

EMBEDDED PLATEAU PROBLEM

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ABSTRACT. We show that if Γ is a simple closed curve bounding an embedded disk in a closed 3-manifold M , then there exists a disk Σ in M with boundary Γ such that Σ minimizes the area among the embedded disks with boundary Γ . Moreover, Σ is smooth, minimal and embedded everywhere except where the boundary Γ meets the interior of Σ . The same result is also valid for homogeneously regular manifolds with sufficiently convex boundary.

1. INTRODUCTION

The Plateau problem asks the existence of an area minimizing disk for a given simple closed curve in a manifold M . This problem was solved for \mathbf{R}^3 by Douglas [Do], and Rado [Ra] in the early 1930s. Later, it was generalized by Morrey for Riemannian manifolds [Mo]. Then, regularity (nonexistence of branch points) of these solutions was shown by Osserman [Os], Gulliver [Gu] and Alt [Al]. However, these area minimizing disks may not be embedded, even though the curves bound an embedded disk in the ambient manifold. They might have self-intersections (see Figure 1).

In the following decades, the question of embeddedness of the area minimizing disk was studied: For which curves are the area minimizing disks embedded? The first such condition ensuring the embeddedness of the disk was due to Rado. In the early 1930s, he showed that if the curve can be projected onto a convex curve in a plane, then it bounds a unique embedded minimal disk which is a graph over the plane. Osserman conjectured that if the curve is extreme (lies in the boundary of its convex hull), then the area minimizing disk spanning the curve must be embedded. In the late 1970s, Gulliver and Spruck proved that if the total curvature of an extreme curve is less than 4π , then the solution to the Plateau problem is embedded [GS]. Later, Almgren-Simon [AS] and Tomi-Tromba [TT] showed the existence of an embedded solution for extreme curves. Then, Meeks and Yau proved the Osserman conjecture in full generality: Any solution to the Plateau problem for an extreme curve must be embedded [MY1]. Recently, Ekholm, White, and Wienholtz generalized Gulliver-Spruck's embeddedness result by removing extremeness condition from the curves [EWW]. Also, recently, Hass, Lagarias and Thurston [HLT] gave interesting results about the isoperimetric inequalities for embedded disks in \mathbf{R}^3 .

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On the other hand, a different version of the Plateau problem was studied after the 1960s. This version asks the existence of an area minimizing surface for a given simple closed curve. If there is no restriction on the topological type of the surface, Geometric Measure Theory gives a positive solution for this question. Federer et al. solved the problem and showed the existence of a surface which minimizes area among all surfaces with the given boundary [Fe]. Moreover, any such surface must be embedded for any simple closed curve.

If we come back to the disk case, there is a relevant result about the same question due to Meeks-Yau [MY3]. They give a necessary condition for a sufficiently smooth simple closed curve in a 3-manifold to bound an embedded minimal disk. In particular, they showed that for a sufficiently smooth simple closed curve Γ in a 3-manifold M , in order to bound a strictly stable embedded minimal disk in M , Γ must be an extreme curve in some sense (see Theorem 4.1).

In this paper, we are approaching the embeddedness question from a different direction. Instead of considering the question that “*for which curves must the area minimizing disks be embedded?*”, we analyze the structure of the surface which minimizes area among the embedded disks whose boundary is any given simple closed curve.

Embedded Plateau problem. Let Γ be a simple closed curve in a manifold M , and let Γ bound an embedded disk. Does there exist an embedded minimal disk which minimizes the area among the embedded disks with boundary Γ ?

This is the most general case for a curve to bound an embedded minimal disk. This is because if Γ does not bound any embedded disk in M , then, of course, there is no embedded minimal disk bounding Γ at all. Our main result is as follows:

Theorem 3.1. *Let Γ be a simple closed curve bounding an embedded disk in a closed 3-manifold M . Then, there exists a disk Σ in M with $\partial\Sigma = \Gamma$ such that Σ minimizes the area among all the embedded disks bounding Γ . Moreover, Σ is minimal and smoothly embedded everywhere except where the boundary Γ meets the interior of Σ .*

In particular, if Σ is as in the theorem, then there is a continuous parametrization $\varphi : D^2 \rightarrow M$ of Σ , with $\varphi(D^2) = \Sigma$ and $\varphi(\partial D^2) = \Gamma$, such that φ is a smooth embedding on $D^2 - \varphi^{-1}(\Gamma)$ and the image $\Sigma - \Gamma$ is a minimal surface (see Figure 1). $\lambda = \Gamma \cap \varphi(\text{int}(D^2))$ is known as *the coincidence set* in the literature (see Remark 3.2). If $\lambda = \emptyset$, then Σ is a smooth embedded minimal disk in M with boundary Γ . We call such a disk Σ a *pseudo-area minimizing disk*.

The outline of the technique is summarized as follows: Let Γ be any simple closed curve bounding an embedded disk in a manifold M . By drilling out a small neighborhood N_i of Γ and changing the metric in a very small neighborhood of the boundary, we can get a manifold M_i with a convex boundary ∂M_i . For a curve $\Gamma_i \subset \partial M_i$ homotopic to Γ in M , there is an area minimizing embedded disk D_i in M_i with $\partial D_i = \Gamma_i$ by [MY2]. When N_i gets smaller, we get a sequence of embedded disks $\{D_i\}$ in M where the areas of the disks are approaching the minimum area for embedded disks, and $\partial D_i = \Gamma_i \rightarrow \Gamma$. Then, the idea is to obtain a limiting surface out of this sequence and to analyze its structure.

