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D-MODULE STRUCTURE OF LOCAL COHOMOLOGY MODULES OF TORIC ALGEBRAS

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Abstract. Let S be a toric algebra over a field $\mathbb{K}$ of characteristic 0 and let I be a monomial ideal of S. We show that the local cohomology modules $H^i_I(S)$ are of finite length over the ring of differential operators $D(S;\mathbb{K})$, generalizing the classical case of a polynomial algebra S. As an application, we compute the characteristic cycles of some local cohomology modules.

1. Introduction

Lyubeznik [Lyu93] introduced an approach of studying local cohomology modules using the theory of D-modules. He obtained many finiteness properties of the local cohomology modules $H^i_I(R)$ when R is a regular ring containing a field of characteristic 0. For example, taking advantage of holonomicity of $H^i_I(R)$ as a D-module, he showed that for any maximal ideal $\mathfrak{m}$ the number of associated prime ideals of $H^i_I(R)$ contained in $\mathfrak{m}$ is finite and that all Bass numbers of $H^i_I(R)$ are finite. When R is a regular local ring of positive characteristic, analogous results were obtained by Huneke and Sharp using the Frobenius functor [HS93]. When R is not regular, the situation is more subtle. There are characteristic-free examples where $H^i_I(R)$ have infinitely many associated primes [SS04]. Also, an example by Hartshorne [Har70] shows that in general the Bass numbers can be infinite (see Example 3.9).

After [Lyu93], there have been several studies on the finiteness properties of R_x and of $H^i_I(R)$ as D-modules for a regular ring R, among them [Bøg95], [Bøg02], [Lyu97], [ÁMBL05]. The first D-finiteness result of R_x for a singular ring R is due to Takagi and Takahashi [TT08] which says R_x is generated by x^{-1} over D when R is a Noetherian graded ring with finite F-representation type. In particular, their theorem applies to the case where R is a normal toric algebra over a perfect field of positive characteristic.

In the present article, we study finiteness properties of the localizations S_f and the local cohomology modules $H^i_I(S)$ as D-modules, where S is a toric algebra (not necessarily normal) over a field $\mathbb{K}$ of characteristic 0, f is a monomial element and I is a monomial ideal of S. In this case, the ring of differential operators $D(S) := D(S;\mathbb{K})$ is much more complicated than the case where S is regular. Using the natural grading of $D(S)$ introduced by Jones [Jon94] and Musson [Mus94], Saito and Traves [ST01, ST04] gave a detailed description of $D(S)$. Based on their results, we prove that any localization $S_f = S[f^{-1}]$ of S is generated by f^{-1} over $D(S)$

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(Theorem 3.1). This implies immediately that $H^i_I(S)$ is $D(S)$-finitely generated if $D(S)$ is left Noetherian. Unfortunately, $D(S)$ is not always left Noetherian [ST09]. Nonetheless, we can show that $H^i_I(S)$ is actually of finite length as a $D(S)$-module (Theorem 3.3). In view of Hartshorne's example (Example 3.9), this result is quite surprising.

As an application, we compute the characteristic cycles of some local cohomology modules $H^i_I(S)$. Characteristic cycles are formal sums of subvarieties (counted with multiplicities) of the characteristic variety of a D-module M. Here, the characteristic variety $\mathrm{Ch}(M)$ is the support of the associated graded module $\mathrm{gr}\, M$ in the spectrum $\mathrm{Spec}(\mathrm{gr}\, D(S))$ of the associated graded ring $\mathrm{gr}\, D(S)$. When S is a polynomial algebra, one can explicitly compute the Bass numbers and the associated primes of $H^i_I(S)$ from its characteristic cycles [ÁM04]. The cohomological dimension of I and the Lyubeznik numbers can also be computed from them [ÁM00]. Through our finiteness results of $H^i_I(S)$, we are able to compute the characteristic cycles of some local cohomology modules. We will show that for normal toric algebras S (in fact, for a more general class of toric algebras) the characteristic variety $\mathrm{Ch}(H^{\dim S}_{\mathfrak{m}}(S))$ of the top local cohomology with maximal support is abstractly isomorphic to the ambient toric variety $\mathrm{Spec}(S)$ (Theorem 4.13).

In section 2, we briefly recall the notions of local cohomology, toric algebras and the ring of differential operators of a commutative algebra over a field. We also describe the structure of rings of differential operators over toric algebras following the notation in [ST01] and [ST04]. In section 3, our main results on the finiteness properties mentioned above are presented. Also, we relate our finiteness results to the notion of sector partition introduced in [SS90] and [MM06]. Some discussions on $\mathrm{gr}\, D(S)$ and the computations of characteristic cycles are in section 4. As suggested by the referee, some relations between our results in section 4 and the recent work of Saito [Sai10] are discussed (see Remarks 4.7, 4.14).

2. Preliminaries

2.1. **Local cohomology.** General facts regarding local cohomology can be found in [ILL+07] or [BS98]. Here, we only recall some basics.

Let R be a Noetherian commutative ring, M an R-module, and I an ideal of R. Define $\Gamma_I(M) := \varinjlim \mathrm{Hom}_R(R/I^k, M)$. Then Γ_I is a left exact R-linear covariant functor and the i-th local cohomology functor H^i_I is defined to be its i-th right derived functor. We call $H^i_I(M)$ the i-th local cohomology module of M supported at the ideal I. If I is generated by $f_1, \dots, f_t$, then $H^i_I(M)$ is the i-th cohomology of the Čech complex

$$0 \to M \to \bigoplus_{i=1}^{t} M_{f_i} \to \bigoplus_{1 \le i < j \le t} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_t} \to 0.$$

2.2. **Toric algebras.** We introduce some notation for later use. For more information on toric algebras, the reader is referred to [Ful93], [MS05] or [ILL+07].

Let A be a $d \times n$ integer matrix with columns $a_1, \dots, a_n$. Assume $\mathbb{Z}A = \mathbb{Z}^d$. For a field $\mathbb{K}$, the semigroup subring $S_{A,\mathbb{K}} := \mathbb{K}[\mathbb{N}A] = \mathbb{K}[t^{a_1}, \dots, t^{a_n}]$ of the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d] = \mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ is called the toric algebra associated to the matrix A. Denote $\widetilde{\mathbb{N}A} := \mathbb{R}_{\ge 0}A \cap \mathbb{Z}^d$ as the saturation of $\mathbb{N}A$. Then $\widetilde{S_{A,\mathbb{K}}} := \mathbb{K}[\widetilde{\mathbb{N}A}]$ is the normalization of $S_{A,\mathbb{K}}$.

2.3. **Rings of differential operators.** For a commutative algebra R over a field $\mathbb{K}$, set $D_0(R;\mathbb{K}) := R$, and for $i > 0$,

$$D_i(R;\mathbb{K}) := \{f \in \mathrm{Hom}_{\mathbb{K}}(R,R) \mid [f,r] \in D_{i-1}(R;\mathbb{K}) \text{ for all } r \in R\}.$$

Then the ring of differential operators is defined to be

$$D(R;\mathbb{K}) := \bigcup_j D_j(R;\mathbb{K}).$$

When R is a polynomial ring over a field $\mathbb{K}$ of characteristic 0, $D(R;\mathbb{K})$ is the usual Weyl algebra. In this paper, a module over $D(R;\mathbb{K})$ means a left $D(R;\mathbb{K})$-module.

Lemma 2.1. *If M is a $D(R;\mathbb{K})$-module and $f \in R$, then the R-module structure on M_f extends uniquely to a $D(R;\mathbb{K})$-module structure such that the natural map $M \to M_f$ is a $D(R;\mathbb{K})$-module homomorphism. In particular, via the Čech complex $H^i_I(M)$ has a natural $D(R;\mathbb{K})$-module structure.*

Proof. See [Lyu00], Example (b). □

When R is a regular algebra over a field $\mathbb{K}$ of characteristic 0, $D(R;\mathbb{K})$ is well understood (see e.g. [Bjö79]). In this case, the local cohomology modules $H^i_I(R)$ are holonomic as $D(R;\mathbb{K})$-modules and hence are of finite length (see [Lyu93]). This essential property enables Lyubeznik to achieve many finiteness results of the local cohomology modules.

Unfortunately, $D(R;\mathbb{K})$ does not behave well when R is singular; we don't have a notion of holonomicity in this case. This complicates the study of $H^i_I(R)$ via the theory of D-modules. On the bright side, when $R = S_{A,\mathbb{K}}$ is a toric algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0, there is a nice combinatorial structure for $D(R;\mathbb{K})$ which we will present in the next subsection. Our finiteness results about local cohomology modules substantially rely on this structure.

