

FIXED POINTS IMPLY CHAOS FOR A CLASS OF DIFFERENTIAL INCLUSIONS THAT ARISE IN ECONOMIC MODELS

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ABSTRACT. We consider multi-valued dynamical systems with continuous time of the form $\dot{x} \in F(x)$, where $F(x)$ is a set-valued function. Such models have been studied recently in mathematical economics. We provide a definition for chaos, ω -chaos and topological entropy for these differential inclusions that is in terms of the natural \mathbb{R} -action on the space of all solutions of the model. By considering this more complicated topological space and its \mathbb{R} -action we show that chaos is the ‘typical’ behavior in these models by showing that near any hyperbolic fixed point there is a region where the system is chaotic, ω -chaotic, and has infinite topological entropy.

1. INTRODUCTION

Many interesting models from mathematical economics give rise to dynamical systems which have the strange property that corresponding to each point in the state space there are many different orbits that each start at that point. We call such dynamical systems multi-valued dynamical systems. Specific multi-valued dynamical systems that we have studied previously have discrete time and a property known as backward dynamics, [8], [9], [7] [12], [13]. In models with backward dynamics on \mathbb{R} there is a (usually noninvertible) continuous function, f , such that if x_t is a state at time t in the model, then $f(x_t) = x_{t-1}$ instead of the expected x_{t+1} (hence the term “backward dynamics”). Thus $x_{t+1} \in f^{-1}(x_t)$, and the set of all possible equilibria in the model forms an inverse limit space. The dynamical system generated by the induced shift homeomorphism on the inverse limit space has been studied in order to describe the dynamics implicit in these backward dynamical systems. By considering the inverse limit space and the shift map we have passed from multi-valued dynamics on \mathbb{R} to well-defined dynamics but on a more complicated metric space.

In this paper we extend our previous work to the continuous time analogue of indeterminate models such as those with backward dynamics. Specifically we consider the space of solutions of specific differential inclusions (see for example [1], [2], [5]). These take the form

$$\dot{x}(t) \in f(x(t)) \text{ a.e.}$$

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with $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^m$ and $f(x(t)) := \{f_\alpha(x(t)) : \alpha \in A\}$ and $f_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is C^1 -smooth for all $\alpha \in A$. To be more precise, by a *differential inclusion* on \mathbb{R}^m we mean a set-valued mapping with domain \mathbb{R}^m of the form

$$f = \{f_\alpha : \alpha \in A\}$$

and

$$f(x) = \{f_\alpha(x) : \alpha \in A\},$$

where each $f_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a C^1 function. A *solution* or *equilibrium* of a differential inclusion is an absolutely continuous function $x : \mathbb{R} \rightarrow \mathbb{R}^m$ such that

$$\dot{x}(t) \in f(x(t)) \text{ a.e.}$$

In 1999, Christiano and Harrison gave an example of such a dynamical system in economics: the one-sector growth model with a production externality [3]. An interesting first question to consider when analyzing such dynamical systems is “What does it mean for this type of system to be chaotic?” Christiano and Harrison use a notion of sensitive dependence on initial conditions in terms of the space of all possible initial states. We give three definitions of chaos in this context. First we give a definition in the spirit of Devaney [4], next we give a definition of ω -chaos which is an extension of the notion of Li-Yorke Chaos [10], [11], and finally we define topological entropy for these differential inclusions [6]. Instead of considering the state space, where the dynamical system is multi-valued, we consider the space, D , of all solutions of the model. This space is connected and metric but is usually more complicated than the original state space. There is a natural \mathbb{R} -action on D that we denote by T . We take the view that if T is ‘chaotic’ on D with its topology, then the original differential inclusion should be called ‘chaotic.’

Christiano and Harrison illustrate the possibility of deterministic and stochastic regime switching equilibria along with equilibria that appear chaotic. However, their analysis of detecting chaotic behavior is statistical and done via partitioning the state space $X = \bigcup_{i=1}^n B_i$ and constructing a single-valued system via $\dot{x} = F(x)$ where $F(x) = f_i(x)$ if $x \in B_i$. This establishes chaos in their sense for one thin slice of the system.

In this paper, instead of considering the action of the differential inclusion on the state space, we examine the space of all possible solutions generated by the model. We provide sufficient conditions for chaos to be present in such a system, and we demonstrate that chaotic behavior is the typical behavior in differential inclusions by showing that these sufficient conditions are almost always satisfied near a fixed point.

Specifically, if we assume that our differential inclusion has at least two branches, f_1 and f_2 , which satisfy the following conditions:

- (1) f_1 has a hyperbolic steady state, x^* , in a region where f_2 has no bounded solutions and
- (2) if
 - x^* is a sink or a source, or
 - x^* is a saddle and $f_2(x^*)$ is not a scalar multiple of an eigenvector of $Df_1(x^*)$,

then the differential inclusion is chaotic in the sense of Devaney, is ω -chaotic, and has infinite topological entropy. In the last section, we apply these results to a concrete example from economics.

2. CHAOS FOR A CLASS OF DIFFERENTIAL INCLUSIONS

Let $f = \{f_1, f_2\}$ be a differential inclusion on \mathbb{R}^2 and let D be the set of all solutions of f . Let ϕ and ψ be the flows generated by f_1 and f_2 . Given an initial condition, $a \in \mathbb{R}^2$, we can visualize all of the solutions, x , in D that start at a via ‘piecing together’ the flows ϕ and ψ . For instance one solution to the differential inclusion that starts at a is given by $x(0) = a$ and $\dot{x}(t) = f_1(t)$ for all $t \in \mathbb{R}$. This solution simply follows the integral curve generated by ϕ through a . More interesting solutions can be constructed by jumping from the integral curves generated by f_1 to integral curves generated by f_2 . For instance, let x be the solution described above and suppose that $b = x(1) \in \mathbb{R}^2$. Define a solution to the differential inclusion, y , by $y(t) = x(t)$ for all $t \leq 1$ but at $b = x(1) = y(1)$ let y ‘switch paths’ to the integral curves generated by f_2 . In other words, $y(0) = a$, $\dot{y}(t) = f_1(t)$ for all $t < 1$, $y(1) = b$, and $\dot{y}(t) = f_2(t)$ for all $t > 1$. It is not hard to see that much more complicated solutions can be constructed from this differential inclusion. In what follows we regularly use existing solutions to the differential inclusion and points in \mathbb{R}^2 to construct new solutions that satisfy the differential inclusion almost everywhere.