By using the standard generalizations on M to ensure the embeddedness of the solutions of the Plateau problem in the Meeks-Yau setting, we also give a

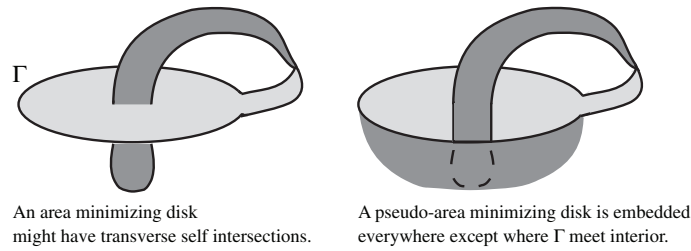


FIGURE 1. For a given Γ , while the area minimizing disks might have transverse self-intersections (left), the pseudo-area minimizing disks have no transverse self-intersection and are embedded everywhere except where the boundary bumps into the interior (right).

generalization of the main result to a homogeneously regular 3-manifold M with sufficiently convex boundary (see Theorem 3.2).

On the other hand, by slightly modifying the pseudo-area minimizing disk Σ with boundary Γ , it is easy to get a smooth, embedded disk Σ' with boundary Γ such that Σ' is minimal everywhere except for a very small region (see Corollary 3.3).

The organization of the paper is as follows: In section 2, we cover some basic results which will be used in the remaining part of the paper. In section 3, we prove the main result. Then in section 4, we make some final remarks.

2. PRELIMINARIES

In this section, we will overview the basic results which we use in the following sections. For more details on these, see [CM] or [HS].

Definition 2.1. Let M be a 3-manifold. A *minimal disk* in M is a disk whose mean curvature vanishes everywhere. An *area minimizing disk* in M is a disk which has the smallest area among the disks in M with the same boundary. A *pseudo-area minimizing disk* in M is a disk which has the smallest area among the embedded disks in M with the same boundary, and has no transverse self-intersection.

Definition 2.2 ([HS]). Let M be a compact Riemannian 3-manifold with boundary. Then M is *mean convex* (or M has sufficiently convex boundary) if the following conditions hold:

- ∂M is piecewise smooth.
- Each smooth subsurface of ∂M has nonnegative curvature with respect to the inward normal.
- There exists a Riemannian manifold N such that M is isometric to a submanifold of N and each smooth subsurface S of ∂M extends to a smooth embedded surface S' in N such that $S' \cap M = S$.

Definition 2.3 ([MY1], [HS]). Let M be a Riemannian 3-manifold. Then M is *homogeneously regular* if the sectional curvature is bounded above and the injectivity radius is bounded away from 0.

In this paper, we will use the following definition for extreme curves. Note that this definition is different from the one in the literature (lying in the boundary of its convex hull), and our definition is more general than the other one.

Definition 2.4. $\Gamma \subset M$ is an *extreme curve* if it is a curve in the boundary of a mean convex submanifold in M .

Now, we state the main facts which we use in the following sections.

Lemma 2.1 ([MY1], [MY2]). *Let M be a compact manifold with sufficiently convex boundary, and let Γ be a nullhomotopic simple closed curve in ∂M . Then, there exists an area minimizing disk D in M with $\partial D = \Gamma$. Moreover, unless $D \subset \partial M$, all such disks are properly embedded (i.e. the boundary of the disk is in the boundary of the manifold) in M and they are pairwise disjoint.*

Now, we state two lemmas due to Hass and Scott [HS], which we use in the following sections.

Lemma 2.2 ([HS, Lemma 3.1]). *Let M be a closed Riemannian 3-manifold. Then, there exists an $\epsilon > 0$ such that for any $x \in M$, the ball $B_\epsilon(x)$ of radius ϵ about x in M has the following property: If $\Gamma \subset \partial B_\epsilon(x)$ is a simple closed curve, and if D is an area minimizing disk in M with $\partial D = \Gamma$, then D is properly embedded in $B_\epsilon(x)$.*

Lemma 2.3 ([HS, Lemma 3.6]). *Let M be a compact Riemannian 3-manifold with strictly convex boundary. Let $\{D_i\}$ be a sequence of properly embedded area minimizing disks in M which have uniformly bounded area. Then there is a subsequence $\{D_{i_j}\}$ which converges to a collection of properly embedded area minimizing disks. If $\{D_i\}$ has a limit point, then the collection is not empty.*

3. MAIN RESULT

In this section, we will prove the main result of the paper.

Let Γ be a simple closed curve which is the boundary of an embedded disk E in a closed Riemannian 3-manifold M .

First, we will construct a sequence of embedded *almost* area minimizing disks $\{D_i\}$ in M with $\partial D_i = \Gamma_i$ and $\Gamma_i \rightarrow \Gamma$ by using the techniques of Calegari and Gabai in [Ga] and [CG]. Then, by taking the limit of this sequence as in [Ga] and [HS], we will get a limit object Δ . Then, by analyzing this object, we will show the main result of the paper.

3.1. The sequence. Take a sequence of open solid tori $\{N_i\}$ which are neighborhoods of Γ . That is, fix a sufficiently small $\epsilon > 0$, and let $N_i = N_{\frac{\epsilon}{i}}(\Gamma)$, where $N_\epsilon(\cdot)$ represents the open ϵ -neighborhood in M . Then $N_1 \supset N_2 \supset \cdots \supset N_i \supset N_{i+1} \supset \cdots$ and $\bigcap_{i=1}^\infty N_i = \Gamma$.

Now, let $M_i = M - N_i$. Clearly, $\{M_i\}$ are compact 3-manifolds with torus boundary. Moreover, $M_1 \subset M_2 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots$ and $\bigcup_{i=1}^\infty M_i = M - \Gamma$. Also, note that for $\delta < \epsilon$, $\overline{\partial N_\delta(\Gamma)}$ is a torus, and let $F : (0, \epsilon] \rightarrow \mathbf{R}$ be a function such that $F(\delta) = |\overline{\partial N_\delta(\Gamma)}|$, where $|\cdot|$ represents the area in M . Since $\overline{\partial N_\delta(\Gamma)}$ degenerates into Γ as $\delta \rightarrow 0$, then $F(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Hence, $|\partial M_i| \rightarrow 0$ as $i \rightarrow \infty$.

Now, we will construct a sequence of area minimizing disks in M . Let E be the disk in M bounding Γ . Modify E if necessary so that E is transverse to ∂M_i . Then, let $\hat{\Gamma}_i = E \cap \partial M_i$. In other words, for each i , let $\hat{\Gamma}_i$ be a simple closed curve in $\partial M_i = \partial N_i$ which is isotopic to the core curve Γ in the solid torus N_i .