2.4. **Rings of differential operators over toric algebras.** In the rest of this paper, we denote $S_A := S_{A,\mathbb{K}}$, where $\mathbb{K}$ is an algebraically closed field of characteristic 0. Following [ST01], the noncommutative ring $D_A := D(S_A,\mathbb{K})$ can be described as a $\mathbb{Z}^d$-graded subring of

$$D(\mathbb{K}[\mathbb{Z}^d];\mathbb{K}) = \mathbb{K}[t_1^{\pm 1},\dots,t_d^{\pm 1}]\langle \partial_1,\dots,\partial_d\rangle,$$

where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij}t_j^{-2}$ and the other pairs of variables commute. More precisely, with the notation $\theta_i := t_i\partial_i$, one has

$$D_A = \bigoplus_{a\in\mathbb{Z}^d} t^a\,\mathbb{I}(\Omega(a)),$$

where $\Omega(a) = \mathbb{N}A \setminus (-a + \mathbb{N}A)$ and $\mathbb{I}(\Omega(a))$ is the vanishing ideal of $\Omega(a)$ in $\mathbb{K}[\theta_1,\dots,\theta_d]$.

3. Finiteness properties of $H^i_I(S_A)$

In this section, $\mathscr{F}$ will be denoted to be the set of all facets of $\mathbb{R}_{\geq 0}A$. For a face τ of $\mathbb{R}_{\geq 0}A$, we write $\mathbb{N}(A\cap\tau) := \mathbb{N}A \cap \mathbb{R}\tau$ and denote $\mathbb{Z}(A\cap\tau)$ as the group generated by $\mathbb{N}(A\cap\tau)$.

We recall some notation in [ST01], which are crucial to the proofs of Theorems 3.1 and 3.3.

For $a \in \mathbb{Z}^d$ and τ a face of $\mathbb{R}_{\geq 0}A$, define

$$E_\tau(a) := \{l \in \mathbb{C}(A \cap \tau) \mid a - l \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}/\mathbb{Z}(A \cap \tau).$$

Notice that

$$E_\tau(a) \subseteq [\mathbb{Z}^d \cap \mathbb{C}(A \cap \tau)]/\mathbb{Z}(A \cap \tau) = \mathbb{Z}\widetilde{(A \cap \tau)}/\mathbb{Z}(A \cap \tau).$$

Here $\mathbb{Z}\widetilde{(A \cap \tau)} := \mathbb{C}(A \cap \tau) \cap \mathbb{Z}^d$ is the saturation of $\mathbb{Z}(A \cap \tau)$, so each $E_\tau(a)$ is a finite set. Consider the ordering $\leq$ on $\mathbb{Z}^d$ defined by

$$[a \leq b] \iff [E_\tau(a) \subseteq E_\tau(b)] \text{ for all faces } \tau \text{ of } \mathbb{R}_{\geq 0}A.$$

This ordering induces an equivalence relation on $\mathbb{Z}^d$ by

$$[a \sim b] \iff [a \leq b \text{ and } b \leq a].$$

Now we are ready for the first main theorem.

Theorem 3.1. *$S_A[f^{-1}] = D_A \cdot f^{-1}$ is cyclic as a left D_A-module for any monomial $f \in S_A$.*

Proof. It's clear that $S_A[f^{-1}] \supseteq D_A \cdot f^{-1}$. Conversely, write $f = t^b$ for some $b \in \mathbb{N}A$ (we may assume $b \neq 0$). Since $S_A[f^{-1}] \subseteq D_A \cdot \{t^{-mb} \mid m \in \mathbb{N}\}$, it suffices to show that $t^{-mb} \in D_A \cdot t^{-b}$ for all $m \in \mathbb{N}$. By Proposition 4.1.5(1) in [ST01], it is enough to prove that $-b \sim -mb$. Indeed, we show that, for any face σ, $E_\sigma(-b) = E_\sigma(-mb)$ as follows:

(1) Let σ be a facet, and suppose $b \notin \sigma$. Then $F_\sigma(-b) < 0$, and hence $F_\sigma(-mb) < 0$. So $E_\sigma(-b) = E_\sigma(-mb) = \emptyset$.

(2) Let τ be a face and suppose that $b \notin \tau$. Then there exists a facet σ containing τ with $b \notin \sigma$. Hence $E_\sigma(-b) = E_\sigma(-mb) = \emptyset$ by (1), and thus $E_\tau(-b) = E_\tau(-mb) = \emptyset$.

(3) Let τ be a face and suppose $b \in \tau$. Then $b \in \mathbb{N}A \cap \tau = \mathbb{N}(A \cap \tau)$. Hence, $\pm b \in \mathbb{Z}(A \cap \tau)$, and thus $E_\tau(-b) = E_\tau(-mb)$ by definition. □

Remark 3.2. Via the Čech complex, it follows immediately from Theorem 3.1 that $H^i_I(S_A)$ is finitely generated over D_A if D_A is left Noetherian. The left Noetherianess of D_A was studied by Saito and Takahashi [ST09]. They proved that D_A is left Noetherian if S_A satisfies Serre's condition (S_2). Serre's condition is, by [Ish88], equivalent to

$$S_A = \bigcap_{\tau:\ \text{facets}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)].$$

Saito and Takahashi also gave a necessary condition (on S_A) for D_A to be left Noetherian. However, D_A is not always left Noetherian.

Nonetheless, we have

Theorem 3.3. *For any i and any monomial ideal I, $H^i_I(S_A)$ is of finite length as a D_A-module.*

Proof. In view of the Čech complex, since any localization $S_A[f^{-1}]$ (with monomial f) is a D_A-submodule of $\mathbb{K}[\mathbb{Z}^d]$, it suffices to show that $\mathbb{K}[\mathbb{Z}^d]$ has a composition series.

Consider the notation in the beginning of this section. For $a, b \in \mathbb{Z}^d$, we will write $a < b$ if $a \leq b$ but $a \nsim b$. Then, for each $a \in \mathbb{Z}^d$, $\bigoplus_{b \geq a} \mathbb{K}t^b$ is generated by t^a as a D_A-module. Moreover,

$$\frac{\bigoplus_{b\geq a} \mathbb{K}t^b}{\bigoplus_{b>a} \mathbb{K}t^b} \cong \bigoplus_{b\in[a]} \mathbb{K}t^b$$

is, by Theorem 4.1.6 in [ST01], a simple D_A-module where $[a] = \{b \in \mathbb{Z}^d \mid b \sim a\}$. Since there are only finitely many faces and since each $E_\tau(a)$ is contained in $\widetilde{\mathbb{Z}(A\cap\tau)}/\mathbb{Z}(A\cap\tau)$, there are finitely many equivalence classes determined by $\sim$; we denote them $[\alpha_1], [\alpha_2], \ldots, [\alpha_k]$. We may rearrange the order so that, for any pair $i < j$, either $\alpha_i > \alpha_j$ or α_i and α_j are incomparable. Denote $T_a := \bigoplus_{b\geq a} \mathbb{K}t^b$. Then the filtration

$$0 \subsetneq T_{\alpha_1} \subsetneq \cdots \subsetneq \Sigma_{l=1}^{i} T_{\alpha_l} \subsetneq \cdots \subsetneq \Sigma_{l=1}^{k} T_{\alpha_l} = \mathbb{K}[\mathbb{Z}^d]$$

is a composition series of D_A-submodules of $\mathbb{K}[\mathbb{Z}^d]$. □

Example 3.4. For 1-dimensional S_A, the composition series of $\mathbb{K}[\mathbb{Z}]$ is easy to describe. In this case, $\mathbb{R}_{\geq 0}A$ has two faces, 0 and $\sigma = \mathbb{R}_{\geq 0}A$. For $a \in \mathbb{Z}$,

$$E_0(a) = \{\ell \in \{0\} \mid a - \ell \in \mathbb{N}A\}/\{0\} \text{ and}$$
$$E_\sigma(a) = \{\ell \in \mathbb{Z} \mid a - \ell \in \mathbb{Z}\}/\mathbb{Z}.$$

Thus $E_0(a) = \{0\}$ if $a \in \mathbb{N}A$, $E_0(a) = \emptyset$ if $a \notin \mathbb{N}A$, and $E_\sigma(a) = \{0\}$ for all $a \in \mathbb{Z}$. Therefore, $[0]$ and $[-1]$ are the two equivalence classes determined by $\sim$, and we have the composition series

$$0 \subsetneq \mathbb{K}[\mathbb{N}A] = T_0 \subsetneq T_{-1} = \mathbb{K}[\mathbb{Z}].$$

Remark 3.5. Theorem 3.1 and Theorem 3.3 also hold for any field $\mathbb{K}$ with characteristic 0 by the isomorphism

$$H_I^i(S_{A,\mathbb{K}}) \otimes \overline{\mathbb{K}} \cong H_I^i(S_{A,\overline{\mathbb{K}}}).$$

Remark 3.6. Suppose $I = \mathfrak{m}$, the maximal graded ideal of S_A. Here we assume that the semigroup $\mathbb{N}A$ is pointed, so that 0 is the only invertible element.