Let $T : D \times \mathbb{R} \rightarrow D$ be the natural \mathbb{R} -action on D . We use

$$T(x, t) = y = T_t(x),$$

where

$$y(s) = x(t + s)$$

for all $s \in \mathbb{R}$. We consider D as a topological space with topology generated by the supremum metric. Specifically, let $\nu_t(x, y) := \min\{d(x(t), y(t)), 1\}$ where $d(\cdot, \cdot)$ is the usual metric on \mathbb{R}^2 . We define a metric on D by

$$\nu(x, y) := \sup_{t \in \mathbb{R}} \left\{ \frac{\nu_t(x, y)}{2^{|t|}} \right\}.$$

For $x \in D$, we let $B_\epsilon(x)$ denote the open ϵ -neighborhood around x in D with respect to the ν metric. For $a \in \mathbb{R}^2$ we let $D_\epsilon(a)$ denote the open ϵ -neighborhood around a with respect to the usual metric on \mathbb{R}^2 . We frequently switch between the two spaces \mathbb{R}^2 , the plane, and D , the space of all solutions of our differential inclusion. That is why we are so careful in our notation for ϵ -neighborhoods.

Let $R \subseteq D$ be T -invariant and closed. We say that (R, T) is *chaotic*, in the sense of Devaney [4], if T has sensitive dependence on initial conditions, is topologically transitive and has a dense set of periodic points on R . We will say that our differential inclusion has a *chaotic region* if there is a closed T -invariant set $R \subseteq D$ such that (R, T) is chaotic.

Next, we establish sufficient conditions for a set of solutions to exhibit chaos in the sense of Devaney. We then show that these sufficient conditions will typically be satisfied near a steady state equilibrium.

Definition 1. Let $a, b \in \mathbb{R}^2$. We say there is a *path from a to b generated by D* provided there exists a solution to our differential inclusion, $w \in D$, and $t_0, t_1 \in \mathbb{R}$ such that $t_0 < t_1$ with $w(t_0) = a$ and $w(t_1) = b$. The path generated by D is $P := \{w(t) : t_0 \leq t \leq t_1\} \subseteq \mathbb{R}^2$.

Definition 2. Let $a, b \in \mathbb{R}^2$. We say there is a *simple path from a to b generated by D* provided there exists a path from a to b generated by D , v with $t_0, t_1 \in T$, $t_0 < t_1$

such that $v(t_0) = a$ and $v(t_1) = b$ and such that \dot{v} has finitely many discontinuities on $[t_0, t_1]$ and also $a \neq v(s) \neq b$ for all $t_0 < s < t_1$. The simple path generated by D is $P := \{v(s) : t_0 \leq s \leq t_1\} \subseteq \mathbb{R}^2$.

Lemma 1. *Let V be a subset of \mathbb{R}^2 such that for every $a, b \in V$ there is a simple path from a to b in V generated by D . Let $V^* := \{x \in D \mid x(t) \in V \text{ for all } t \in \mathbb{R}\}$. Then $T|_{V^*}$ has a dense set of periodic points.*

Proof. Let $x \in V^*$ and $\epsilon > 0$. Pick S such that $(1/2)^S < \epsilon$. For $t \in [-S, S]$ let $y(t) = x(t)$. This implies $y \in B_\epsilon(x)$. If $y(S) = y(-S)$, then repeating this path gives a periodic y with $y \in B_\epsilon(x)$ (here the ϵ -ball is defined in terms of the metric ν). If $y(S) \neq y(-S)$, then there is a path in V from $y(S)$ to $y(-S)$. Adding this path to $\{y(t) : -S \leq t \leq S\}$, we get a path from $y(-S)$ to $y(S)$ and back to $y(-S)$. Repeating this path gives a function $y : [-S, \infty) \rightarrow \mathbb{R}^2$ which is periodic and (except for not being defined on $(-\infty, -S)$) is a solution of our differential inclusion. For each $n \in \mathbb{N}$ we can define $y : [-nS, \infty) \rightarrow \mathbb{R}^2$ by following the periodic path ‘backwards’ n -times. This gives a solution y of our differential inclusion that is periodic and $y \in B_\epsilon(x)$. Hence there is a dense set of periodic points under T in V^* . \square

Lemma 2. *Let V be a subset of \mathbb{R}^2 such that for every $a, b \in V$ there is a simple path from a to b in V generated by D . Let $V^* := \{x \in D \mid x(t) \in V \text{ for all } t \in \mathbb{R}\}$. Then $T|_{V^*}$ is topologically transitive.*

Proof. Let $U, W \subset V^*$ be nonempty open sets in V^* (recall that we use the topology on the solution space generated by the metric ν). Let $x \in U$ and $y \in W$. Pick $\epsilon > 0$ sufficiently small so that $B_\epsilon(x) \cap V^* \subset U$ and $B_\epsilon(y) \cap V^* \subseteq W$. Pick S so that $(1/2)^S < \epsilon$ and let $z \in V^*$ be defined piecewise such that $z(t) = x(t)$ for $-\infty < t \leq S$. There exists a simple path from $x(S) = z(S)$ to $y(-S)$ in V generated by D , call it $\{w(t) : 0 \leq t \leq Q\}$. Define $z(t)$ for $t \in [S, S+Q]$ by $z(t) = w(t-S)$. Then for $t \geq S+Q$, let $z(t) = y(t-2S-Q)$. Then since z is defined piecewise to agree with solutions for our differential inclusion we see that z is a solution as well. Moreover $z \in V^*$ because each of its pieces is. Let $z' = T(z, 2S+Q)$. Then $z'(-S) = y(-S)$ and $z'(t) = y(t)$ for all $-S \leq t \leq S$. Hence $z' \in B_\epsilon(y)$. Thus $T(U, 2S+Q) \cap W \neq \emptyset$. Hence $T|_{V^*}$ is topologically transitive. \square

Lemma 3. *Let V be a subset of \mathbb{R}^2 such that*

- (1) *for every $a, b \in V$ there is a simple path from a to b in V generated by D ;*
- (2) *there is a solution $w \in D$ such that $w(t) \in V$ for all $t \in \mathbb{R}$ and $\{w(t) : t \in \mathbb{R}\}$ is not dense in V .*

