H_1$-action and $\ker \psi_1 = SU(l_1)$.

**Lemma 5.11.** Let $N_1$ be a torus manifold with $G_2$-action, $A$ be a closed codimension two submanifold of $N_1$, $\psi_1 \in \text{Hom}(SU(l_1) \times U(1), Z(G_2))$ such that $\text{im} \psi_1$ acts trivially on $A$ and $\ker \psi_1 = SU(l_1)$. Also let

\[
H_0 = SU(l_1 + 1) \times \text{im} \psi_1, \\
H_1 = S(U(l_1) \times U(1)) \times \text{im} \psi_1, \\
H_2 = \{ (g, \psi_1(g)); g \in S(U(l_1) \times U(1)) \}.
\]

(1) Then $H_1$ acts on $N_1$ by $(g, t)x = \psi_1(g)^{-1}tx$, where $x \in N_1$ and $(g, t) \in H_1$.

(2) Assume that $Z(G_2)$ acts effectively on $N_1$ and let $y \in A$ and $V$ be the one-dimensional complex $H_1$-representation $N_y(A, N_1)$. Then $V$ extends to an $l_1 + 1$-dimensional complex representation of $H_0$. Therefore there is an $l_1 + 1$-dimensional complex $G$-vector bundle $E'$ over $A$ which contains $N(A, N_1)$ as a subbundle.

(3) Then the normal bundle of $H_0/H_1 \times A$ in $H_0 \times H_1 N_1$ is isomorphic to the tautological bundle over $P_{\mathbb{C}}(E' \oplus \{ 0 \})$.

The lemma guarantees together with the discussion at the end of section 4 that one can remove $H_0/H_1 \times A$ from $H_0 \times H_1 N_1$ and replace it by $A$ to get a torus manifold with $G$-action $M$ such that $M^{H_0} = A$. The blow up of $M$ along $A$ is $H_0 \times H_1 N_1$.

**Proof.** (1) is trivial.

(2) For $i = 1, \ldots, l_1 + 1$ let

\[
\lambda_i : T_1 \to S^1 \quad \begin{pmatrix} g_1 \\ \vdots \\ g_{l_1+1} \end{pmatrix} \mapsto g_i
\]

and $\mu : \text{im} \psi_1 \to S^1$ be the character of the im $\psi_1$ representation $N_y(A, N_1)$. Then $\mu$ is an isomorphism.

Also, by [4, p. 176] the character ring of the maximal torus $T_1 \times \text{im} \psi_1$ of $H_1 = S(U(l_1) \times U(1)) \times \text{im} \psi_1$ is given by

\[
R(T_1 \times \text{im} \psi_1) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{l_1+1}, \mu, \mu^{-1}]/(\lambda_1 \cdots \lambda_{l_1+1} - 1).
\]

With this notation, the character of $V$ is given by $\mu \lambda_{l_1+1}^{\pm 1}$. Therefore the $H_0$-representation $W$ with the character $\mu \sum_{i=1}^{l_1+1} \lambda_i^{\pm 1}$ is $l_1 + 1$-dimensional and $V \subset W$.

Let $G_2 = G_2' \times \text{im} \psi_1$ and $E'' = N(A, N_1)$ be equipped with the action of $G_2'$ but without the action of $H_1$. Then $E' = E'' \otimes_{\mathbb{C}} W$ is a $G$-vector bundle with the required features.
Now we turn to (3). The normal bundle of $H_0/H_1 \times A$ in $H_0 \times H_1 N_1$ is given by $H_0 \times H_1 N(A,N_1)$.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
H_0 \times H_1 N(A,N_1) & \xrightarrow{f} & P_{C}(E' \oplus \{0\}) \times E' \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
H_0/H_1 \times A & \xrightarrow{g} & P_{C}(E' \oplus \{0\})
\end{array}
\]

where the vertical maps are the natural projections and $f, g$ are given by

\[
f([h_1, h_2] : m) = ([m \otimes h_2 h_1 e_1], m \otimes h_2 h_1 e_1)
\]

and

\[
g([h_1, h_2], q) = [m_q \otimes h_2 h_1 e_1],
\]

where $e_1 \in W - \{0\}$ is fixed such that for all $g' \in S(U(l_1) \times U(1))$, $\psi_1(g')g' e_1 = e_1$ and $m_q \neq 0$ is some element of the fiber of $N(A,N_1)$ over $q \in A$.

The map $f$ induces an isomorphism of the normal bundle of $H_0/H_1 \times A$ in $H_0 \times H_1 N_1$ and the tautological bundle over $P_{C}(E' \oplus \{0\})$.

5.3. Admissible triples. Now we are in the position to state our first classification theorem. To do so, we need the following definition.

**Definition 5.12.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. Then a triple $(\psi, N, A)$ with

- $\psi \in \text{Hom}(S(U(l_1) \times U(1)), Z(G_2))$,
- $N$ a torus manifold with $G_2$-action,
- $A$ the empty set or a closed codimension two submanifold of $N$, such that
- $\text{im} \psi$ acts trivially on $A$ and $\ker \psi = SU(l_1)$ if $A \neq \emptyset$

is called **admissible** for $(\tilde{G}, G_1)$. We say that two admissible triples $(\psi, N, A)$, $(\psi', N', A')$ for $(\tilde{G}, G_1)$ are equivalent if there is a $G_2$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$ and

\[
\psi = \begin{cases} 
\psi' & \text{if } l_1 > 1, \\
\psi'^{\pm 1} & \text{if } l_1 = 1.
\end{cases}
\]

**Theorem 5.13.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. There is a one-to-one-correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-action such that $G_1$ is elementary and the equivalence classes of admissible triples for $(\tilde{G}, G_1)$.

**Proof.** Let $M$ be a torus manifold with $\tilde{G}$-action such that $G_1$ is elementary. Then, by Corollaries 5.7 and 5.9 $(\psi_1, N_1, M^{H_0})$ is an admissible triple, where $\psi_1$ is defined as in Lemma 5.3 and $N_1$ is defined as in Lemma 5.1.

Let $(\psi, N, A)$ be an admissible triple for $(\tilde{G}, G_1)$. If $A \neq \emptyset$, then, by Lemma 5.11 the blow down of $H_0 \times H_1 N$ along $H_0/H_1 \times A$ is a torus manifold with $\tilde{G}$-action. If $A = \emptyset$, then we have the torus manifold $H_0 \times H_1 N$. 


We show that these two operations are inverse to each other. Let $M$ be a torus manifold with $G$-action. If $M^{H_0} = \emptyset$, then, by Corollary 5.10 we have $M = H_0 \times H_1 \cdot N_1$. If $M^{H_0} \neq \emptyset$, then by the discussion before Lemma 5.11 $M$ is the blow down of $H_0 \times H_1 \cdot N_1$ along $H_0/H_1 \times M^{H_0}$.

Now assume $l_1 > 1$. Let $(\psi, N, A)$ be an admissible triple with $A \neq \emptyset$ and $M$ be the blow down of $H_0 \times H_1 \cdot N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11 we have $A = M^{H_0}$. By Lemma 5.10 and Corollary 4.3 we have $N = N_1$. With Lemmas 4.3 and 5.3 one sees that $\psi = \psi_1$, where $\psi_1$ is the homomorphism defined in Lemma 5.3 for $M$.

Now let $(\psi, N, \emptyset)$ be an admissible triple and $M = H_0 \times H_1 \cdot N$. Then we have $M^{H_0} = \emptyset$. By Lemma 5.10 we have $N = N_1$. As in the first case one sees $\psi = \psi_1$.

Now assume $l_1 = 1$. Let $(\psi, N, A)$ be an admissible triple with $A \neq \emptyset$ and $M$ be the blow down of $H_0 \times H_1 \cdot N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11 $A = M^{H_0}$. By Lemma 5.9 we have two choices for $N_1$ and $\psi = \psi_1^{\pm 1}$. Because the two choices for $N_1$ lead to equivalent admissible triples we recover the equivalence class of $(\psi, N, A)$. In the case $A = \emptyset$ a similar argument completes the proof of the theorem.

Corollary 5.14. Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. Then the torus manifolds with $G$-action such that $G_1$ is elementary and $M^{G_1} \neq \emptyset$ are given by blow downs of fiber bundles over $\mathbb{C}P^{l_1}$ with fiber some torus manifold with $G_2$-action along a submanifold of codimension two.

Now we specialise our classification result to special classes of torus manifolds.

Theorem 5.15. Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$, $M$ be a torus manifold with $G$-action and $(\psi, N, A)$ be the admissible triple for $(\tilde{G}, G_1)$ corresponding to $M$. Then $H^\ast(M; \mathbb{Z})$ is generated by its degree two part if and only if $H^\ast(N; \mathbb{Z})$ is generated by its degree two part and $A$ is connected.

Proof. To make the notation simpler we omit the coefficients of the cohomology in the proof. If $H^\ast(M)$ is generated by its degree two part, then $H^\ast(N)$ is generated by its degree two part by [15, p. 716]. Moreover, $A$ is connected by [15, p. 738] and Corollary 5.4.

Now assume that $H^\ast(N)$ is generated by its degree two part and $A = \emptyset$. Then by Poincaré duality $H_{\text{odd}}(N) = 0$. Therefore by a universal coefficient theorem $H^\ast(N) = \text{Hom}(H_\ast(N, \mathbb{Z}))$ is torsion free. By Corollary 4.6 $M$ is a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber $N$. Because the Serre spectral sequence of this fibration degenerates, we have

$$H^\ast(M) \cong H^\ast(\mathbb{C}P^{l_1}) \otimes H^\ast(N)$$

as a $H^\ast(\mathbb{C}P^{l_1})$-modul. Because $H^\ast(N)$ is generated by its degree two part, it follows that the cohomology of $M$ is generated by its degree two part.

Now we turn to the general case $A \neq \emptyset$. Then, by [15, p. 716], $H^\ast(A)$ is generated by its degree two part. Moreover, $H^\ast(N) \rightarrow H^\ast(A)$ is surjective. Let $\tilde{M}$ be the blow up of $M$ along $A$ and $F : \tilde{M} \rightarrow M$ be the map defined in section 4.

Because, by Lemma 4.2 $F$ is the identity outside some open tubular neighborhood of $A \times \mathbb{C}P^{l_1}$, the induced homomorphism $F^\ast : H^\ast(M, A) \rightarrow H^\ast(\tilde{M}, A \times \mathbb{C}P^{l_1})$ is an isomorphism by excision. Furthermore, the push forward $F_\ast : H^\ast(M) \rightarrow H^\ast(\tilde{M})$ is a section of $F^\ast : H^\ast(M) \rightarrow H^\ast(\tilde{M})$. Therefore $F^\ast : H^\ast(M) \rightarrow H^\ast(\tilde{M})$ is injective and $H^{\text{odd}}(M)$ vanishes.
Because $A$ is connected, we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
H^2(\tilde{M}, A \times \mathbb{C}P^l_1) & \longrightarrow & H^2(\tilde{N}, A) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^2(\mathbb{C}P^l_1) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & H^2(N) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^2(\mathbb{C}P^l_1) & \longrightarrow & H^2(A \times \mathbb{C}P^l_1) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^3(\tilde{M}, A \times \mathbb{C}P^l_1) & \longrightarrow & 0 & \longrightarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
$$

Now from the snake lemma it follows that

$$H^2(M, A) \cong_{F^*} H^2(\tilde{M}, A \times \mathbb{C}P^l_1) \cong H^2(N, A)$$

and

$$H^3(M, A) \cong_{F^*} H^3(\tilde{M}, A \times \mathbb{C}P^l_1) \cong 0.$$

Because $\iota_{NM} = F \circ \iota_{\tilde{N}\tilde{M}}$, where $\iota_{NM}, \iota_{N\tilde{M}}$ are the inclusions of $N$ in $M$ and $\tilde{M}$, the left arrow in the following diagram is an isomorphism:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(M, A) & \longrightarrow & H^2(M) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\
\iota_{NM} & \downarrow & \iota_{NM} & \downarrow & \downarrow & \downarrow & \downarrow & \iota_{NM} & \downarrow \\
0 & \longrightarrow & H^2(N, A) & \longrightarrow & H^2(N) & \longrightarrow & H^2(A) & \longrightarrow & 0.
\end{array}
$$

Therefore it follows from the five lemma that

$$H^2(M) \cong H^2(N)$$

and

$$H^2(\tilde{M}) \cong H^2(\mathbb{C}P^l_1) \oplus H^2(N) \cong H^2(\mathbb{C}P^l_1) \oplus H^2(M).$$

Let $t \in H^2(\mathbb{C}P^l_1)$ be a generator of $H^2(\mathbb{C}P^l_1)$ and $x \in H^*(\tilde{M})$. Then, because $H^*(\tilde{M})$ is generated by its degree two part, there are sums of products $x_i \in H^*(M)$ of elements of $H^2(M)$ such that

$$x = F_1 F^*(x) = F_1 \left( \sum F^*(x_i) t^i \right) = \sum x_i F_1(t^i).$$

Therefore it remains to show that $F_1(t^i)$ is a product of elements of $H^2(M)$.

The $l_1 + 1$ characteristic submanifolds $\tilde{M}_1, \ldots, \tilde{M}_{l_1+1}$ of $\tilde{M}$ which are permuted by $W(G_1)$ are the preimages of the characteristic submanifolds of $\mathbb{C}P^l_1$ under the projection $\tilde{M} \rightarrow \mathbb{C}P^l_1$. Therefore they can be oriented in such a way that $t$ is the Poincaré dual of each of them.
Figure 1. The orbit space of a blow down

Because \( F \) restricts to a diffeomorphism \( \tilde{M} - A \times \mathbb{C}P^1 \to M - A \) and \( F(\tilde{M}) = M \), \( F(t^i), i \leq l_1 \), is the Poincaré dual \( PD\left(\bigcap_{1 \leq k \leq i} M_k\right) \) of the intersection \( \bigcap_{1 \leq k \leq i} M_k \) of characteristic submanifolds of \( M \), which belong to \( \mathfrak{F}_1 \). Therefore for \( i \leq l_1 \) we have

\[
F(t^i) = PD\left(\bigcap_{1 \leq k \leq i} M_k\right) = F(t^i).
\]

Because \( t^i = 0 \) for \( i > l_1 \), the statement follows. \( \square \)

**Theorem 5.16.** Let \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SU(l_1 + 1) \), \( M \) be a torus manifold with \( \tilde{G} \)-action and \( (\psi, N, A) \) be the admissible triple for \( (\tilde{G}, G_1) \) corresponding to \( M \). Then \( M \) is quasitoric if and only if \( N \) is quasitoric and \( A \) is connected.

**Proof.** At first assume that \( M \) is quasitoric. Then \( N \) is quasitoric and \( A \) is connected because all intersections of characteristic submanifolds of \( M \) are quasitoric and connected.

Now assume that \( N \) is quasitoric and \( A \subset N \) is connected. Then, by Theorem 5.15 and [15, p. 738], the \( T \)-action on \( M \) is locally standard and \( M/T \) is a homology polytope. We have to show that \( M/T \) is face preserving homeomorphic to a simple polytope.

