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NON-INTEGRATED DEFECT RELATION FOR MEROMORPHIC MAPS OF COMPLETE KÄHLER MANIFOLDS INTO $\mathbb{P}^n(\mathbb{C})$ INTERSECTING HYPERSURFACES

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Abstract. In this paper, we establish a non-integrated defect relation for a meromorphic map of a complete Kähler manifold whose universal covering is biholomorphic to a ball in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position, as well as an application to the Gauss map of a closed regular submanifold of $\mathbb{C}^m$. The result provides a complement to the recent result of Ru (2004) on a defect relation for meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position.

1. Introduction

Relating to the study of the value distribution of the Gauss maps of complete minimal surfaces in $\mathbb{R}^m$, H. Fujimoto (see [4]) introduced the notion of the non-integrated defect for a holomorphic map of an open Riemann surface M into $\mathbb{P}^n(\mathbb{C})$ and obtained some results analogous to the classical defect relation. In [5], he generalized these results to the case of a meromorphic map of a complete Kähler manifold M whose universal covering is biholomorphic to a ball in $\mathbb{C}^m$ into $\mathbb{P}^{n_1}(\mathbb{C})\times\cdots\times\mathbb{P}^{n_k}(\mathbb{C})$ and the meromorphic map satisfies a certain growth condition (see the condition in Theorem 1.1).

Let M be an m-dimensional complex Kähler manifold. Let f be a meromorphic map of M into $\mathbb{P}^n(\mathbb{C})$, μ_0 be a positive integer and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d with $f(M)\not\subset D$. We denote the intersection multiplicity of the image of f and D at $f(p)$ by $\nu^f(D)(p)$ and the pull-back of the normalized Fubini-Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by Ω_f. The *non-integrated defect of f with respect to D cut by μ_0* is defined by

$$\delta^f_{\mu_0}(D) := 1-\inf\{\eta\geq 0 : \eta \text{ satisfies condition } (\star)\}.$$

Here, the condition $(\star)$ means that there exists a bounded non-negative continuous function h on M with zeros of order not less than $\min(\nu^f(D),\mu_0)$ such that

$$d\eta\Omega_f+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log h^2\geq[\min(\nu^f(D),\mu_0)],$$

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where d is the degree of D and we mean by $[\nu]$ the $(1,1)$-current associated with a divisor ν. Note that the condition $(\star)$ also means that, for each holomorphic function ϕ ($\not\equiv 0$) on an open subset U of M with $\nu_\phi = \min(\nu^f(D), \mu_0)$ outside an analytic set of codimension ≥ 2, the function $u := \log(h^2\|f\|^{2d\eta}/|\phi|^2)$ is continuous and plurisubharmonic on U, where $\|f\|^2 = |f_0|^2 + \cdots + |f_n|^2$, and $f = [f_0 : \cdots : f_n]$ is a (local) reduced representation of f. So, similar to the classical Nevanlinna's defect, we have the following properties.

- $0 \leq \delta^f_{\mu_0}(D) \leq 1$. To see $\delta^f_{\mu_0}(D) \geq 0$, take $\eta = 1$ and $h = |Q(f)|/\|f\|^d$, where Q is the homogeneous polynomial defining D.
- If $f(M) \cap D = \emptyset$, then, by taking $\eta = 0, h = 1$, we have that $\delta^f_{\mu_0}(D) = 1$.
- If $\nu^f(D)(p) \geq \mu$ for all $p \in f^{-1}(D)$, with some positive integer $\mu \geq \mu_0$, then $\delta^f_{\mu_0}(D) \geq 1 - \mu_0/\mu$ by taking $\eta = \mu_0/\mu$ and $h = (|Q(f)|/\|f\|^d)^{\mu_o/\mu}$.

Let $\omega = \frac{\sqrt{-1}}{2}\sum_{ij} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$ be the Kähler form of M. We define

$$\text{Ric } \omega = dd^c \log(\det(h_{ij})),$$

where $d = \partial + \bar{\partial}$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

We assume the following growth condition for f: there exists a non-zero bounded continuous real-valued function h on M such that $\rho\Omega_f + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h^2 \geq \text{Ric } \omega$ for some non-negative constant ρ. In this paper, we extend the results in [5] to the case where $f(M)$ intersects with hypersurfaces instead of hyperplanes; namely, we prove the following main theorem.

Theorem 1.1. *Let M be an m-dimensional complete Kähler manifold and $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is algebraically non-degenerate (i.e., its image is not contained in any proper subvariety of $\mathbb{P}^n(\mathbb{C})$). Assume that the universal covering of M is biholomorphic to a ball in $\mathbb{C}^m$. Let $D_1, ..., D_q$ be hypersurfaces of degree d_j in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $d =$ l.c.m.$\{d_1, \ldots, d_q\}$ (the least common multiple of $\{d_1, \ldots, d_q\}$). Assume that there exists a non-zero bounded continuous real-valued function h on M such that $\rho\Omega_f + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h^2 \geq \text{Ric } \omega$ for some non-negative constant ρ. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta^f_{l-1}(D_j) \leq n + 1 + \epsilon + \frac{\rho l(l-1)}{d},$$

where $l \leq 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n$ and $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x.

We remark that in the case $M = \mathbb{C}^m$, we also have the following statement (see Theorem 4.5 and Corollary 4.6) which is essentially due to Min Ru (see [10] and [9]) without the truncation and to An and Phuong with the truncation (see [1]).

Theorem 1.2. *Let $f : \mathbb{C}^m \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is algebraically non-degenerate. Let $D_1, ..., D_q$ be hypersurfaces of degree d_j in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $d =$ l.c.m.$\{d_1, \ldots, d_q\}$. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta^{f,\star}_{l-1}(D_j) \leq n + 1 + \epsilon,$$

where $l \leq 2^{n^2+4n}e^n d^{2n}(nI(\epsilon^{-1}))^n$, $I(x) := \min\{k \in \mathbb{N} : k > x\}$ *for a positive real number* x, *and* $\delta^{f,\star}_{l-1}(D)$ *is the classical Nevanlinna's (truncated) defect of* f *with respect to* D.

Note that, from the discussion below, we have that $\delta^{f}_{l-1}(D) \leq \delta^{f,*}_{l-1}(D)$ (see Proposition 2.1). Thus, Theorem 1.1 and Theorem 1.2 are complementing each other.

The main tools of the proof are the following fundamental result of Yau: "*Let* M *be a complete Riemannian manifold of infinite volume and* u *a non-negative function satisfying* $\triangle \log u = 0$ *almost everywhere. Then* $\int_M u^p = \infty$ *for* $p > 0$," as well as the techniques developed in [10] and [11] of dealing with hypersurfaces in general position. The proof gets down to the construction of such a function u on M which contradicts Yau's result mentioned above if the theorem fails.

2. Preliminaries

First of all, throughout the article, we'll use the common letter K to denote a constant, even when it should be replaced by a new constant. Let h be a non-constant holomorphic function on an open domain $G \subset \mathbb{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of integers $\alpha_i \geq 0$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $D^\alpha h = D_1^{\alpha_1} \cdots D_m^{\alpha_m} h$, where $D_i h = (\partial/\partial z_i)h$, for $i = 1, \ldots, m$. We define $\nu^0_h : G \to \mathbb{Z}$ by

$$\nu^0_h(z) := \max\{k : D^\alpha h(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < k\} \ (z \in G).$$

By a divisor on a domain G in $\mathbb{C}^m$ we mean a map ν of G into $\mathbb{Z}$ such that, for each $z_0 \in G$, there are non-zero holomorphic functions h and g on a connected neighborhood $U(\subset G)$ of z_0 so that $\nu(z) = \nu^0_h(z) - \nu^0_g(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$.