Consider the manifolds with torus boundary $M_i \subset M_{i+1}$. Change the metric of M_{i+1} in $M_{i+1} - M_i$ so that M_{i+1} becomes a compact manifold with sufficiently convex boundary, say \widehat{M}_{i+1} (M_{i+1} with a new metric). Note that the new metric is the same as the original metric of M in the M_i part. As \widehat{M}_{i+1} is mean convex, by Lemma 2.1, there exists an area minimizing disk \widehat{D}_{i+1} in \widehat{M}_{i+1} with $\partial\widehat{D}_{i+1} = \widehat{\Gamma}_{i+1}$ (see Figure 2).

Now, consider the intersection $\partial M_i \cap \widehat{D}_{i+1}$. By modifying M_i if necessary, we can assume the intersection is transverse, and it is a collection of simple closed curves. By construction, the curves in the intersection are either essential in ∂M_i and isotopic to Γ in M , or not essential in Γ . Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be essential curves, and let $\{\beta_1, \beta_2, \dots, \beta_m\}$ be the nonessential ones. By construction, we know that $n \geq 1$ and $m \geq 0$. Without loss of generality, let α_1 be the innermost curve in \widehat{D}_{i+1} among $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Let E_i be the subdisk in \widehat{D}_{i+1} with $\partial E_i = \alpha_1$.

Now, consider that $\partial E_i \subset \partial M_i$ and $E_i \subset M_{i+1}$. If $E_i \subset M_i$, then define the i^{th} element of the desired sequence $D_i = E_i$. Otherwise, $E_i \cap (M_{i+1} - M_i) \neq \emptyset$ and consists of planar surfaces whose boundary is some curves in $\{\beta_1, \beta_2, \dots, \beta_m\}$. Now replace these planar surfaces in E_i with the isotopic surface in ∂M_i with the same boundary. Then, smooth out the corners into M_i and push the parts in ∂M_i into M_i so that surgered E_i , say \widehat{E}_i , becomes a disk in M_i . Hence, define Γ_i as α_1 , and D_i as \widehat{E}_i .

By construction $\{D_i\}$ has the following properties:

- D_i is a properly embedded smooth disk in M_i , i.e. $\partial D_i = \Gamma_i = D_i \cap \partial M_i$.
- $|D_i| < T_i + 2|\partial M_i|$, where T_i is the infimum of the areas of the embedded disks in M with boundary Γ_i .

By construction, the first property is clear. To see the second property, first consider that E_i is the area minimizing disk in \widehat{M}_{i+1} , and the metric of \widehat{M}_{i+1} coincides with the original metric on M in M_i . Clearly, $T'_i < T_i + |\partial M_i|$, where T'_i is the infimum of the areas of the embedded disks in M_i with boundary Γ_i , as D_i is obtained by modifying $E_i \cap M_i$, by patching the missing parts from ∂M_i and smoothing out corners. By [MY2], smoothing out the folding curves decreases the area; hence $|D_i| \leq |E_i \cap M_i| + |\partial M_i|$. Since $|E_i \cap M_i| \leq T'_i$, the second property follows.

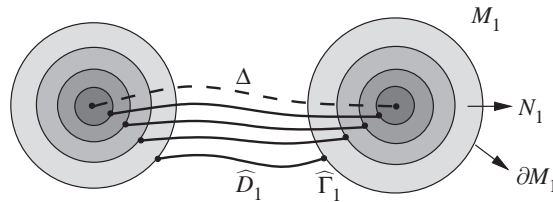


FIGURE 2. For any i , $M_i = M - N_i$; then $N_i \supset N_{i+1}$, and $M_i \subset M_{i+1}$. $\widehat{\Gamma}_i \subset \partial M_i$ is a simple closed curve isotopic to Γ in M ($\{\widehat{\Gamma}_i\}$ are shown as pairs of points, and Γ is shown as the innermost pair of points in the figure). \widehat{D}_i is an area minimizing disk in \widehat{M}_i (M_i with modified metric) with $\partial\widehat{D}_i = \widehat{\Gamma}_i$.

It is clear that D_i may not be area minimizing in M ; however, with the second property, they can be thought of as *almost area minimizing* in M . Note that $E_i \cap M_i \subset D_i$ is area minimizing in M , so only the patched parts of D_i (replacements of $E_i \cap (M_{i+1} - M_i)$) are not area minimizing, which are a very small part of D_i . This is because it is reasonable to think of D_i as almost area minimizing.

Now, let T be the infimum of the areas of the embedded disks in M with boundary Γ . Let A_i be the infimum of the areas of embedded annuli with boundary $\Gamma \cup \Gamma_i$. By construction, $T_i \leq T + A_i$ and $T \leq T_i + A_i$. Since $\Gamma_i \rightarrow \Gamma$, $A_i \rightarrow 0$ as $i \rightarrow \infty$. This implies $T_i \rightarrow T$ as $i \rightarrow \infty$. Hence, by the second property and $|\partial M_i| \rightarrow 0$, $|D_i| \rightarrow T$ as $i \rightarrow \infty$.

To sum up, we constructed a sequence of embedded smooth disks $\{D_i\}$ in M such that $\partial D_i = \Gamma_i \subset \partial M_i$ with $\Gamma_i \rightarrow \Gamma$ and $|D_i| \rightarrow T$ as $i \rightarrow \infty$, where T is the infimum of the areas of the embedded disks in M with boundary Γ .

3.2. The limit. In this section, by using the techniques of [Ga] and [HS], from the sequence $\{D_i\}$ of *almost* area minimizing embedded disks in M , we will get a limit object Δ which can be thought of as a special topological limit. Then, by using this object, we will prove the main result of the paper. The following is a modified version of a definition of a special topological limit due to Gabai [Ga].