Let \( T_2 = T \cap G_2 \). Then the orbit space \( N/T_2 \) is face preserving homeomorphic to a simple polytope \( P \). Because \( A \) is connected, \( A/T_2 \) is a facet \( F_1 \) of \( P \).

With the notation from Lemma 5.11 let

\[
B = \{(z_0 : 1) \in P_{\mathbb{C}}(E' \oplus \mathbb{C}); z_0 \in E', |z_0| \leq 1\}.
\]

Then the orbit space of the \( T \)-action on \( B \) is given by \( F_1 \times \Delta^{l_1+1} \).

Let \( B' \) be a closed \( \tilde{G} \)-invariant tubular neighborhood of \( H_0/H_1 \times A \) in \( H_0 \times H_1 \). \( N \).

Then the bundle projection \( \partial B' \to H_0/H_1 \) extends to an equivariant map

\[
H_0 \times H_1 \to H_0/H_1 \times A \to \hat{N},
\]

which induces a face preserving homeomorphism

\[
\left(H_0 \times H_1 \to H_0/H_1 \times A\right)/T \cong P \times \Delta^{l_1}.
\]

Now \( M \) is given by gluing \( B \) and \( H_0 \times H_1 \to H_0/H_1 \times A \) in the boundaries \( \partial B' \), \( \partial B' \). The corresponding gluing of the orbit spaces is illustrated in Figure 1 for the case
dim $N = 2$ and $l_1 = 1$. Because the gluing map $f : \partial B \to \partial B'$ is $\tilde{G}$-equivariant and $G_1$ acts transitively on the fibers of $\partial B \to A$ and $\partial B' \to A$, it induces a map

$$\tilde{f} : F_1 \times \Delta^l \to \partial B / T \to \partial B' / T = F_1 \times \Delta^l,$$

where $\tilde{f}_1 : F_1 \to F_1$ is a face preserving homeomorphism and $\hat{f}_2 : F_1 \times \Delta^l \to \Delta^l$ such that, for all $x \in F_1$, $\hat{f}_2(x, \cdot)$ is a face preserving homeomorphism of $\Delta^l$.

Now fix embeddings

$$\Delta^l \to \mathbb{R}^{l+1} \quad \text{and} \quad P \to \mathbb{R}^{n-l_1} \times [0, 1]$$

such that $\Delta^l \subset \mathbb{R}^{l_1} \times \{1\}$, $\Delta^l \supset \text{conv}(0, \Delta^l)$ and $P \cap \mathbb{R}^{n-l_1-1} \times \{0\} = F_1$.

Denote by $p_1 : \mathbb{R}^{l_1+1} \to \mathbb{R}$ and $p_2 : \mathbb{R}^{n-l_1} \to \mathbb{R}$ the projections on the last coordinate. For $\epsilon > 0$ small enough, $P$ and $P \cap \{p_2 \geq \epsilon\}$ are combinatorially equivalent. Therefore there is a face preserving homeomorphism

$$g_1 : P \to P \cap \{p_2 \geq \epsilon\}$$

such that $g_1(F_1) = P \cap \{p_2 = \epsilon\}$ and $g_1(F_1) = F_1 \cap \{p_2 \geq \epsilon\}$ for the other facets of $P$. The map

$$g_2 : F_1 \times [0, 1] \to P \cap \{p_2 = \epsilon\},$$

$$(x, y) \mapsto x(1-y) + yg_1(x)$$

is a face preserving homeomorphism with $p_2 \circ g_2(x, y) = \epsilon y$ for all $(x, y) \in F_1 \times [0, 1]$.

Now let

$$\hat{P} = P \times \Delta^{l_1+1} \cap \{p_1 = p_2\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1},$$

$${\tilde{P}}_1 = P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \geq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1},$$

$${\tilde{P}}_2 = P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \leq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}.$$

Then there are face preserving homeomorphisms

$$h_1 : P \times \Delta^l \to {\tilde{P}}_1, \quad (x, y) \mapsto (g_1(x), p_2(g_1(x))y)$$

and

$$h_2 : F_1 \times \Delta^{l_1+1} \to {\tilde{P}}_2, \quad (x, y) \mapsto (g_2(x, p_1(y)), \epsilon y).$$

We claim that $\hat{P}$ and $M/T$ are face preserving homeomorphic. This is the case if

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2 : F_1 \times \Delta^l \to F_1 \times \Delta^{l_1+1}$$

extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. Now for $(x, y) \in F_1 \times \Delta^l$ we have

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2(x, y) = \hat{f}^{-1} \circ h_1^{-1}(g_2(x, p_1(y)), \epsilon y)$$

$$= \hat{f}^{-1} \circ h_1^{-1}(g_2(x, 1), \epsilon y)$$

$$= \hat{f}^{-1}(g^{-1}_1 \circ g_2(x, 1), y)$$

$$= (\hat{f}_1^{-1}(x), (\hat{f}_2(x, \cdot))^{-1}(y)).$$

Because $\Delta^{l_1+1}$ is the cone over $\Delta^l$, this map extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$.

\begin{lemma}
Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$, $M$ be a torus manifold with $\tilde{G}$-action and $(\psi, N, A)$ be the admissible triple for $(\tilde{G}, G_1)$ corresponding to $M$. Then there is an isomorphism $\pi_1(N) \to \pi_1(M)$.
\end{lemma}
Proof: Let \( \tilde{M} \) be the blow up of \( M \) along \( A \). Then, by [16, p. 270], there is an isomorphism \( \pi_1(\tilde{M}) \to \pi_1(M) \).

Now, by Corollary 5.3, \( \tilde{M} \) is the total space of a fiber bundle over \( \mathbb{C}P^{l_1} \) with fiber \( N \). Therefore there is an exact sequence

\[
\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \to \pi_1(N) \to \pi_1(\tilde{M}) \to 0.
\]

Because the torus action on \( N \) has fixed points, there is a section in this bundle. Hence, \( \pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \) is surjective. \( \square \)

6. The Case \( G_1 = SO(2l_1) \)

In this section we study torus manifolds with \( G \)-action, where \( \tilde{G} = G_1 \times G_2 \) and \( G_1 = SO(2l_1) \) is elementary. It turns out that the restriction of the action of \( G_1 \) to \( U(l_1) \) on such a manifold has the same orbits as the action of \( SO(2l_1) \). Therefore the results of the previous section may be applied to construct invariants for such manifolds. For simply connected torus manifolds with \( G \)-action these invariants determine their \( \tilde{G} \)-equivariant diffeomorphism type.

Let \( \tilde{G} = G_1 \times G_2 \), where \( G_1 = SO(2l_1) \) is elementary, and let \( M \) be a torus manifold with \( G \)-action. Then, by Lemmas 6.1 and 6.3, one sees that the principal orbit type of the \( G_1 \)-action is given by \( SO(2l_1)/SO(2l_1 - 1) \). Therefore the \( G_1 \)-action has only three orbit types, \( SO(2l_1)/SO(2l_1 - 1), SO(2l_1)/SO(2l_1 - 1) \times O(1) \) and \( SO(2l_1)/SO(2l_1 - 1) \times O(1) \). The induced action of \( U(l_1) \) has the same orbits, which are of types \( U(l_1)/U(l_1 - 1), U(l_1)/U(l_1 - 1, \mathbb{Z}_2) \) and \( U(l_1)/U(l_1) \), respectively. Here \( U(l_1 - 1, \mathbb{Z}_2) \) denotes the subgroup of \( U(l_1) \), which is generated by \( U(l_1 - 1) \) and the diagonal matrix with all entries equal to \(-1\).

Let \( S = S^1 \). Then there is a finite covering

\[
SU(l_1) \times S \to U(l_1)
\]

So we may replace the factor \( G_1 \) of \( \tilde{G} \) by \( SU(l_1) \) and \( G_2 \) by \( S \times G_2 \) to reach the situation of the previous section.

Let \( x \in M^T \) and \( T_2 = T \cap G_2 \). Then we may assume by Lemma 5.4 that the \( G_1 \times T_2 \)-representation \( T_x M \) is given by

\[
T_x M = V \oplus W;
\]

where \( V \) is a complex representation of \( T_2 \) and \( W \) is the standard real representation of \( G_1 \). Therefore

\[
T_2 M = V \oplus V_0 \otimes \mathbb{C} W_0
\]

as a \( SU(l_1) \times S \times T_2 \)-representation, where \( V_0 \) is the standard complex one-dimensional representation of \( S \) and \( W_0 \) is the standard complex representation of \( SU(l_1) \).

Therefore the group homomorphism \( \psi_1 \) and the groups \( H_0, H_1, H_2 \) introduced in Lemma 5.3 have the following form:

\[
\text{im } \psi_1 = S
\]

and

\[
H_0 = SU(l_1) \times S,
\]

\[
H_1 = S(U(l_1 - 1) \times U(1)) \times S,
\]

\[
H_2 = \left\{ (g, g_{l_1 + 1}^{-1}) \in H_1 : g = \begin{pmatrix} A & 0 \\ 0 & g_{l_1 + 1} \end{pmatrix} \text{ with } A \in U(l_1 - 1) \right\}.
\]
Lemma 6.2. Let $N_1$ be the intersection of $l_1 - 1$ characteristic submanifolds of $M$ belonging to $\mathfrak{g}_1$ as defined in Lemmas 5.4 and 5.10. Then, by Lemma 5.3, we know that $N_1$ is a component of $M^{H_2}$ and $M = H_0N_1$. Therefore we have $N_1 = M^{H_2}$ if, for all $H_0$-orbits $O$, $O^{H_2}$ is connected. Because all orbits are of type $H_0/H_0$, $H_0/H_2$, $H_0/(H_2, \mathbb{Z}_2)$ and

$$
(H_0/H_2)^{H_2} = N_{H_0}H_2/H_2 = H_1/H_2,
$$

$$
(H_0/(H_2, \mathbb{Z}_2))^{H_2} = N_{H_0}H_2/(H_2, \mathbb{Z}_2) = H_1/(H_2, \mathbb{Z}_2),
$$

it follows that $N_1 = M^{H_2}$.

The projection $H_1 \to H_1/H_2$ induces an isomorphism $S \to H_1/H_2$. Therefore $S$ acts freely on $(H_0/H_2)^{H_2}$. Hence, $S$ acts effectively on $N_1$.

By Corollary 5.7, $N_S^S = M^{H_0}$ has codimension two in $N_1$.

After these general remarks we first discuss the case where there are no exceptional $SO(2l_1)$-orbits. That means the case where there are no orbits of type $SO(2l_1)/S(O(2l_1 - 1) \times O(1))$. Then the induced $U(l_1)$-action also has no exceptional orbits. Moreover, by Corollary 5.7, $M$ is a special $SO(2l_1)$- $U(l_1)$-manifold in the sense of Jänich [9].

At first we discuss the question under which conditions the action of $U(l_1) \times G_2$ on a torus manifold satisfying the above conditions on the $U(l_1)$-orbits and having no exceptional $U(l_1)$-orbits extends to an action of $SO(2l_1) \times G_2$.

Let $X$ be the orbit space of the $U(l_1)$-action on $M$. Then, by [9, p. 303], $X$ is a manifold with boundary such that the interior $\tilde{X}$ of $X$ corresponds to orbits of type $U(l_1)/U(l_1 - 1)$ and the boundary $\partial X$ to the fixed points. The action of $G_2$ on $M$ induces a natural action of $G_2$ on $X$.

Following Jänich [9] we may construct from $M$ a manifold $M \ominus M^{U(l_1)}$ with boundary, on which $U(l_1) \times G_2$ acts such that all orbits of the $U(l_1)$-action on $M \ominus M^{U(l_1)}$ are of types $U(l_1)/U(l_1 - 1)$ and $(M \ominus M^{U(l_1)})/U(l_1) = X$. Denote by $P_M$ the $G_2$-equivariant principal $S^1$-bundle

$$
\left( M \ominus M^{U(l_1)} \right)^{U(l_1-1)} \to X.
$$

Lemma 6.1. Let $M$ be a torus manifold with $U(l_1) \times G_2$-action such that all $U(l_1)$-orbits are of type $U(l_1)/U(l_1 - 1)$ or $U(l_1)/U(l_1)$. Then the action of $U(l_1) \times G_2$ on $M$ extends to an action of $SO(2l_1) \times G_2$ if and only if there is a $G_2$-equivariant $\mathbb{Z}_2$-principal bundle $P'_M$ such that

$$
P_M = S^1 \times _{\mathbb{Z}_2} P'_M,
$$

where the action of $G_2$ on $S^1$ is trivial.

Proof. If the action extends to an $SO(2l_1) \times G_2$-action, then $SO(2l_1) \times G_2$ acts on $M \ominus M^{U(l_1)}$. Therefore $P'_M = (M \ominus M^{U(l_1)})^{SO(2l_1-1)} \to X$ is such a $G_2$-equivariant $\mathbb{Z}_2$-principal bundle.

If there is such a $G_2$-equivariant $\mathbb{Z}_2$-bundle $P'_M$, then by a $G_2$-equivariant version of Jänich’s Klassifikationssatz [9] there is a torus manifold $M'$ with $SO(2l_1) \times G_2$-action with $M'/U(l_1) = X$ and $P_M = S^1 \times _{\mathbb{Z}_2} P'_M = P'_M$. Therefore $M'$ and $M$ are $U(l_1) \times G_2$-equivariantly diffeomorphic. □

Lemma 6.2. Let $M, M'$ be torus manifolds with $SO(2l_1) \times G_2$-action such that there are no exceptional $SO(2l_1)$-orbits and $H_1(M; \mathbb{Z})$ and $H_1(M'; \mathbb{Z})$ are torsion.
If there is a $U(l_1) \times G_2$-equivariant diffeomorphism $f : M \to M'$, then there is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$. Moreover, $g$ and $f$ induce the same map on $M/U(l_1) - B$, where $B$ is a collar of $\partial(M/U(l_1))$.

Proof. The map $f$ induces a $G_2$-equivariant diffeomorphism $\hat{f} : X = M/\text{SO}(2l_1) \to M'/\text{SO}(2l_1)$. We use this map to identify these spaces. It follows from [3, p. 91] and the equality $H_1(X; \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ that $H_1(X; \mathbb{Z})$ is torsion. Hence, $H^1(X; \mathbb{Z}) = 0$.

Recall that for the universal principal $\mathbb{Z}_2$-bundle $P \to \mathbb{R}P^\infty$, the first Chern-class of the principal $S^1$-bundle $S^1 \times_{\mathbb{Z}_2} P \to \mathbb{R}P^\infty$ is given by $\delta w_1(P)$, where $\delta : H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^2(\mathbb{R}P^\infty; \mathbb{Z})$ is the Bockstein homomorphism and $w_1(P)$ is the first Stiefel-Whitney class of $P$. By naturality, this relation also holds for any principal $\mathbb{Z}_2$-bundle over $X$. Because $H^1(X; \mathbb{Z}) = 0$, the Bockstein homomorphism $\delta : H^1(X; \mathbb{Z}_2) \to H^2(X; \mathbb{Z})$ is injective.