(1) Recall that $H^i_{\mathfrak{m}}(S_A)$ can be computed as the i-th cohomology of the Ishida complex [Ish88] or [ILL+07]. Therefore, $H^1_{\mathfrak{m}}(S_A)$ is finitely generated as an S_A-module. Indeed, it suffices to observe that

$$H^1_{\mathfrak{m}}(S_A) \twoheadleftarrow \bigcap_{\sigma:\text{ rays}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A\cap\sigma)] \subseteq \bigcap_{\tau:\text{ facets}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A\cap\tau)] \subseteq \widetilde{S_A}$$

and the fact that $\widetilde{S_A}$ is finite over S_A. Moreover, $H^d_{\mathfrak{m}}(S_A)$ is cyclic as a left D_A-module. This is because the d-th module in the Ishida complex is $\mathbb{K}[\mathbb{Z}^d]$, which is cyclic by Theorem 3.1.

(2) In general, Schäfer and Schenzel [SS90] showed that there is a partition of $\mathbb{Z}^d$ with respect to which $H^i_{\mathfrak{m}}(S_A)$ can be written as a finite direct sum of $\mathbb{K}$-vector spaces. This decomposition coincides with the sector partition appearing in [MM06] (see also [HM05] for a more general notion of sector partition). More precisely, let $\mathrm{Conv}(A)$ be the set of all faces of $\mathbb{R}_{\geq 0}A$, and for any filter (cocomplex) ∇ of $\mathrm{Conv}(A)$, denote

$$P_\nabla = \bigcap_{A\cap\tau\in\nabla} [\mathbb{N}A + \mathbb{Z}(A\cap\tau)] \setminus \bigcup_{A\cap\tau\notin\nabla} [\mathbb{N}A + \mathbb{Z}(A\cap\tau)].$$

Then the P_∇'s form a partition (sector partition) of $\mathbb{Z}^d$ and

$$H^i_{\mathfrak{m}}(S_A) = \bigoplus_{\nabla} \mathbb{K}[P_\nabla] \otimes_{\mathbb{K}} H^i(\mathrm{Conv}(A), \mathrm{Conv}(A) \setminus \nabla; \mathbb{K}).$$

On the other hand, for $a \in \mathbb{Z}^d$ denote

$$\nabla(a) := \{\text{face } \tau \text{ of } \mathbb{R}_{\geq 0}A \mid a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}$$

and consider the equivalence relation $a \equiv a' \Leftrightarrow \nabla(a) = \nabla(a')$. Then $P_{\nabla(a)}$ is the equivalence class containing a. Notice that P_∇ could be empty and that $[a \in P_\nabla] \Leftrightarrow [P_\nabla = P_{\nabla(a)}]$.

Theorem 6 in [MM06] shows that the partition determined by the equivalence relation $\sim$ in the proof of Theorem 3.3 is finer than the sector partition determined by $\equiv$.

Notice that each $\mathbb{K}[P_\nabla]$ is naturally a left $\mathbb{Z}^d$-graded D_A-module because each $\mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)]$ is as well. If S_A is normal, the D_A-module $\mathbb{K}[P_\nabla]$ is simple. In fact, we have

Theorem 3.7. *If S_A is normal and I is a monomial ideal in S_A, then every simple subquotient of $H^i_I(S_A)$ is of the form $\mathbb{K}[P_\nabla]$ coming from the sector partition.*

Proof. By Theorem 3.3 and Remark 3.6(2), we only have to show that $\sim$ and $\equiv$ define the same equivalence relation on $\mathbb{Z}^d$. Note that the normality of S_A implies that

$$E_\tau(a) = \{0\} \text{ if } a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \text{ and}$$
$$E_\tau(a) = \emptyset \text{ if } a \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau).$$

So we have

$$\begin{aligned} & a \sim b \\ \Leftrightarrow & E_\tau(a) = E_\tau(b) \text{ for all } \tau \\ \Leftrightarrow & E_\tau(a) = \{0\} \text{ if and only if } E_\tau(b) = \{0\} \\ \Leftrightarrow & a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \text{ if and only if } b \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \\ \Leftrightarrow & a \equiv b. \end{aligned}$$

Therefore, the simple subquotients of $\mathbb{K}[\mathbb{Z}^d]$ are precisely the D_A-modules

$$\mathbb{K}[P_\nabla] = \bigoplus_{b \in [a]} \mathbb{K}t^b \cong \frac{\bigoplus_{b \geq a} \mathbb{K}t^b}{\bigoplus_{b > a} \mathbb{K}t^b}. \qquad \square$$

Example 3.8. Consider $A = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right)$. Then $S_A = \mathbb{K}[t, ts, ts^2]$ is a 2-dimensional normal toric algebra. Let $I = (ts)$ be the ideal of S_A generated by ts. We shall describe a composition series of $H^1_I(S_A)$. By Čech complex, $H^1_I(S_A) = \mathbb{K}[\mathbb{Z}^d]/S_A$. Following the notation in the proof of Theorem 3.3 and Remark 3.6, let

$$\nabla_0 = \{0, \sigma_1, \sigma_2, \mathbb{R}_{\geq 0}A\},\ \nabla_1 = \{\sigma_1, \mathbb{R}_{\geq 0}A\},\ \nabla_2 = \{\sigma_2, \mathbb{R}_{\geq 0}A\},$$
$$\nabla_{12} = \{\sigma_1, \sigma_2, \mathbb{R}_{\geq 0}A\},\ \nabla_A = \{\mathbb{R}_{\geq 0}A\},$$

where $\sigma_1 = \mathbb{R}_{\geq 0}\binom{1}{0}$ and $\sigma_2 = \mathbb{R}_{\geq 0}\binom{1}{2}$. Then

$$\begin{aligned}
&P_{\nabla_0} = P_{\nabla(a_0)} = \mathbb{N}A,\\
&P_{\nabla_1} = P_{\nabla(-a_1)} = [\mathbb{N}A + \mathbb{Z}(A\cap\sigma_1)] \setminus [\mathbb{N}A + \mathbb{Z}(A\cap\sigma_2)],\\
&P_{\nabla_2} = P_{\nabla(-a_2)} = [\mathbb{N}A + \mathbb{Z}(A\cap\sigma_2)] \setminus [\mathbb{N}A + \mathbb{Z}(A\cap\sigma_1)],\\
&P_{\nabla_{12}} = \emptyset, \text{ and}\\
&P_{\nabla_A} = P_{\nabla(-a_3)} = \mathbb{Z}^2 \setminus \big[(\mathbb{N}A + \mathbb{Z}(A\cap\sigma_1)) \cup (\mathbb{N}A + \mathbb{Z}(A\cap\sigma_2))\big],
\end{aligned}$$

where $a_0 = \binom{0}{0}$ and a_i, $i = 1, 2, 3$, is the i-th column of A. In terms of notation in Theorem 3.3,

$$\begin{aligned}
T_{a_0} &= \mathbb{K}[P_{\nabla_0}] = S_A,\\
T_{-a_1} &= \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_1}],\\
T_{-a_2} &= \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_2}],\\
T_{-a_3} &= \mathbb{K}[\mathbb{Z}^2] = \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \oplus \mathbb{K}[P_{\nabla_A}].
\end{aligned}$$

So $0 \subset T_{a_0} \subset T_{-a_1} \subset T_{-a_1} + T_{-a_2} \subset T_{-a_3} = \mathbb{K}[\mathbb{Z}^2]$ is a composition series of $\mathbb{K}[\mathbb{Z}^2]$. Quotienting out S_A, we obtain a composition series of $H^1_I(S_A)$:

$$0 \subset \mathbb{K}[P_{\nabla_1}] \subset \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \subset \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \oplus \mathbb{K}[P_{\nabla_A}].$$

Example 3.9. This example is essentially due to Hartshorne [Har70]. We adopt its combinatorial description which can be found in [ILL+07] or [MS05].

Consider $A = \begin{pmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix}$. Then $S_A = \mathbb{K}[r, rs, rt, rst]$ is a normal toric algebra. Consider the ideal $I = (r, rs)$ of S_A. Then the socle $\mathrm{Hom}_{S_A}(S_A/\mathfrak{m}, H^2_I(S_A))$ is infinite dimensional, where $\mathfrak{m} = (r, rs, rt, rst)$ is the maximal graded ideal of S_A. However, according to Theorem 3.3 $H^2_I(S_A)$ is of finite length over D_A.