Take a non-zero meromorphic function φ on a domain G in $\mathbb{C}^m$. For each $z_0 \in G$, we choose non-zero holomorphic functions g and h on a neighborhood $U(\subset G)$ of z_0 such that $\varphi = \frac{g}{h}$ on U and $\dim(f^{-1}(0) \cup g^{-1}(0)) \leq m-2$. We define $\nu^\infty_\varphi := \nu_h$, $\nu^a_\varphi := \nu_{g-ah}$ for $a \in \mathbb{C}$ and $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$, which are independent of the choices of h and g and so are globally well-defined on G. Let f be a meromorphic map of $B(R_0) \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, $0 < R_0 \leq \infty$. We take holomorphic functions $f_0, f_1, \ldots, f_n$ such that $I_f := \{z \in B(R_0), f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $f(z) = [f_0(z) : \cdots : f_n(z)]$ on $B(R_0) - I_f$ in terms of homogeneous coordinates $[w_0 : \cdots : w_n]$ on $\mathbb{P}^n(\mathbb{C})$. We call such a representation $f(z) = [f_0(z) : \cdots : f_n(z)]$ a reduced representation of f.

For $z = (z_1, ..., z_m) \in \mathbb{C}^m$ we set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ and define $B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}$, $\quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\}$ for $0 < r \leq +\infty$, where we mean $B(\infty) = \mathbb{C}^m$ and $S(\infty) = \emptyset$. Define

$$\sigma_m := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \qquad \text{on} \quad \mathbb{C}^m - \{0\},$$
$$v_l := (dd^c\|z\|^2)^l \qquad \text{for} \quad 1 \leq l \leq m.$$

Let $f(z) = [f_0(z) : \cdots : f_n(z)]$ be a reduced representation of f. Set $\|f\| := \left(|f_0|^2 + \cdots + |f_n|^2\right)^{1/2}$. Then the pull-back of the normalized Fubini-Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by f is given by

$$\Omega_f = dd^c \log\|f\|^2.$$

Fixing $r_0 < R_0$, the characteristic function of f is defined by

$$T_f(r,r_0) = \int_{r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} \Omega_f \wedge v_{m-1} \qquad (0 < r_0 < r < R_0).$$

We then have (see [12], pp. 251-255),

$$T_f(r,r_0) = \int_{S(r)} \log\|f\|\sigma_m - \int_{S(r_0)} \log\|f\|\sigma_m.$$

Let μ_0 be a positive integer or ∞ and ν be a divisor on a domain $B(R_0) \subset \mathbb{C}^m$. Set $|\nu| = \overline{\{z \in B(R_0) : \nu(z) \neq 0\}}$. We define the counting function of ν truncated by μ_0 by

$$N_{\nu}^{[\mu_0]}(r_0,r) = \int_{r_0}^{r} \frac{n^{[\mu_0]}(t)}{t} dt,$$

where

$$n^{[\mu_0]}(t) = \frac{1}{t^{2m-2}} \int_{|\nu| \cap B(t)} \min\{\nu, \mu_0\} v_{m-1} \quad \text{if } m \geq 2,$$
$$n^{[\mu_0]}(t) = \sum_{|z| \leq t} \min\{\nu(z), \mu_0\} \quad \text{if } m = 1.$$

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d, and let Q be the homogeneous polynomial defining D. We define the divisor $\nu(f,D)(z) := \nu^0_{Q \circ f}(z)$ $(z \in B(R_0))$ which is rephrased as the intersection multiplicity of the image of f and D at $f(z)$. We define the (truncated) counting function of D for f by, for fixed $r_0 < R_0$,

$$N_f^{[\mu_0]}(r,D) := N_{\nu(f,D)}^{[\mu_0]}(r_0,r).$$

By the first main theorem,

$$N_f^{[\mu_0]}(r,D) \leq dT_f(r,r_0) + O(1).$$

The classical Nevanlinna's defect $\delta_{\mu_0}^{f,\star}(D)$ of f with respect to D cut by μ_0 is defined by

$$\delta_{\mu_0}^{f,\star}(D) = \limsup_{r \to R_0} \left(1 - \frac{N_f^{[\mu_0]}(r,D)}{dT_f(r,r_0)}\right).$$

The relationship between the non-integrated defect and the classical Nevanlinna's defect is given as follows.

Proposition 2.1. *If* $\lim_{r \to R_0} T_f(r,r_0) = \infty$, *then*

$$0 \leq \delta_{\mu_0}^{f}(D) \leq \delta_{\mu_0}^{f,\star}(D) \leq 1,$$

where $\delta^{\star}$ *is the classical Nevanlinna's defect.*

Proof. Take η satisfying the condition $(\star)$ in the definition of $\delta_{\mu_0}^{f}(D)$. The function

$$v := d\eta \log\|f\| + \log h - \log|\varphi|$$

is then plurisubharmonic, where h is bounded and φ is holomorphic on $B(R_0)$ with $\nu_\varphi = \min(\nu^f(D), \mu_0)$ outside an analytic set of codimension ≥ 2. Therefore,

$$\begin{aligned} 0 &\leq \int_{S(r)} v\sigma_m - \int_{S(r_0)} v\sigma_m \\ &= d\eta \int_{S(r)} \log\|f\|\sigma_m + \int_{S(r)} \log h\sigma_m - \int_{S(r)} \log|\varphi|\sigma_m + K \\ &\leq d\eta T_f(r, r_0) - N_f^{[\mu_0]}(r, D) + K, \end{aligned}$$

where K is a constant, because h is bounded from above. This implies that

$$\frac{N_f^{[\mu_0]}(r, D)}{dT_f(r, r_0)} \leq \eta + \frac{K}{T_f(r, r_0)}.$$

As $r \to R_0$, we obtain $\delta^\star_{\mu_0}(D) \geq 1 - \eta$. Hence $\delta^\star_{\mu_0}(D) \geq \delta^f_{\mu_0}(D)$. □

3. The generalized Wronskians and the lemma of the logarithmic derivative

We first recall the following lemma of the logarithmic derivative.