Therefore, the principal $S^1$-bundle $P_M \to X$ has up to isomorphism at most one restriction of structure group to $\mathbb{Z}_2$. Therefore the two restrictions of the structure group induced by the $SO(2l_1)$-actions on $M, M'$ are the same up to a $G_2$-equivariant isomorphism.

Therefore, by the proof of Jänich’s Klassifikationssatz, there is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$, which induces the same map as $f$ outside a neighborhood of $\partial X$.

Now we turn to the case where there are exceptional $SO(2l_1)$-orbits. Then we have:

**Theorem 6.3.** Let $M, M'$ be two simply connected torus manifolds with $SO(2l_1) \times G_2$-action. Then $M$ and $M'$ are $SO(2l_1) \times G_2$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.

Proof. In this proof we take all cohomology groups with coefficients in $\mathbb{Z}$. Let $f : M \to M'$ be a $U(l_1) \times G_2$-equivariant diffeomorphism. Moreover, let $A, A'$ be the union of the exceptional $U(l_1)$-orbits in $M, M'$, respectively. Because the $U(l_1)$-representation $N_x(M^{U(l_1)}, M)$ is the standard representation for all $x \in M^{U(l_1)}$, there are invariant neighborhoods of $M^{U(l_1)}$ and $M^{U(l_1)}$ which do not contain any exceptional orbit. Hence, $A, A'$ are closed submanifolds of $M, M'$.

Denote by $D, D'$ the unit disc bundle in $N(A, M)$ and $N(A', M')$, respectively. Let $h : D \to B \subset M$ and $h' : D' \to B' \subset M'$ be $SO(2l_1) \times G_2$-equivariant tubular neighborhoods of $A$ and $A'$.

Then, by Theorems 4.6 and 8.3 of [10] pp. 10, 19], we may assume that $f(B) = B'$ and that $h^{-1} \circ f \circ h$ is a linear map.

It is sufficient to show the following:

1. There is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M - \hat{B} \to M' - \hat{B}'$ such that $g$ and $f$ induce the same maps on $(\partial B)/U(l_1)$.

2. The map $g$ extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $M \to M'$.

If $H_1(M - \hat{B})$ is torsion, we may apply the arguments from the proof of Lemma 6.2 to show (1). Therefore we show that $H_1(M - \hat{B})$ is torsion.

Let $A_1, \ldots, A_k$ be the orientable components of $A$ of codimension two in $M$. We fix orientations for each of these components and for $M$. Let $\tau_1, \ldots, \tau_k \in H^2(M)$ be the Poincaré duals for $A_1, \ldots, A_k$. Because $H_1(M) = 0$, it follows from a universal
coefficient theorem and Poincaré duality that
\[ H^2(M) \cong \text{Hom}(H_2(M), \mathbb{Z}) \cong \text{Hom}(H^{2n-2}(M), \mathbb{Z}), \]
where an isomorphism is given by
\[ \alpha \mapsto (\beta \mapsto \langle \beta \alpha, [M] \rangle). \]
Here we have \( \dim M = 2n \). In particular, \( H^2(M) \) is torsion free.

We claim that the \( \tau_1, \ldots, \tau_k \) are linear independent. Let \( a_1, \ldots, a_k \in \mathbb{Z} \) such that
\[ 0 = \sum_{i=1}^{k} a_i \tau_i. \]
Then we have \( 0 = a_i t^*_{A_i} \tau_i \), where \( t_{A_i} : A_i \to M \) is the inclusion. By restricting to an orbit \( O \) contained in \( A_i \), we get
\[ 0 = a_i t^*_{O, A_i} \tau_i \in H^2(SO(2l_1)/SO(2l_1 - 1) \times O(1)) = \mathbb{Z}_2. \]
Because \( N(A_i, M)|_O = SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 \mathbb{R}^2 \) with \( \mathbb{Z}_2 \) acting on \( \mathbb{R}^2 \) by multiplication with \( -1 \), it follows that \( \partial^*_{O, A_i} \tau_i \neq 0 \). Therefore \( a_i \) is divisible by two.

Hence, we may replace \( a_i \mapsto \frac{1}{2}a_i \) in (6.1). Since the above arguments then hold for the new \( a_i \), we see that the original \( a_i \) are divisible by arbitrary high powers of two. Therefore they must vanish.

There is an exact sequence
\[ H^{2n-2}(M) \to H^{2n-2}(A) \to H^{2n-1}(M, A) \to 0. \]
Because, by [3, p. 185], there are no components of \( A \) which have codimension one in \( M \), there is an isomorphism
\[ H^{2n-2}(A) \cong \mathbb{Z}^k \oplus (\mathbb{Z}_2)^{k_1}, \]
where \( k_1 \) is the number of non-orientable components of codimension two of \( A \). Let
\[ \phi : H^{2n-2}(A) \to \mathbb{Z}^k \]
\[ \alpha \mapsto (\langle \alpha, [A_1] \rangle, \ldots, \langle \alpha, [A_k] \rangle). \]
Because the \( \tau_1, \ldots, \tau_k \) are linear independent, it follows that \( \phi \circ t^* : H^{2n-2}(M) \to \mathbb{Z}^k \) has rank \( k \).

Therefore, from the exactness of the above sequence, it follows that \( H^{2n-1}(M, A) \) is torsion. By Poincaré duality and excision, it follows that \( H_1(M - B) \) is torsion. Hence we have proven (1).

Now we prove (2). By Theorem 9.4 of [10, p. 24], it is sufficient to show that
\[ k = h^{l-1} \circ g \circ h : \partial D \to \partial D' \]
extends to an \( SO(2l_1) \times G_2 \)-equivariant diffeomorphism \( D \to D' \).

Let \( O \) be an \( SO(2l_1) \)-orbit in \( A \) and \( S \to O \) be the restriction of the sphere bundle \( \partial D \to A \). Because \( f \) and \( g \) induce the same maps on the orbit space \( (\partial B)/U(l_1) \) and \( S \) is \( SO(2l_1) \)-invariant, we have \( k(S) = h^{l-1} \circ f \circ h(S) = S' \). Because \( h^{l-1} \circ f \circ h : D \to D' \) is a linear map, we see that \( S' \) is the restriction of the sphere bundle \( \partial D' \to A' \) to an \( SO(2l_1) \)-orbit \( O' \).

We may choose \( SO(2l_1) \)-equivariant bundle isomorphisms
\[ k_1 : SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 S^n \to S \]
and
\[ k'_1 : SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 S^n \to S'. \]
Because $f$ and $g$ induce the same maps on the orbit space $S/SO(2l_1) = S^m/\mathbb{Z}_2 = \mathbb{R}P^m$ and $h^{-1} \circ f \circ h$ is a linear map, it follows that $k_{i-1} \circ k \circ k_1$ is of the form

$$[gSO(2l_1 - 1), x] \mapsto [gSO(2l_1 - 1), \pm Ax] = [gSO(2l_1 - 1), \pm Ax],$$

where $z \in S(O(2l_1 - 1) \times O(1))/SO(2l_1 - 1) = \mathbb{Z}_2$ and $A \in O(m + 1)$. Therefore $k$ is linear on each fiber. Hence, it extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $D \to D'$.

Let $M$ be a simply connected torus manifold with $SO(2l_1) \times G_2$-action. By Theorem 6.3 there is an admissible triple $(\psi, N, A)$ corresponding to $M$ equipped with the action of $SU(l_1) \times S \times G_2$ as above. The admissible triple $(\psi, N, A)$ determines the $SU(l_1) \times S \times G_2$-equivariant diffeomorphism type of $M$. With Theorem 6.3 we see that the $SO(2l_1) \times G_2$-equivariant diffeomorphism type of $M$ is determined by $(\psi, N, A)$.

**Lemma 6.4.** Let $M$ be a torus manifold with $G_1 \times G_2$-action, where $G_1 = SO(2l_1)$ is elementary and $G_2$ is a not necessarily connected Lie group. If $M^{SO(2l_1)}$ is connected, then $G_2$ acts orientation preserving on $N(M^{SO(2l_1)}, M)$. Therefore $G_2$ acts orientation preserving on $M$ if and only if it acts orientation preserving on $M^{SO(2l_1)}$.

**Proof.** Let $g \in G_2$, $x \in M^{SO(2l_1)}$ and $y = gx \in M^{SO(2l_1)}$. Because $M^{SO(2l_1)}$ is connected there is an orientation preserving $SO(2l_1)$-invariant isomorphism

$$N_x(M^{SO(2l_1)}, M) \cong N_y(M^{SO(2l_1)}, M).$$

Therefore $g : N_x(M^{SO(2l_1)}, M) \to N_y(M^{SO(2l_1)}, M)$ induces an automorphism $\phi$ of the $SO(2l_1)$-representation $N_x(M^{SO(2l_1)}, M)$ which is orientation preserving if and only if $g$ is orientation preserving.

Because, by Lemma 6.3, $N_x(M^{SO(2l_1)}, M)$ is just the standard real representation of $SO(2l_1)$, its complexification $N_x(M^{SO(2l_1)}, M) \otimes \mathbb{C}$ is an irreducible complex representation. Therefore, by Schur’s Lemma, there is a $\lambda \in \mathbb{C} - \{0\}$ such that for all $a \in N_x(M^{SO(2l_1)}, M)$,

$$\phi(a) \otimes 1 = \phi c(a \otimes 1) = a \otimes \lambda.$$

This equation implies that $\lambda \in \mathbb{R} - \{0\}$ and $\phi(a) = \lambda a$. Therefore $\phi$ is orientation preserving.

7. **The case $G_1 = SO(2l_1 + 1)$**

In this section we discuss actions of groups, which have a covering group, whose action on $M$ factors through $\hat{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$ elementary. In the case $G_1 = SO(3)$ we also assume $\# \mathbb{S}_1 = 1$ or that the principal orbit type of the $SO(3)$-action on $M$ is given by $SO(3)/SO(2)$.

It is shown that a torus manifold with $\hat{G}$-action is a product of a sphere and a torus manifold with $G_2$-action or the blow up along the fixed points of $G_1$ is a fiber bundle over a real projective space.

We assume that $T_1 = T \cap G_1$ is the standard maximal torus of $G_1$. 


7.1. The $G_1$-action on $M$.

**Lemma 7.1.** Let $\hat{G} = G_1 \times G_2$ with $G_1 = SO(2l_1+1)$ and let $M$ be a torus manifold with $G$-action such that $G_1$ is elementary. If $l_1 > 1$ there is, by Lemma 3.3, a component $N_1$ of $\bigcap_{M_i \in \mathcal{S}_1} M_i$ with $N_1^T \neq \emptyset$. If $l_1 = 1$ let $N_1$ be a characteristic submanifold belonging to $\mathcal{S}_1$. Then:

1. $N_1$ is a component of $M^{SO(2l_1)}$.
2. $M = G_1 N_1$.

**Proof.** Let $x \in N_1^T$. Then, by Lemmas 3.1, 3.4 and Remark 3.2, $G_1 x = SO(2l_1)$. Let $T_2$ be the maximal torus $T \cap G_2$ of $G_2$. On the tangent space of $M$ in $x$ we have the $SO(2l_1) \times T_2$-representation

$$T_x M = N_x (G_1 x, M) \oplus T_x G_1 x.$$  

By Lemma 3.1, $T_2$ acts trivially on $G_1 x$. Moreover, $T_2$ acts almost effectively on $N_x (G_1 x, M)$. Therefore it follows by dimensional reasons that $N_x (G_1 x, M)$ splits as a sum of complex one-dimensional $SO(2l_1) \times T_2$-representations. If $l_1 > 1$, $SO(2l_1)$ has no non-trivial one-dimensional complex representations. Therefore we have

$$T_x M = \bigoplus V_i \oplus W,$$

where the $V_i$ are one-dimensional complex representations of $T_2$ and $W$ is the standard real representation of $SO(2l_1)$.

If $l_1 = 1$ and $\# \mathcal{S}_1 = 2$, then $SO(2l_1)$ acts trivially on $N_x (G_1 x, M)$ because $SO(3)/SO(2)$ is the principal orbit type of the $SO(3)$-action on $M$, p. 181.

If $l_1 = 1$ and $\# \mathcal{S}_1 = 1$, then, by the discussion leading to Convention 3.5, $SO(2)$ acts trivially on $N_x (G_1 x, M)$. Therefore in these cases $T_2 M$ splits as in (7.1).

Because $N_x (G_1 x, M)$ is the tangent space of $N_1$ in $x$ the maximal torus $T_1$ of $G_1$ acts trivially on $N_1$. Therefore $N_1$ is the component of $M^{T_1}$ which contains $x$. Because $T_x N_1 = (T_x M)^{T_1} = (T_x M)^{SO(2l_1)}$, $N_1$ is a component of $M^{SO(2l_1)}$.

Now we prove (2). Let $y \in N_1$. Then there are the following possibilities:

- $G_{1y} = G_1$,
- $G_{1y} = SO(2l_1) \times O(1)$ and dim $G_1 y = 2l_1$.
- $G_{1y} = SO(2l_1)$ and dim $G_1 y = 2l_1$.

If $g \in G_1$ such that $g y \in N_1$, then

$$gG_1 y (g)^{-1} = G_1 y \in \{SO(2l_1) \times O(1), SO(2l_1), G_1\}$$

and

$$g \in N_{G_1} G_{1y} = \begin{cases} G_1, & \text{if } y \in M^{G_1}, \\ SO(2l_1) \times O(1), & \text{if } y \notin M^{G_1}. \end{cases}$$

Therefore $G_1 y \cap N_1 \subset SO(2l_1) \times O(1)y$ contains at most two elements. If $y$ is not fixed by $G_1$, then one sees as in the proof of Lemma 3.3 that $G_1 y$ and $N_1$ intersect transversely in $y$.

Therefore $G_1 (N_1 - N_1^{G_1})$ is open in $M - M^{G_1}$ by Lemma A.5. Because $M^{G_1}$ has codimension at least three, $M - M^{G_1}$ is connected. But

$$G_1 \left( N_1 - N_1^{G_1} \right) = G_1 N_1 \cap (M - M^{G_1})$$

is also closed in $M - M^{G_1}$. Hence

$$M - M^{G_1} = G_1 \left( N_1 - N_1^{G_1} \right) = G_1 N_1 - N_1^{G_1}.$$
Therefore one sees as in the proof of Lemma \cite{5.3} that
\[ M = G_1 N_1 \Pi \left( M^{G_1} - N_1^{G_1} \right). \]
Because \( G_1 N_1 \) and \( M^{G_1} - N_1^{G_1} \) are closed in \( M \), the statement follows. \( \square \)

**Corollary 7.2.** If in the situation of Lemma \cite{5.4} the \( G_1 \)-action on \( M \) has no fixed point in \( M \), then \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \) or \( M = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 \), where \( \mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1) \).

In the second case the \( \mathbb{Z}_2 \)-action on \( N_1 \) is orientation reversing.

If \( l_1 = 1 \) and \( \# Z_1 = 1 \), then we have \( M = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N_1 \). If \( l_1 = 1 \) and \( \# Z_1 = 2 \), then we have \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \).