In fact, $H^2_I(S_A)$ is D_A-simple. To see this, using the notation in Remark 3.6 we consider the filter $\nabla = \{\sigma_{12}, \mathbb{R}_{\geq 0}A\}$, where $\sigma_{12} = \mathbb{R}_{\geq 0}\begin{pmatrix} 1&1\\ 0&1\\ 0&0 \end{pmatrix}$ is a facet of $\mathbb{R}_{\geq 0}A$. We claim that

$$P_\nabla = [\mathbb{N}A + \mathbb{Z}(A\cap\sigma_{12})] \setminus [(\mathbb{N}A + \mathbb{Z}a_1) \cup (\mathbb{N}A + \mathbb{Z}a_2)],$$

where $a_1 = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 1\\1\\0 \end{pmatrix}$. Therefore, in view of the Čech complex we have the isomorphism

$$H^2_I(S_A) \cong \mathbb{K}[P_\nabla],$$

which is D_A-simple by Theorem 3.7.

The claim is equivalent to the equality

$$\begin{aligned}
&\big[\mathbb{N}A + \mathbb{Z}(A\cap\sigma_{12})\big] \cap \Big[\bigcup_{\sigma=\sigma_{13},\sigma_{24},\sigma_{34}} (\mathbb{N}A + \mathbb{Z}(A\cap\sigma))\Big]\\
=&\big[\mathbb{N}A + \mathbb{Z}(A\cap\sigma_{12})\big] \cap \big[(\mathbb{N}A + \mathbb{Z}a_1) \cup (\mathbb{N}A + \mathbb{Z}a_2)\big],
\end{aligned}$$

where $\sigma_{13} = \mathbb{R}_{\geq 0}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_{24} = \mathbb{R}_{\geq 0}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\sigma_{34} = \mathbb{R}_{\geq 0}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. This equality can be verified by the following data:

$$\mathbb{N}A + \mathbb{Z}a_1 = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid y \geq 0 \text{ and } z \geq 0\},$$
$$\mathbb{N}A + \mathbb{Z}a_2 = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq y \text{ and } z \geq 0\},$$
$$\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{12}) = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid z \geq 0\},$$
$$\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{13}) = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid y \geq 0\},$$
$$\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{24}) = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq y\},$$
$$\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{34}) = \{{}^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq z\}.$$

Remark 3.10. Helm and Miller [HM03] studied the Bass numbers of local cohomology modules over toric algebras. As a generalization of Hartshorne's example, their main result ([HM03], Theorem 7.1) implies that for a Gorenstein normal toric algebra S_A, $\mathbb{N}A$ is not simplicial if and only if there exists an $\mathbb{N}A$-graded prime $\mathfrak{p}$ of dimension 2 such that $H^{d-1}_{\mathfrak{p}}(S_A)$ has infinite-dimensional scole.

4. Associated graded rings gr D_A and characteristic cycles

4.1. **Associated graded rings** gr D_A**.** Let R be a $\mathbb{K}$-algebra as in subsection 2.3. The definition of $D(R;\mathbb{K})$ gives an order filtration of $D(R;\mathbb{K})$:

$$0 \subseteq R = D_0(R;\mathbb{K}) \subseteq D_1(R;\mathbb{K}) \subseteq D_2(R;\mathbb{K}) \subseteq \cdots .$$

Define the associated graded ring of $D(R;\mathbb{K})$ to be

$$\operatorname{gr} D(R;\mathbb{K}) := D_0 \oplus (D_1/D_0) \oplus (D_2/D_1) \oplus \cdots ,$$

where $D_i := D_i(R;\mathbb{K})$. From the definition of $D(R;\mathbb{K})$, $\operatorname{gr} D(R;\mathbb{K})$ is a commutative R-algebra and we have the natural embedding $R \hookrightarrow \operatorname{gr} D(R;\mathbb{K})$. For example, if $R = \mathbb{K}[t_1, \ldots, t_d]$ is a polynomial algebra over $\mathbb{K}$, then $D(R;\mathbb{K}) = \mathbb{K}[t_1, \ldots, t_d]\langle \partial_1, \ldots, \partial_d \rangle$ is the Weyl algebra. The associated graded ring $\operatorname{gr} D(R;\mathbb{K}) = \mathbb{K}[t_1, \ldots, t_d, \xi_1, \ldots, \xi_d]$ is a $2d$-dimensional polynomial algebra over $\mathbb{K}$, where ξ_i is the image of ∂_i in the associated graded ring $\operatorname{gr} D(R;\mathbb{K})$. In what follows, we will use the description

$$\operatorname{gr} D(\mathbb{K}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}];\mathbb{K}) = \mathbb{K}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, \Theta_1, \ldots, \Theta_d],$$

where $\Theta_i = t_i \xi_i$ is the image of $\theta_i = t_i \partial_i$ in the associated graded ring $\operatorname{gr} D(R;\mathbb{K})$.

When R is a regular algebra over $\mathbb{K}$, $\operatorname{Spec}(\operatorname{gr} D(R;\mathbb{K}))$ can be identified as the cotangent bundle of the variety $\operatorname{Spec} R$ with the projection

$$\pi : \operatorname{Spec}(\operatorname{gr} D(R;\mathbb{K})) \to \operatorname{Spec} R$$

induced by the embedding $R \hookrightarrow \operatorname{gr} D(R;\mathbb{K})$. The fiber of π over a closed point of $\operatorname{Spec} R$ is the cotangent space over that point which is isomorphic to the affine space $\mathbb{K}^{\dim R}$.

In this subsection, we discuss the map π for a certain class of toric algebras. We shall see that in some cases the fibers of π behave nicely (Theorems 4.5, 4.8). We also give an example (Example 4.6) of a more complicated nature.

To begin with, consider the natural order filtration of D_A inherited from that of $D(\mathbb{K}[\mathbb{Z}^d])$. With respect to this filtration, one can regard $\operatorname{gr} D_A$ as a commutative subalgebra of $\operatorname{gr} D(\mathbb{K}[\mathbb{Z}^d]) = \mathbb{K}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, \Theta_1, \ldots, \Theta_d]$. When $\operatorname{gr} D_A$ is finitely generated over $\mathbb{K}$, Musson showed that it has dimension $2d$ [Mus87]. Saito and

Traves proved that $\mathrm{gr}\, D_A$ is finitely generated over $\mathbb{K}$ if and only if the semigroup $\mathbb{N}A$ is scored [ST04]. By definition, a semigroup $\mathbb{N}A$ is scored if

$$\mathbb{N}A = \bigcap_{\sigma:\ \text{facet}} \{a \in \mathbb{Z}^d \mid F_\sigma(a) \in F_\sigma(\mathbb{N}A)\}$$

or, equivalently, $\widetilde{\mathbb{N}A} \setminus \mathbb{N}A$ is a union of finitely many hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$. The scored condition implies Serre's condition (S_2) (see Remark 3.2).

We should remark that if S_A is normal, then $\mathrm{gr}\, D_A$ is Gorenstein ([Mus87], Theorem D). In general, even for a 1-dimensional semigroup ring (which is always scored), the associated graded ring can have bad singularities. In fact, we have the following.

Proposition 4.1. *If S_A is a 1-dimensional toric algebra which is not normal, then $\mathrm{gr}\, D_A$ is not Cohen-Macaulay.*

Proof. Using the formula in Lemma 4.3, we see that $\mathrm{gr}\, D_A$ is again a toric algebra over $\mathbb{K}$. Indeed, notice that 0 is the only facet and that $n_{0,w} = |\Omega(w)|$. Note also that $|\Omega(-w)| = w + |\Omega(w)|$ by Lemma 4.4. So by Lemma 4.3,

$$\mathrm{gr}\, D_A = \mathbb{K}\left[t\xi, t^{|\Omega(w)|}\xi^{|\Omega(-w)|} : |w| \in \{a_1, \dots, a_n\} \cup \mathrm{Hole}(\mathbb{N}A)\right].$$

Therefore $\mathrm{gr}\, D_A$ is a two-dimensional toric algebra over $\mathbb{K}$.

We claim that

$$\dim_{\mathbb{K}} \frac{\mathbb{K}[t,\xi]}{\mathrm{gr}\, D_A} < \infty. \tag{4.1}$$

Take ℓ to be the maximal number of $2|\Omega(-w)|$ for $w \in \mathrm{Hole}(\mathbb{N}A)$. To prove (4.1), it is enough to show that $t^u\xi^v \in \mathrm{gr}\, D_A$ for all pairs $u, v \in \mathbb{N}$ satisfying $u+v \geq \ell$. Since $[t^u\xi^v \in \mathrm{gr}\, D_A \Leftrightarrow t^v\xi^u \in \mathrm{gr}\, D_A]$, by symmetry we may assume $w_0 := u - v \geq 0$.

- If $w_0 \in \mathrm{Hole}(\mathbb{N}A)$, $2u \geq u + v \geq \ell \geq 2|\Omega(-w_0)|$. So $(t\xi)^{u-|\Omega(-w_0)|} \in \mathrm{gr}\, D_A$, and hence $t^u\xi^v = (t\xi)^{u-|\Omega(-w_0)|}t^{|\Omega(-w_0)|}\xi^{|\Omega(w_0)|} \in \mathrm{gr}\, D_A$.
- If $w_0 \in \mathbb{N}A$ we have $t^u\xi^v = (t\xi)^v t^{w_0} = (t\xi)^v t^{|\Omega(-w_0)|} \in \mathrm{gr}\, D_A$.