Definition 3.1. A collection of pairwise disjoint embedded surfaces Δ in a Riemannian manifold M is called the *topological limit* of the sequence $\{D_i\}$ of embedded surfaces in M if

- $\Delta = \{x = \lim x_i \mid x_i \in D_i, \{x_i\} \text{ is a convergent sequence in } M\}$
- $\Delta = \{x = \lim x_{n_i} \mid x_i \in D_i, \{x_i\} \text{ has a convergent subsequence } \{x_{n_i}\} \text{ in } M\}$

In other words, the sequence $\{D_i\}$ is such that the set of the limits of all $\{x_i\}$ with $x_i \in M$ and the set of the limits of the subsequences are the same. This is an essential condition on Δ to be a collection of *pairwise disjoint* embedded surfaces. Otherwise, one might simply take a sequence such that $D_{2i+1} = \Sigma_1$ and $D_{2i} = \Sigma_2$, where Σ_1 and Σ_2 are intersecting disks. Then, without the first condition (Δ being just the union of limit points), $\Delta = \Sigma_1 \cup \Sigma_2$ in this case, which is not a collection of pairwise disjoint embedded disks. However, the first condition forces Δ to be either Σ_1 or Σ_2 , not the union of them. By similar reasons, this condition is also important to ensure the embeddedness of the disks in the collection Δ .

Now, we will show that there is a subsequence of the sequence constructed in the previous part which gives a topological limit Δ . Then, by showing that the limit Δ is a collection of disks, we will prove the main result of the paper.

Theorem 3.1. *Let Γ be a simple closed curve bounding an embedded disk in a closed 3-manifold M . Then, there exists a disk Σ in M with $\partial \Sigma = \Gamma$ such that Σ minimizes the area among the embedded disks bounding Γ . Moreover, Σ is minimal and smoothly embedded everywhere except where the boundary Γ meets the interior of Σ .*

Proof. Let $\{D_i\}$ be as defined in the previous part. We abuse the notation by taking $\{D_i\}$ as a sequence of open disks. Our aim is to get a convergence as in Definition 3.1 for an appropriate subsequence of $\{D_i\}$.

Step 0. After passing to a subsequence of $\{D_i\}$, the following holds:

$$\begin{aligned}\Delta &= \{x = \lim x_i \mid x_i \in D_i, \{x_i\} \text{ is a convergent sequence in } M\} \\ &= \{x = \lim x_{n_i} \mid x_i \in D_i, \{x_i\} \text{ has a convergent subsequence } \{x_{n_i}\} \text{ in } M\}.\end{aligned}$$

Proof. For each j subdivide M into a finite number of closed regions such that the $j + 1^{\text{st}}$ subdivision is a subdivision of the j^{th} one. Also, let the mesh of these subdivisions converge to 0. In other words, let B_k^j be the k^{th} region of the j^{th} subdivision and $M = \bigcup_{k=1}^{n_j} B_k^j$, where $B_i^{j-1} = B_{i_1}^j \cup \dots \cup B_{i_r}^j$. Also, for any j, k , $\text{diam}(B_k^j) < C_j$, where $C_j \rightarrow 0$ as $j \rightarrow \infty$.

Now, choose a subsequence of $\{D_i\}$ such that if $i \geq j$ and $D_i \cap B_k^j \neq \emptyset$, then for any $k > i$, $D_k \cap B_r^j \neq \emptyset$. By abuse of notation, replace this subsequence with the original sequence $\{D_i\}$. Now, for this new sequence, for any $x = \lim x_{n_i}$ where $x_{n_i} \in D_{n_i}$, by construction, there is a convergent sequence $\{x_i\}$ with $x_i \in D_i$ such that $x = \lim x_i$. Hence, Step 0 follows. \square

Step 1. Δ is not empty.

Proof. In order to show that Δ is nonempty, it will suffice to construct a convergent sequence $\{x_{n_i}\}$ in M with $x_{n_i} \in D_{n_i}$. Now, consider a meridian curve γ' of ∂M_1 (think of Γ_1 as the longitude of ∂M_1). Push γ' into M_1 a little bit (a small isotopy), and denote the new curve by γ in M_1 . Now, γ links Γ and Γ_i in M for all i . Hence, for any i , $D_i \cap \gamma$ is not empty. Let $x_i \in D_i \cap \gamma$. Since γ is a simple closed curve, the sequence $\{x_i\}$ must have a convergent subsequence. Hence, Δ is nonempty. \square

Step 2. Let $Z = \{z_i\}$ be a countable dense subset of Δ , where $Z \cap \Gamma = \emptyset$. Then, after passing to a subsequence of $\{D_j\}$, the following holds. For any i , there exists a sequence $\{E_j^i\}$ of embedded disks $E_j^i \subset D_j$ which converges to a smoothly embedded disk E_i such that $z_i \in E_i$.

Proof. As M is a closed manifold, by Lemma 2.2, there exists a $\rho > 0$ such that for any $x \in M$, if $\Gamma \subset \partial B_\rho(x)$, then any area minimizing disk D in M with boundary Γ must be in $B_\rho(x)$.

Now, let $Z_1 = \{z_i \in Z \mid d(z_i, \Gamma) < 2\rho\}$ and $Z_2 = Z - Z_1$. Also, let $\rho_i = \frac{d(z_i, \Gamma)}{2}$ for any $z_i \in Z_1$. Now, we claim that for any $z_i \in Z_1$, there is an embedded disk E_i in Δ with $z_i \in E_i$ and $\partial \bar{E}_i \cap B_{\rho_i}(z_i) = \emptyset$. We also claim that for $z_i \in Z_2$, there is an embedded disk E_i in Δ with $z_i \in E_i$ and $\partial \bar{E}_i \cap B_\rho(z_i) = \emptyset$.