Theorem 3.1 (See [5],Theorem 3.1)**.** *Let ϕ be a non-zero a meromorphic function on $B(R_0) \subset \mathbb{C}^m$, $0 < R_0 \leq \infty$, and let $\alpha = (\alpha_1, ..., \alpha_m) \neq (0, ..., 0), 0 < r_0 < R_0$, and take positive numbers p, p' such that $0 < p|\alpha| < p' < 1$. Then, for $r_0 < r < R < R_0$,*

$$\int_{S(r)} \Big|z^\alpha (D^\alpha \phi/\phi)(z)\Big|^p \sigma_m(z) \leq K\Big(\frac{R^{2m-1}}{R-r} T_\phi(R, r_0)\Big)^{p'},$$

where K is a constant not depending on each r and R, and $z^\alpha := z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ for $z = (z_1, \ldots, z_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Let F be a meromorphic map of $B(R_0) \subset \mathbb{C}^m$ $(0 < R_0 \leq \infty)$ into $\mathbb{P}^{l-1}(\mathbb{C})$. Take a reduced representation $F = [F_1 : \cdots : F_l]$. We shall say that F is *linearly non-degenerate* if $F(B(R_0)) \not\subset H$ for every hyperplane H in $\mathbb{P}^{l-1}(\mathbb{C})$, or equivalently, $F_1, ..., F_l$ are linearly independent on $\mathbb{C}^m$. Take an arbitrary set $\alpha^i = (\alpha_{i1}, ..., \alpha_{il}), 1 \leq i \leq l$, of non-negative integers. We define the generalized Wronskian of F by

$$W_{\alpha^1 \ldots \alpha^l}(F) = \det\left(D^{\alpha^1} F, \ldots, D^{\alpha^l} F\right).$$

Proposition 3.2 (See [5], Proposition 4.5)**.** *Let $F : B(R_0) \to \mathbb{P}^{l-1}(\mathbb{C})$ be a linearly non-degenerate meromorphic map. Then there exist $\alpha^j = (\alpha_{j1}, \ldots, \alpha_{jl})$ with $\alpha_{ji} \geq 0$ being integers, $|\alpha^j| \leq l - 1$ for $1 \leq j \leq l$, and $|\alpha^1| + \cdots + |\alpha^l| \leq l(l-1)/2$ such that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^l}(F) \not\equiv 0$.*

Take such $\alpha^j = (\alpha_{j1}, \ldots, \alpha_{jl}), 1 \leq j \leq l$, so that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^l}(F) \not\equiv 0$. Let $L_1, \ldots, L_l$ be linear forms of l variables and assume that they are linearly independent. Theorem 3.1 implies the following proposition.

Proposition 3.3 (See [5], Proposition 6.1)**.** *In the above situation, set $l_0 = |\alpha^1| + \cdots + |\alpha^l|$ and take t, p' with $0 < tl_0 < p' < 1$. Then, for $0 < r_0 < R_0$ there exists a* *positive constant K such that for $r_0 < r < R < R_0$,*

$$\int_{S(r)} \Big| z^{\alpha^1+\cdots+\alpha^l} \frac{W_{\alpha^1\ldots\alpha^l}(F)}{L_1(F)\cdots L_l(F)}\Big|^t \sigma_m \leq K\Big(\frac{R^{2m-1}}{R-r}T_F(R,r_0)\Big)^{p'}.$$

Proof. By the property of the Wronskian (see [5], Proposition 4.9),

$$W_{\alpha^1\ldots\alpha^l}(F_1,...,F_l) = CW_{\alpha^1\ldots\alpha^l}(L_1(F),...,L_l(F)),$$

where C is a constant depending only on $L_1,\ldots,L_l$. Hence, we only need to estimate

$$I := \int_{S(r)} \Big| z^{\alpha^1+\cdots+\alpha^l} \frac{W_{\alpha^1\ldots\alpha^l}(L_1(F),...,L_l(F))}{L_1(F)\cdots L_l(F)}\Big|^t \sigma_m.$$

Let

$$\chi = \frac{W_{\alpha^1\ldots\alpha^l}(L_1(F),...,L_l(F))}{L_1(F)\cdots L_l(F)}$$

and set $\varphi_j := L_j(F)/L_1(F), 1 \leq j \leq l$. Again, by the property of the Wronskian (see [5], Proposition 4.9), we can write χ as

$$\chi = \frac{W_{\alpha^1\ldots\alpha^l}(\varphi_1,\varphi_2,...,\varphi_l)}{\varphi_1\cdots\varphi_l} = \sum_{\sigma_{i_1\cdots i_l}} \mathrm{sgn}(\sigma_{i_1\cdots i_l})\frac{D^{\alpha^1}\varphi_{i_1}}{\varphi_{i_1}}\cdots\frac{D^{\alpha^l}\varphi_{i_l}}{\varphi_{i_l}},$$

where $\sigma_{i_1\cdots i_l}$ is the permutation from $(1,2,\ldots,l)$ to $(i_1,\ldots,i_l)$. Therefore, the above integrand can be estimated from above by a positive constant multiple of the sum of some functions of the type

$$\psi_{i_1\ldots i_l} := \Big| z^{\alpha^1+\cdots+\alpha^l}\frac{D^{\alpha^1}\varphi_{i_1}}{\varphi_{i_1}}\cdots\frac{D^{\alpha^l}\varphi_{i_l}}{\varphi_{i_l}}\Big|^t.$$

Set $p_j = \frac{|\alpha^j|}{|\alpha^1|+\cdots+|\alpha^l|}$ for $1 \leq j \leq l$. By the Hölder inequality, we obtain

$$\int_{S(r)} \psi_{i_1\ldots i_l}\sigma_m \leq \prod_{j=1}^{l}\Big[\int_{S(r)}\Big| z^{\alpha_j}\frac{D^{\alpha_j}\varphi_{i_j}}{\varphi_{i_j}}\Big|^{t/p_j}\sigma_m\Big]^{p_j}.$$

Since $(t/p_j)|\alpha^j| = (|\alpha_1|+\cdots+|\alpha_l|)t < p' < 1$ for $1 \leq j \leq l$, we can apply Theorem 3.1 to get

$$\int_{S(r)} \psi_{i_1\ldots i_l}\sigma_m \leq K\prod_{j=1}^{l}\Big(\frac{R^{2m-1}}{R-r}T_{\varphi_{i_j}}(R,r_0)\Big)^{p'p_j}.$$

On the other hand, by (2.5) in [5],

$$T_{\varphi_j}(r,r_0) \leq T_F(r,r_0) + K$$

for every $i = 1,...,l$. Therefore, we conclude that

$$I \leq K\Big(\frac{R^{2m-1}}{R-r}T_F(R,r_0)\Big)^{p'}.$$

This completes the proof of the proposition. □

4. Proof of the main theorem

In this section, we prove the main theorem (Theorem 1.1). Before doing that, we first prove some preparation lemmas.

Let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d, located in general position, and let $Q_j, 1 \le j \le q$, be homogeneous polynomials defining D_j. Let N be a large integer (to be determined later), and let V_N be the space of homogeneous polynomials of $n+1$ variables of degree N. Pick n distinct polynomials $\gamma_1, ..., \gamma_n \in \{Q_1, \dots, Q_q\}$. Arrange the n-tuples $\mathbf{i} = (i_1, ..., i_n)$ of non-negative integers by lexicographic order. Define, for the n-tuples $\mathbf{i} = (i_1, ..., i_n)$ of non-negative integers with $\sigma(\mathbf{i}) := \sum_j i_j \le N/d$, the spaces $W_{\mathbf{i}} := W_{N,\mathbf{i}}$ by

$$W_{N,\mathbf{i}} = \sum_{\mathbf{e} \ge \mathbf{i}} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0,...,0)} = V_N$ and $W_{\mathbf{i}} \supset W_{(\mathbf{i}')}$ if $\mathbf{i}' \ge \mathbf{i}$, so that the $\{W_{\mathbf{i}}\}$ in fact define a filtration of V_N. We recall the following lemma due to [3].