So the claim is proved. Now, applying the criterion in Remark 3.2 to the toric algebra $\mathrm{gr}\, D_A$, we see that $\mathrm{gr}\, D_A$ doesn't satisfy Serre's condition (S_2). Hence, $\mathrm{gr}\, D_A$ is not Cohen-Macaulay. □

Remark 4.2. The claim (4.1) holds true for any affine curve with injective normalization (see the proof of Theorem 3.12 in [SS88]). In fact, this codimension is known to be the Letzter–Makar-Limanov invariant, which plays an important role in the theory of Calogero-Moser space. For more information, see the work by Berest and Wilson [BW99].

Now, let $\mathfrak{m}$ be the maximal graded ideal of S_A corresponding to the closed point 0 of the toric variety $\mathrm{Spec}(S_A)$. Let $I = \sqrt{\mathfrak{m}\mathrm{gr}D_A}$ be the radical of the extended ideal of $\mathfrak{m}$ under the embedding $S_A \hookrightarrow \mathrm{gr}D_A$. We are going to show that if $\mathbb{N}A$ is simplicial and scored, then $\mathrm{gr}D_A/I$ is isomorphic to S_A as $\mathbb{K}$-algebras. This implies that the reduced induced structure of the fiber of $\pi : \mathrm{Spec}(\mathrm{gr}D_A) \to \mathrm{Spec}(S_A)$ over the point 0 is isomorphic to the ambient toric variety.

The following two lemmas are needed:

Lemma 4.3 ([ST04])**.** *For scored* $\mathbb{N}A$*,*

$$\operatorname{gr} D_A = \bigoplus_{a\in\mathbb{Z}^d} t^a \mathbb{K}[\Theta_1, \ldots, \Theta_d] \cdot P_a \textit{ where}$$

$$P_a = \prod_{\sigma\in\mathscr{F}} F_\sigma(\Theta)^{n_{\sigma,a}} \textit{ and } n_{\sigma,a} = \#\{F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(a) + F_\sigma(\mathbb{N}A)]\}.$$

Lemma 4.4. *Let* $\mathbb{N}A$ *be scored. Then for any* $a \in \mathbb{Z}^d$ *and* $\sigma \in \mathscr{F}$*,*

$$n_{\sigma,-a} = n_{\sigma,a} + F_\sigma(a).$$

In particular,

1. *if* σ *is a facet with the property that* $F_\sigma(a) \leq 0$*, then* $n_{\sigma,ka} \leq k \cdot n_{\sigma,a}$ *for large* $k \in \mathbb{N}$*, and furthermore* $P_a^k = P_{ka} \cdot P$ *for some* $P \in \mathbb{K}[\Theta]$*;*
2. *if* $F_\sigma(a) \leq 0$ *for all* $\sigma \in \mathscr{F}$ *and* $-a \notin \mathbb{N}A$*, then* $n_{\sigma,ka} < k \cdot n_{\sigma,a}$ *for some* $\sigma \in \mathscr{F}$ *and large* $k \in \mathbb{N}$*;*
3. *if* $a \in \mathbb{N}A$*, then* $n_{\sigma,-a} = F_\sigma(a)$*;*
4. *if* $F_\sigma(a) > 0$*, then* $n_{\sigma,ka} = 0$ *for large* $k \in \mathbb{N}$*.*

Proof. To prove $n_{\sigma,-a} = n_{\sigma,a} + F_\sigma(a)$ for any $a \in \mathbb{Z}^d$ and $\sigma \in \mathscr{F}$, it's enough to show the case where $F_\sigma(a) > 0$. Set $N = F_\sigma(\mathbb{N}A)$ and $n = F_\sigma(a) > 0$. Then

$$\begin{aligned} n_{\sigma,-a} &= \#\{N \setminus (n + N)\} \\ &= \#\{b \in N \mid b - n \notin N\} \\ &= n + \#\{b \in N \mid b - n \notin N \text{ but } b - kn \in N \text{ for some } k \geq 2\} \\ &= n + \#\{c \in N \mid c + n \notin N\} \\ &= n + \#\{N \setminus (-n + N)\} = n + n_{\sigma,a}. \end{aligned}$$

Note that for the second equality we need the assumption that $\mathbb{N}A$ is scored.

Now, we prove the four additional statements:

1. Notice that $n_{\sigma,ka} = kF_\sigma(-a)$ for large k and that $n_{\sigma,a} = n_{\sigma,-a} + F_\sigma(-a)$ where $n_{\sigma,-a} \geq 0$.
2. By assumption, $-a$ lies on a hyperplane parallel to some facet, say σ_0. Then $n_{\sigma_0,-a} > 0$ and hence $n_{\sigma_0,ka} < k \cdot n_{\sigma_0,a}$ by (1).
3. $a \in \mathbb{N}A$ implies $n_{\sigma,a} = 0$.
4. This follows from the definition. Indeed, since $\mathbb{N}A$ is scored, $(F_\sigma(ka)+\mathbb{N}_0) \subseteq F_\sigma(\mathbb{N}A)$ for large k. Then $F_\sigma(ka)+F_\sigma(\mathbb{N}A) \subseteq F_\sigma(\mathbb{N}A)$, and hence $n_{\sigma,ka} = 0$.

$\square$

Theorem 4.5. *If* $\mathbb{N}A$ *is a simplicial scored semigroup, then*

$$\operatorname{gr} D_A/I = \mathbb{K}\left[\overline{t^{-a_i} \cdot P_{-a_i}}; i = 1, \ldots, n\right] \cong S_A,$$

where $I = \sqrt{\mathfrak{m} \operatorname{gr} D_A}$ *and* $\overline{t^{-a_i} \cdot P_{-a_i}}$ *is the image of* $t^{-a_i} \cdot P_{-a_i}$ *in* $\operatorname{gr} D_A/I$*.*

Proof. We sketch how the proof of the left equality goes. Let $\mathscr{F} = \{\sigma_1, \ldots, \sigma_d\}$ be the set of all facets of $\mathbb{R}_{\geq 0}A$, and let

$$C = -\widetilde{\mathbb{N}A} = \{a \in \mathbb{Z}^d \mid F_\sigma(a) \leq 0 \text{ for all } \sigma \in \mathscr{F}\}.$$

We will prove the left equality in three steps. The first step shows $\Theta_i \in I$ for $i = 1, \ldots, d$. The second step shows $t^a \cdot P_a \in I$ for all $a \in \mathbb{Z}^d \setminus C$. Finally, the third step shows that $t^a \cdot P_a \in I$ if $a \in C \setminus (-\mathbb{N}A)$ and that $t^a \cdot P_a$ is a product of some $t^{-a_i} \cdot P_{-a_i}$, $i = 1, \ldots, n$, if $a \in C \cap (-\mathbb{N}A \setminus \{0\})$.

(1) For each $i = 1, \ldots, d$, consider the following subset of $\mathbb{Z}^d$:

$$\{F_{\sigma_i}(\Theta) = -1\} \cap \left[\bigcap_{j \neq i} \{F_{\sigma_j}(\Theta) = 0\} \right].$$

Since $\mathbb{N}A$ is simplicial, this is a one point set for each i, say $\{u_i\}$. Notice that since $t^{-u_i} \in I$, $F_{\sigma_i}^{n_{\sigma_i, u_i}} = P_{u_i} = t^{-u_i} \cdot t^{u_i} P_{u_i} \in I$, where $n_{\sigma_i, u_i} > 0$. Therefore, $F_{\sigma_i} \in I$ for each i. Since $F_{\sigma_1}, \ldots, F_{\sigma_d}$ are linearly independent, we conclude that $\Theta_i \in I$ for $i = 1, \ldots, d$.

(2) For $a \in \mathbb{Z}^d \setminus C$, $F_\sigma(a) > 0$ for some $\sigma \in \mathscr{F}$. By Lemma 4.4(4), choose k large so that

$$P_{ka} = \prod_{F_\sigma(a) < 0} F_\sigma^{n_{\sigma, ka}}.$$

Now, consider as in (1) the one-point set

$$\left[\bigcap_{F_\sigma(a) < 0} \{F_\sigma(\Theta) = F_\sigma(ka)\} \right] \cap \left[\bigcap_{F_\sigma(a) \geq 0} \{F_\sigma(\Theta) = 0\} \right] = \{b\}.$$

We have $P_b = P_{ka}$ and $t^{ka-b} \in I$. By Lemma 4.4(1)

$$(t^a P_a)^k = t^{ka} P_a^k = t^{ka} P_{ka} \cdot P = (t^b P_b) \cdot P \cdot t^{ka-b} \in I.$$

Therefore, $t^a P_a \in I$ as desired.

(3) Let $a \in C$.