Now, fix $z_i \in Z_2$. Since $z_i \in \Delta$, there exists a sequence $\{x_j^i\}$ with $x_j^i \in D_j$ and $x_j^i \rightarrow z_i$. By deleting the tangential points in the intersection, and by modifying ρ if necessary, we can assume that $\partial B_\rho(z_i)$ is transverse to $\{D_j\}$. Let $E_j^i = B_\rho(z_i) \cap D_j$. For sufficiently large j , being away from ∂M_j , the almost area minimizing disk D_j is area minimizing near z_i and hence E_j^i is an area minimizing disk in $B_\rho(z_i)$ by Lemma 2.2. Hence $|E_j^i| < \frac{1}{2}|\partial B_\rho(z_i)|$ for any j , and the sequence of properly embedded area minimizing disks in $B_\rho(z_i)$ have uniformly bounded area. Also, since M is closed, we can assume that $B_\rho(z_i)$ has a strictly convex boundary, by taking a smaller ρ if necessary (since M is compact, there exists R such that for any $0 < r < R$, $B_r(x)$ has a strictly convex boundary for any $x \in M$). Hence, by Lemma 2.3, $\{E_j^i\}$ has a subsequence converging to E_i , where E_i is the area minimizing disk in M with $z_i \in E_i$. By using the diagonal subsequence argument, we can modify our sequence $\{D_i\}$ accordingly. In other words, for each z_i , use the

above argument and get a diagonal subsequence, and call this new sequence $\{D_i\}$ again (abuse of notation), and define Δ for this new sequence.

Similarly, fix $z_i \in Z_1$. As above, let $\{x_j^i\}$ be a sequence with $x_j^i \in D_j$ and $x_j^i \rightarrow z_i$, and $E_j^i = B_{\rho_i}(z_i) \cap D_j$. By construction, $\rho_i \leq \rho$ for any i . Again for sufficiently large j , being away from ∂M_j , an almost area minimizing disk D_j is area minimizing near z_i and hence E_j^i is an area minimizing disk in $B_{\rho_i}(z_i)$ by Lemma 2.2. As before, by Lemma 2.3, $\{E_j^i\}$ has a subsequence converging to E_i , where E_i is the area minimizing disk in M with $z_i \in E_i$. Continuing with the diagonal subsequence argument for each z_i , we get a subsequence of $\{D_i\}$ with the required properties. \square

Hence, with the above construction, we get $\bigcup_{z_i \in Z} E_i = \Delta - \Gamma$. Now, we will analyze the structure of the limit object Δ . In particular, we will prove that Δ is a union of “disks” which minimize area among the embedded disks. Now, clearly $\Gamma \subset \Delta$. We want to specify some parts of Γ which bump into the interior of Δ . In other words, define *the coincidence set* $\lambda \subset \Gamma$ such that

$$\lambda = \{x \in \Gamma \mid \exists \rho_x > 0, \exists \{y_j\} \text{ with } y_j \in D_j \text{ and } \hat{d}_j(y_j, \Gamma_j) > \rho_x \text{ such that } y_j \rightarrow x\}.$$

Here, \hat{d}_j is the induced path metric on D_j . In other words, a point in the boundary Γ is in the coincidence set λ if there exists a sequence $\{z_j\}$ in Δ which is away from the boundary Γ and $z_j \rightarrow x$. Hence by definition, the coincidence set λ corresponds to some closed subsegments of Γ which meet the interior part of Δ . It can be thought of as the *defective* parts of the embedded minimal disks we are constructing. Note that λ might be empty.

Step 3. $\Delta - \Gamma$ is a minimal surface in M .

Proof. By Step 2, for each $z_i \in Z$, there is an embedded disk $E_i \subset \Delta$. Now, if $x \in E_i \cap E_j$, then they must coincide in a neighborhood of x . Otherwise, since E_i and E_j are minimal disks, they must cross transversely near x [HS]. However, by construction, this would imply that D_i is not embedded for sufficiently large i . Hence, the E_i 's are either pairwise disjoint or locally coincide. As $\{z_i\}$ is a dense subset of Δ , by Step 2, for any $x \in \Delta - \Gamma$, we can find a neighborhood of x in Δ , say E_x , which is an open minimal disk. Minimality comes from being locally area minimizing. Hence, $\Delta - \Gamma = \bigcup_{x \in \Delta - \Gamma} E_x$. This shows that each component Σ'_i of $\Delta - \Gamma$ is a surface. \square

Step 4. Δ is a collection of disks which minimize area among the embedded disks.

Proof. Now, we will show that for each component Σ'_i of $\Delta - \Gamma$, $\Sigma_i = \Sigma'_i \cup \Gamma$ is a pseudo-area minimizing disk in M where $\Sigma_i - \Gamma$ is a smooth minimal surface. In other words, we will show that there is a continuous map $\varphi_i : D^2 \rightarrow M$ such that $\varphi(\partial D^2) = \Gamma$ and $\varphi(D^2) = \Sigma_i$, where φ is an embedding except at $\varphi^{-1}(\lambda)$. That is, $\Sigma_i \subset M$ is the image of a continuous map from a disk where it fails to be an embedding only at the coincidence set λ .

Let α be a simple closed curve in Σ'_i . Let A be a neighborhood of α which is a very thin annulus in Σ'_i . Now, let $g : D \rightarrow A$ be an isometric immersion of a disk D into Σ_i , where D is a very long thin rectangle with $g(D) = A$. Also, assume that $|D| > C$, where C is a constant with $C > |D_i|$ for any i . Existence of such a C comes from the construction as D_i is an area minimizing disk in M_i , and

$|D_i| < |A_i| + |D_1|$ where A_i is a very thin annulus with $\partial A_i = \Gamma_1 \cup \Gamma_i$. Since we can find a uniform bound for $|A_i|$, the existence of C follows.

Now, as $D_i \rightarrow \Delta$, and $A \subset \Delta - \Gamma$, we can find isometric immersions $g_i : D^2 \rightarrow D_i$ such that $g_i \rightarrow g$ in the C^∞ topology. Now, there are two cases. Either $g_i(D^2) \subset D_i$ is also a thin annulus A_i in D_i approximating A , or $g_i(D^2)$ is an embedded disk in D_i which is spiraling around A . In the latter case, this would mean that $|D_i| > |D| > C$, which is a contradiction. Hence, $g_i(D^2) = A_i$. Then, we can choose a suitable essential simple closed curve β_i in each annulus A_i (β_i is a core curve of the annulus A_i) such that the sequence $\{\beta_i\}$ converges to α , i.e. $\beta_i \rightarrow \alpha$. Let F_i be the disks in D_i with $\partial F_i = \beta_i$.