Lemma 4.1 (See [3], Proposition 3.3)**.** *For any non-negative integer N and any $\{\gamma_1, ..., \gamma_n\} \subset \{Q_1, \dots, Q_q\}$, the dimension of the vector space $\frac{V_N}{(\gamma_1, \dots, \gamma_n) \cap V_N}$ is equal to the number of n-tuples $(\mathbf{i}) = (i_1, ..., i_n) \in \mathbb{Z}^n_{\ge 0}$ such that $i_1 + \cdots + i_n \le N$ and $0 \le i_1, ..., i_n \le d-1$. In particular, for all $N \ge n(d-1)$, we have*

$$\dim \frac{V_N}{(\gamma_1, ..., \gamma_n) \cap V_N} = d^n.$$

Lemma 4.2 (See [10], Lemma 3.2)**.** *There is an isomorphism*

$$\frac{W_{\mathbf{i}}}{W_{\mathbf{i}'}} \cong \frac{V_{N-d\sigma(\mathbf{i})}}{(\gamma_1, ..., \gamma_n) \cap V_{N-d\sigma(\mathbf{i})}},$$

where $\mathbf{i}' > \mathbf{i}$ are consecutive n-tuples with $W'_{\mathbf{i}} \subset W_{\mathbf{i}}$.

Let $\triangle_{\mathbf{i}} = \dim(W_{\mathbf{i}}/W_{\mathbf{i}'})$, where $\mathbf{i}' > \mathbf{i}$ are consecutive n-tuples with $W_{\mathbf{i}'} \subset W_{\mathbf{i}}$. By Lemma 4.1, $\triangle_{\mathbf{i}} = d^n$ for every $\mathbf{i}$ such that $N - d\sigma(\mathbf{i}) \ge n(d-1)$. Moreover, Lemma 4.1 implies that $\triangle_{\mathbf{i}}$ is independent of the choice of $\gamma_1, ..., \gamma_n$. Hence, $\sum_{\mathbf{i}} i_j \triangle_{\mathbf{i}}$ is independent of the choice of $\gamma_1, ..., \gamma_n$ and j for $j = 1, \dots, n$. Set, for $1 \le j \le n$,

$$\triangle := \sum_{\mathbf{i}} i_j \, \triangle_{\mathbf{i}} \, . \tag{4.1}$$

Lemma 4.3. *With $N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1) + nd$ for any $\epsilon > 0$, we have*

$$\frac{lN}{\triangle} \le d(n+1) + \epsilon/2 \tag{4.2}$$

where $l = \binom{N+n}{n}$. Moreover, l satisfies the following estimate:

$$l \le 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n, \tag{4.3}$$

where $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x.

Proof. First notice that

$$l = \binom{N+n}{n} = \frac{(n+N)(n+N-1)\cdots(n+1)N!}{N!n!} \leq \frac{(N+n)^n}{n!}. \tag{4.4}$$

Now since N is divisible by d, it follows from Lemma 4.1 that

$$\begin{aligned}
\triangle = \sum_{\sigma(\mathbf{i})\leq N/d} i_j \triangle_{\mathbf{i}} \geq \sum_{\sigma(\mathbf{i})\leq N/d-n} i_j \triangle_{\mathbf{i}} = d^n \sum_{\sigma(\mathbf{i})\leq N/d-n} i_j \\
&= \frac{d^n}{n+1} \sum_{\sigma(\hat{\mathbf{i}})=N/d-n} \sum_{j=1}^{n+1} i_j \\
&= \frac{d^n}{n+1} \sum_{\sigma(\hat{\mathbf{i}})=N/d-n} (N/d-n) \\
&= \frac{d^n}{n+1} \binom{N/d}{n} (N/d-n) \\
&= \frac{N(N-d)\cdots(N-dn)}{d(n+1)!},
\end{aligned}$$

where $\hat{\mathbf{i}} = (i_1, \ldots, i_{n+1})$ and, in the above, we used the fact that the number of non-negative integer m-tuples with sum $\leq T$ for a positive integer T is equal to the number of non-negative integer $(m+1)$-tuples with sum exactly T, which is $\binom{T+m}{m}$.

For every integer $j \leq n$, $(N-dj) \geq (N-dn)$; so

$$\prod_{j=1}^{n} \frac{1}{N-dj} \leq \Big(\frac{1}{N-dn}\Big)^n$$

and thus

$$\frac{lN}{\triangle} \leq d(n+1)\Big(\frac{N+n}{N-nd}\Big)^n.$$

Using

$$N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1) + nd,$$

one finds that

$$\begin{aligned}
\Big(\frac{N+n}{N-nd}\Big)^n &= \Big(1 + \frac{n+nd}{N-nd}\Big)^n \\
&= 1 + \sum_{r=1}^{n} \binom{n}{r} \Big(\frac{nd+n}{N-nd}\Big)^r \\
&\leq 1 + (2^n-1)\frac{nd+n}{N-nd} \\
&\leq 1 + \frac{\epsilon}{2d(n+1)}.
\end{aligned}$$

Therefore

$$\frac{lN}{\triangle} \leq d(n+1) + \epsilon/2.$$

To estimate l, we use the following inequality:

$$\begin{pmatrix} x+y \\ y \end{pmatrix} \leq \frac{(x+y)^{x+y}}{x^x y^y} = \Big(1+\frac{y}{x}\Big)^x \Big(1+\frac{x}{y}\Big)^y = \Big(e(1+\frac{x}{y})\Big)^y$$

for positive integers x, y. Hence, with $N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1)+nd$, we have

$$\begin{aligned} l = \begin{pmatrix} N+n \\ n \end{pmatrix} &\leq e^n \Big(1+\frac{N}{n}\Big)^n \\ &\leq e^n \big(1+2d(n+1)(d+1)(2^n-1)(I(\epsilon^{-1})+1)+d\big)^n \\ &\leq 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n. \end{aligned}$$

□

We now prove the main theorem. Since the universal covering of M is the unit ball in $\mathbb{C}^m$, by lifting f to the covering, we may assume that $M = B(1) \subset \mathbb{C}^m$. So we let $f : B(1) \longrightarrow \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate map. The proof of the main theorem breaks into the following two cases: the case

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty$$

and the case

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} = \infty.$$

We first deal with the case when

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty.$$

Let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_1, \dots, d_q$, located in general position. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the l.c.m. (the least common multiple) of the d_j's, we can assume that $Q_1, \dots, Q_q$ have the same degree d. For $N \in \mathbb{N}$, let V_N be the space of homogeneous polynomials of $n+1$ variables of degree N and fix a (arbitrary) basis $\phi_1, ..., \phi_l$, where $l = \dim V_N$. Set $F = [\phi_1(f) : \cdots : \phi_l(f)]$. Then $F : B(1) \rightarrow \mathbb{P}^{l-1}(\mathbb{C})$ is linearly non-degenerate. By Proposition 3.2, there exist $\alpha^j = (\alpha_{j1}, \dots, \alpha_{jl})$ with $\alpha_{ji} \geq 0$ being integers, $|\alpha^j| \leq l-1$ for $1 \leq j \leq l$, and $|\alpha^1| + \cdots + |\alpha^l| \leq l(l-1)/2$ such that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^l}(F) \not\equiv 0$.