If $-a \notin \mathbb{N}A$, by Lemma 4.4(2) $(t^a P_a)^k = t^{ka} P_{ka} \cdot P$ for some nonconstant $P \in \mathbb{K}[\Theta]$. Since P is a product of some F_σ's, $P \in I$ by (1), and hence $t^a P_a \in I$.

If $-a \in \mathbb{N}A \setminus \{0\}$, write $-a = \sum m_i a_i$. By Lemma 4.4(3),

$$n_{\sigma, a} = F_\sigma(-a) = \sum m_i F_\sigma(a_i) = \sum m_i n_{\sigma, -a_i},$$

and therefore

$$t^a P_a = t^{\sum m_i(-a_i)} \cdot \prod_{\sigma \in \mathscr{F}} F_\sigma^{\sum m_i n_{\sigma, -a_i}} = \prod (t^{-a_i} P_{-a_i})^{m_i}.$$

To complete the proof, we establish the right isomorphism. First, recall that if $R \to S$ is a homomorphism of commutative rings and Q is a prime ideal in S lying over a prime ideal q of R, then $\dim(S_Q/qS_Q) \geq \mathrm{ht}Q - \mathrm{ht}q$. On the other hand, since $\mathrm{gr}\, D_A$ is finitely generated as a $\mathbb{K}$-algebra which is also a domain, each maximal ideal of $\mathrm{gr}\, D_A$ has height $2d$. (Here, we use the fact that $\dim \mathrm{gr}\, D_A = 2d$.) Therefore, $\dim(\mathrm{gr} D_A/I) \geq d$. Now, consider the surjection from the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ to $\mathbb{K}\left[\overline{t^{-a_i} \cdot P_{-a_i}}; i = 1, \ldots, n\right]$. By Lemma 4.4(3), $P_{-a_i} = \prod_{\sigma \in \mathscr{F}} F_\sigma^{F_\sigma(a_i)}$. Observe that $\overline{t^{-a_i} \cdot P_{-a_i}}, i = 1, \ldots, n$, satisfy the relations in the toric ideal $I_A = \{x^u - x^v | Au = Av\}$ (where for $u \in \mathbb{Z}^n$, $x^u := x_1^{u_1} \cdots x_n^{u_n}$).

Hence we have a surjection

$$S_A \cong \mathbb{K}[x_1,\dots,x_n]/I_A \longrightarrow \mathbb{K}\left[\overline{t^{-a_i}\cdot P_{-a_i}}; i=1,\dots,n\right],$$

which is an isomorphism by comparing the dimensions. □

Example 4.6. Consider

$$A=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\end{pmatrix}.$$

$S_A=\mathbb{K}[s,t,u,stu^{-1}]$ is a 3-dimensional normal toric algebra which is isomorphic to the toric algebra appearing in Example 3.9. By 4.1, 4.6, and 6.3 in [ST04],

$$\begin{aligned}\operatorname{gr} D_A = \mathbb{K}[&s,t,u,stu^{-1},\Theta_s,\Theta_t,\Theta_u,s^{-1}\Theta_s(\Theta_s+\Theta_u),t^{-1}\Theta_t(\Theta_t+\Theta_u),s^{-1}t^{-1}u\Theta_s\Theta_t\\ &u^{-1}(\Theta_s+\Theta_u)(\Theta_t+\Theta_u),tu^{-1}(\Theta_s+\Theta_u),t^{-1}u\Theta_t,su^{-1}(\Theta_t+\Theta_u),s^{-1}u\Theta_s].\end{aligned}$$

Set

$$\begin{aligned}&a=s,b=t,c=u,d=stu^{-1},e=\Theta_s,f=\Theta_t,g=\Theta_u,h=s^{-1}\Theta_s(\Theta_s+\Theta_u),\\ &i=t^{-1}\Theta_t(\Theta_t+\Theta_u),j=u^{-1}(\Theta_s+\Theta_u)(\Theta_t+\Theta_u),k=s^{-1}t^{-1}u\Theta_s\cdot\Theta_t,\\ &l=tu^{-1}(\Theta_s+\Theta_u),m=t^{-1}u\Theta_t,n=su^{-1}(\Theta_t+\Theta_u),o=s^{-1}u\Theta_s.\end{aligned}$$

Consider the surjection $\phi:\mathbb{K}[a,\dots,o]\to\operatorname{gr} D_A$. Using Macaulay 2, we see that a primary decomposition of $\sqrt{\mathfrak{m}\operatorname{gr} D_A}$ is the intersection of the two ideals $(o,n,d,a,c,f+g,e,b,fj-il,fi+jm,f^2+lm,fk+hm,hi-jk,fh-kl)$ and $(m,l,d,a,c,f,e+g,b,gk-io,gi+kn,g^2+no,gj+hn,hi-jk,gh-jo)$ modulo Kerϕ. Therefore, the fiber $\pi^{-1}(0)$ has two components, each of which is 4-dimensional.

Remark 4.7. The left equality of Theorem 4.5 and Example 4.6 can be achieved alternatively by a result of Saito.

By Proposition 4.14 in [Sai10], $\pi^{-1}(\mathfrak{m})=\{\mathfrak{P}(\mathfrak{q},\nu)\mid \nu\cap\mathbb{R}_{\geq 0}A=\{0\}\}$. If $\mathbb{N}A$ is simplicial, then $\pi^{-1}(\mathfrak{m})=\{\mathfrak{P}(\mathfrak{m}_0,-\mathbb{R}_{\geq 0}A)\}$, which is the left equality of Theorem 4.5. On the other hand, consider Example 4.6. If $\mathfrak{P}(\mathfrak{q},\nu)\in\pi^{-1}(\mathfrak{m})$, then by [Sai10], Proposition 4.14, $\mathfrak{q}\supseteq(\Theta_s,\Theta_t+\Theta_u)$ or $(\Theta_t,\Theta_s+\Theta_u)$. If $\mathfrak{q}=(\Theta_s,\Theta_t+\Theta_u)$, then $\nu=\{\Theta_s\leq 0,\Theta_t+\Theta_u\leq 0\}$. If $\mathfrak{q}=(\Theta_t,\Theta_s+\Theta_u)$, then $\nu=\{\Theta_t\leq 0,\Theta_s+\Theta_u\leq 0\}$. They are the two minimal primes mentioned in Example 4.6.

As a corollary of Theorem 4.5, we can describe the fibers $\pi^{-1}(p)$ for every nonzero closed point p in $\operatorname{Spec} S_A$. Let p be in the torus orbit O_τ for some e-dimensional face τ of $\mathbb{R}_{\geq 0}A$, so p corresponds to a semigroup homomorphism $f_p:\mathbb{N}A\to\mathbb{K}$ with $f_p(a_i)=c_i$, where $c_i=0$ if and only if $a_i\notin\mathbb{N}(A\cap\tau)$. Then p corresponds to the maximal ideal $\mathfrak{m}_p=(t^{a_1}-c_1,\dots,t^{a_n}-c_n)$ of S_A. The following theorem gives the reduced induced structure of $\pi^{-1}(p)$.

Theorem 4.8. *Under the hypotheses of Theorem* 4.5*, we have*

$$\frac{\operatorname{gr} D_A}{\sqrt{\mathfrak{m}_p\operatorname{gr} D_A}}\cong S_B\otimes\mathbb{K}[\delta_1,\dots,\delta_e],$$

where S_B is the toric algebra generated by a simplicial scored semigroup

$$\mathbb{N}B\cong\frac{\mathbb{N}A+\mathbb{Z}(A\cap\tau)}{\mathbb{Z}(A\cap\tau)}$$

and $\mathbb{K}[\delta_1,\dots,\delta_e]$ is a polynomial ring in $e(=\dim\tau)$ variables.