**Proof.** Let \( g \in S(O(2l_1) \times O(1)) = N_{G_1} SO(2l_1) \). Then \( gN_1 \) is a component of \( M^{SO(2l_1)} \). Because \( N_1 \subset M^{SO(2l_1)} \), \( gN_1 \) only depends on the class \( gSO(2l_1) \in S(O(2l_1) \times O(1))/SO(2l_1) = \mathbb{Z}_2 \).

Therefore there are two cases:

1. There is a \( g \in S(O(2l_1) \times O(1)) \) such that \( gN_1 \neq N_1 \).
2. The submanifold \( N_1 \) is \( S(O(2l_1) \times O(1)) \)-invariant, i.e. \( gN_1 = N_1 \) for all \( g \in S(O(2l_1) \times O(1)) \).

If \( l_1 = 1 \) and \( \# Z_1 = 1 \), then \( N_1 \) is the only characteristic submanifold of \( M \) belonging to \( Z_1 \). Therefore only the second case occurs.

If \( l_1 = 1 \) and \( \# Z_1 = 2 \), then there is a \( g_1 \in N_{G_1}, T_1 \) such that \( N_1 \neq g_1 N_1 \).

Therefore we are in the first case.

In general we have \( M = G_1 \times N_1 / \sim \) with
\[
(g_1, y_1) \sim (g_2, y_2) \iff g_1 y_1 = g_2 y_2 \iff g_2^{-1} g_1 y_1 = y_2 \iff g_2^{-1} g_1 \in S(O(2l_1) \times O(1)) \text{ and } g_2^{-1} g_1 y_1 = y_2.
\]

In case \( 1 \) the last statement is equivalent to
\[
g_2^{-1} g_1 \in SO(2l_1) \text{ and } g_2^{-1} g_1 y_1 = y_2.
\]

Therefore we get \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \).

In case \( 2 \) we have as in the proof of Corollary \cite{5.6}
\[
M = SO(2l_1 + 1) \times S(O(2l_1) \times O(1)) N_1 = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N_1.
\]

This equation implies that \( M \) is the orbit space of a diagonal \( \mathbb{Z}_2 \)-action on \( SO(2l_1 + 1)/SO(2l_1) \times N_1 \).

Because \( M \) is orientable this action has to be orientation preserving. But the \( \mathbb{Z}_2 \)-action on \( SO(2l_1 + 1)/SO(2l_1) \) is orientation reversing. Therefore the \( \mathbb{Z}_2 \)-action on \( N_1 \) is also orientation reversing. \( \square \)

**Corollary 7.3.** In the situation of Lemma \cite{7.1} \( M^{G_1} \subset N_1 \) is empty or has codimension one in \( N_1 \).
Proof. By Lemma 7.1 it is clear that \( M^{G_1} \subset N_1 \). For \( y \in M^{G_1} \) consider the \( G_1 \) representation \( T_y M \). Because \( N_1 \) is a component of \( M^{SO(2l_1)} \), the restriction of \( T_y M \) to \( SO(2l_1) \) equals the \( SO(2l_1) \)-representation \( T_x M \), where \( x \in N_1^T \).

Because, by Lemma 3.4, \( T_x M \) is a direct sum of a trivial representation and the standard real representation of \( SO(2l_1) \) and \( T_1 \subset SO(2l_1), T_y M \) is a sum of a trivial and the standard real representation of \( SO(2l_1 + 1) \) by [1, p. 167]. Therefore \( M^{G_1} \subset N_1 \) has codimension one.

\[ \square \]

7.2. Blowing up along \( M^{G_1} \). As in section 5 we discuss the question as to when a manifold of the form given in Corollary 7.2 is a blow up.

If \( M \) is the blow up of \( M \) along \( M^{G_1} \), then there is an equivariant embedding of \( P_\mathbb{R}(N(M^{G_1}, M)) \) into \( M \). Therefore the \( G_1 \)-action on \( M \) has an orbit of type \( SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \). This fact shows that \( M \) is of the form

\[
SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} \tilde{N}_1,
\]

where \( \tilde{N}_1 \) is the proper transform of \( N_1 \). By Corollary 4.5, \( \tilde{N}_1 \) and \( N_1 \) are \( G_2 \)-equivariantly diffeomorphic. Because \( M^{G_1} \) has codimension one in \( N_1 \), the \( \mathbb{Z}_2 \)-action on \( N_1 \) has a fixed point component of codimension one.

The following lemma shows that these two conditions are sufficient.

Lemma 7.4. Let \( N_1 \) be a torus manifold with \( G_2 \)-action. Assume that there are a non-trivial orientation reversing action of \( \mathbb{Z}_2 = S((O(2l_1) \times O(1))/SO(2l_1)) \) on \( N_1 \), which commutes with the action of \( G_2 \), and a closed codimension one submanifold \( A \) of \( N_1 \), on which \( \mathbb{Z}_2 \) acts trivially.

Let \( E' = N(A, N_1) \) be equipped with the action of \( G_2 \) induced from the action on \( N_1 \) and the trivial action of \( \mathbb{Z}_2 \). Denote by \( W \) the standard real representation of \( SO(2l_1 + 1) \). Then:

1. \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1 \) is orientable.
2. The normal bundle of \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1 \) is isomorphic to the tautological bundle over \( P_\mathbb{R}(E \otimes W \cup \{0\}) \).

The lemma guarantees, together with the discussion at the end of section 3 that one may remove \( SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A \) from \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1 \) and replace it by \( A \) to get a torus manifold with \( G \)-action \( M \) such that \( M^{SO(2l_1 + 1)} = A \). The blow up of \( M \) along \( A \) is \( SO(2l_1 + 1)/S(O(2l_1)) \times_{\mathbb{Z}_2} N_1 \).

\[ \text{Proof.} \] The diagonal \( \mathbb{Z}_2 \)-action on \( SO(2l_1 + 1)/SO(2l_1) \times N_1 \) is orientation preserving. Therefore \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1 \) is orientable.

The normal bundle of \( SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A \) in \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1 \) is given by \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N(A, N_1) \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N(A, N_1) & \xrightarrow{f} & P_\mathbb{R}(E' \otimes W) \times E' \otimes W \\
\pi_1 & & \pi_1 \\
SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A & \xrightarrow{g} & P_\mathbb{R}(E' \otimes W),
\end{array}
\]

where the vertical maps are the natural projections and \( f, g \) are given by

\[
f([hSO(2l_1) : m]) = ([m \otimes he_1], m \otimes he_1)
\]
and
\[ g(hS(O(2l_1) \times O(1)), q) = [m_q \otimes h e_1], \]
where \( e_1 \in W - \{0\} \) is fixed such that for all \( g' \in SO(2l_1) \), \( g'e_1 = e_1 \) and \( m_q \neq 0 \) is some element of the fiber of \( E' \) over \( q \).

The map \( f \) induces an isomorphism of the normal bundle of the normal bundle of 
\[ SO(2l_1)/SO(2l_1) \times O(1)) \times A \]
in \( SO(2l_1)/SO(2l_1) \times \mathbb{Z}_2 N_1 \) and the tautological bundle over \( P_\mathbb{Z}(E' \otimes W + \{0\}). \quad \square \]

**Lemma 7.5.** If \( l_1 > 1 \), in the situation of Lemma [7.1], then \( \bigcap_{M_i \in \mathfrak{F}_1} M_i = M^{SO(2l_1)} \) has at most two components. It has two components if and only if \( M = S^{2l_1} \times N_1 \).

**Proof.** If \( M = S^{2l_1} \times N_1 \), then \( \bigcap_{M_i \in \mathfrak{F}_1} M_i = \{N, S\} \times N_1 \), where \( N, S \) are the north and the south poles of the sphere, respectively. Otherwise the blow up of \( M \) along \( M^{SO(2l_1+1)} \) is given by \( S^{2l_1} \times \mathbb{Z}_2 N_1 \), which is a fiber bundle over \( RP^{2l_1} \).

The characteristic submanifolds of \( S^{2l_1} \times \mathbb{Z}_2 N_1 \), which are permuted by \( W(G_1) \), are given by the preimages of the following submanifolds of \( RP^{2l_1} \):
\[ \mathbb{R} P^{2l_1-2} = \{(x_1 : x_2 : \cdots : x_{2i-2} : 0 : 0 : x_{2i+1} : \cdots : x_{2l_1+1}) : (x_1 \in \mathbb{R} P^{2l_1}) \}, \quad i = 1, \ldots, l_1. \]

These characteristic submanifolds are also given by the proper transforms \( \tilde{M}_i \) of the characteristic submanifolds \( M_i \in \mathfrak{F}_1 \) of \( M \). Because
\[ \bigcap_{i=1}^{l_1} \mathbb{R} P^{2l_1-2} = \{(0 : 0 : \cdots : 0 : 1)\}, \]
it follows that
\[ \bigcap_{M_i \in \mathfrak{F}_1} \tilde{M}_i = \tilde{N}_1 = \tilde{M}^{SO(2l_1)}. \]

Therefore, with Lemma 4.3 and Corollary 7.3
\[ \bigcap_{M_i \in \mathfrak{F}_1} M_i = N_1 = M^{SO(2l_1)} \]
follows. In particular, \( \bigcap_{M_i \in \mathfrak{F}_1} M_i \) is connected. \quad \square

**Lemma 7.6.** If \( l_1 = 1 \), in the situation of Lemma 7.1, then the following statements are equivalent:
- \( M^{SO(2)} \) has two components.
- \( \#\mathfrak{F}_1 = 2 \).
- \( M = S^2 \times N_1 \).

If \( l_1 = 1 \) and \( \#\mathfrak{F}_1 = 1 \), then \( M^{SO(2)} \) is connected.

**Proof.** At first we prove that all components of \( M^{SO(2)} \) are characteristic submanifolds of \( M \) belonging to \( \mathfrak{F}_1 \). By Lemma 7.1, \( N_1 \) is a characteristic submanifold of \( M \) and a component of \( M^{SO(2)} \) such that \( G_1 N_1 = M \). Therefore, if \( x \in M^{SO(2)} \), then there is a \( g \in N_{G_1} SO(2) \) such that \( g^{-1} x \in N_1 \). This implies \( x \in g N_1 \). Because \( gN_1 \) is a characteristic submanifold belonging to \( \mathfrak{F}_1 \) and a component of \( M^{SO(2)} \), it follows that \( M^{SO(2)} \) is a union of characteristic submanifolds of \( M \) belonging to \( \mathfrak{F}_1 \).

Now assume that \( \#\mathfrak{F}_1 = 1 \). Then we have \( M^{SO(2)} = N_1 \). Therefore \( M^{SO(2)} \) is connected.
Now assume that \( M = SO(3)/SO(2) \times N_1 \). Then it is clear that \( M^{SO(2)} \) has two components.

Now assume that \( M^{SO(2)} \) has two components. Because these components are characteristic submanifolds belonging to \( \mathfrak{g}_1 \), it follows that \( \# \mathfrak{g}_1 = 2 \).

Now assume that \( \# \mathfrak{g}_1 = 2 \). If there is no \( G_1 \)-fixed point, then it follows from Corollary 7.2.

\[
M = SO(3)/SO(2) \times N_1.
\]

Assume that there is a \( G_1 \)-fixed point in \( M \). Then the blow up of \( M \) along \( M^{G_1} \) contains an orbit of type \( SO(3)/S(O(2) \times O(1)) \). Now Corollary 7.2 implies \( \# \mathfrak{g}_1 = 1 \). Therefore there is no \( G_1 \)-fixed point if \( \# \mathfrak{g}_1 = 2 \).

### 7.3. Admissible pairs

We are now in the position to state another classification theorem. To do so, we use the following definition.

**Definition 7.7.** Let \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SO(2l_1 + 1) \). Then a pair \((N, A)\) with

- \( N \) a torus manifold with \( G_2 \times \mathbb{Z}_2 \)-action such that the \( \mathbb{Z}_2 \)-action is orientation reversing or trivial,
- \( A \subset N \) the empty set or a closed \( G_2 \times \mathbb{Z}_2 \)-invariant submanifold of codimension one, on which \( \mathbb{Z}_2 \) acts trivially, such that if \( A \neq \emptyset \), then \( \mathbb{Z}_2 \) acts non-trivially on \( N \),

is called admissible for \((\tilde{G}, G_1)\).

We say that two admissible pairs \((N, A), (N', A')\) are equivalent if there is a \( G_2 \times \mathbb{Z}_2 \)-equivariant diffeomorphism \( \phi : N \to N' \) such that \( \phi(A) = A' \).

**Theorem 7.8.** Let \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SO(2l_1 + 1) \). There is a one-to-one correspondence between the \( \tilde{G} \)-equivariant diffeomorphism classes of torus manifolds with \( \tilde{G} \)-actions such that \( G_1 \) is elementary and equivalence classes of admissible pairs for \((\tilde{G}, G_1)\).

**Proof.** Let \( M \) be a torus manifold with \( \tilde{G} \)-action. If \( \bigcap_{M_i \in \mathfrak{g}_1} M_i \) has two components and \( l_1 > 1 \) or \( \# \mathfrak{g}_1 = 2 \) and \( l_1 = 1 \), then we assign to \( M \) the admissible pair \( \Phi(M) = (N_1, \emptyset) \), where \( N_1 \) is a component of \( \bigcap_{M_i \in \mathfrak{g}_1} M_i \) or a characteristic submanifold belonging to \( \mathfrak{g}_1 \) in the case \( l_1 = 1 \). The action of \( \mathbb{Z}_2 \) is trivial in this case.

If \( \bigcap_{M_i \in \mathfrak{g}_1} M_i \) is connected and \( l_1 > 1 \) or \( \# \mathfrak{g}_1 = 1 \) and \( l_1 = 1 \), then we assign to \( M \) the pair

\[
\Phi(M) = \left( \bigcap_{M_i \in \mathfrak{g}_1} M_i, M^{SO(2l_1 + 1)} \right).
\]

Because \( \bigcap_{M_i \in \mathfrak{g}_1} M_i = M^{SO(2l_1)} \) there is a non-trivial action of

\[
\mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1)
\]
on \( \bigcap_{M_i \in \mathfrak{g}_1} M_i \).

Now let \((N, A)\) be an admissible pair for \((\tilde{G}, G_1)\). If the \( \mathbb{Z}_2 \)-action on \( N \) is trivial, we have \( A = \emptyset \) and assign to \((N, \emptyset)\) the torus manifold with \( \tilde{G} \)-action \( \Psi((N, \emptyset)) = S^{2l_1} \times N \).

If the \( \mathbb{Z}_2 \)-action on \( N \) is non-trivial, we assign to \((N, A)\) the blow down \( \Psi((N, A)) \) of \( SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N \) along \( SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A \).

By Lemma 7.4 it is clear that this construction gives a one-to-one correspondence between torus manifolds with \( \tilde{G} \)-action such that \( \bigcap_{M_i \in \mathfrak{g}_1} M_i \) has two components.
and $l_1 > 1$ and admissible pairs with trivial $\mathbb{Z}_2$-action. With Lemma 7.6 we see that an analogous statement holds for $l_1 = 1$ and $\#\tilde{S}_1 = 2$.