Let $V^ := \{x \in D \mid x(t) \in V \text{ for all } t \in \mathbb{R}\}$. Then $T|_{V^*}$ has sensitive dependence on initial conditions.*

Proof. Using condition (2) of the lemma, let $w \in D$ be the solution that is not dense in V . Then let $U \subset V$ be a nonempty open set in V (in the usual topology on \mathbb{R}^2) such that $w(t) \in V \setminus U$ for all $t \in \mathbb{R}$. Let $z \in V$ and $0 < \delta < 1$ such that $D_\delta(z) \subset U$ (recall that $D_\delta(z)$ denotes the δ -neighborhood of z in the usual topology on \mathbb{R}^2). We show that $T|_{V^*}$ has sensitive dependence on initial conditions by showing that for any $x \in V^*$ and $\epsilon > 0$ there is a solution y and $t \in \mathbb{R}$ such that $T(y, t)$ and $T(x, t)$ are more than $\frac{\delta}{2}$ apart and $y \in B_\epsilon(x)$ (recall that $B_\epsilon(x)$ denotes the ϵ -neighborhood of x with respect to the ν metric on the solution space D). Let $x \in V^*$ and $\epsilon > 0$ be given. There are two cases to consider.

First we suppose that there exists a t_0 such that $x(t) \notin D_{\delta/2}(z)$ for all $t \geq t_0$. In this case pick S such that $S > t_0$ and $(1/2)^S < \epsilon$. Let $y(t) = x(t)$ for $t \in (-\infty, S]$. Let $\{a(t) : 0 \leq t \leq M\}$ be a path from $x(S)$ to z in V generated by D . For $t \in [S, S + M]$ let $y(t) = a(t - S)$. For $t > S + M$ let $y(t)$ follow any solution, v , in V^* that has $v(0) = z$. We then have $y \in B_\epsilon(x)$ and

$$\nu(T(x, S + M), T(y, S + M)) \geq \min\{d(x(S + M), y(S + M)), 1\} \geq \delta/2.$$

For the second case suppose that for each $n \in \mathbb{N}$, there is a $t_n \geq n$ such that $x(t_n) \in D_{\delta/2}(z)$. Pick S such that $(1/2)^S < \epsilon$. Let $y(t) = x(t)$ for $t \in (-\infty, S]$, so $y \in B_\epsilon(x)$. Let $\{b(t) : 0 \leq t \leq N\}$ be a path from $x(S)$ to $w(0)$ in V generated by D (recall that w is a solution that never enters U). Let $y(t) = b(t - S)$ for $t \in [S, S + N]$ and let $y(t) = w(t - S - N)$ for $t > S + N$. Let $m \in \mathbb{N}$ with $m > S + N$. Then we have

$$\nu(T(x, t_m), T(y, t_m)) \geq \min\{d(x(t_m), y(t_m)), 1\} \geq \delta/2.$$

Combining these two cases we see that $T|_{V^*}$ has sensitive dependence on initial conditions. □

From Lemmas 1–3, we have the following theorem.

Theorem 1. *Let V be a subset of \mathbb{R}^2 such that*

- (1) *for every $a, b \in V$ there is a simple path from a to b in V generated by D ;*
- (2) *there is a solution $w \in D$ such that $w(t) \in V$ for all $t \in \mathbb{R}$ and $\{w(t) : t \in \mathbb{R}\}$ is not dense in V .*

Let $V^ := \{x \in D \mid x(t) \in R \text{ for all } t \in \mathbb{R}\}$. Then $T|_{V^*}$ is chaotic in the sense of Devaney.*

A natural question is whether such a set V and V^* satisfying conditions (1) and (2) of the theorem can exist. The answer is yes. In fact such sets are quite easy to construct, but we leave a discussion of their existence to Section 4.

3. EMBEDDED SHIFT SPACES

In this section we assume that the differential inclusion admits a region $V \subseteq \mathbb{R}^2$ such that for every $a, b \in V$ there is a simple path in V generated by D from a to b . We show that, under this assumption, the differential inclusion displays other forms of chaos by showing that, for each $n \in \mathbb{N}$, there are uncountably many different closed T -invariant subsets that are conjugate to a full shift on n symbols. We use these subspaces to demonstrate that (D, T) is ω -chaotic and has infinite topological entropy.

First we give some definitions adopted to our setting of a space of solutions to a differential inclusion. Let $x \in D$ be a solution to our differential inclusion. The *omega-limit set of x under T* is the set of limit points of $orb_T(x)$ in D . We denote this set by $\omega_T(x)$ or $\omega(x)$.

A set S with at least two points is *scrambled* provided for all $x, y \in S$ with $x \neq y$ we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \nu(T_t(x), T_t(y)) &> 0, \\ \liminf_{t \rightarrow \infty} \nu(T_t(x), T_t(y)) &= 0. \end{aligned}$$

We say that (D, T) is *chaotic in the sense of Li and Yorke* or has *Li-Yorke chaos* provided there is an uncountable scrambled set $S \subseteq D$, [11].

A stronger notion than Li-Yorke chaos is that of ω -chaos. A set $S \subseteq D$ (having at least two points) is called an ω -scrambled set provided for any $x, y \in S$ with $x \neq y$ we have

- (1) $\omega(x) \setminus \omega(y)$ is uncountable;
- (2) $\omega(x) \cap \omega(y)$ is nonempty;
- (3) $\omega(x)$ is not contained in the set of periodic points.

We say that $T : D \times \mathbb{R} \rightarrow D$ is ω -chaotic provided there exists an uncountable ω -scrambled set in D , [10].

Let $N \in \mathbb{N}$ and let $t_N \in \mathbb{R}^+$. In the upcoming we compute $h_{top}(T)$ and we find ω -scrambled sets for T by first considering the discrete time action given by $T_{t_N} : D \times \mathbb{Z} \rightarrow D$, where $T_{t_N}(x, n) = T_{n \cdot t_N}(x)$. We denote this system by $(D, [T_{t_N}])$. We find closed $[T_{t_N}]$ -invariant subsets of D , B_N , such that $(B_N, [T_{t_N}])$ is conjugate to a full-shift on N letters.

Let V be a subset of \mathbb{R}^2 such that for every $a, b \in V$ there is a simple path from a to b in V generated by D . We begin by constructing a closed invariant subset of D that is conjugate to the full 2-shift. Let $q \in V$ and let P_0 be a simple path in V generated by D from q to q . Since P_0 is a finite union of arcs we can find a point $r \neq q$ such that $r \notin P_0$. Let

- (1) P_0 be the simple path from q to q in V generated by D described above;
- (2) P_1 be a simple path from r to r in V generated by D ;
- (3) U be a simple path from q to r in V generated by D , and
- (4) W be a simple path from r to q in V generated by D .