Proof. First, notice that t^a, $a \in \mathbb{N}(A \cap \tau)$, acts as a unit on $\frac{\operatorname{gr} D_A}{\sqrt{\mathfrak{m}_p \operatorname{gr} D_A}}$ because $p \in O_\tau$. So, by abusing the notation

$$\frac{\operatorname{gr} D_A}{\sqrt{\mathfrak{m}_p \operatorname{gr} D_A}} \cong \frac{\operatorname{gr} D_A[\tau^{-1}]}{\sqrt{\mathfrak{m}_p \operatorname{gr} D_A[\tau^{-1}]}},$$

where τ^{-1} means we invert t^a for all $a \in \mathbb{N}(A \cap \tau)$. Note also that

$$\operatorname{gr} D_A[\tau^{-1}] \cong \operatorname{gr} D(S_A[\tau^{-1}]) = \operatorname{gr} D(\mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)]).$$

Next, choose a simplicial scored semigroup $\mathbb{N}B$ so that

$$\mathbb{N}A + \mathbb{Z}(A \cap \tau) = \mathbb{N}B \oplus \mathbb{Z}(A \cap \tau).$$

To do this, let's first assume $\mathbb{N}A$ is normal. By an exercise of section 1.2 in [Ful93], $\frac{\mathbb{R}_{\geq 0}A + \mathbb{R}\tau}{\mathbb{R}\tau}$ is a simplicial rational polyhedral cone with facets $\frac{\gamma + \mathbb{R}\tau}{\mathbb{R}\tau}$, where the γ's are the facets of $\mathbb{R}_{\geq 0}A$ containing τ. So $\frac{\mathbb{N}A + \mathbb{Z}(A\cap\tau)}{\mathbb{Z}(A\cap\tau)}$ is a simplicial normal semigroup in $\mathbb{Z}^d/\mathbb{Z}(A \cap \tau)$. $\mathbb{N}B$ can be obtained by choosing suitable elements in $[\mathbb{N}A + \mathbb{Z}(A \cap \tau)] \setminus \mathbb{Z}(A \cap \tau)$. For the general simplicial scored semigroup $\mathbb{N}A$, we just have to notice that $\widetilde{\mathbb{N}A} \setminus \mathbb{N}A$ is a union of hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$. So $\mathbb{Z}(\widetilde{A \cap \tau}) = \mathbb{Z}(A \cap \tau)$ and $[\mathbb{N}A + \mathbb{Z}(\widetilde{A \cap \tau})] \setminus [\mathbb{N}A + \mathbb{Z}(A \cap \tau)]$ is a union of hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$ containing τ.

Now,

$$\operatorname{gr} D_A[\tau^{-1}] \cong \operatorname{gr} D(\mathbb{K}[\mathbb{N}B \oplus \mathbb{Z}(A \cap \tau)]) \cong \operatorname{gr} D_B \otimes \operatorname{gr} D_{\mathbb{Z}(A\cap\tau)}.$$

Therefore,

$$\begin{aligned} \frac{\operatorname{gr} D_A}{\sqrt{\mathfrak{m}_p \operatorname{gr} D_A}} &\cong \frac{\operatorname{gr} D_B \otimes \operatorname{gr} D_{\mathbb{Z}(A\cap\tau)}}{\sqrt{\mathfrak{m}_p \operatorname{gr} D_B \otimes \operatorname{gr} D_{\mathbb{Z}(A\cap\tau)}}} \\ &\cong \frac{\operatorname{gr} D_B}{\sqrt{\mathfrak{m}_B \operatorname{gr} D_B}} \otimes \mathbb{K}[\delta_1, \ldots, \delta_e] \\ &\cong S_B \otimes \mathbb{K}[\delta_1, \ldots, \delta_e] \end{aligned}$$

by Theorem 4.5, where $\delta_1, \ldots, \delta_e$ are the standard derivations of $\mathbb{K}[\mathbb{Z}(A \cap \tau)]$. □

4.2. **Characteristic cycles.** Let $D := D(R; \mathbb{K})$ as defined in section 2. Let M be a D-module with a filtration $\{M_i\}$ such that $D_i M_j \subseteq M_{i+j}$. The associated graded module $\operatorname{gr} M := \bigoplus M_i/M_{i-1}$ has the natural $\operatorname{gr} D$-module structure. We call $\{M_i\}$ a good filtration if $\operatorname{gr} M$ is finitely generated over $\operatorname{gr} D$. If M is finitely generated over D by $x_1, \ldots, x_n$, then the filtration $\{\sum_{j=1}^n D_i x_j\}$ is good.

From now on, we assume that $\operatorname{gr} D$ is Noetherian. This is always the case when R is regular.

For a D-module M with a good filtration $\{M_i\}$, define the characteristic variety $\operatorname{Ch}(M)$ of M to be the support of the $\operatorname{gr} D$-module $\operatorname{gr} M$,

$$\text{i.e. } \operatorname{Ch}(M) = \operatorname{Var}(\operatorname{ann}_{\operatorname{gr} D} \operatorname{gr} M) \subseteq \operatorname{Spec}(\operatorname{gr} D).$$

The characteristic cycle $\operatorname{CC}(M)$ of M is the formal sum of the irreducible components V_i of $\operatorname{Ch}(M)$ counted with multiplicity. More precisely,

$$\operatorname{CC}(M) = \sum m_i V_i,$$

where the multiplicity m_i is the length of the $(\operatorname{gr} D)_{p_i}$-module $(\operatorname{gr} M)_{p_i}$ and p_i is the prime ideal corresponding to V_i.

$\mathrm{Ch}(M)$ and $\mathrm{CC}(M)$ do not depend on the choice of good filtration. A more detailed discussion about characteristic varieties can be found in [Gin86].

Example 4.9. R is naturally a D-module generated by the identity 1. With the filtration $\{D_i \cdot 1\}$, $\mathrm{gr}\, R = R$ is the $\mathrm{gr}\, D$-module generated by 1. So $R \cong \mathrm{gr}\, D/\mathrm{ann}_{\mathrm{gr}\, D}(1)$, and therefore $\mathrm{Ch}(R) = \mathrm{Var}(\mathrm{ann}_{\mathrm{gr}\, D}(1))$ is abstractly isomorphic to the ambient variety $\mathrm{Spec}\, R$.

As we mentioned in the introduction, many invariants of $H^i_I(R)$ can be computed via the characteristic cycles when R is a polynomial algebra over $\mathbb{K}$ (see e.g. [ÁM00], [ÁM04]). In this subsection, we compute the characteristic cycles of some local cohomology modules $H^i_I(S_A)$ using our results of finiteness properties in section 3. By the main result in [ST04], $\mathrm{gr}\, D_A$ is finitely generated over $\mathbb{K}$ if and only if $\mathbb{N}A$ is scored. In particular, $\mathrm{gr}\, D_A$ is Noetherian when $\mathbb{N}A$ is scored, so it makes sense to talk about characteristic cycles in this case.

Example 4.10. For a 1-dimensional toric algebra $S_A = \mathbb{K}[t^{a_i} \mid i = 1, \dots, n]$, $\mathrm{Ch}(H^1_I(S_A))$ is particularly simple. As in Proposition 4.1,

$$\mathrm{gr}\, D_A = \mathbb{K}\left[t\xi, t^{|\Omega(w)|}\xi^{|\Omega(-w)|} : |w| \in \{a_1, \dots, a_n\} \cup \mathrm{Hole}(\mathbb{N}A)\right].$$

For any monomial ideal $I \neq 0$ of S_A, $H^1_I(S_A) = \frac{D_A \cdot (1/t)}{S_A}$ by Theorem 3.1. Notice that $t^{a_i} \in \sqrt{\mathrm{ann}_{\mathrm{gr}\, D_A} \mathrm{gr}(H^1_I(S_A))}$ and that $\xi^{a_i} \notin \sqrt{\mathrm{ann}_{\mathrm{gr}\, D_A} \mathrm{gr}(H^1_I(S_A))}$. So by Theorem 4.5,

$$\frac{\mathrm{gr}\, D_A}{\sqrt{\mathrm{ann}_{\mathrm{gr}\, D_A} \mathrm{gr}(H^1_I(S_A))}} = \mathbb{K}[\delta_i \mid i = 1, \dots, n] \cong S_A,$$

where δ_i is the image of ξ^{a_i} in $\frac{\mathrm{gr}\, D_A}{\sqrt{\mathrm{ann}_{\mathrm{gr}\, D_A} \mathrm{gr}(H^1_I(S_A))}}$. Therefore, $\mathrm{Ch}(H^1_I(S_A))$ is abstractly isomorphic to the ambient toric variety $\mathrm{Spec}(S_A)$. Furthermore, in view of the exact sequence

$$0 \to S_A \to D_A \cdot (1/t) \to H^1_I(S_A) \to 0,$$

we have

$$\mathrm{CC}(S_A[1/f]) = \mathrm{Ch}(S_A) + \mathrm{Ch}(H^1_I(S_A)) \text{ for any monomial } f \in S_A$$

by the additivity of the characteristic cycles on exact sequences. In particular, $\mathrm{Ch}(S_A[1/f])$ has two components, each of which is abstractly isomorphic to $\mathrm{Spec}(S_A)$.

Example 4.11. Consider the toric algebra in Example 3.8. Let $p = \Theta_t$ and $q = \Theta_s$. Then we have

$$\mathrm{gr}\, D_A = \mathbb{K}[t, ts^2, ts, t^{-1}(2p-q)^2, s(2p-q), t^{-1}s^{-1}(2p-q)q, t^{-1}s^{-2}q^2, s^{-1}q, p, q].$$

Set

$$a = t, b = ts^2, c = ts, d = t^{-1}(2p-q)^2, e = s(2p-q),$$
$$f = t^{-1}s^{-1}(2p-q)q, g = t^{-1}s^{-2}q^2, h = s^{-1}q, i = p, j = q,$$

and consider the surjection $\phi : \mathbb{K}[a, \dots, j] \to \mathrm{gr}\, D_A$. Table 1 gives the information about the characteristic cycles. Again, notice that each M is cyclic by Theorem 3.1, so $\mathrm{ann}_{\mathrm{gr}\, D_A} \mathrm{gr}\, M$ is easy to compute.