Now let $(N, A)$ be an admissible pair such that $\mathbb{Z}_2$ acts non-trivially on $N$. Then the discussion after Lemma 7.4 shows that $\Phi(\Psi((N, A)))$ is equivalent to $(N, A)$.

If $M$ is a torus manifold with $G_1 \times G_2$-action such that $G_1$ is elementary and $N_1 = \bigcap_{M_i \in \tilde{S}_1} M_i$ is connected, the blow up of $M$ along $M^{SO(2l_1+1)}$ is given by

$$SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N_1.$$ 

Therefore we find that $\Psi(\Phi(M))$ is equivariantly diffeomorphic to $M$. \hfill \Box

8. Classification

Here we use the results of the previous sections to state a classification of torus manifolds with $G$-action. We do not consider actions of groups, which have $SO(2l_1)$ as an elementary factor, because as explained in section 6 these factors may be replaced by $SU(l_1) \times S^1$. We get the classification by iterating the constructions given in Theorem 5.13 and Theorem 7.8.

We illustrate this iteration in the case that all elementary factors of $G$ are isomorphic to $SU(l_i + 1)$. Let $\tilde{G} = \prod_{i=1}^k G_i \times T_0$ and $M$ be a torus manifold with $\tilde{G}$-action such that all $G_i$ are elementary and isomorphic to $SU(l_i + 1)$.

In Theorem 5.13 we constructed a triple $(\psi_1, N_1, A_1)$, which determines the $\tilde{G}$-equivariant diffeomorphism type of $M$. Here $N_1$ is a torus manifold with $\prod_{i=2}^k G_i \times T_0$-action. Therefore there is a triple $(\psi_2, N_2, A_2)$ which determines the $\prod_{i=2}^k G_i \times T_0$-equivariant diffeomorphism type of $N_1$. Because $N_2 \subset N_1$ such that $G_2 N_2 = N_1$ and $A_1$ is $G_2$-invariant, we have $G_2(A_1 \cap N_2) = A_1$. Therefore the $G$-equivariant diffeomorphism type of $M$ is determined by

$$(\psi_1 \times \psi_2, N_2, A_1 \cap N_2, A_2).$$

Continuing in this manner leads to a triple

$$(\psi, N, (A_1, \ldots, A_k)),$$

where $\psi \in \text{Hom}\left(\prod_{i=1}^k S(U(l_i) \times U(1)), T_0\right)$, $N$ is a $2l_0$-dimensional torus manifold and the $A_i$ are codimension two submanifolds of $N$ or empty.

The iteration becomes more complicated if there are more than one elementary factors of $\tilde{G}$ isomorphic to $SO(2l_i + 1)$. To illustrate what happens here, we discuss the case $\tilde{G} = G_1 \times G_2 \times T_0$, where the $G_i$ are elementary and isomorphic to $SO(2l_i + 1)$.

Then, by Theorem 7.8 there is an admissible pair $(N_1, B_1)$ for $(\tilde{G}, G_1)$ corresponding to $M$, where $N_1$ is a torus manifold with $G_2 \times T_0 \times (\mathbb{Z}_2)_1$-action. By Lemmas 7.3 and 7.6 we have two cases:

1. $N_1^{SO(2l_2)}$ has two components.
2. $N_1^{SO(2l_2)}$ is connected.

In the first case we have $N_1 = SO(2l_2 + 1)/SO(2l_2) \times N_2$, where $N_2$ is a $2l_0$-dimensional torus manifold. The action of $(\mathbb{Z}_2)_1$ on $N_1$ commutes with the action of $G_2 \times T_0$. Therefore the action of $(\mathbb{Z}_2)_1$ on $N_1$ splits as a product of an action on $SO(2l_2 + 1)/SO(2l_2)$ and an action on $N_2$. Because there is only one non-trivial action of $\mathbb{Z}_2$ on $SO(2l_2 + 1)/SO(2l_2)$ which commutes with the
action of $SO(2l_1 + 1)$, the $G_2 \times T^{l_0} \times (\mathbb{Z}_2)^1$-equivariant diffeomorphism type of $N_1$ is completely determined by a pair $(N_2, a_{12})$, where $N_2$ is equipped with the action of $T^{l_0} \times (\mathbb{Z}_2)^1$ and $a_{12} \in \{0, 1\}$ is non-zero if and only if the $(\mathbb{Z}_2)^1$-action on $SO(2l_1 + 1)/SO(2l_2)$ is non-trivial.

In the second case the $G_2 \times T^{l_0}$-equivariant diffeomorphism type of $N_1$ is determined by a pair $(N_2, B_2)$, where $N_2 = N_1^{SO(2l_2)}$. Because $N_2$ is $(\mathbb{Z}_2)^1$-invariant in this case, $N_2$ is a torus manifold with $T^{l_0} \times (\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2$-action, where $(\mathbb{Z}_2)^2 = S(O(2l_2) \times O(1))/SO(2l_2)$. We put $a_{12} = 0$ in this case.

As in the case where there are only elementary factors isomorphic to $SU(l_1 + 1)$, one sees that the $G_1 \times G_2 \times T^{l_0}$-equivariant diffeomorphism type of $M$ is determined by

$$(N_2, (N_2 \cap B_1, B_2), a_{12}).$$

There are some relations between $a_{12}$ and $B_1$. For example, if $a_{12} = 1$, then there are no $(\mathbb{Z}_2)^1$-fixed points in $N_1$. Therefore $B_1$ has to be empty.

If there are more than two elementary factors of $\hat{G}$ isomorphic to $SO(2l_1 + 1)$, we have to introduce more numbers $a_{ij}$. There are some relations between the $a_{ij}$ coming from the fact that $M$ is required to be orientable. This will be explained in the proof of Lemma 8.3.

8.1. Admissible 5-tuples. We use the following definition to make the above constructions more formal.

**Definition 8.1.** Let $\hat{G} = \prod_{i=1}^{k} G_i \times G'$ with

$$G_i = \begin{cases} 
SU(l_i + 1) & \text{if } i \leq k_0, \\
SO(2l_i + 1) & \text{if } i > k_0
\end{cases}$$

and $k_0 \in \{0, \ldots, k\}$. Then a 5-tuple

$$(\psi, N, (A_i)_{i=1,\ldots,k_0}, (B_i)_{i=k_0+1,\ldots,k}, (a_{ij})_{k_0+1 \leq i < j \leq k})$$

with

1. $\psi \in \text{Hom}(\prod_{i=1}^{k_0} SU(l_i) \times U(1), Z(G'))$ and $\psi_i = \psi|_{SU(l_i) \times U(1)}$,
2. $N$ a torus manifold with $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-action,
3. $A_i \subseteq N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant closed submanifold of codimension two, on which $\psi_i$ acts trivially, such that if $A_i \neq \emptyset$, then $\ker \psi_i = SU(l_i)$,
4. $B_i \subseteq N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant closed submanifold of codimension one, on which $(\mathbb{Z}_2)_i$ acts trivially, such that if $B_i \neq \emptyset$, then the action of $(\mathbb{Z}_2)_i$ on $N$ is non-trivial,
5. $a_{ij} \in \{0, 1\}$ such that
   (a) if $a_{ij} = 1$, then:
      (i) the action of $(\mathbb{Z}_2)_j$ on $N$ is trivial,
      (ii) $a_{jk} = 0$ for $k > j$,
      (iii) $B_i = \emptyset$,
   (b) if the action of $(\mathbb{Z}_2)_i$ on $N$ is non-trivial, then it is orientation preserving if and only if $\sum_{j>i} a_{ij}$ is odd,
Remark 8.2. By Lemma B.1 two submanifolds $A_1, A_2$ of $N$ satisfying condition (3) intersect transversely if and only if no component of $A_1$ is a component of $A_2$.

By Lemma B.3 two submanifolds $A_1, A_2$ of $N$ satisfying conditions (3) and (4), respectively, always intersect transversely.

By Lemma B.5 two submanifolds $B_1, B_2$ of $N$ satisfying condition (4) intersect transversely if and only if no component of $B_1$ is a component of $B_2$.

Lemma 8.3. Let $\tilde{G}$ be as above. Then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples

$$(\psi, N, (A_i)_{i=1, \ldots, k_0}, (B_i)_{i=1, \ldots, k_0}, (a_{ij})_{0 \leq i < j \leq k_0})$$

for $(\tilde{G}, \prod_{i=1}^k G_i)$ and the equivalence classes of admissible 5-tuples

$$(\psi', N', (A_i')_{i=1, \ldots, k_0}, (B_i')_{i=1, \ldots, k_0}, (a_{ij}')_{0 \leq i < j \leq k_0})$$

for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is elementary for the $G_k \times G'$-action on $N'$.

Proof. At first assume that $G_k = SU(l_k + 1)$. Let $(\psi, N, (A_i)_{i=1, \ldots, k-1}, \emptyset, \emptyset)$ be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is elementary for the $G_k \times G'$-action on $N$.

Let $(\psi_k, N_k, A_k)$ be the admissible triple for $(G_k \times G', G_k)$ which corresponds to $N$ under the correspondence given in Theorem 5.13. Then $N_k$ is a submanifold of $N$. By Lemma B.1 $A_i, i = 1, \ldots, k-1$, intersects $N_k$ transversely. Therefore $N_k \cap A_i$ has codimension 2 in $N_k$. Because $A_i = G_k(N_k \cap A_i)$, $N_k \cap A_i$ has no component which is contained in $A_k$ or $N_k \cap A_j, j \neq i$. Therefore by

$$(\psi \times \psi_k, N_k, (A_1 \cap N_k, \ldots, A_{k-1} \cap N_k, A_k), \emptyset, \emptyset)$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$ is given.

Now let

$$(\psi \times \psi_k, N_k, (A_1, \ldots, A_k), \emptyset, \emptyset)$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. Let $H_0 = G_k \times \text{im} \psi_k$ and $H_1 = S(U(l_k) \times U(1)) \times \text{im} \psi_k$. 

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Then, by Lemma 5.11, the blow down to-one correspondence. preserving on \( \psi, N \)
\[ (8.2) \]
(\( \psi, N, (G_k F(A_1), \ldots, G_k F(A_{k-1})), \emptyset, \emptyset \))

an admissible triple for \((\tilde{G}, \prod_{i=1}^{k-1} G_i)\) is given.

As in the proof of Theorem 6.19 one sees that this construction leads to a one-to-one correspondence.

Now assume that \( G_k = SO(2l_k + 1) \). Let
\[ (8.1) \]
(\( \psi, N, (A_i)_{i=1, \ldots, k_0}, (B_i)_{i=k_0+1, \ldots, k-1}, (a_{ij})_{k_0+1 \leq i < j \leq k-1} \))

be an admissible 5-tuple for \((\tilde{G}, \prod_{i=1}^{k-1} G_i)\) such that \( G_k \) is elementary for the \( G_k \times G' \)-action on \( N \).

At first assume that, for the \( G_k \)-action on \( N \), \( N^{SO(2l_k)} \) is connected. Let \((N_k, B_k)\) be the admissible pair for \((G_k \times G', G_k)\) which corresponds to \( N \) under the correspondence given in Theorem 7.8. Then \( N_k \) is a submanifold of \( N \) which is invariant under the action of \( G' \times \prod_{i=1}^{k-1} (Z_2)_i \), where \((Z_2)_k = SO(2l_k) \times O(1))/SO(2l_k). \)

For \( i < k \), let \( a_{ik} = 0 \).

We claim that by
\[ (8.2) \]
(\( \psi, N, (A_1 \cap N_k, \ldots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \ldots, B_{k-1} \cap N_k, B_k), (a_{ij}) \))

an admissible 5-tuple for \((\tilde{G}, \prod_{i=1}^{k} G_i)\) is given.

At first note that, for \( i = 1, \ldots, k - 1 \), the \( A_i \) and \( B_i \) intersect \( N_k \) transversely by Lemmas 8.1 and 8.3. Therefore \( A_i \cap N_k \) and \( B_i \cap N_k \) has codimension two or one, respectively, in \( N_k \).

One sees as in the case \( G_k = SU(l_k + 1) \) that the \( N_k \cap A_i \) and \( N_k \cap B_i \) intersect pairwise transversely.

Now we verify condition (5) of Definition 8.1 for the 5-tuple \((8.2)\). By Lemma 6.4, \((Z_2)_i, i < k \), acts orientation preserving on \( N \) if and only if it acts orientation preserving on \( N_k \). This proves (5b) because (8.1) is an admissible 5-tuple and \( a_{ik} = 0 \).

Because, by Lemma 7.4, \( G_k N_k = N \), \((Z_2)_i, i < k \), acts trivially on \( N_k \) if and only if it acts trivially on \( N \). This proves (5c) and (5(a)i) because (5a) and (5(a)i) hold for the admissible 5-tuple \((8.1)\) and \( a_{ik} = 0 \).

Because \( a_{ik} = 0 \), (5(a)iii) and (5(a)iiii) are clear.

Now assume that \( N^{SO(2l_k)} \) is non-connected. Then, by Lemmas 6.2 and 7.6 we have
\[ N = SO(2l_k + 1)/SO(2l_k) \times N_k. \]

In this case the \((Z_2)_i\)-action, \( i < k \), on \( N \) commutes with the action of \( SO(2l_k + 1) \). Therefore it splits in a product of an action on \( SO(2l_k + 1)/SO(2l_k) \) and an action on \( N_k \). We put \( a_{ik} = 1 \) if the \((Z_2)_i\)-action on \( SO(2l_k + 1)/SO(2l_k) \) is non-trivial and \( a_{ik} = 0 \) otherwise. Because there is only one non-trivial action of \( Z_2 \) on \( SO(2l_k + 1)/SO(2l_k) \) which commutes with the action of \( SO(2l_k + 1) \), we may recover the action of \((Z_2)_i, i < k \), on \( N \) from the action on \( N_k \) and \( a_{ik} \).
We identify $SO(2l_k)/SO(2l_k) \times N_k$ with $N_k$ and equip it with the trivial action of $(\mathbb{Z}_2)_k = S(O(2l_k) \times O(1))/SO(2l_k)$. We claim that by
\begin{equation}
(\psi, N_k, (A_1 \cap N_k, \ldots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \ldots, B_{k-1} \cap N_k, \emptyset), (a_{ij}))
\end{equation}
an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$ is given.

Conditions $\mathbf{a}$ and $\mathbf{b}$ of Definition 8.1 and the transversality condition are verified as in the previous cases.  

Therefore we only have to verify condition $\mathbf{c}$. Because the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k + 1)/SO(2l_k)$ is orientation reversing, the $(\mathbb{Z}_2)_i$-action, $i < k$ on $N_k$ has the same orientation behavior as the action on $N$ if and only if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k + 1)/SO(2l_k)$ is trivial. By the definition of $a_{ik}$, this is the case if and only if $a_{ik} = 0$. Therefore $\mathbf{c}$ follows because $\mathbf{a}$ is an admissible 5-tuple and $(Z_2)_k$ acts trivially on $N_k$.