Given $z \in B(1)$ there exists a numbering $\{i_1, ..., i_q\}$ of the indices $1, ..., q$ such that

$$|Q_{i_1} \circ f(z)| \leq \cdots \leq |Q_{i_q} \circ f(z)|. \tag{4.5}$$

Since $Q_1, ..., Q_q$ are in general position, the Hilbert Nullstellensatz implies that for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk}(x_0, ..., x_n) Q_{i_j}(x_0, ..., x_n),$$

where $b_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, are homogeneous forms with coefficients in $\mathbb{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\},$$

where c_1 is a positive constant depending only on the coefficients of b_{jk}, thus depending only on the coefficients of Q_j. Therefore,

$$\|f(z)\|^d \leq c_1 \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\}. \tag{4.6}$$

By (4.5) and (4.6), we get

$$\prod_{j=1}^{q} \frac{\|f(z)\|^d}{|Q_j(f)(z)|} \leq c_1^{q-n} \prod_{k=1}^{n} \frac{\|f(z)\|^d}{|Q_{i_k}(f)(z)|}. \tag{4.7}$$

Take $\gamma_1 = Q_{i_1}, \cdots, \gamma_n = Q_{i_n}$ and let $V_N = W_{\mathbf{0}} \supset \cdots \supset W_{\mathbf{i}} \supset W_{\mathbf{i}'} \supset \cdots$ be the filtration of V_N, associated to $\{\gamma_1, \ldots, \gamma_n\}$ as discussed earlier. We now choose a basis $\psi_1, ..., \psi_l$ for V_N in the following way: We start with the last non-zero $W_{\mathbf{i}}$ and pick a basis of it. Then, we continue inductively as follows: suppose $\mathbf{i}' > \mathbf{i}$ are consecutive n-tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leq N$ and assume that we have chosen a basis of $W_{(\mathbf{i}')}$. It follows directly from the definition that we may pick representatives in $W_{\mathbf{i}}$ for the quotient space $W_{\mathbf{i}}/W_{\mathbf{i}'}$, of the form $\gamma_1^{i_1} \cdots \gamma_n^{i_n} \eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. We extend the previously constructed basis in $W_{\mathbf{i}'}$ by adding these representatives. In particular we have obtained a basis for $W_{\mathbf{i}}$ and our induction procedure may go on unless $W_{\mathbf{i}} = V_N$. Note that if we let ψ be an element of the basis constructed with respect to $W_{\mathbf{i}}/W_{\mathbf{i}'}$, then we may write $\psi = \gamma_1^{i_1} \cdots \gamma_n^{i_n} \eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. Thus we have a bound

$$|\psi(f)(z)| \leq c_2 |\gamma_1(f)(z)|^{i_1} \cdots |\gamma_n(f)(z)|^{i_n} \|f(z)\|^{N-d\sigma(i)}, \tag{4.8}$$

where c_2 is a positive constant which depends only on f and $Q_1, \ldots, Q_q$. Observe that there are precisely $\triangle_{\mathbf{i}}$ such functions ψ in our basis. Write $\psi_1, ..., \psi_l$ as linear forms $L_1, ..., L_l$ in $\phi_1, ..., \phi_l$ so that $\psi_t(f) = L_t(F)$, where $F = [\phi_1(f) : \cdots : \phi_l(f)]$. Then (4.8) implies that

$$\prod_{t=1}^{l} \Big| L_t(F(z)) \Big| \leq K \left(\prod_{\mathbf{i}=(i_1,\ldots,i_n)} |\gamma_1^{i_1}(f(z)) \cdots \gamma_n^{i_n}(f(z))|^{\triangle_{\mathbf{i}}} \right) \|f(z)\|^{lN - d\sum_{\mathbf{i}} \sigma(\mathbf{i})\triangle_{\mathbf{i}}},$$

where, as we noted earlier, K is a constant depending only on f and $D_1, \ldots, D_q$ which may be different each time. So

$$\frac{\|f(z)\|^{d\sum_{\mathbf{i}} \sigma(\mathbf{i})\triangle_{\mathbf{i}}}}{\prod_{\mathbf{i}} |\gamma_1^{i_1\triangle_{\mathbf{i}}}(f(z))| \cdots |\gamma_n^{i_n\triangle_{\mathbf{i}}}(f(z))|} \leq K \frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l} \Big| L_t(F(z)) \Big|};$$

thus, using (4.1),

$$\frac{\|f(z)\|^{dn\triangle}}{|\gamma_1^{\triangle}(f(z))| \cdots |\gamma_n^{\triangle}(f(z))|} \leq K \frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l} \Big| L_t(F(z)) \Big|}.$$

With $\gamma_1 = Q_{i_1}, \cdots, \gamma_n = Q_{i_n}$, this gives

$$\frac{\|f(z)\|^{dn\triangle}}{|Q_{i_1}^{\triangle}(f(z)) \cdots Q_{i_n}^{\triangle}(f(z))|} \leq K \frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l} \Big| L_t(F(z)) \Big|}. \tag{4.9}$$

On the other hand, from (4.7), we get

$$\frac{\|f(z)\|^{dq\triangle}}{|Q_1^{\triangle}(f(z)) \cdots Q_q^{\triangle}(f(z))|} \leq K \frac{\|f(z)\|^{dn\triangle}}{|Q_{i_k}^{\triangle}(f)(z) \cdots Q_{i_n}^{\triangle}(f)(z)|}. \tag{4.10}$$

Combining (4.9) and (4.10), we derive that

$$\frac{\|f(z)\|^{dq\triangle}}{|Q_1^{\triangle}(f(z))\cdots Q_q^{\triangle}(f(z))|}\le K\frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l}\Big|L_t(F(z))\Big|}.$$

Hence,

$$\frac{\|f(z)\|^{dq\triangle-lN}|W_{\alpha^1\dots\alpha^l}(F)(z)|}{|Q_1^{\triangle}(f(z))\cdots Q_q^{\triangle}(f(z))|}\le K\frac{|W_{\alpha^1\dots\alpha^l}(F)(z)|}{|L_1(F(z))\cdots L_l(F(z))|}. \tag{4.11}$$

Note that although $L_1,\dots,L_l$ depend on z, there are only finitely many such choices since there are only finitely many choices of $\{\gamma_1,\dots,\gamma_n\}\subset\{Q_1,\dots,Q_q\}$.

We continue with the proof of the main theorem by absurdity. We assume that

$$\rho\Omega_f+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log h^2\ge \text{Ric }\omega \tag{4.12}$$

and

$$\sum_{j=1}^{q}\delta_{l-1}^{f}(D_j)>(n+1)+\epsilon+\frac{\rho l(l-1)}{d}. \tag{4.13}$$

Then, from the discussion earlier, there exist constants $\eta_j\ge 0$ and continuous plurisubharmonic functions $\tilde u_j(\not\equiv-\infty)$ such that $e^{\tilde u_j}|\varphi_j|\le\|f\|^{d\eta_j}$ for $j=1,...,q$, and

$$q-\sum_{j=1}^{q}\eta_j>n+1+\epsilon+\frac{\rho l(l-1)}{d}, \tag{4.14}$$

where φ_j is a non-zero holomorphic function with $\nu^0_{\varphi_j}=\min(\nu^f(D_j),l-1)$. Let $u_j=\tilde u_j+\log|\varphi_j|$. Then, u_j $(\not\equiv-\infty),1\le j\le q$, are continuous plurisubharmonic functions,

$$e^{u_j}\le\|f\|^{d\eta_j}, \tag{4.15}$$

and $u_j-\log|\varphi_j|$ is plurisubharmonic, where φ_j is a non-zero holomorphic function with $\nu^0_{\varphi_j}=\min(\nu^f(D_j),l-1)$. Let

$$v:=\log\Big|z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\dots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\Big|+\sum_{j=1}^{q}\triangle u_j, \tag{4.16}$$

where $\triangle$ is the integer defined in (4.1). We now show that v is plurisubharmonic on $M=B(1)$. To do so, we need the following lemma.