Now, further assume that α separates Σ_i into two parts, say S_1 and let S_2 , and let $\Gamma \subset S_2$. In other words, $S_1 \subset \Sigma_i$ and $\partial S_1 = \alpha$ with $S_1 \cap \lambda = \emptyset$. Being away from Γ , hence from the coincidence set λ , this implies that the disks $\{F_i\}$ are area minimizing disks (not almost area minimizing). Hence, as in Lemma 3.3 of [Ga], the sequence of disks $\{F_i\}$ converges to a disk Ω in Σ_i with $\partial\Omega = \alpha$; i.e. $S_1 = \Omega$ is a smooth minimal disk in M . Hence, we show that for any separating simple closed curve α in Σ_i with $S_1 \cap \lambda = \emptyset$, there is a smooth disk Ω in Σ_i with $\partial\Omega = \alpha$.

Now, by choosing a suitable sequence of simple closed curves $\{\alpha_n\}$ in Σ_i , we can exhaust Σ_i with disks Ω_n such that $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_n \subset \dots$ with $\Sigma_i = \bigcup_n \Omega_n$, where Ω_n is a disk in Σ_i with $\partial\Omega_n = \alpha_n$ and $\Omega_n \cap \lambda = \emptyset$. This means that $\alpha_n \rightarrow \Gamma \cup l \cup \lambda$, where l is a collection of line segments $\{l_j^k\}$ in Σ_i which connects Γ with one of the endpoints of the components $\{\lambda_k\}$ of λ . In particular, for each component λ_k of λ , let n_k be the number of line segments $\{l_1^k, l_2^k, \dots, l_{n_k}^k\}$ connecting the component λ_k to the Γ . Then n_k is the number of the *local sheets* of Σ_i near λ_k , i.e., the components of $N(\lambda_k) \cap \Omega_n$, where $N(\lambda_k)$ is a sufficiently small neighborhood of λ_k in M and n is sufficiently large.

Since the sequence of the disks $\{\Omega_n\}$ exhausting Σ_i does not contain λ , λ is in the component $\Sigma_i - \Omega_n$ which also contains Γ . Hence, for sufficiently large n , $\Sigma_i - \Omega_n$ is the union of some thin neighborhood of Γ and some thin strips around line segments $\{l_j^k\}$ connecting Γ to λ , as Ω_n is exhausting Σ_i . Hence, α_n curves approach $l \cup \lambda$ from both sides and $\partial\Omega_n = \alpha_n \rightarrow \Gamma \cup l \cup \lambda$. Hence, we can get a continuous parametrization $\hat{\varphi} : D^2 \rightarrow M$ with $\hat{\varphi}(D^2) = \Sigma_i$ and $\hat{\varphi}(\partial D^2) = \Gamma \cup l \cup \lambda$.

Now, our aim is to get a continuous map $\varphi : D^2 \rightarrow M$ with $\varphi(D^2) = \Sigma_i$ and $\varphi(\partial D^2) = \Gamma$ by modifying $\hat{\varphi}$. Fix k_0 and j_0 . Consider $l_{j_0}^{k_0} \cup \lambda_{k_0}$. Let $U = N(l_{j_0}^{k_0} \cup \lambda_{k_0})$ be a very small neighborhood of $l_{j_0}^{k_0} \cup \lambda_{k_0}$ in Σ_i such that $U \cap l_j^k = \emptyset$ for any $(k, j) \neq (k_0, j_0)$. Consider $\hat{\varphi}^{-1}(U) \subset D^2$. Let V be the component of $\hat{\varphi}^{-1}(U)$ in D^2 which contains the segment $\hat{\varphi}^{-1}(l_{j_0}^{k_0}) \subset \partial D^2$. Hence, $\hat{\varphi}|_V : V \rightarrow U$ and $\hat{\varphi}(V \cap \partial D^2) \subset \Gamma \cup l_{j_0}^{k_0} \cup \lambda_{k_0}$.

Now, we can continuously modify $\hat{\varphi}$ in V into a continuous map φ so that $\varphi|_V : V \rightarrow U$ is a continuous embedding with $\varphi(\partial \bar{V} \cap \text{int}(D^2)) = \hat{\varphi}(\partial \bar{V} \cap \text{int}(D^2))$ and $\varphi(V \cap \partial D^2) \subset \Gamma$. To see this, one can define a continuous family of maps $\hat{\varphi}_t : V \rightarrow U$ with $0 \leq t \leq 1$ such that $\hat{\varphi}_0 = \hat{\varphi}|_V$ and $\hat{\varphi}_t(V \cap \partial D^2) \subset \Gamma \cup (l_{j_0}^{k_0} \cup \lambda_{k_0})_t$. Here, $(l_{j_0}^{k_0} \cup \lambda_{k_0})_t$ is a subsegment in $l_{j_0}^{k_0} \cup \lambda_{k_0}$ which is getting smaller as t increases, and finally $(l_{j_0}^{k_0} \cup \lambda_{k_0})_1$ is the endpoint of $l_{j_0}^{k_0} \cup \lambda_{k_0}$ in Γ . Intuitively, this continuous deformation of the parametrizations corresponds to pushing $l_{j_0}^{k_0} \cup \lambda_{k_0}$ into Γ in Σ_i . Since, $U = N(l_{j_0}^{k_0} \cup \lambda_{k_0})$ is disjoint from other line segments l_j^k , one can modify $\hat{\varphi}$

for each l_j^k . Finally, we get a continuous map $\varphi : D^2 \rightarrow M$ with $\varphi(D^2) = \Sigma_i$ and $\varphi(\partial D^2) = \Gamma$.