If the $(\mathbb{Z}_2)_i$-action on $N_k$ is trivial and non-trivial on $SO(2l_k + 1)/SO(2l_k)$, then the $(\mathbb{Z}_2)_i$-action on $N$ is orientation reversing. Therefore $\sum_{j > i} a_{ij}$ is odd.

The $(\mathbb{Z}_2)_i$-actions on $N_k$ and $SO(2l_k + 1)/SO(2l_k)$ are trivial if and only if the $(\mathbb{Z}_2)_i$-action on $N$ is trivial. Therefore $\sum_{j > i} a_{ij}$ is odd or trivial. This verifies $\mathbf{c}$.

If there is a $j < i$ such that $a_{ij} = 1$, then $(\mathbb{Z}_2)_i$ acts trivially on $N$ because the admissible 5-tuple $\mathbf{a}$ satisfies $\mathbf{a}$. Therefore $a_{ik} = 0$. This proves $\mathbf{c}$.

If the $(\mathbb{Z}_2)_i$-action on $SO(2l_k + 1)/SO(2l_k)$ is non-trivial, the action on $N$ has no fixed points. Therefore $B_i = \emptyset$. This proves $\mathbf{c}$. The property $\mathbf{c}$ is clear.

Now let
\begin{equation}
(\psi, N_k, (A_1, \ldots, A_{k_0}), (B_{k_0+1}, \ldots, B_k), (a_{ij}))
\end{equation}
be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. At first assume that $(\mathbb{Z}_2)_k$ acts non-trivially on $N_k$. Then the blow down $N$ of $\tilde{N} = SO(2l_k + 1)/SO(2l_k) \times (\mathbb{Z}_2)_k N_k$ along $SO(2l_k + 1)/SO(2l_k) \times (\mathbb{Z}_2)_k B_k$ is a torus manifold with $G_k \times G' \times \prod_{i=k_0+1}^{k-1} (\mathbb{Z}_2)_i$-action. As in the case $G_k = SU(l_k + 1)$ one sees that
\begin{equation}
(\psi, N, (G_k F(A_1), \ldots, G_k F(A_{k_0})), (G_k F(B_{k_0+1}), \ldots, G_{k-1} F(B_{k-1})), (a_{ij}))
\end{equation}
is an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$.

If $(\mathbb{Z}_2)_k$ acts trivially on $N_k$, then put
\begin{equation}
N = SO(2l_k + 1)/SO(2l_k) \times N_k.
\end{equation}

Here $(\mathbb{Z}_2)_i$, $i < k$, acts by the product action of the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k + 1)/SO(2l_k)$ and the action on $N_k$ if $a_{ik} = 1$. Otherwise $(\mathbb{Z}_2)_i$, acts by the product action of the trivial action on $SO(2l_k + 1)/SO(2l_k)$ and the action on $N_k$. Now by
\begin{align*}
(\psi, N, (SO(2l_k + 1)/SO(2l_k) \times A_1, \ldots, SO(2l_k + 1)/SO(2l_k) \times A_{k_0}),)
& \quad (SO(2l_k + 1)/SO(2l_k) \times B_{k_0+1}, \ldots, SO(2l_k + 1)/SO(2l_k) \times B_{k-1}), (a_{ij}))
\end{align*}
an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem $\mathbf{a}$ one sees that this construction leads to a one-to-one correspondence.\qed
Let $\tilde{G} = \prod_i G_i \times T^{l_0}$ and

$$(\psi, M, (A_i), (B_i), (a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is an elementary factor of $\prod_{i \geq k} G_i \times T^{l_0}$ for the action on $M$. Furthermore, let

$$(\psi', N, (A'_i), (B'_i), (a'_{ij}))$$

be the admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ corresponding to $(\psi, M, (A_i), (B_i), (a_{ij}))$. Then the following lemma shows that $G_i$, $i > k$, is an elementary factor of $\prod_{i \geq k} G_i \times T^{l_0}$ for the action on $M$ if and only if it is an elementary factor of $\prod_{i \geq k+1} G_i \times T^{l_0}$ for the action on $N$.

**Lemma 8.4.** Let $\tilde{G} = G_1 \times G' \times G''$, $M$ be a torus manifold with $\tilde{G}$-action and $N$ be a component of an intersection of characteristic submanifolds of $M$ which is $G_1 \times G'$-invariant and contains a $T$-fixed point $x$ such that $G_1$ acts non-trivially on $N$. Furthermore, assume that $G''$ is a product of elementary factors for the action on $M$.

Then $N$ is a torus manifold with $G_1 \times G' \times T^{l_0}$-action for some $l_0 \geq 0$ and $G_1$ is an elementary factor of $\tilde{G}$, with respect to the action on $M$, if and only if it is an elementary factor of $G_1 \times G' \times T^{l_0}$, with respect to the action on $N$.

**Proof.** Assume that $G_1$ is an elementary factor for one of the two actions on $M$ and $N$. Then $G_1$ is isomorphic to a simple group or Spin(4). If $G_1$ is simple and not isomorphic to $SU(2)$, then the statement is clear.

Therefore there are two cases, $G_1 = SU(2), \text{Spin}(4)$.

If $x$ is not fixed by $G_1$, then $G_1 = SU(2)$ is elementary for both actions on $N$ and $M$ by Lemma 3.1. Therefore we may assume that $x \in N^{G_1} \subset M^{G_1}$. Then there is a bijection

$$\tilde{\mathfrak{f}}_{x,M} \to \tilde{\mathfrak{f}}_{x,N} \coprod \tilde{\mathfrak{f}}_{x,N}^k,$$

where

$$\tilde{\mathfrak{f}}_{x,M} = \{\text{characteristic submanifolds of } M \text{ containing } x\},$$
$$\tilde{\mathfrak{f}}_{x,N} = \{\text{characteristic submanifolds of } N \text{ containing } x\},$$
$$\tilde{\mathfrak{f}}_{x,N}^k = \{\text{characteristic submanifolds of } M \text{ containing } N\}.$$

This bijection is compatible with the actions of the Weyl group of $G_x$.

At first assume that $G_1 = SU(2)$ is elementary for the action on $M$ but not for the action on $N$. Then there is another simple factor $G_2 = SU(2)$ of $G_1 \times G' \times T^{l_0}$ such that $G_1 \times G_2$ is elementary for the action on $N$. At first assume that $G_2$ is elementary for the action on $M$.

Let $w_i \in W(G_1)$, $i = 1, 2$, be generators. Then there are two non-trivial $W(G_1 \times G_2)$-orbits $\tilde{\mathfrak{f}}_1, \tilde{\mathfrak{f}}_2$ in $\tilde{\mathfrak{f}}_{x,M}$. We have:

- $\# \tilde{\mathfrak{f}}_i = 2$, $i = 1, 2$,
- $w_1$, $i = 1, 2$, acts non-trivially on $\tilde{\mathfrak{f}}_i$ and trivially on the other orbit.
But because $G_1 \times G_2$ is elementary for the action on $N$, there is exactly one non-trivial $W(G_1 \times G_2)$-orbit $\mathfrak{F}_1'$ in $\mathfrak{F}_{xN}$. We have:

- $\#\mathfrak{F}_1' = 2$,
- $w_i$, $i = 1, 2$, acts non-trivially on $\mathfrak{F}_1'$.

This is a contradiction.

If $G_2$ is not elementary, then $G_2$ is a simple factor of an elementary factor. In this case the action of $W(G_1 \times G_2)$ on $\mathfrak{F}_{xM}$ behaves as in the first case. Therefore we also get a contradiction in this case.

Under the assumption that $G_1 = \text{Spin}(4)$ is elementary for the action on $M$, a similar argument shows that $G_1$ is elementary for the action on $N$.

Therefore $G_1$ is elementary for the action on $N$ if it is elementary for the action on $M$.

If $G_1$ is elementary for the action on $N$ but not elementary for the action on $M$, then it is a simple factor of an elementary factor $G_1' \neq G_1$ of $\tilde{G}$ or a product $G_2' \times G_3'$ of elementary factors $G_2'$ and $G_3'$ of $\tilde{G}$. But because $G''$ is a product of elementary factors, it contains all elementary factors of $\tilde{G}$ which have non-trivial intersection with $G''$. Because $G_1$ is not contained in $G''$, it follows that $G_1', G_2'$ and $G_3'$ are subgroups of $G_1 \times G'$. Therefore, by the above argument, $G_1'$ or $G_2'$ and $G_3'$ are elementary for the action on $N$. Because elementary factors cannot contain each other, we get a contradiction to the assumption that $G_1$ is elementary for the action on $N$. \hfill $\square$

Recall from section 3 that if $M$ is a torus manifold with $G$-action, then we may assume that all elementary factors of $G$ are isomorphic to $SU(l_i + 1)$, $SO(2l_i + 1)$ or $SO(2l_i)$. That means $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^{k_0}$. Because, as described in section 6, we may replace elementary factors isomorphic to $SO(2l_i)$ by $SU(l_i) \times S^1$, the following theorem may be used to construct invariants of torus manifolds with $\tilde{G}$-action. By Theorem 5.3, these invariants determine the $\tilde{G}$-equivariant diffeomorphism type of simply connected torus manifolds with $\tilde{G}$-action.

**Theorem 8.5.** Let $\tilde{G} = \prod_{i=1}^k G_i \times T^{k_0}$ with

$$G_i = \begin{cases} 
SU(l_i + 1) & \text{if } i \leq k_0, \\
SO(2l_i + 1) & \text{if } i > k_0
\end{cases}$$

and $k_0 \in \{0, \ldots, k\}$. Then there is a one-to-one correspondence between the equivalence classes of admissible $5$-tuples for $\tilde{G}$ and the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-action such that all $G_i$ are elementary.

**Proof.** This follows from Lemma 5.3 and Lemma 5.4 by induction. \hfill $\square$

Using Lemma 4.8 and Theorem 5.10 we get the following result for quasitoric manifolds.

**Theorem 8.6.** Let $\tilde{G} = \prod_{i=1}^k G_i \times T^{k_0}$ with $G_i = SU(l_i + 1)$. Then there is a one-to-one correspondence between the equivalence classes of admissible $5$-tuples for $\tilde{G}$ of the form

$$(\psi, N, (A_i)_{1 \leq i \leq k}, \emptyset, \emptyset)$$

with $N$ quasitoric, $A_i$, $1 \leq i \leq k$, connected and the $\tilde{G}$-equivariant diffeomorphism classes of quasitoric manifolds with $\tilde{G}$-action.
Remark 8.7. Remark 2.9 and Theorem 5.15 lead to a similar result for torus manifolds with $G$-actions whose cohomologies are generated by their degree two parts.

Corollary 8.8. Let $\tilde{G} = \prod_{i=1}^{k} G_i \times T^{l_0}$ with $G_i$ elementary and $M$ a torus manifold with $G$-action. Then $M/G$ has dimension $l_0 + \# \{G_i; \ G_i = SO(2l_i)\}$.

Proof. At first we discuss the case where all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i + 1)$ or $SU(l_i + 1)$, i.e. $\# \{G_i; \ G_i = SO(2l_i)\} = 0$. By Lemma 4.7 replacing $M$ by the blow up $\tilde{M}$ of $M$ along the fixed points of $G_1$ does not change the orbit space. Therefore, by Corollaries 5.6 and 7.2, we have up to finite coverings

$$M/G = (M/G_1)/\left(\prod_{i \geq 2} G_i \times T^{l_0}\right) = (\tilde{M}/G_1)/\left(\prod_{i \geq 2} G_i \times T^{l_0}\right) = ((H_0 \times H_1)/G_1)/\left(\prod_{i \geq 2} G_i \times T^{l_0}\right) = N_1/\left(\prod_{i \geq 2} G_i \times T^{l_0}\right),$$

where $N_1$ is the $\prod_{i \geq 2} G_i \times T^{l_0}$-manifold from the admissible 5-tuple for $(\tilde{G}, G_1)$ corresponding to $M$. Here $H_0, H_1$ are defined as in Lemma 5.3 if $G_1 = SU(l_1 + 1)$. If $G_1 = SO(2l_1 + 1)$, we have $H_0 = SO(2l_1 + 1)$ and $H_1 = SO(2l_1 \times O(1))$.

By iterating this argument we find that $M/G = N/T^{l_0}$ up to finite coverings, where $N$ is the $T^{l_0}$-manifold from the admissible 5-tuple for $\tilde{G}$ corresponding to $M$.

Now we study the case $l_0' = \# \{G_i; \ G_i = SO(2l_i)\} \neq 0$. As discussed in section 6 the orbits of the $G$-action on $M$ do not change if we replace an elementary factor isomorphic to $SO(2l_i)$ by $SU(l_i) \times S^1$. Therefore this replacement does not change the dimension of the orbit space, but it increases $l_0$ by one and decreases $l_0'$ by one. Therefore the statement follows by induction on $l_0'$.

8.2. Applications. Now we apply our classification results to special cases. We first discuss the case where $M$ is a torus manifold with $G$-action such that $G$ is semi-simple and $H^*(M; \mathbb{Z})$ is generated by its degree two part.

Corollary 8.9. If $G$ is semi-simple and $M$ is a torus manifold with $G$-action such that $H^*(M; \mathbb{Z})$ is generated by its degree two part, then

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1)$$

and

$$M = \prod_{i=1}^{k} \mathbb{C}P^{l_i},$$

where each $SU(l_i + 1)$ acts in the usual way on $\mathbb{C}P^{l_i}$ and trivially on $\mathbb{C}P^{l_j}$, $j \neq i$.

Proof. By Lemma 2.8 and Remark 2.9 all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i + 1)$. Because $G$ is semi-simple, there is only one admissible 5-tuple for $\tilde{G}$, namely (const, pt, $\emptyset, \emptyset$). It corresponds to a product of complex projective spaces.

Next we discuss torus manifolds $M$ with $G$-action such that $\dim M/G \leq 1$. With Theorem 5.5 we recover the following two results of S. Kuroki [15, 11]:

Corollary 8.10. Let $M$ be a simply connected torus manifold with $G$-action such that $M$ is a homogeneous $G$-manifold. Then $M$ is a product of even-dimensional spheres and complex projective spaces.
Proof. By Corollary 8.8, the center of $G$ is zero-dimensional. Moreover, all elementary factors of $G$ are isomorphic to $SU(l_i + 1)$ or $SO(2l_i + 1)$. Therefore the admissible 5-tuple corresponding to $M$ is given by

$$(\text{const, pt, } 0, 0, (a_{ij})),$$

where the $a_{ij} \in \{0, 1\}$ are unknown. In particular, no elementary factor of $G$ has a fixed point in $M$. Therefore, by Corollaries 5.6 and 7.2, $M$ splits into a direct product of complex projective spaces and even-dimensional spheres.

**Corollary 8.11.** If the $G$-action on the simply connected torus manifold $M$ has an orbit of codimension one, then $M$ is the projectivisation of a complex vector bundle or a sphere bundle over a product of complex projective spaces and even-dimensional spheres.