Let $s_0 > 0$ be defined so that $P_0(s_0) = P_0(0) = q$. Let $s_1 > 0$ be defined such that $P_1(s_1) = P_1(0) = r$. Let $s_2 > 0$ be such that $U(0) = q$ and $U(s_2) = r$, and finally let $s_3 > 0$ be such that $W(0) = r$ and $W(s_3) = q$. Let $t_2 = s_0 + s_1 + s_2 + s_3$.

Define a new path from q to q in W generated by D , Q_0 , by

- (1) $Q_0(0) = q$;
- (2) $Q_0(t) = P_0(t)$ for $0 \leq t \leq s_0$;
- (3) $Q_0(t) = U(t - s_0)$ for $s_0 \leq t \leq s_0 + s_2$;
- (4) $Q_0(t) = P_1(t - s_0 - s_2)$ for $s_0 + s_2 \leq t \leq s_0 + s_2 + s_1$, and
- (5) $Q_0(t) = W(t - s_1)$ for $s_0 + s_2 + s_1 \leq t \leq t_2$.

Notice that $Q_0(0) = Q_0(t_2) = q$. Similarly define Q_1 from q to q in V generated by D by

- (1) $Q_1(0) = q$;
- (2) $Q_1(t) = U(t)$ for $0 \leq t \leq s_2$;
- (3) $Q_1(t) = P_1(t - s_2)$ for $s_2 \leq t \leq s_2 + s_1$;
- (4) $Q_1(t) = W(t - s_2 - s_1)$ for $s_2 + s_1 \leq t \leq s_2 + s_1 + s_3$, and
- (5) $Q_1(t) = P_0(t - s_2 - s_1 - s_3)$ for $s_2 + s_1 + s_3 \leq t \leq t_2$.

Again we have $Q_1(0) = Q_1(t_2) = q$. Both Q_0 and Q_1 are paths from q to q with length t_2 but they go in different directions. We will use these two paths and the discretized action \hat{T}_{t_2} to define a closed invariant subset of D that is conjugate to the full 2-shift.

Let $2^{\mathbb{Z}}$ denote the space of all bi-infinite strings of 0s and 1s with the usual metric topology. Specifically if $\alpha = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$ and $\beta = (\dots, \beta_{-1}, \beta_0, \beta_1, \dots)$ have n with $|n|$ minimal such that $\beta_n \neq \alpha_n$, then the metric ρ is defined by $\rho(\alpha, \beta) =$

$2^{-|n|}$. Let σ_2 denote the shift map on $2^{\mathbb{Z}}$:

$$\sigma_2(\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots) = (\dots, \alpha_0, \alpha_1, \alpha_2, \dots).$$

Let $\alpha = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots) \in 2^{\mathbb{Z}}$. Define x_α by the following:

- (1) $x_\alpha(0) = q$;
- (2) $x_\alpha(t) = Q_{\alpha_0}(t)$ for $0 \leq t \leq t_2$;
- (3) $x_\alpha(t) = Q_{\alpha_n}(t)$ for $nt_0 \leq t \leq (n + 1)t_2$.

Since x_α is just the piecewise concatenation of simple paths from q to q in V generated by D we see that x_α is both well-defined and an element of D . Let $B_2 = \{x_\alpha : \alpha \in 2^{\mathbb{Z}}\}$.

Lemma 4. *Let $\delta = \max\{d(Q_0(t), Q_1(t)) : 0 \leq t \leq t_2\}$. Let $t_\delta \in [0, t_2]$ be such that $d(Q_0(t), Q_1(t)) = \delta$. Suppose that $x_\alpha, x_\beta \in B_2$ with*

$$\nu(x_\alpha, x_\beta) < \frac{\delta}{2^{nt_2}}.$$

Then $\alpha_i = \beta_i$ for all $-n \leq i \leq n$.

Proof. Since $\nu(x_\alpha, x_\beta) < \frac{\delta}{2^{nt_2}}$ we see that

$$\frac{d(x_\alpha(t), x_\beta(t))}{2^{|t|}} \leq \frac{\delta}{2^{nt_2}}$$

for all $t \in \mathbb{R}$. Let $m \in \mathbb{Z}$ with $\alpha_m \neq \beta_m$ be such that $|m|$ is minimal with respect to this property. Then by definition,

$$x_\alpha(t) = x_\beta(t)$$

for all $-(m - 1)t_2 \leq t \leq (m - 1)t_2$, but

$$d(x_\alpha((m - 1)t_2 + t_\delta), x_\beta((m - 1)t_2 + t_\delta)) = \delta$$

by the definition of δ and t_δ . This implies that

$$\frac{\delta}{2^{|m-1|t_2+t_\delta}} \leq \nu(x_\alpha, x_\beta) < \frac{\delta}{2^{nt_2}}$$

and since $t_\delta \leq t_2$, we see that

$$\frac{\delta}{2^{|m|t_2}} < \frac{\delta}{2^{|m-1|t_2+t_\delta}},$$

which implies

$$\frac{\delta}{2^{|m|t_2}} < \frac{\delta}{2^{nt_2}}.$$

Hence $|m| > n$. □

Proposition 1. (B_2, \hat{T}_{t_2}) is topologically conjugate to $(2^{\mathbb{Z}}, \sigma_2)$.

Proof. Let $h : B_2 \rightarrow 2^{\mathbb{Z}}$ be defined so that $h(x_\alpha) = \alpha$. Clearly h is one-to-one and onto. It follows from Lemma 4 and the definition of the metric on $2^{\mathbb{Z}}$ that h is continuous.