Even though we obtained φ from $\hat{\varphi}$, which may not be smooth along $l \cup \lambda$, since the choice of l is arbitrary, and $\Sigma_i - \Gamma$ is smooth by previous steps, φ can be chosen as a smooth embedding on $D^2 - \varphi^{-1}(\Gamma)$ by construction. If the coincidence set $\lambda = \Gamma \cap \varphi(\text{int}(D^2))$ is empty, then Σ_i is an embedded minimal disk in M with $\partial \Sigma_i = \Gamma$. Otherwise, the disk Σ_i might fail to be smooth on λ . Note also that the restriction of φ to the interior of D^2 may not be an embedding either, if $n_k > 1$ for some k (see Section 4).

Now, we claim that Σ_i is minimizing area among the embedded disks with boundary Γ . Otherwise, there is a compact subdisk E of Σ_i which is not area minimizing among the embedded disks. Then there is an embedded disk E' in M with smaller area. Let $|E| - |E'| = \xi$. By construction, since $D_i \rightarrow \Delta$, we can find sufficiently close disks in the sequence $\{D_i\}$ to Σ_i such that there is a subdisk $E'' \subset D_i$ with $||E| - |E''|| < \frac{\xi}{2}$ and $|\mathcal{A}| < \frac{\xi}{2}$, where \mathcal{A} is an annulus with boundary $\partial E \cup \partial E''$. However, this implies that $|E'| + |\mathcal{A}| < |E''|$. This is a contradiction as $E' \subset D_i$ is area minimizing among the embedded disks with the same boundary. So, Σ_i is also area minimizing among the embedded disks with the same boundary. Hence, Step 4 follows. \square

Hence, by taking one of the Σ_i pieces in Δ given by Step 4, we get a continuous map $\varphi : D^2 \rightarrow M$ with $\varphi(D^2) = \Sigma_i$ and $\varphi(\partial D^2) = \Gamma$. Moreover, φ is a smooth embedding on $D^2 - \varphi^{-1}(\Gamma)$, and $\Sigma_i - \Gamma$ is an embedded minimal surface. The proof follows. \square

Remark 3.1. This theorem shows that for a given simple closed curve Γ bounding a disk in a closed 3-manifold, there exists a pseudo-area minimizing disk Σ in M with $\partial \Sigma = \Gamma$. Moreover, the theorem gives the structure of the pseudo-area minimizing disks: A pseudo-area minimizing disk Σ may not be an embedded disk, but it can only fail embeddedness if the boundary Γ bumps into the interior of Σ ($\varphi(\text{int}(D^2))$). Also, Γ can only intersect the interior of Σ nontransversely; i.e., Σ has no transverse self-intersection. A pseudo-area minimizing disk Σ is smooth and minimal everywhere except where Γ meets the interior of Σ .

On the other hand, this result is true for more general manifolds, namely homogeneously regular 3-manifolds with sufficiently convex boundary. In Section 6 of [HS], one might find the reasons why we need the conditions of being homogeneously regular and being sufficiently convex to extend these results.

Theorem 3.2. *Let Γ be a simple closed curve bounding an embedded disk in a homogeneously regular 3-manifold M with sufficiently convex boundary. Then, there exists a disk Σ in M with $\partial \Sigma = \Gamma$ such that Σ minimizes the area among the embedded disks bounding Γ . Moreover, Σ is embedded in the interior, and it is smooth and minimal everywhere except where the boundary Γ meets the interior of Σ .*

Proof. Since Lemma 2.1 is valid for these new ambient manifolds, we can still construct the sequence $\{D_i\}$ as before. By replacing the lemmas we used in the previous theorem with their analogs in the new ambient manifold as in Section 6 of [HS], the same proof would work. \square

In other words, this theorem applies to closed 3-manifolds, compact 3-manifolds with sufficiently convex boundary, homogeneously regular noncompact 3-manifolds, and homogeneously regular noncompact 3-manifolds with sufficiently convex boundary.

The structure of the pseudo-area minimizing disks given by the main theorem also tells us how to construct nearby special smoothly embedded disks with the same boundary.

Corollary 3.3. *Let M be a closed 3-manifold or a homogeneously regular 3-manifold with sufficiently convex boundary. Let Γ be a simple closed curve bounding an embedded disk in M . Then, for any given $\epsilon > 0$, there exist a smooth, embedded disk Σ_ϵ in M with $\partial\Sigma_\epsilon = \Gamma$ such that $|\Sigma_\epsilon| < C_\Gamma + \epsilon$ and there exists a small region R in Σ_ϵ where $|R| < \epsilon$ with $\Sigma_\epsilon - R$ is a minimal surface. Here $|\cdot|$ represents the area, and C_Γ is the infimum of the areas of the embedded disks in M with boundary Γ .*

Proof. According to Theorem 3.1, the pseudo-area minimizing disk Σ can only fail embeddedness along the coincidence set λ where the boundary Γ bumps into the interior of Σ . Hence, for a given ϵ , take a small neighborhood of this segment in Σ , and push every sheet slightly away from λ so that we get a smooth, embedded disk. Since we can choose the neighborhood as small as we want, the corollary follows. \square

Remark 3.2 (Regularity near the coincidence set and the thin obstacle problem). Another interesting question on the problem is the regularity of φ near the coincidence set. This is a well-known problem in the literature and it's known as the *Thin obstacle problem* or *Signorini problem*. Clearly, φ may not be smooth along the coincidence set, but it may have some regularity when restricted to one side of the coincidence set. In the classical setting of the problem in dimension 2, let Ω be a bounded open subset in \mathbf{R}^2 and A be a line segment in Ω . Let $\psi : A \rightarrow \mathbf{R}$ and $g : \partial\Omega \rightarrow \mathbf{R}$ be given where $g \geq \psi$ on $\partial\Omega \cap A$ and where ψ, g are smooth. Then the question is to find $\varphi : \Omega \rightarrow \mathbf{R}$, where $\varphi|_{\partial\Omega} = g$ and $\varphi|_A \geq \psi$, and $\varphi(\Omega)$ has minimum area. In the classical works [Fr], [Ni] on the problem, the authors showed the Lipschitz continuity of the solutions in all dimensions, and in dimension 2, Richardson showed that the solutions are $C^{1, \frac{1}{2}}$, which is the optimal regularity [Ri]. Recently, Guillen generalized Richardson's result to any dimension [Gui].