Proposition 4.4. *In the above situation, set*

$$\psi=\frac{W_{\alpha^1\cdots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}.$$

Then

$$\nu_\psi^\infty\le\sum_{j=1}^{q}\triangle\min\{\nu^0_{Q_j(f)},l-1\}$$

outside an analytic set of codimension at least two.

Proof. Let I_F be the indeterminacy set of F, and take $a \in B(1)\backslash I_F$. We first show the following claim: *If h is a holomorphic function around a, assume that $D^\alpha h \not\equiv 0$ around a. Then* $\nu^0_{D^\alpha h}(a) = \max\{0, \nu^0_h(a) - |\alpha|\}$. To see this, take a system of holomorphic local coordinates $z = (z_1, \ldots, z_m)$ in a neighborhood of a such that $z(a) = 0$ and h can be written as $h = z_1^{\nu^0_h(a)}\tilde{h}$, and $\tilde{h}$ has no zero in a neighborhood of a. From this representation of h, we can easily conclude the claim.

Now for each $a \in B(1)\backslash I_F$, without loss of generality, we may assume that $Q_j(f)$ vanishes at a for $1 \leq j \leq q_1$ and that $Q_j(f)$ does not vanish at a for $j > q_1$. By the assumption that the Q_j's are in general position, we know that $q_1 \leq n$.

For $\{Q_1, ..., Q_n\} \subset \{Q_1, ..., Q_q\}$, consider the filtration $V_N = W_{\mathbf{0}} \supset \cdots \supset W_{\mathbf{i}} \supset W_{\mathbf{i}'} \supset \cdots$, associated to $\{Q_1, \ldots, Q_n\}$ as discussed earlier, and take a basis $\psi_1, ..., \psi_l$ of V_N according to this filtration. Then, there are linearly independent linear forms $L_1, ..., L_l$ such that $\psi_t(f) = L_t(F), 1 \leq t \leq l$. Denote by $W := W_{\alpha^1 \ldots \alpha^l}(F)$, the generalized Wronskian of F. From the basic properties of the generalized Wronskian (see [5] Proposition 4.9),

$$W = W_{\alpha^1 \ldots \alpha^l}(F) = CW_{\alpha^1 \ldots \alpha^l}(L_1(F), \ldots, L_l(F)) = CW_{\alpha^1 \ldots \alpha^l}(\psi_1(f), ..., \psi_l(f)),$$

where C is some constant. Let ψ be an element of the basis $\{\psi_1, \ldots, \psi_l\}$. As we discussed earlier, we may write $\psi = Q_1^{i_1} \cdots Q_n^{i_n}\eta$ with $\eta \in V_{N-d\sigma(i)}$. Therefore

$$\psi(f) = (Q_1(f))^{i_1} \cdots (Q_n(f))^{i_n}\eta(f),$$

and note that there are $\triangle_{\mathbf{i}}$ such that ψ is our basis. Assume that $\nu^0_{Q_j(f)}(a) \geq l-1$ for $1 \leq j \leq q_0$ and $\nu^0_{Q_j(f)}(a) < l-1$ for $q_0 < j \leq q_1$. Since, from above, $W = C\det(D^{\alpha^i}(\psi_j(f)))_{1\leq i,j\leq l}$, by the claim (note that there are $\triangle_{\mathbf{i}}$ such that ψ is our basis), and noticing that $|\alpha^j| \leq l-1$ for $1 \leq j \leq l$,

$$\begin{aligned}
\nu^0_W(a) &\geq \sum_{\mathbf{i}} \left(\sum_{j=1}^{q_0} i_j(\nu^0_{Q_j}(a) - (l-1)) \right) \triangle_{\mathbf{i}} \\
&= \sum_{j=1}^{q_0} \left(\sum_{\mathbf{i}} i_j \triangle_{\mathbf{i}} \right) (\nu^0_{Q_j}(a) - (l-1)) = \triangle \sum_{j=1}^{q_0} (\nu^0_{Q_j}(a) - (l-1)).
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{q} \nu^0_{Q_j(f)}(a) = \sum_{j=1}^{n} \nu^0_{Q_j(f)}(a) = \sum_{j=1}^{q_0} \nu^0_{Q_j(f)}(a) + \sum_{j=q_0}^{q_1} \nu^0_{Q_j(f)}(a).$$

Hence, $\nu^\infty_\psi(a) \leq \sum_{j=0}^{q} \triangle \min\{\nu_{Q_j(f)}(a), l-1\}$. □

From the above proposition, by the definition of v (see (4.16)), and using the fact that $u_j - \log|\varphi_j|$ is plurisubharmonic and $\nu^0_{\varphi_j} = \min(\nu^f(D_j), l-1)$, we see that v is plurisubharmonic on $M = B(1)$.

We now continue our proof. By the growth condition of f (see (4.12)), there exists a continuous plurisubharmonic function $w \not\equiv -\infty$ on $B(1)$ such that

$$e^w dV \leq \|f\|^{2\rho} v_m. \tag{4.17}$$

Set

$$t = \frac{2\rho}{qd \triangle - lN - \triangle d(\eta_1 + \cdots + \eta_q)} \tag{4.18}$$

and

$$\chi := z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\ldots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}.$$

Define

$$u := w + tv.$$

Then u is plurisubharmonic and so subharmonic on the Kähler manifold M.

By the result of S.-T. Yau ([13]) and L. Karp ([8]), we have necessarily

$$\int_{B(1)} e^u dV = \infty,$$

because $B(1)$ has infinite volume with respect to the given complete Kähler metric (cf. [8], Theorem B). Now, from (4.15), (4.17) and (4.18),

$$\begin{aligned} e^u dV &= e^{w+tv}dV \le e^{tv}\|f\|^{2\rho}v_m \\ &= |\chi|^t\Big(\prod_{j=1}^{q} e^{t\triangle u_j}\Big)\|f\|^{2\rho}v_m \le |\chi|^t\Big(\prod_{j=1}^{q}\|f\|^{t\triangle d\eta_j}\Big)\|f\|^{2\rho}v_m \\ &= |\chi|^t\|f\|^{2\rho+td\triangle\sum_{j=1}^{q}\eta_j}v_m = |\chi|^t\|f\|^{t(dq\triangle-lN)}v_m. \end{aligned}$$

The contradiction will appear if we can show that

$$\int_{B(1)} e^u dV < \infty.$$

From Lemma 4.3, $\frac{lN}{\triangle} \le d(n+1)+\epsilon$. Thus $qd - \frac{lN}{\triangle} \ge d(q-(n+1+\epsilon))$. So, using (4.14),

$$dq\triangle - lN - \triangle\sum_{j=1}^{q} d\eta_j \ge d\triangle(q-(n+1+\epsilon)) - \triangle\sum_{j=1}^{q} d\eta_j > \triangle\rho l(l-1).$$