We show that h^{-1} is continuous. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ large enough so that $\frac{1}{2^{Nt_2}} < \epsilon$. Let $\alpha, \beta \in 2^{\mathbb{Z}}$ such that $\alpha_i = \beta_i$ for all $-N \leq i \leq N$. Then $x_\alpha(t) = x_\beta(t)$ for all $t \in [-Nt_2, Nt_2]$. Thus

$$\nu(x_\alpha, x_\beta) \leq \frac{1}{2^{Nt_2}} < \epsilon.$$

We now show that $h^{-1} \circ \sigma_2 \circ h = \hat{T}_{t_2}$. Let $\alpha = (\dots, \alpha_{-1} \cdot \alpha_0, \alpha_1, \dots) \in 2^{\mathbb{Z}}$ and x_α be the corresponding point in B_2 . Then $h(x_\alpha) = \alpha$ and $\sigma_2(\alpha) = (\dots, \alpha_0 \cdot \alpha_1, \alpha_2, \dots)$. Consider $x_{\sigma_2(\alpha)}$ and $\hat{T}_{t_2}(x_\alpha)$. Let $t \in \mathbb{R}$ and choose $n \in \mathbb{Z}$ such that $nt_2 \leq t < (n+1)t_2$. By definition of $x_{\sigma_2(\alpha)}$,

$$x_{\sigma_2(\alpha)}(t) = Q_{\sigma_2(\alpha)_n}(t) = Q_{\alpha_{n+1}}(t)$$

(recall that $\sigma_2(\alpha)_n$ denotes the n th coordinate of $\sigma_2(\alpha)$) but

$$\hat{T}_{t_2}(x_\alpha)(t) = x_\alpha(t_2 + t).$$

Since n was chosen so that $nt_2 \leq t < (n+1)t_2$ we see that $(n+1)t_2 \leq t_2 + t < (n+2)t_2$. Thus

$$\hat{T}_{t_2}(x_\alpha)(t) = x_\alpha(t_2 + t) = Q_{\alpha_{n+1}}(t).$$

This holds for all $t \in \mathbb{R}$. So $h^{-1} \circ \sigma_2 \circ h = \hat{T}_{t_2}$. □

Let $B_2^* = orb_T(B_2)$. Then B_2^* is closed and T -invariant.

Proposition 2. *Let $y \in B_2^*$. Then the following are equivalent:*

- (1) $y \in \omega_T(x_\alpha)$;
- (2) *there exists some $x_\beta \in \omega_{\hat{T}_{t_2}}(x_\alpha)$ such that $T_i(x_\beta) = y$ and $\beta \in \omega_{\sigma_2}(\alpha)$.*

Proof. Suppose that $y \in \omega_T(x_\alpha)$. Then there is a sequence $s_i \rightarrow \infty$ of real numbers with $T_{s_i}(x_\alpha) \rightarrow y$ in $B_2^* \subseteq D$. Thus

$$\nu(T_{s_i}(x_\alpha), y) \rightarrow 0$$

as $i \rightarrow \infty$. Hence

$$d(x_\alpha(s_i), y(0)) \rightarrow 0$$

as $i \rightarrow \infty$. So choose $J \in \mathbb{N}$ large enough so that for all $j \geq J$,

$$\nu(T_{s_j}(x_\alpha), y) < \delta.$$

Recall that $\delta = \max\{d(Q_0(t), Q_1(t))\}$. This implies that

$$d(x_\alpha(s_j), y(0)) < \delta,$$

and so for each $j \geq J$ there is some $0 \leq u_j < t_2$ such that $x_\alpha(s_j + u_j) = y(0)$. Let $v_j = s_j + u_j$. Then we still have $T_{v_j}(x_\alpha) \rightarrow y$. For each $j \geq J$, let $v_j = k_j t_2 + r_j$, where $0 \leq r_j < t_2$. Since $y(0)$ is a point in the path made up of P_0, U, P_1 , and W and $x_\alpha(v_j) = y(0)$ we see that x_α ‘passes through’ $y(0)$ at time v_j either while following the path Q_0 or the path Q_1 which are identical as subsets of \mathbb{R}^2 , but they follow the paths P_0, U, P_1 and V in different orders. When x_α is following Q_0 at $x_\alpha(v_j)$ we will get one value of r_j , call it r_0 , and when x_α is following Q_1 at $x_\alpha(v_j)$ we will get a different value for r_j , call it r_1 . Since we have infinitely many v_j ’s, we see that infinitely many of them must agree. Without loss of generality, assume that infinitely many agree and equal r_0 .

Consider $T_{v_j - r_0}(x_\alpha) \rightarrow T_{-r_0}(y)$ since T is continuous. By our choice of r_0 , $T_{-r_0}(y) = x_\beta$ for some $\beta \in 2^{\mathbb{Z}}$. By definition of $r_j = r_0$ we see that $v_j - r_0 = k_j t_2$. Hence

$$\hat{T}_{t_2}^{k_j}(x_\alpha) = T_{v_j - r_0}(x_\alpha) \rightarrow x_\beta.$$

Thus $x_\beta \in \omega_{\hat{T}_{t_2}}(x_\alpha)$, and since \hat{T}_{t_2} is topologically conjugate with σ_2 , we see that $\beta \in \omega_{\sigma_2}(\alpha)$. □

It is known that $(2^{\mathbb{Z}}, \sigma_2)$ is ω -chaotic, [10]. We use that fact and the previous proposition to establish that (D, T) is ω -chaotic.

Theorem 2. (D, T) is ω -chaotic.

Proof. Let $\hat{A} \subseteq 2^{\mathbb{Z}}$ be an uncountable ω -scrambled set in $2^{\mathbb{Z}}$. Let $A \subseteq B_2$ be defined so that $x_\alpha \in A$ if, and only if, $\alpha \in \hat{A}$. Since (B_2, \hat{T}_{t_2}) is conjugate with $(2^{\mathbb{Z}}, \sigma_2)$ we see that A is an uncountable ω -scrambled set for (B_2, \hat{T}_{t_2}) . This implies that for every $x_\alpha, x_\beta \in A$ we have

- (1) $\omega_{\hat{T}_{t_2}}(x_\alpha) \setminus \omega_{\hat{T}_{t_2}}(x_\beta)$ is uncountable;
- (2) $\omega_{\hat{T}_{t_2}}(x_\alpha) \cap \omega_{\hat{T}_{t_2}}(x_\beta)$ is nonempty;
- (3) $\omega_{\hat{T}_{t_2}}(x_\alpha)$ is not contained in the set of periodic points.

In other words A is an uncountable scrambled set for \hat{T}_{t_2} . We will show that A is an uncountable ω -scrambled set for T .