TABLE 1

M	$J=\phi^{-1}(\mathrm{ann}_{\mathrm{gr}\,D_A}\,\mathrm{gr}\,M)$	primary decomposition of $\sqrt{J}$
$S_A[1/t]$	$(f,g,h,i,j)+\mathrm{Ker}\phi$	$(a,c,f,g,h,i,f,bd-e^2)\cap(d,e,f,g,h,i,j,ab-c^2)$
$H^1_{(t)}(S_A)$	$(f,g,h,i,j,a)+\mathrm{Ker}\phi$	$(a,c,f,g,h,i,f,bd-e^2)$
$S_A[1/ts^2]$	$(d,e,f,i,j)+\mathrm{Ker}\phi$	$(j,i,h,g,f,e,d,ab-c^2)\cap(j,i,f,c,b,e,d,ag-h^2)$
$H^1_{(ts^2)}(S_A)$	$(d,e,f,i,j,b)+\mathrm{Ker}\phi$	$(j,i,f,e,d,c,b,ag-h^2)$
$S_A[1/ts]$	$(i,j)+\mathrm{Ker}\phi$	$(j,i,h,c,a,b,e,dg-f^2)\cap(j,i,h,g,f,c,a,bd-e^2)$ $\cap(j,i,h,g,f,e,d,ab-c^2)\cap(j,i,f,c,b,e,d,ag-h^2)$
$H^2_{(t,ts^2)}(S_A)$	$(a,b,c,e,h,i,j)+\mathrm{Ker}\phi$	$(a,b,c,e,h,i,j,dg-f^2)$

Example 4.12. Consider the toric algebra in Example 4.6 and use the notation there. By Theorem 3.1, we see that

$$J:=\phi^{-1}(\mathrm{ann}_{\mathrm{gr}\,D_A}\,\mathrm{gr}(H^1_{(s)}(S_A)))=(a,e,f,g,i,j,k,m,n)+\mathrm{Ker}\phi$$

and the primary decomposition of $\sqrt{J}$ is

$$\begin{aligned}\sqrt{J}=&(o,n,m,k,j,i,c,a,f,g,h,e)\cap(n,m,l,k,j,i,d,a,f,g,h,e)\\&\cap(n,m,k,j,i,d,c,a,f,g,e,bh-lo).\end{aligned}$$

So $\mathrm{CC}(H^1_{(s)}(S_A))$ is a sum of varieties which are not all isomorphic.

To compute $\mathrm{Ch}(H^3_{\mathfrak{m}}(S_A))$, we need

$$J':=\phi^{-1}(\mathrm{ann}_{\mathrm{gr}\,D_A}\,\mathrm{gr}(H^3_{\mathfrak{m}}(S_A)))=(a,b,c,d,e,f,g,l,m,n,o)+\mathrm{Ker}\phi$$

and $\sqrt{J'}=(a,b,c,d,e,f,g,l,m,n,o,hi-jk)$.

Inspired by Examples 4.10, 4.11, and 4.12, we have the following.

Theorem 4.13. *For any scored pointed semigroup $\mathbb{N}A$, the characteristic variety $\mathrm{Ch}(H^d_{\mathfrak{m}}(S_A))$ is abstractly isomorphic to the ambient toric variety* $\mathrm{Spec}(S_A)$.

Proof. Since $\mathbb{N}A$ is pointed, by the Ishida complex

$$H^d_{\mathfrak{m}}(S_A)=\frac{\mathbb{K}[\mathbb{Z}^d]}{\mathbb{K}[\bigcup_{\sigma:\mathrm{facet}}(\mathbb{N}A+\mathbb{Z}(A\cap\sigma))]}. \tag{4.2}$$

By Theorem 3.1, $\mathbb{K}[\mathbb{Z}^d]=D_A\cdot(1/t^\alpha)$ for some interior point α, i.e. for some $\alpha\in\mathbb{N}A\setminus[\bigcup_{\sigma:\mathrm{facet}}\mathbb{N}(A\cap\sigma)]$. Consider the expression of $\mathrm{gr}\,D_A$ in Lemma 4.3. Denote $J:=\mathrm{ann}_{\mathrm{gr}\,D_A}\,\mathrm{gr}(H^d_{\mathfrak{m}}(S_A))$. Notice that:

(1) For any $a\in\mathbb{Z}^d\setminus(-\widetilde{\mathbb{N}A})$ and for any facet σ with $F_\sigma(a)>0$, there exists $n\in\mathbb{N}$ so that $na-\alpha\in\mathbb{N}A+\mathbb{Z}(A\cap\sigma)$. Therefore,
$$t^a\mathbb{K}[\Theta_1,\dots,\Theta_d]\cdot P_a\subseteq\sqrt{J}$$
for all $a\in\mathbb{Z}^d\setminus(-\widetilde{\mathbb{N}A})$.

(2) For $a=0\in\mathbb{Z}^d$, $P_a=1$ and $\theta_i\cdot t^{-\alpha}=-\alpha_i t^{-\alpha}\in H^d_{\mathfrak{m}}(S_A)$. So by considering the order filtration, $\Theta_i\in J$ for all $i=1,\dots,d$.

(3) By exactly the same argument as in item (3) in the proof of Theorem 4.5, we have $t^aP_a\in\sqrt{J}$ for $a\in[-\widetilde{\mathbb{N}A}\setminus(-\mathbb{N}A)]$. Also, for $a\in-\mathbb{N}A\setminus\{0\}$, t^aP_a is a product of some $t^{-a_i}P_{-a_i}$.

(4) For $a \in -\mathbb{N}A \setminus \{0\}$, $t^a P_a \cdot t^{-\alpha} = P(-\alpha)t^{a-\alpha} \neq 0$ in $\operatorname{gr}(H^d_{\mathfrak{m}}(S_A))$ because α is an interior point and because $a - \alpha \in \mathbb{Z}^d \setminus \bigcup_{\sigma:\text{facet}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \sigma)]$. So $t^a P_a \notin \sqrt{J}$ for $a \in -\mathbb{N}A \setminus \{0\}$.

Therefore,

$$\frac{\operatorname{gr} D_A}{\sqrt{J}} = \mathbb{K}\left[\overline{t^{-a_i} P_{-a_i}} \middle| a_i : \text{ columns of } A\right],$$

which is isomorphic to S_A by a similar argument as in the final part of the proof of Theorem 4.5. □

Remark 4.14. We mention that the description of $\sqrt{J}$ in Theorem 4.13 can be obtained alternatively using Theorem 6.2(2) in [Sai10].

For β, let $\mathfrak{m}_\beta$ be the ideal $(\theta_1 - \beta_1, \ldots, \theta_d - \beta_d)$ of the polynomial ring $\mathbb{K}[\theta_1, \ldots, \theta_d]$. Note that $\mathfrak{m}_\beta = \mathfrak{m}_0 + \beta$ in the notation of [Sai10]. Then $L(\mathfrak{m}_\beta) = D_A/I(\mathfrak{m}_\beta)$ is an irreducible D_A-module (see Theorem 4.1.6 in [ST01] and (5.2) in [Sai10]).

Take $\alpha \in \mathbb{Z}^d$ as in the proof of Theorem 4.13. Then we claim that

$$H^d_{\mathfrak{m}_A}(S_A) \cong L(\mathfrak{m}_{-\alpha}). \tag{4.3}$$

Since $\tau(\mathfrak{m}_0 - \alpha) = -\mathbb{R}_{\geq 0}A$, $\sqrt{J} = \sqrt{\operatorname{gr} I(\mathfrak{m}_{-\alpha})} = \mathfrak{P}(\mathfrak{m}_0, -\mathbb{R}_{\geq 0}A)$ by (4.3) and Theorem 6.2(2) in [Sai10].

The claim (4.3) can be proved as follows: By Theorem 3.1, $H^d_{\mathfrak{m}_A}(S_A) = D_A \cdot t^{-\alpha}$. Since $\mathbb{N}A$ is scored, $\mathbb{Z}^d \cap \mathbb{Q}(A \cap \tau) = \mathbb{Z}(A \cap \tau)$ for all faces τ. Hence, for $\beta \in \mathbb{Z}^d$, $\beta \not\sim -\alpha$ if and only if $\beta \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)$ for some face τ if and only if $\beta \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)$ for some facet σ. Therefore, we see that $I(\mathfrak{m}_{-\alpha}) \cdot t^{-\alpha} = 0$ by (4.2). Thus there exists a surjective D_A-homomorphism from $L(\mathfrak{m}_{-\alpha})$ to $H^d_{\mathfrak{m}_A}(S_A)$. Since $L(\mathfrak{m}_{-\alpha})$ is irreducible, it is an isomorphism.

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