This implies that $tl(l-1)/2 < 1$. Since $|\alpha^1|+\cdots+|\alpha^l| \le l(l-1)/2$, we can choose p' such that $t(|\alpha^1|+\cdots+|\alpha^l|) \le tl(l-1)/2 < p' < 1$. By the help of the identity (cf. [12], p.226)

$$v_m = (dd^c|z|^2)^m = 2m|z|^{2m-1}\sigma_m \wedge d|z|,$$

we have

$$\begin{aligned} \int_{B(1)} e^u dV &\le \int_{B(1)} |\chi|^t\|f\|^{t(dq\triangle-lN)}v_m \\ &\le 2m\int_0^1 r^{2m-1}\Big(\int_{S(r)} |\chi|^t\|f\|^{t(dq\triangle-lN)}\sigma_m\Big)dr \end{aligned}$$

$$= 2m\int_0^1 r^{2m-1}\Big(\int_{S(r)}\Big|z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\ldots\alpha^l}(F)\|f\|^{(dq\triangle-lN)}}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\sigma_m\Big|^t\Big)dr. \tag{4.19}$$

On the other hand, by (4.11),

$$\frac{|W_{\alpha^1\ldots\alpha^l}(F)|\|f\|^{(dq\triangle-lN)}}{|Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)|} \le K\sum_{L_1,\ldots,L_l}\left(\frac{|W_{\alpha^1\ldots\alpha^l}(F)|}{|L_1(F)\cdots L_l(F)|}\right), \tag{4.20}$$

where the summation is taken for all the possible linear forms choices of the linear forms $L_1,\ldots,L_l$. Note that the set of linear forms $\{L_1,\ldots,L_l\}$ comes from the filtration of V_N associated to the $\{\gamma_1,\dots,\gamma_n\}\subset\{Q_1,\dots,Q_q\}$. Hence the number of choices of the sets $\{L_1,\dots,L_l\}$ is the same as the number of the choices of the sets $\{\gamma_1,\dots,\gamma_n\}$, which is finite. Hence the summation in (4.20) is a finite sum whose number of terms depends only on f and $Q_1,\dots,Q_q$. By Proposition 3.3, for each $L_1,\dots,L_l$,

$$\int_{S(r)}\Big|z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\dots\alpha^l}(F)}{L_1(F)\cdots L_l(F)}\Big|^t\sigma_m\le K\Big(\frac{R^{2m-1}}{R-r}T_F(R,r_0)\Big)^{p'}. \tag{4.21}$$

Combining (4.20) and (4.21) thus gives

$$\int_{S(r)}\Big|z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\dots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\Big|^t\|f\|^{t(dq\triangle-lN)}\sigma_m\le K\Big(\frac{R^{2m-1}}{R-r}T_F(R,r_0)\Big)^{p'}, \tag{4.22}$$

for $r_0<r<R<1$, where, as we noted, we use the letter K to denote a constant depending only on f and $D_1,\dots,D_q$ even when it should be replaced by a new constant. According to Lemma 2.4 in [7], if we choose $R:=r+(1-r)/eT_F(r,r_0)$, then

$$T_F(R,r_0)\le 2T_F(r,r_0)\le 2dT_f(r,r_0)$$

outside a set E with $\int_E 1/(1-r)dr<\infty$. If

$$\limsup_{r\longrightarrow 1}\frac{T_f(r,r_0)}{\log 1/(1-r)}<\infty,$$

then (4.22) becomes

$$\int_{S(r)}\Big|z^{\alpha^1+\cdots+\alpha^l}\frac{W_{\alpha^1\dots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\Big|^t\|f\|^{t(dq\triangle-lN)}\sigma_m\le\frac{K}{(1-r)^{p'}}\Big(\log\frac{1}{1-r}\Big)^{p'} \tag{4.23}$$

for all $r\in[0,1)$ outside a set E with $\int_E 1/(1-r)dr<\infty$. Varying a constant K slightly, we may assume that the above inequality holds for all $r\in[0,1)$ because of Proposition 5.5 in [5]. Therefore, by (4.19) and (4.23), we have

$$\int_{B(1)}e^u dV\le K\int_0^1\frac{r^{2m-1}}{(1-r)^{p'}}\Big(\log\frac{1}{1-r}\Big)^{p'}dr<\infty,$$

since $p'<1$. This contradicts the result of S.-T. Yau ([13]) and L. Karp ([8]) mentioned earlier. This completes the proof for the first case.

We now deal with the case where

$$\limsup_{r\longrightarrow 1}\frac{T_f(r,r_0)}{\log 1/(1-r)}=\infty.$$

This case is similar to the standard Nevanlinna theory. We use the logarithmic derivative lemma and the previous discussions to prove the following refinement of the second main theorem (see [10]).

Theorem 4.5. *Let $f: B(R_0)\longrightarrow\mathbb{P}^n(\mathbb{C})$, $0<R_0\le\infty$, be a meromorphic map which is algebraically non-degenerate and $D_1,...,D_q$ be hypersurfaces of degree $d_j, 1\le j\le q$, in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then, for every $\epsilon>0$,*

$$(q-(n+1+\epsilon))T_f(r,r_0)\le\sum_{j=1}^q d_j^{-1}N_f^{[l-1]}(r,D_j)+S(r),$$

where $l \le 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n$, $d = l.c.m.\{d_1, ..., d_q\}$, *and* $S(r)$ *is evaluated as follows:*

(1) In the case $R_0 < \infty$,

$$S(r) \le K\Big(\log^+ \frac{1}{R_0 - r} + \log^+ T_f(r, r_0)\Big)$$

for every $r \in [0, R_0)$ *excluding a set* E *with* $\int_E \frac{1}{R_0 - t} dt < \infty$.

(2) In the case $R_0 = \infty$,

$$S(r) \le K(\log^+ T_f(r, r_0) + \log r)$$

for every $r \in [0, R_0)$*excluding a set* E' *with* $\int_{E'} dt < \infty$.

Proof. Without loss of generality, we assume that $d_1 = \cdots = d_q = d$. Similar to the proof of (4.22), by using (4.20) and Proposition 3.3, we have

$$\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \le K\Big(\frac{R^{2m-1}}{R - r} T_F(R, r_0)\Big)^{p'}, \tag{4.24}$$

for $r_0 < r < R < R_0$. Hence, by virtue of the concavity of the logarithm, the above inequality implies that

$$\begin{aligned} & t \int_{S(r)} \log |z^{\alpha^1 + \cdots + \alpha^l}| \sigma_m + t \int_{S(r)} \log \Big| \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big| \sigma_m \\ & + t(dq \triangle - Nl) \int_{S(r)} \log \|f\| \sigma_m \\ & \le K\Big(\log^+ \frac{R}{R - r} + \log^+ T_F(R, r_0)\Big) + O(1), \end{aligned} \tag{4.25}$$

for $r_0 < R < R_0$. But, by the Jensen formula (see [5], p.236),

$$\int_{S(r)} \log \Big| \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big| \sigma_m = N_{\nu^0_{W_{\alpha^1 \ldots \alpha^l}(F)}}(r_0, r) - \triangle \sum_{j=1}^{q} N_f(r, D_j) + O(1).$$