By the previous proposition we see that for each $x_\gamma \in \omega_{\hat{T}_{t_2}}(x_\alpha) \setminus \omega_{\hat{T}_{t_2}}(x_\beta)$, $x_\gamma \in \omega_T(x_\alpha) \setminus \omega_T(x_\beta)$. Hence (1) holds for A with respect to T . The previous proposition implies that $\omega_{\hat{T}_{t_2}}(x_\alpha) \cap \omega_{\hat{T}_{t_2}}(x_\beta) \subseteq \omega_T(x_\alpha) \cap \omega_T(x_\beta)$. Hence (2) holds for A with respect to T . Finally, (3) implies that there is a nonperiodic $\gamma \in 2^{\mathbb{Z}}$ such that $x_\gamma \in \omega_{\hat{T}_{t_2}}(x_\alpha)$. The previous proposition guarantees that $\omega_{\hat{T}_{t_2}}(x_\alpha) \subseteq \omega_T(x_\alpha)$, and hence (3) holds for A with respect to T . Therefore A is an uncountable ω -scrambled set for T , and (D, T) is ω -chaotic. \square

To define the *topological entropy* of T we begin with the definition for the topological entropy of a continuous function $f : X \rightarrow X$ on a metric space X with metric μ . We define a new metric on f -orbit segments in (X, f) . Let $n \in \mathbb{N}$, and define

$$\mu_n^f(x, y) = \max_{0 \leq t \leq n} \{\mu(f^t(x), f^t(y))\}.$$

We denote the open ϵ -ball centered at $x \in X$ in this metric by

$$B_f(x, \epsilon, n) = \{y \in X : \mu_n^f(x, y) < \epsilon\}.$$

A set $E \subseteq X$ is said to be (n, ϵ) -spanning provided

$$X \subseteq \bigcup_{x \in E} B_f(x, n, \epsilon).$$

Let $S_\mu(f, \epsilon, n)$ denote the minimal cardinality of an (n, ϵ) -spanning set. Finally define the *topological entropy of f* by

$$h_{top}(f) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log S_\mu(f, \epsilon, n).$$

It is well known that topological entropy is an invariant of topological conjugacy [17].

We extend the notion of topological entropy in the standard way to our system $T : D \times \mathbb{R} \rightarrow D$ by redefining the metric on the orbit segments [6]. Let $S \in \mathbb{R}^+$. For each $x, y \in D$, define

$$\nu_S^T(x, y) = \max_{-S \leq t \leq S} \{\nu(T_t(x), T_t(y))\}.$$

The rest of the definition of topological entropy in this case follows exactly as above (see [6]).

It is well known that the topological entropy of $(2^{\mathbb{Z}}, \sigma_2)$ is $\log 2$ [17]. Hence $h_{top}(\hat{T}_{t_2}|_{B_2}) = \log 2$.

Lemma 5. $h_{top}(\hat{T}_{t_2}|_{B_2}) \leq h_{top}(T|_{B_2^*})$.

Proof. Let $E \subseteq B_2^*$ be an (S, ϵ) -spanning set for T . For each $y \in E$ there is some $t_y \in \mathbb{R}$ and $\beta_y \in 2^{\mathbb{Z}}$ such that $T_{t_y}(x_{\beta_y}) = y$. The set $\hat{E} = \{x_{\beta_y} : y \in E\}$ is an (S, ϵ) -spanning set of B_2 . Moreover, $|\hat{E}| \leq |E|$. So $S_\nu(\hat{T}_{t_2}, \epsilon, S) \leq S_\nu(T, \epsilon, S)$. The result follows. \square

Since B_2^* is an invariant subset of D we see that

$$\log 2 \leq h_{top}(T|_{B_2^*}) \leq h_{top}(T).$$

The above construction can be extended from $2^{\mathbb{Z}}$ and B_2 to $N^{\mathbb{Z}}$ and B_N by starting with N many points, q_i , and simple paths from q_i to q_i in V generated by D and a careful description of Q_i . Since $h_{top}(\sigma_N) = \log N$ we see that $h_{top}(T) \geq \log N$ for each $N \in \mathbb{N}$. Thus:

Theorem 3. $h_{top}(T) = \infty$.

4. CONSTRUCTING V

In this section we show that for differential inclusions with at least two branches, $F = \{f_1, f_2\}$, with the property that

- (1) f_1 has a hyperbolic steady state, a^* , in a region where f_2 has no bounded solutions and
- (2) if
 - a^* is a sink or a source, or
 - a^* is a saddle and $f_2(a^*)$ is not a scalar multiple of an eigenvector of $Df_1(a^*)$,

we can construct a set V satisfying

- (1) for every $a, b \in V$ there is a simple path from a to b in V generated by D ;
- (2) there is a solution $w \in D$ such that $w(t) \in V$ for all $t \in \mathbb{R}$ and $\{w(t) : t \in \mathbb{R}\}$ is not dense in V .

It follows from the previous two sections that $(V^*, T|_{V^*})$ is chaotic in the sense of Devaney, (D, T) is ω -chaotic, and T has infinite topological entropy.

Let $K \subset \mathbb{R}^2$ be a nonempty compact set such that f_2 has no bounded solutions in K , i.e. for any $x \in D$ with $\dot{x}(t) = f_2(t)$ for all $t \in \mathbb{R}$ there is some $t_0 < 0 < t_1$ such that $x(t_0), x(t_1) \notin K$. Let $a \in K$ and suppose that there exists a simple path from a to a generated by D , which we call P , such that $P \subset K$. In general, P is compact and is a finite union of arcs. It is easy to see that $\mathbb{R}^2 - P$ has only one unbounded component, and so we can write $\mathbb{R}^2 \setminus P$ as

$$\left(\bigcup_{\alpha \in A} C_\alpha \right) \cup C_0,$$

where each C_α and C_0 are components of $\mathbb{R}^2 \setminus P$ and C_0 is the unique unbounded component. Let $R := \bigcup_{\alpha \in A} C_\alpha \cup P$. Note that this is the same as $\mathbb{R}^2 \setminus C_0$. Since C_0 is open, we know that R is closed. We show that any set R constructed in this way satisfies conditions (1) and (2) of the theorem. See Figure 1 for a possible configuration of these sets.