Hence, when we apply these results to our case, when $M = \mathbf{R}^3$, and given that a simple closed curve $\Gamma \subset \mathbf{R}^3$ is smooth, then the above results imply that if φ is as in the main theorem, then for each local sheet of Σ , φ is $C^{1, \frac{1}{2}}$ on either side of the coincidence set λ . Also, when M is a hyperbolic 3-manifold and Γ is a geodesic, then Calegari-Gabai's result when applied to our case implies C^1 regularity of φ on either side of the coincidence set (Lemma 1.31 in [CG]).

On the other hand, assuming that Γ is smooth enough, it is possible to show that φ is $H^{1,2}$ near the coincidence set by arguing as in [CG] (Section 1.7). In particular, if Γ is a smooth simple closed curve in M , then it can be shown that $\varphi : D^2 \rightarrow M$ given in the main theorem is in the Sobolev space $H^{1,2}$; i.e., the derivative $d\varphi$ is defined and L^2 in the sense of distribution. The idea is basically the same if $\varphi_i : D^2 \rightarrow M$ with $\varphi(D^2) = D_i$, where $\{D_i\}$ is the sequence of area minimizing disks constructed in section 3.1 with modified metric. Then each φ_i induces a conformal structure on D^2 . The L^2 norm of the derivatives can only

blow up along a neck pinch. In our case, we work with D^2 , and hence we cannot have a neck pinch by the energy minimizing property of φ_i . So, we get the limit of these conformal structures, and the L^2 norm of $d\varphi$ can be bounded in terms of the L^2 norms of $d\varphi_i$. In other words, the idea is the same with the argument in [CG] except we work with disks instead of surfaces.

There are also relevant results in the literature concerning the structure of the coincidence set. When $M = \mathbf{R}^3$ and Γ is analytic, then the coincidence set is a finite union of points and intervals in certain cases [Le], [At] (see Remark 1.32 in [CG]).

4. CONCLUDING REMARKS

In this paper, we studied the embeddedness question of minimal disks in 3-manifolds. Unlike the many results considering the question “for which curves must the area minimizing disks be embedded?” in the literature, we analyzed the structure of the surface which minimizes area among the embedded disks whose boundary is any given simple closed curve. Hence, we showed that among all the embedded disks with fixed boundary, the area minimizer (pseudo-area minimizing disk) exists; however, it may not be an embedded disk. It is a disk in the manifold, and it only fails to be an embedding along the coincidence set where the boundary bumps nontransversely into the interior. Other than this exceptional part, the disk is a smoothly embedded minimal surface. Hence, for any simple closed curve in a 3-manifold M , we construct a canonical *almost* embedded disk in M among the embedded disks bounding the given curve.

Intuitively, one might think of the pseudo-area minimizing disk Σ in the following way. Let D be the area minimizing disk M with boundary Γ . As in Figure 1, D may not be embedded, and it might have self-intersection. Then, one can get Σ from D by pushing the self-intersection into the boundary. The interesting fact here is that Σ is smooth, minimal, and embedded everywhere except where Γ meets the interior of Σ , say λ . An alternative way to see the picture is that if you push from the coincidence set λ in Σ into the convex part, you can reduce the area of Σ (like a folding curve in [MY2]), but you create a transverse self-intersection in the interior. This cannot happen as Σ is minimizing area among the *embedded* disks. In other words, the coincidence set λ behaves like a barrier to embeddedness, even though you can reduce the area by going in that direction.

Also, one might ask whether φ is an embedding on the whole $\text{int}(D^2)$ or not. This is not true in general. The reason is that the interior might have nontransverse self-intersection with itself at the coincidence set λ . For example, if Σ has more than one local sheet near the coincidence set λ , then $\varphi|_{\text{int}(D^2)}$ cannot be an embedding. To construct such an example, one can take two “parallel” close embedded area minimizing disks Σ_1 and Σ_2 . Then, by adding a tiny bridge β between them one can get another area minimizing disk Σ with new boundary which is close to $\Sigma_1 \cup \Sigma_2 \cup \beta$. Now, if you make a thin and long horn from the part close to Σ_1 which intersects both Σ_1 and Σ_2 transversely as in Figure 1, then the area minimizing disk $\hat{\Sigma}$ which minimizes area for the new boundary curve is an example for such a situation. One needs to push the intersection of the horn with both Σ_1 and Σ_2 to the boundary, and the interiors of them will meet in the boundary. Hence, φ cannot be an embedding on $\text{int}(D^2)$ in general.

On the other hand, there is a relevant result about the same question due to Meeks-Yau [MY3]. They give a necessary condition for a sufficiently smooth simple closed curve in a 3-manifold to bound a strictly stable embedded minimal disk.

Theorem 4.1 ([MY3, Theorem 3]). *Let Γ be a $C^{2,\alpha}$ simple closed curve in a 3-manifold M . Then, Γ bounds a strictly stable minimal disk Σ in M if and only if there exists a sufficiently convex codimension-0 submanifold N in M , whose topological type is a 3-ball, and $\Gamma \subset \partial N$.*

Hence, by combining this result with ours for $C^{2,\alpha}$ smooth simple closed curves, we conclude that if Γ is a $C^{2,\alpha}$ simple closed curve in a 3-manifold M , and there is no sufficiently convex domain N as in the above theorem with $\Gamma \subset \partial N$, then the pseudo-area minimizing disk Σ given by our main theorem is not embedded up to the boundary. In other words, if $\varphi : D^2 \rightarrow M$ parameterizes Σ with $\varphi(D^2) = \Sigma$, $\varphi(\partial D^2) = \Gamma$, then $\Gamma \cap \varphi|_{\text{int}(D^2)} \neq \emptyset$. However, neither our result nor the above theorem of Meeks-Yau says anything about unstable minimal disks. It is still possible for such a simple closed curve in a 3-manifold M to bound an embedded unstable minimal disk in M .

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