By Proposition 4.4, $\triangle \sum_{j=1}^q N_f(r, D_j) - N_{\nu^0_{W_{\alpha^1 \ldots \alpha^l}(F)}}(r_0, r) \le \triangle \sum_{j=1}^q N_f^{[l-1]}(r, D_j)$ and therefore (4.25) becomes

$$(dq \triangle - Nl) T_f(r) \le \sum_{j=1}^{q} \triangle N_f^{[l-1]}(r, D_j) + K\Big(\log^+ \frac{R}{R - r} + \log^+ T_F(R, r_0)\Big) + O(1).$$

By Lemma 4.3, with $N = 2d(n+1)(nd+n)(2^n - 1)(I(\epsilon^{-1}) + 1) + nd$ for any $\epsilon > 0$, we have

$$\frac{lN}{\triangle} \le d(n+1) + \epsilon,$$

and, moreover, l satisfies $l \le 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n$. Hence,

$$(q-(n+1+\epsilon))T_f(r) \le \sum_{j=1}^{q} d^{-1} N_f^{[l-1]}(r, D_j) + K\Big(\log^+ \frac{R}{R-r} + \log^+ T_F(R, r_0)\Big)$$

$$\le \sum_{j=1}^{q} d^{-1} N_f^{[l-1]}(r, D_j) + K\Big(\log^+ \frac{R}{R-r} + \log^+ T_f(R, r_0)\Big). \tag{4.26}$$

Since $T_f(r, r_0)$ is continuous, increasing and we may assume $T_f(r, r_0) \ge 1$, we can apply Lemma 2.4 in [7] to show that

$$T_f\Big(r + \frac{R_0 - r}{eT_f(r, r_0)}, r_0\Big) \le 2T_f(r, r_0)$$

outside a set E of r such that $\int_E 1/(R_0 - r)dr < \infty$ in the case $R_0 < \infty$ and

$$T_f\Big(r + \frac{1}{T_f(r, r_0)}, r_0\Big) \le 2T_f(r, r_0)$$

outside a set E' of r such that $\int_{E'} dr < \infty$ in the case $R_0 = \infty$. Substituting $R = r + \frac{R_0 - r}{eT_f(r, r_0)}$ if $R_0 < \infty$ and $R = r + 1/T_f(r, r_0)$ if $R_0 = \infty$ in (4.26) proves the theorem. □

Corollary 4.6. *In the same situation as in Theorem* 4.5, *if*

$$(i) \quad \limsup_{r \to R_0} \frac{T_f(r, r_0)}{\log(1/R_0 - r)} = \infty$$

or

$$(ii) \quad R_0 = \infty,$$

then

$$\sum_j \delta_{l-1}^f (D_j) \le \sum_j \delta_{l-1}^{f,\star}(D_j) \le n + 1 + \epsilon,$$

where $\delta^{f,\star}$ *is the classical Nevanlinna's defect defined by*

$$\delta_{l-1}^{f,\star}(D_j) = \limsup_{r \to R_0} \left(1 - \frac{N_f^{[l-1]}(r, D_j)}{dT_f(r, r_0)}\right).$$

Corollary 4.6 gives the proof of the second case. The proof of the main theorem (Theorem 1.1) is thus complete.

5. Value distribution of the Gauss map of a complete regular submanifold of $\mathbb{C}^m$

Let $f = (f_1, ..., f_m) : M \to \mathbb{C}^m$ be a regular submanifold of $\mathbb{C}^m$; namely, let M be a connected complex manifold and f be a holomorphic map of M into $\mathbb{C}^m$ such that rank $d_p f = \dim M$ for every point $p \in M$.

To each point $p \in M$, we assign the tangent space $T_p(M)$ of M at p which may be regarded as an n-dimensional linear subspace of $T_{f(p)}\mathbb{C}^m$. On the other hand, each $T_p(\mathbb{C}^m)$ is identified with $T_0(\mathbb{C}^m) = \mathbb{C}^m$ by a parallel translation. Therefore, to each $T_p(M)$ corresponds a point $G(p)$ in the complex Grassmannian manifold $G(n, m)$ of all n-dimensional linear subspaces of $\mathbb{C}^m$, where $n = \dim M$.

Definition 5.1. We call the map $G : M \to G(n,m)$ the Gauss map of $f : M \to \mathbb{C}^m$.

The space $G(n,m)$ is canonically embedded in $\mathbb{P}^N(\mathbb{C})$=$\mathbb{P}(\bigwedge^n \mathbb{C}^m)$, where $N = \binom{m}{n} -1$. The Gauss map G may be identified with a holomorphic map of M into $\mathbb{P}^N(\mathbb{C})$ given as follows:

Taking holomorphic local coordinates $(z_1, ..., z_n)$ defined on an open set U, we consider the map

$$\bigwedge := D_1 f \wedge \cdots \wedge D_n f : U \to \bigwedge^n \mathbb{C}^m - \{0\},$$

where $D_i f = ((\partial/\partial z_i) f_1, \cdots, (\partial/\partial z_i) f_{N+1})$. Then, locally we have

$$G = \pi \circ \bigwedge,$$

where $\pi : \mathbb{C}^{N+1} - \{0\} \longrightarrow \mathbb{P}^N(\mathbb{C})$ is the canonical projection map.

A regular submanifold M of $\mathbb{C}^m$ is considered a Kähler manifold with the metric ω induced from the standard flat metric on $\mathbb{C}^m$. By dV we denote the volume form on M. We can see that, for arbitrarily holomorphic coordinates $z_1, ..., z_n$,

$$dV = |\bigwedge|^2 \Big(\frac{\sqrt{-1}}{2}\Big)^n dz_1 \wedge d\bar{z_1} \wedge \cdots \wedge dz_n \wedge d\bar{z_n},$$

where

$$|\bigwedge|^2 = \sum_{1 \le i_1 < \cdots < i_n \le m} \Big| \frac{\partial (f_{i_1}, ..., f_{i_n})}{\partial (z_1, ..., z_n)} \Big|^2.$$

Therefore, for a regular submanifold $f : M \to \mathbb{C}^m$, the Gauss map $G : M \to \mathbb{P}^N(\mathbb{C})$ satisfies the following growth condition:

$$\Omega_G + dd^c \log h^2 = dd^c \log |\bigwedge|^2 = \text{Ric } (\omega),$$

where $h = 1$.

As a direct consequence of Theorem 1.1, we have

Theorem 5.2. *Let $f : M \to \mathbb{C}^m$ be a complete regular submanifold such that the universal covering of M is biholomorphic to $B(R_0)$ $(0 < R_0 \le +\infty)$. If the Gauss map $G : M \to \mathbb{P}^N(\mathbb{C})$ is algebraically non-degenerate, then for all hypersurfaces $D_1, ..., D_q$ of degree d_j, $j = 1, ..., q$ in general position, by letting d =l.c.m. $\{d_1, \ldots, d_q\}$ (the least common multiple of $\{d_1, \ldots, d_q\}$), we have, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta^G_{l-1}(D_j) \le N + 1 + \epsilon + \frac{l(l-1)}{d},$$

where $l \le 2^{N^2+4N} e^N d^{2N} (NI(\epsilon^{-1}))^N$, $n = \dim M$, and $N = \binom{m}{n} - 1$.

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