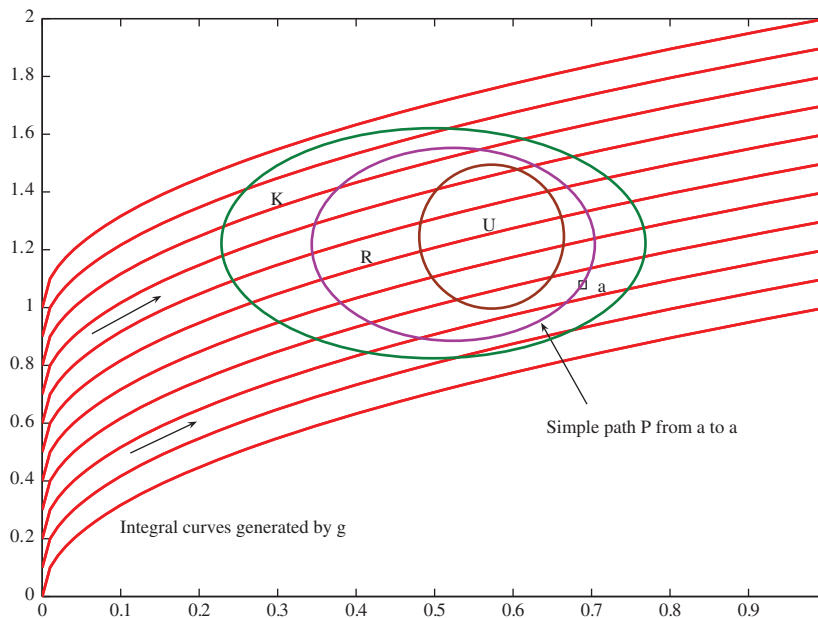


FIGURE 1

Lemma 6. Let $K \subset \mathbb{R}^2$ be a nonempty compact set such that f_2 has no bounded solutions in K . Let $a \in K$ and suppose that there exists a simple path generated by D, P , from a to a such that $P \subset K$. Let $\mathbb{R}^2 \setminus P$ be

$$\left(\bigcup_{\alpha \in A} C_\alpha \right) \cup C_0,$$

where each C_α and C_0 are components of $\mathbb{R}^2 \setminus P$ and C_0 is the unique unbounded component. Let $R := \bigcup_{\alpha \in A} C_\alpha \cup P$, and let $R^* = \{x \in D : x(t) \in D \text{ for all } t \in \mathbb{R}\}$. Then $(R^*, T|_{R^*})$ is chaotic in the sense of Devaney, (D, T) is ω -chaotic and T has infinite topological entropy.

Proof. Let $c, d \in R$. Since $P \subseteq K, R \subseteq K$. Since f_2 has no bounded solutions in K , any solution, x , with $x(0) = c$ and $\dot{x}(t) = f_2(t)$ for all $t \in \mathbb{R}$ must eventually leave K . Thus there is some $t_0 > 0$ such that $x(t_0) \in P$ (since P contains the boundary of R). Following this line of reasoning we see that there exists a simple path from c to some point $z \in P$ and a simple path from some $w \in P$ to d . So there is a simple path from c to z , from z to a , from a to w and from w to d . Thus there is a path from c to d in R and condition (1) of Theorem 1 is established. This is enough to satisfy the assumptions of Section 3. Hence we can conclude that (D, T) is ω -chaotic and that T has infinite topological entropy.

Suppose $R^\circ \neq \emptyset$. Since P is a finite union of arcs in $\mathbb{R}^2, P^\circ = \emptyset$. Thus there exists $U \subset R$ open in \mathbb{R}^2 such that $P \cap U = \emptyset$. So in this case P itself satisfies condition (2) of Theorem 1.

Suppose instead that $R^\circ = \emptyset$. Then P contains no simple closed curves. Recall that ϕ and ψ are the flows generated by f_1 and f_2 respectively. Then (without loss

of generality) there exists $0 < t_1, t_2$ such that $\psi^t(a) \in P$ for all $0 \leq t \leq t_1$ and $\phi^t(\psi^{t_1}(a)) \in P$ for $0 \leq t \leq t_2$ with $\phi^{t_2}(\psi^{t_1}(a)) = a$. Since ϕ “doubles back” on ψ in this case we see that there is some $0 < t_3 \leq t_2$ such that $P_2 := \{\phi^t(\psi^{t_1}(a)) : 0 \leq t \leq t_3\} = \{\psi^s(a) : 0 \leq s \leq t_1\} =: P_1$. Let $b \in P_1$ such that $b \neq a$ and $b \neq \psi^{t_1}(a)$. Let $t_4 \in (0, t_1)$ such that $\psi^{t_4}(a) = b$ and $t_5 \in (0, t_3)$ such that $\phi^{t_5}\psi^{t_1}(a) = b$. Let $y \in D$ be the solution (with period $t_4 + t_3 - t_5$) such that $y(0) = a$, $y(s) = \psi^s(a)$ for $0 \leq s \leq t_4$ and $y(s) = \phi^{t_5+(s-t_4)}\psi^{t_4}(a)$ for $t_4 \leq s \leq t_4 + (t_3 - t_5)$. Then in the relative topology there is a nonempty open set in R that this periodic solution misses, specifically a small open set around $\psi^{t_1}(a)$. Hence there is a solution $w \in R^*$ such that $\{w(t) : t \in \mathbb{R}\}$ is not dense in R^* , and so condition (2) of Theorem 1 is satisfied. Hence $(V^*, T|_{V^*})$ is chaotic in the sense of Devaney. \square

In [14] we show that if f_1 has a hyperbolic steady state in a region K which has no bounded solutions for f_2 , then often there is such a path P from a point a back to itself, as described above. Thus if we are in this case, then our differential inclusion gives rise to a region in the solution space on which T is chaotic in the sense of Devaney, T is ω -chaotic and T has infinite topological entropy.

More specifically, let $a^* \in \mathbb{R}^2$ with $f_1(a^*) = 0$ and $f_2(a^*) \neq 0$. Let $A = Df_1(a^*)$ with eigenvalues λ_1, λ_2 and corresponding eigenvectors e_1, e_2 . We show the following:

Theorem 4 (Sink/Source [14]). *If there exists $\epsilon > 0$ such that a^* is a sink (asymptotically stable) or a source (asymptotically unstable) for f_1 on $B_\epsilon(a^*)$, then a^* is contained in a region K in which f_2 has no bounded solutions, and there is a point $b \in K$ and a simple path, P , generated by D from b to b in K .*

Theorem 5 (Saddle [14]). *Suppose $\lambda_1 < 0$ and $\lambda_2 > 0$ with $f_2(a^*) \neq \alpha e_1$ and $f_2(a^*) \neq \beta e_2$ for all α, β . Then a^* is contained in a region K in which f_2 has no bounded solutions, and there is a point $b \in K$ and a simple path, P , generated by D from b to b in K .*

Thus if we assume that our differential inclusion has at least two branches, f_1 and f_2 , which satisfy the following conditions:

- (1) f_1 has a hyperbolic steady state, a^* , in a region where f_2 has no bounded solutions and
- (2) if
 - a^* is a sink or a source, or
 - a^* is a saddle and $f_2(a^*)$ is not a scalar multiple of an eigenvector of $Df_1(a^*)$,

then Theorem 1 combined with Theorems 4 and 5 imply that the differential inclusion is chaotic in the sense of Devaney on a region near a^* , (D, T) is ω -chaotic, and T has infinite topological entropy.