Proof. By Corollary 8.8, we may assume that there is a covering group $\tilde{G} = S^1 \times \prod_i G_i$ of $G$ with $G_i$ elementary and $G_i = SU(l_i + 1)$ or $G_i = SO(2l_i + 1)$. We assume that the $G_i$ are sorted in such a way that

- $G_i = SO(2l_i + 1)$ and $G_i$ has no fixed point in $M$ if $i \leq k_0$,
- $G_i = SU(l_i + 1)$ and $G_i$ has no fixed point in $M$ if $k_0 + 1 \leq i \leq k_1$,
- $G_i = SU(l_i + 1), SO(2l_i + 1)$ and $G_i$ has fixed points in $M$ if $i \geq k_1 + 1$,

where $k_0 \leq k_1$ are some constants.

By Corollaries 5.6 and 7.2, we know that $M$ is of the form

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times H_{0k_0+1} \times H_{1k_0+1} \left( H_{0k_0+2} \times H_{1k_0+2} \left( \ldots \left( H_{0k_1} \times H_{1k_1} M' \right) \ldots \right) \right),$$

where

$$H_{0i} = SU(l_i + 1) \times \text{im } \psi_i,$n
$$H_{1i} = S(U(l_i + 1) \times U(1)) \times \text{im } \psi_i,$$

for $i = k_0 + 1, \ldots, k_1$, and $M'$ is a torus manifold with $\tilde{G}'$-action, where $\tilde{G}' = \prod_{i \geq k_1 + 1} G_i \times S^1$.

Because the action of $H_{1i}$ on $H_{0j}$, $j > i$, is trivial and the actions of the $H_{1i}$ on $M'$ commute, $M$ may be written as

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times \left( \prod_{i=k_0+1}^{k_1} H_{0i} \times \prod_{i=1}^{k_1} H_{1i} M' \right).$$

Therefore $M$ is a fiber bundle over a product of even-dimensional spheres and complex projective spaces with fiber $M'$.

Let $(\psi, N', (A_i), (B_i), (a_{ij}))$ be the admissible 5-tuple for $\tilde{G}'$ corresponding to $M'$. Because $\dim N' = 2$ and all $G_i$, $i > k_1$, have fixed points in $M'$, we have

$$N' = S^2, \quad A_i \neq \emptyset, \quad B_i \neq \emptyset.$$

Because the $S^1$-action on $S^2$ has only two fixed points, $N$ and $S$, there are at most two elementary factors isomorphic to $SU(l_i + 1)$. The orientation reversing involutions of $S^2$ which commute with the $S^1$-action and have fixed points are given by “reflections” at $S^1$-orbits. Therefore there is at most one elementary
factor isomorphic to $SO(2l_i + 1)$. If there is such a factor, then there is at most one $G_i$ isomorphic to $SU(l_i + 1)$ because $N$ is mapped to $S$ by such a reflection. Let 

$$\phi_i : S(U(l_i) \times U(1)) \to U(1) \quad \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix} \mapsto g \quad (A \in U(l_i), g \in U(1)).$$

Then we have the following admissible 5-tuples:

<table>
<thead>
<tr>
<th>$G'$</th>
<th>5-tuple</th>
<th>$M'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^4$</td>
<td>$(\emptyset, S^4, \emptyset, \emptyset, \emptyset)$</td>
<td>$S^2$</td>
</tr>
<tr>
<td>$S^4 \times SU(l_1 + 1)$</td>
<td>$(\phi_1^{\emptyset, S^4, ({N}, \emptyset, \emptyset)}$</td>
<td>$\mathbb{C}P^{l_1 + 1}$</td>
</tr>
<tr>
<td>$S^4 \times SO(2l_1 + 1)$</td>
<td>$(\emptyset, S^4, \emptyset, \emptyset, \emptyset)$</td>
<td>$S^{2l_1 + 2}$</td>
</tr>
<tr>
<td>$S^4 \times SU(l_1 + 1) \times SU(l_2 + 1)$</td>
<td>$(\phi_1^{\emptyset, S^4, ({N}, \emptyset, \emptyset)}$</td>
<td>$\mathbb{C}P^{l_1 + l_2 + 1}$</td>
</tr>
<tr>
<td>$S^4 \times SU(l_1 + 1) \times SO(2l_2 + 1)$</td>
<td>$(\phi_1^{\emptyset, S^4, ({N}, \emptyset, \emptyset)}$</td>
<td>$S^{2l_1 + 2l_2 + 2}$</td>
</tr>
</tbody>
</table>

Therefore the statement follows.

Now we turn to the case where $M$ is a torus manifold with $G$-action such that $G$ is semi-simple and has exactly two elementary factors $G_1, G_2$. We start with a discussion of the case where $G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)$.

**Corollary 8.12.** Let $G = G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)$ with $G_1$ and $G_2$ elementary of rank $l_1, l_2$, respectively, and let $M$ be a torus manifold with $G$-action. Then $M$ is one of the following:

$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \mathbb{C}P^{l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2} \times \mathbb{Z}_2, S^{2l_1} \times S^{2l_2} \times \mathbb{Z}_2, S^{2l_2}, S^{2l_1 + 2l_2}$.

Here $S^1_1$ denotes the l-sphere together with the $\mathbb{Z}_2$-action generated by the anti-podal map and $S^2_2$ the l-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

Furthermore, the $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

**Proof.** First assume that $G_1, G_2 \neq SO(2l)$. Then we have the following possibilities for the admissible 5-tuple of $M$:

<table>
<thead>
<tr>
<th>$G_1$</th>
<th>$G_2$</th>
<th>5-tuple</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(l_1 + 1)$</td>
<td>$SU(l_2 + 1)$</td>
<td>$(\text{const}, \emptyset, \emptyset, \emptyset)$</td>
<td>$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}$</td>
</tr>
<tr>
<td>$SU(l_1 + 1)$</td>
<td>$SO(2l_2 + 1)$</td>
<td>$(\text{const}, \emptyset, \emptyset, \emptyset)$</td>
<td>$\mathbb{C}P^{l_1} \times S^{2l_2}$</td>
</tr>
<tr>
<td>$SO(2l_1 + 1)$</td>
<td>$SO(2l_2 + 1)$</td>
<td>$(\emptyset, \emptyset, \emptyset, a_{12} = 0)$</td>
<td>$S^{2l_1} \times S^{2l_2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(\emptyset, \emptyset, \emptyset, a_{12} = 1)$</td>
<td>$S^{2l_1} \times S^{2l_2}$</td>
</tr>
</tbody>
</table>

If $G_1 = SU(l_1 + 1)$ and $G_2 = SO(2l_2)$, then by Corollary 8.10 there is one admissible triple for $(G, G_1)$, namely $(\text{const}, S^{2l_2}, \emptyset)$. It corresponds to $\mathbb{C}P^{l_1} \times S^{2l_2}$.

Now assume that $G_1 = SO(2l_1 + 1)$ and $G_2 = SO(2l_2)$. Let $(N, B)$ be the admissible pair for $(G, G_1)$ corresponding to $M$. Then, by Corollary 8.6 we have $N = S^{2l_2}$. Up to equivariant diffeomorphism there are two orientation reversing involutions on $S^{2l_2}$ which commute with the action of $G_2$, the anti-podal map and a reflection at a hyperplane in $\mathbb{R}^{2l_2 + 1}$. Therefore we have four possibilities for $M$:

$S^{2l_1} \times S^{2l_2}, S^{2l_1 + 2l_2}, S^{2l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2}$.

\(\square\)
For the discussion of the case $G_1 \times G_2 = SO(2l_1) \times SO(2l_2)$ we need the following lemma.

**Lemma 8.13.** Let $\tilde{G} = SO(2l_1) \times S^1$ and $M$ be a simply connected torus manifold with $G$-action such that $SO(2l_1)$ is an elementary factor of $\tilde{G}$, $S^1$ acts effectively on $M$ and $M^{S^1}$ has codimension two in $M$.

Then $M$ is equivariantly diffeomorphic to $\#_i(S^2 \times S^{2l_1})_i$ or $S^{2l_1+2}$.

Here the action of $\tilde{G}$ on $S^{2l_1+2}$ is given by the restriction of the usual $SO(2l_1+3)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^2 \times S^{2l_1}$ is the product action of the usual action of $S^1$ and $SO(2l_1)$ on $S^2$ and $S^{2l_1}$, respectively. Moreover, the connected sums are equivariant.

**Proof.** As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S \times S^1$. Let $(\psi, N, A)$ be the admissible triple corresponding to $M$. Then $\psi$ is completely determined by the discussion in section 6 and $A = N^S = M^{SU(l_1)}$. Furthermore $S$ and $S^1$ act effectively on $N$. All components of $N^S$ and $N^{S^1}$ have codimension two in $N$.

By Lemma 5.17, $N$ is simply connected.

Denote by $\tilde{M}$ the blow up of $M$ along $A$. Because all $T$-fixed points of $M$ are contained in $A$, we have $I \# MT = \# MT$. On the other hand, $\tilde{M}$ is a fiber bundle with fiber $N$ over $CP^{l_1-1}$. Therefore we have $I \# N^{S \times S^1} = \# \tilde{M}$.

From this $\# MT = \# N^{S \times S^1}$ follows.

Because $S$ and $S^1$ both act effectively on $N$ such that their fixed point sets have codimension two, it follows from the classification of simply connected four-dimensional $T^2$-manifolds given in [20] pp. 547, 549 that the $T$-equivariant diffeomorphism type of $N$ is determined by $\# MT$ and that $\# MT$ is even.

Therefore the $S \times S^1 \times SU(l_1)$-equivariant diffeomorphism type of $M$ is uniquely determined by $\# MT = \chi(M)$. It follows from Theorem 6.3 that the $SO(2l_1) \times S^1$-equivariant diffeomorphism type of $M$ is uniquely determined by $\chi(M)$. Because

$$M_k = \begin{cases} \#_{k-1}(S^2 \times S^{2l_1})_i & \text{if } k \geq 1, \\ S^{2l_1+2} & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k + 2$, the statement follows. \hfill $\square$

**Corollary 8.14.** Let $\tilde{G} = SO(2l_1) \times SO(2l_2)$ and $M$ be a simply connected torus manifold with $G$-action such that $SO(2l_1)$, $SO(2l_2)$ are elementary factors of $\tilde{G}$.

Then $M$ is equivariantly diffeomorphic to $\#_i(S^{2l_1} \times S^{2l_2})_i$, or $M = S^{2l_1+2l_2}$.

Here the action of $\tilde{G}$ on $S^{2l_1+2}$ is given by the restriction of the usual $SO(2l_1+2l_2+1)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^{2l_1} \times S^{2l_2}$ is the product action of the usual action of $SO(2l_1)$ and $SO(2l_2)$ on $S^{2l_1}$ and $S^{2l_2}$, respectively. Moreover, the connected sums are equivariant.

**Proof.** As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S \times SO(2l_2)$. Let $(\psi, N, A)$ be the admissible triple for $(SU(l_1) \times S \times SO(2l_2), SU(l_1))$ corresponding to $M$. Then $\psi$ is completely determined by the discussion in section 6 and $A = N^S$. Furthermore, $S$ acts effectively on $N$ such that $N^S$ has codimension two.

By Lemma 5.17, $N$ is simply connected. Therefore, by Lemma 8.13, the equivariant diffeomorphism type of $N$ is uniquely determined by $\chi(N) \in 2Z$. Because all other parts of the triple $(\psi, N, A)$ are determined by the discussion in section 6 and the equivariant diffeomorphism type of $N$, it follows that the equivariant diffeomorphism type of $M$ is determined by $\chi(N)$. Let $T_2$ be the maximal torus $T \cap SO(2l_2)$
of $SO(2l_2)$. Then as in the proof of Lemma 8.13 one sees that

$$\chi(M) = \#M^T = \#N^{S \times T_2} = \chi(N).$$

Therefore the equivariant diffeomorphism type of $M$ is uniquely determined by $\chi(M) \in 2\mathbb{Z}$. Because

$$M_k = \begin{cases} \#i\chi-1(S^{2l_1} \times S^{2l_2}), & \text{if } k \geq 1, \\ S^{2l_1 + 2l_2}, & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k + 2$, the statement follows. $\square$

At the end of this section we give a classification of four-dimensional torus manifolds with $G$-action.

**Corollary 8.15.** Let $M$ be a four-dimensional torus manifold with $G$-action and $G$ be a non-abelian Lie group of rank two. Then $M$ is one of the following:

- $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $S^4$, $S^2_1 \times_{\mathbb{Z}_2} S^2_1$, $S^2_1 \times_{\mathbb{Z}_2} S^2_2$
- or an $S^2$-bundle over $\mathbb{C}P^1$. Here $S^2_2$ denotes the two-sphere together with the $\mathbb{Z}_2$-action generated by the anti-podal map and $S^2_1$ the two-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

**Proof.** Let $\tilde{G}$ be a covering group of $G$. Then there are the following possibilities using Convention 3.5:

$$\tilde{G} = SU(3),\ SU(2) \times SU(2),\ SU(2) \times S^1,$$

$$SU(2) \times SO(3),\ SO(3) \times SO(3),\ SO(3) \times S^1,\ Spin(4),\ SO(5).$$

If $\tilde{G} = \text{Spin}(4)$, we replace it by $SU(2) \times S^1$ as before.

Then we have the following admissible 5-tuples:

<table>
<thead>
<tr>
<th>$\tilde{G}$</th>
<th>5-tuple</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(3)$</td>
<td>(const, pt, $\emptyset$, $\emptyset$)</td>
<td>$\mathbb{C}P^2$</td>
</tr>
<tr>
<td>$SU(2) \times SU(2)$</td>
<td>(const, pt, $\emptyset$, $\emptyset$)</td>
<td>$\mathbb{C}P^1 \times \mathbb{C}P^1$</td>
</tr>
<tr>
<td>$SU(2) \times S^1$</td>
<td>$(\psi, S^2, \emptyset, 0, 0)$, $(\psi, S^2, N, 0, 0)$</td>
<td>$S^2$-bundle over $\mathbb{C}P^1$</td>
</tr>
<tr>
<td></td>
<td>$(\psi, S^2, {N, S}, 0, 0)$</td>
<td>$\mathbb{C}P^2$</td>
</tr>
<tr>
<td>$SU(2) \times SO(3)$</td>
<td>(const, pt, $\emptyset$, $\emptyset$)</td>
<td>$\mathbb{C}P^1 \times S^2$</td>
</tr>
<tr>
<td>$SO(3) \times SO(3)$</td>
<td>$(0, pt, \emptyset, 0, 0, a_{12} = 1)$, $(0, pt, \emptyset, 0, 0, a_{12} = 0)$</td>
<td>$S^2_1 \times_{\mathbb{Z}_2} S^2_1$, $S^2 \times S^2$</td>
</tr>
<tr>
<td></td>
<td>$(0, S^2, \emptyset, 0, 0)$, $(0, S^2, \emptyset, 0, 0)$</td>
<td>$S^2 \times S^2$</td>
</tr>
<tr>
<td></td>
<td>$(0, S^2_1, \emptyset, 0, 0)$, $(0, S^2_1, \emptyset, 0, 0)$</td>
<td>$S^2_1 \times_{\mathbb{Z}<em>2} S^2_1$, $S^2_1 \times</em>{\mathbb{Z}_2} S^2_2$</td>
</tr>
<tr>
<td></td>
<td>$(0, S^2_2, \emptyset, 0, 0)$, $(0, S^2_2, \emptyset, 0, 0)$</td>
<td>$S^2_2 \times S^2_2$</td>
</tr>
<tr>
<td>$SO(5)$</td>
<td>$(0, pt, \emptyset, 0, 0)$</td>
<td>$S^4$</td>
</tr>
</tbody>
</table>

Here $\psi$ is a group homomorphism $S(U(1) \times U(1)) \to S^1$. $\square$
Lemma A.1. Let \( l > 1 \). Then \( S(U(l) \times U(1)) \) is a maximal subgroup of \( SU(l+1) \).