5. ECONOMIC MODEL

In this section we describe a continuous-time dynamic model from economics [14] based on [3] where the dynamical system characterizing the equilibria in the model is generated by a differential inclusion (for more models, see [16, 15]). The model consists of a large number (a continuum indexed by the unit interval) of households and firms. All the households are identical and all the firms are identical

as well. Each household consumes and supplies labor with preferences over (non-negative) paths for consumption $\mathbf{c} := \{c_t\}$ and labor $\mathbf{n} := \{n_t\}$ given by a real-valued function:

$$(1) \quad W(\mathbf{c}, \mathbf{n}) := \int_0^\infty e^{-\rho t} [\log(c_t) + \sigma \log(1 - n_t)] dt,$$

with $\rho, \sigma > 0$. The function W ranks different paths for consumption and labor for each household. We say that a household prefers $\{\hat{\mathbf{c}}, \hat{\mathbf{n}}\}$ to $\{\bar{\mathbf{c}}, \bar{\mathbf{n}}\}$ if and only if $W(\hat{\mathbf{c}}, \hat{\mathbf{n}}) > W(\bar{\mathbf{c}}, \bar{\mathbf{n}})$. Output in the economy, denoted by Y , is produced by each firm using capital k and labor n and depends on y , the average level of production across all firms in the economy. Each firm's level of production is given by the following function: $Y = f(y, k, n) := y^\gamma k^\alpha n^{1-\alpha}$, where $0 < \alpha < 1, 0 < \gamma < 1$.

Each household seeks to maximize (1) by the choice of $\{k_t, c_t, n_t\}$ subject to its budget constraint

$$(2) \quad \dot{k}_t = w_t n_t + (r_t - \delta)k_t - c_t,$$

taking as given $\{w_t, r_t\}$ and k_0 . The parameter $0 < \delta < 1$ is the depreciation rate. We will refer to this utility maximization problem as the household's problem. Each firm rents capital (owned by the households) and hires labor (also offered by the households). Each firm seeks to maximize profit $f(y_t, k_t, n_t) - w_t n_t - r_t k_t$ by the choice of $k_t, n_t \geq 0$ taking $y_t, w_t, r_t > 0$ given. We will refer to this profit maximization problem as the firm's problem.

An equilibrium in the model consists of prices $\mathbf{w} := \{w_t\}, \mathbf{r} := \{r_t\}$ and plans $\mathbf{c} := \{c_t\}, \mathbf{k} := \{k_t\}, \mathbf{n} := \{n_t\}$ such that given prices, $\{\mathbf{w}, \mathbf{r}\}$, the plan $\{\mathbf{c}, \mathbf{k}, \mathbf{n}\}$ solves the household's problem and the plan $\{\mathbf{k}, \mathbf{n}\}$ solves the firm's problem. It is also required that the supply of goods equals the demand for goods: $c_t + \delta k_t + \dot{k}_t = k_t^{\alpha_k} n_t^{\alpha_n}$.

Let Λ_t be the Lagrange multiplier on (2) and $\mathbf{\Lambda} := \{\Lambda_t\}$. It can be shown that one has paths $\{\mathbf{k}, \mathbf{\Lambda}, \mathbf{n}\}$, where $k_t, \lambda_t, n_t > 0, \mathbf{k}$ and $\mathbf{\Lambda}$ are continuous and piecewise differentiable (a.e.) and \mathbf{n} is piecewise continuous (a.e.) that satisfy the following:

$$(3) \quad \dot{k}_t = g^k(k_t, \Lambda_t, n_t) := k_t^{\alpha_k} n_t^{\alpha_n} - 1/\Lambda_t - \delta k_t,$$

$$(4) \quad \dot{\Lambda}_t = g^\Lambda(k_t, \Lambda_t, n_t) := \Lambda_t(\rho + \delta - \alpha k_t^{\alpha_k - 1} n_t^{\alpha_n}),$$

$$(5) \quad 0 = M(k_t, \Lambda_t, n_t) := \Lambda_t(1 - \alpha)k_t^{\alpha_k} n_t^{\alpha_n - 1} - \sigma/(1 - n_t),$$

$$(6) \quad 0 = \lim_{t \rightarrow \infty} e^{-\rho t} \Lambda_t k_t,$$

where $\alpha_k := \alpha/(1 - \gamma)$ and $\alpha_n := (1 - \alpha)/(1 - \gamma)$. (6) is a sufficient terminal condition for establishing a solution to the infinite-horizon optimization problems solved by the households.

Given $k_t > 0$ and $\Lambda_t > 0$, one uses equation (5) to solve for n_t .

In [14], we show that there are either 2 or 0 solutions for n_t strictly between 0 and 1 if and only if $\alpha_n > 1$. Suppose $\alpha_n > 1$. Let the solutions at time t be denoted by $0 < n^1(k_t, \Lambda_t) < n^2(k_t, \Lambda_t) < 1$.

The dynamics in this model are given by the following differential inclusion:

$$(7) \quad \begin{bmatrix} \dot{k}_t \\ \dot{\Lambda}_t \end{bmatrix} \in \{G^1(k_t, \Lambda_t), G^2(k_t, \Lambda_t)\},$$

where G^i is defined by (3) and (4):

$$G^i(k, \Lambda) := \begin{bmatrix} g^k(k, \Lambda, n^i(k, \Lambda)) \\ g^\Lambda(k, \Lambda, n^i(k, \Lambda)) \end{bmatrix}.$$

In [14], we show that Euler equation branching occurs in a neighborhood of the fixed point and therefore the results apply. Specifically, this model has infinite topological entropy, ω -chaos and Devaney chaos.

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