Proof. Let \( H \) be a subgroup of \( SU(l+1) \) with \( S(U(l) \times U(1)) \subset H \subset SU(l+1) \).

Because \( SU(l) \times U(1) \) is a maximal connected subgroup of \( SU(l+1) \), the identity component of \( H \) has to be \( S(U(l) \times U(1)) \). Therefore \( H \) is contained in the normalizer of \( SU(l) \times U(1) \). Because \( l > 1 \),

\[
N_{SU(l+1)}S(U(l) \times U(1))/S(U(l) \times U(1))
\]

\[
= (SU(l+1)/S(U(l) \times U(1)))^{SU(l) \times U(1)} = (\mathbb{CP}^l)^{SU(l) \times U(1)}
\]

is just one point. Therefore \( H = S(U(l) \times U(1)) \) follows. \( \square \)

Lemma A.2. Let \( \psi : S(U(l) \times U(1)) \to S^1 \) be a non-trivial group homomorphism and

\[
H_0 = SU(l+1) \times S^1,
H_1 = S(U(l) \times U(1)) \times S^1,
H_2 = \{(g, \psi(g)), g \in S(U(l) \times U(1))\}.
\]

Then \( H_1 \) is the only connected proper closed subgroup of \( H_0 \) which contains \( H_2 \) properly.

Proof. Let \( H_2 \subset H \subset H_0 \) be a closed connected subgroup. Then we have

\[
\text{rank } H_0 \geq \text{rank } H \geq \text{rank } H_2 = \text{rank } H_0 - 1.
\]

At first assume that \( \text{rank } H = \text{rank } H_0 \). Then we have by [18, p. 297]

\[
H = H' \times S^1,
\]

where \( H' \) is a connected subgroup of maximal rank of \( SU(l+1) \). Let \( \pi_1 : H_0 \to SU(l+1) \) be the projection to the first factor. Because \( H' = \pi_1(H) \supset \pi_1(H_2) = S(U(l) \times U(1)) \) and \( S(U(l) \times U(1)) \) is a maximal connected subgroup of \( SU(l+1) \), we have by Lemma A.1 that \( H = H_1 \) or \( H = H_0 \).

Now assume that \( \text{rank } H = \text{rank } H_2 \). Then there is a non-trivial group homomorphism \( H \to S^1 \). Therefore locally \( H \) is a product \( H' \times S^1 \), where \( H' \) is a simple group which contains \( SU(l) \) as a maximal rank subgroup. By [2, p. 219], we have

\[
H' = E_7, E_8, G_2, SU(l).
\]

If \( H' = SU(l) \), then we have \( H = H_2 \). Therefore we have to show that the other cases do not occur.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( \dim H_0 )</th>
<th>( \dim H' \times S^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>81</td>
<td>( \dim E_7 \times S^1 = 134 )</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>( \dim E_8 \times S^1 = 249 )</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>( \dim G_2 \times S^1 = 15 )</td>
</tr>
</tbody>
</table>

Therefore the first two cases do not occur. Because there is no \( G_2 \)-representation of dimension less than seven, the third case does not occur. \( \square \)

Lemma A.3. Let \( T \) be a torus and \( \psi_1, \psi_2 : S(U(l) \times U(1)) \to T \) be two group homomorphisms. Furthermore, let, for \( i = 1, 2 \),

\[
H_i = \{(g, \psi_i(g)) \in SU(l+1) \times T; g \in S(U(l) \times U(1))\}
\]
be the graph of $\psi_1$. Then:

1. If $l > 1$, then $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$ if and only if $\psi_1 = \psi_2$.
2. If $l = 1$, then $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$ if and only if $\psi_1 = \psi_2^+$. 

Proof. At first assume that $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$. Let $g' \in SU(l + 1) \times T$ such that 

$$H_1 = g'H_2g'^{-1}.$$ 

Because $T$ is contained in the center of $SU(l + 1) \times T$, we may assume that $g' = (g, 1) \in SU(l + 1) \times \{1\}$. Let $\pi_1 : SU(l + 1) \times T \to SU(l + 1)$ be the projection on the first factor. Then:

$$S(U(l) \times U(1)) = \pi_1(H_1) = g\pi_1(H_2)g^{-1} = gS(U(l) \times U(1))g^{-1}.$$

By Lemma A.3 it follows that 

$$g \in N_{SU(l+1)}S(U(l) \times U(1)) = \begin{cases} S(U(l) \times U(1)) & \text{if } l > 1, \\ N_{SU(2)}S(U(1) \times U(1)) & \text{if } l = 1. \end{cases}$$

Now for $h \in S(U(l) \times U(1))$ we have 

$$(h, \psi_1(h)) = g'(g^{-1}hg, \psi_1(h))g'^{-1}.$$ 

Now $(g^{-1}hg, \psi_1(h))$ lies in $H_2$. Therefore we may write 

$$g'(g^{-1}hg, \psi_1(h))g'^{-1} = g'(g^{-1}hg, \psi_2(g^{-1}hg))g'^{-1} = (h, \psi_2(g^{-1}hg)).$$

If $l > 1$ we have 

$$\psi_2(g^{-1}hg) = \psi_2(g)^{-1}\psi_2(h)\psi_2(g) = \psi_2(h).$$

Otherwise we have 

$$\psi_2(g^{-1}hg) = \psi_2(h^{\pm 1}) = \psi_2(h)^{\mp 1}.$$ 

The other implications are trivial. Therefore the statement follows. \[\Box\]

**Lemma A.4.** Let $l \geq 1$. Spin(2l) is a maximal connected subgroup of Spin(2l + 1). Its normalizer consists of two components.

Proof. By [2] p. 219, Spin(2l) is a maximal connected subgroup of Spin(2l + 1) and 

$$N_{\text{Spin}(2l+1)} \text{Spin}(2l)/\text{Spin}(2l) = (\text{Spin}(2l + 1)/\text{Spin}(2l))^{\text{Spin}(2l)} = (S^{2l})^{\text{Spin}(2l)}$$

consists of two points. Therefore the second statement follows. \[\Box\]

**Lemma A.5.** Let $G$ be a Lie group which acts on the manifold $M$. Furthermore, let $N \subset M$ be a submanifold. If the intersection of $Gx$ and $N$ is transverse in $x$ for all $x \in N$, then $GN$ is open in $M$.

Proof. We will show that $f : G \times N \to M$, $(h, x) \mapsto hx$ is a submersion. Because a submersion is an open map, it follows that $GN = f(G \times N)$ is open in $M$. For
$g \in G$, let
\[ l_g : G \times N \to G \times N, \]
\[(h, x) \mapsto (gh, x) \]
and
\[ l'_g : M \to M, \]
\[ x \mapsto gx. \]

Then we have for all $g \in G$
\[ f = l'_g \circ f \circ l_{g^{-1}}. \]

Now for $(g, x) \in G \times N$ we have
\[ D_{(g, x)}f = D_x l'_g D_{(e, x)}f D_{(g, x)}l_{g^{-1}}. \]

Because $Gx$ and $N$ intersect transversely in $x$, the differential $D_{(e, x)}f$ is surjective. Because $l'_g, l_{g^{-1}}$ are diffeomorphisms, it follows that $D_{(g, x)}f$ is surjective. Therefore $f$ is a submersion. \(\square\)

**Appendix B. Generalities on torus manifolds**

**Lemma B.1.** Let $M$ be a torus manifold and $M_1, \ldots, M_k$ be pairwise distinct characteristic submanifolds of $M$ with $N = M_1 \cap \cdots \cap M_k = \emptyset$. Then each $M_i$ intersects transversely with $\bigcap_{j=1}^{k-1} M_j$. Therefore $N$ is a submanifold of $M$ with codim $N = 2k$ and $\dim(\lambda(M_1), \ldots, \lambda(M_k)) = k$. Furthermore, $N$ is the union of some components of $M^{(\lambda(M_1), \ldots, \lambda(M_k))}$.

**Proof.** We prove the lemma by induction on $k$. Let $k \geq 1$ and $x \in N$. Then we have
\[ T_x M = \bigcap_{i=1}^k T_x M_i \oplus \bigoplus_j V_j, \]

where the $V_j$ are one-dimensional complex $\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle$-representations. Since the $M_i$ have codimension two in $M$, each $\lambda(M_i)$ acts non-trivially on exactly one $V_j$.

If $\text{codim} \bigcap_{i=1}^k T_x M_i < 2k$, then there are $i_1$ and $i_2$ such that $V_{j_{i_1}} = V_{j_{i_2}}$. Therefore
\[ T_x M_{i_1} = T_x M_{i_2} = T_x M^{(\lambda(M_{i_1}), \lambda(M_{i_2}))} \]

has codimension two.

Since $\langle \lambda(M_{i_1}), \lambda(M_{i_2}) \rangle$ has dimension two, it does not act almost effectively on $M$. This is a contradiction. Therefore $\bigcap_{i=1}^k T_x M_i$ has codimension $2k$. By induction hypothesis $\bigcap_{i=1}^{k-1} M_i$ is a submanifold of codimension $2k - 2$ and $T_x \bigcap_{i=1}^{k-1} M_i = \bigcap_{i=1}^{k-1} T_x M_i$. Thus, $M_k$ and $\bigcap_{i=1}^{k-1} M_i$ intersect transversely. Therefore $N$ is a submanifold of $M$ of codimension $2k$.

If $\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle$ has dimension smaller than $k$, then the weights of the $V_j$ are linear dependent. Therefore there is $(a_1, \ldots, a_k) \in \mathbb{Z}^k - \{0\}$ such that
\[ C = V_1^{a_1} \otimes \cdots \otimes V_k^{a_k}, \]

where $C$ denotes the trivial $\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle$-representation. This gives a contradiction because each $\lambda(M_i)$ acts non-trivially on exactly one $V_j$. 

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Because $\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle$ has dimension $k$, $M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle}$ has dimension at most $2n - 2k$. But $N$ is contained in $M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle}$ and has dimension $2n - 2k$. Therefore it is the union of some components of $M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle}$.

Lemma B.2. Let $M$ be a torus manifold of dimension $2n$ and $N$ be a component of the intersection of $k(\leq n)$ characteristic submanifolds $M_1, \ldots, M_k$ of $M$ with $N^T \neq \emptyset$. Then $N$ is a torus manifold. Moreover, the characteristic submanifolds of $N$ are given by the components of intersections of characteristic submanifolds $M_i \neq M_1, \ldots, M_k$ of $M$ with $N$, which contain a $T$-fixed point.

Proof. Let $M_i \neq M_1, \ldots, M_k$ be a characteristic submanifold of $M$ with $(M_i \cap N)^T \neq \emptyset$. Then, by Lemma B.1 each component of $M_i \cap N$ which contains a $T$-fixed point has codimension two in $N$. This means that they are characteristic submanifolds of $N$.

Now let $N_1 \subset N$ be a characteristic submanifold and $x \in N_1^T$. Then we have

$$T_xN = T_x(M_1 \cap \cdots \cap M_k).$$

There are $n$ characteristic submanifolds $M_1, \ldots, M_n$ which intersect transversely in $x$. Therefore we have

$$T_xN = T_x(M_1 \cap \cdots \cap M_k).$$

We may assume that there is a $1 \leq k \leq n$ such that $T'$ acts trivially on $N_x(M_i, M)$ for $i > k$ and non-trivially on $N_x(M_i, M)$ for $i \leq k$. Then we have

$$T_xN = (T_xM)^{T'} = N_x(M_{k+1}, M) \oplus \cdots \oplus N_x(M_n, M) = T_x(M_1 \cap \cdots \cap M_k).$$

Lemma B.4. Let $M$ be a torus manifold with $T^n \times \mathbb{Z}_2$-action such that $\mathbb{Z}_2$ acts non-trivially on $M$. Furthermore, let $B \subset M$ be a submanifold of codimension one on which $\mathbb{Z}_2$ acts trivially and let $N$ be the intersection of characteristic submanifolds $M_1, \ldots, M_k$ of $M$. Then $B$ and $N$ intersect transversely.

Proof. Let $x \in B \cap N$; then we have the $\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \times \mathbb{Z}_2$-representation $T_xM$. It decomposes as the sum of the eigenspaces of the non-trivial element of $\mathbb{Z}_2$. Because $B$ has codimension one the eigenspace to the eigenvalue $-1$ is
one-dimensional. Because the irreducible non-trivial torus representations are two-dimensional, we have
\[ T_x N = (T_x M)^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} = T_x M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \times \mathbb{Z}_2 \oplus N_x(B, M)^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \]
\[ = T_x M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \times \mathbb{Z}_2 \oplus N_x(B, M). \]
This means that the intersection is transverse. \( \square \)

**Lemma B.5.** Let \( M^{2n} \) be a \( (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2 \)-manifold such that \( (\mathbb{Z}_2)_i \) acts non-trivially on \( M \). Furthermore, let \( B_i \subset M \), \( i = 1, 2 \), be closed connected submanifolds of codimension one such that \( (\mathbb{Z}_2)_i \) acts trivially on \( B_i \). Then the following statements are equivalent:

1. \( B_1, B_2 \) intersect transversely,
2. \( B_1 \neq B_2 \),
3. \( (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2 \) acts effectively on \( M \) or \( B_1 \cap B_2 = \emptyset \).

**Proof.** Denote by \( V_i \) the non-trivial real irreducible representation of \( (\mathbb{Z}_2)_i \). Let \( x \in B_1 \cap B_2 \). Then for the \( (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2 \)-representation \( T_x M \) there are two possibilities:
\[ T_x M = \begin{cases} \mathbb{R}^{2n-1} \oplus V_1 \oplus V_2, \\ \mathbb{R}^{2n-2} \oplus V_1 \oplus V_2. \end{cases} \]
In the first case \( B_i \), \( i = 1, 2 \), is the component of \( M^{(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2} \) containing \( x \) and \( (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2 \) acts non-effectively on \( M \). In the second case \( (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2 \) acts effectively on \( M \) and \( B_1, B_2 \) intersect transversely in \( x \).
All conditions given in the lemma imply that we are in the second case or \( B_1 \cap B_2 = \emptyset \). Therefore they are equivalent. \( \square \)

**Remark B.6.** Lemmas B.1 [B.4] also hold if we do not require that a characteristic manifold contains a \( T \)-fixed point.

**References**


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