ON SYMMETRIC PRODUCTS OF CURVES

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ABSTRACT. Let C be a smooth complex projective curve of genus g and let $C^{(2)}$ be its second symmetric product. This paper concerns the study of some attempts at extending to $C^{(2)}$ the notion of gonality. In particular, we prove that the degree of irrationality of $C^{(2)}$ is at least $g - 1$ when C is generic and that the minimum gonality of curves through the generic point of $C^{(2)}$ equals the gonality of C. In order to produce the main results we deal with correspondences on the k-fold symmetric product of C, with some interesting linear subspaces of $\mathbb{P}^n$ enjoying a condition of Cayley-Bacharach type, and with monodromy of rational maps. As an application, we also give new bounds on the ample cone of $C^{(2)}$ when C is a generic curve of genus $6 \leq g \leq 8$.

1. INTRODUCTION

Let C be a smooth irreducible complex projective curve of genus $g \geq 0$. The *gonality* of C is the minimum positive integer d such that C admits a covering $f : C \longrightarrow \mathbb{P}^1$ of degree d, and we denote it by $\mathrm{gon}(C)$. The gonality is an important invariant of curves, and it has been studied since the nineteenth century. Throughout we will deal with two attempts at extending the notion of gonality to varieties of higher dimension. In particular, we treat this topic on the second symmetric product $C^{(2)}$ of the curve C, which is the smooth surface parametrizing the unordered pairs of points of the curve. We would like to point out that symmetric products of curves are very concrete projective varieties that are naturally defined by C. Moreover, they somehow reflect the geometry of the curve and are deeply involved in the classical theory of curves. So it seems natural and interesting to study the problem of generalizing the notion of gonality on such varieties.

It is worth noting that to compute the gonality of curves is a quite difficult task. Indeed, besides examples of morphisms reaching the expected minimum degree, one has to provide non-existence results for lower degrees. As it shall be clear in the following, similar remarks shall hold for both the notions we are going to introduce.

The most natural extension of gonality is probably the degree of irrationality. Initially, it has been introduced in an algebraic context by Moh and Heinzer in [17], whereas its geometric interpretation has been deeply studied by Yoshihara [20, 21, 22]. Given an irreducible complex projective variety X of dimension n, the

degree of irrationality of X is defined to be the integer

$$\mathrm{d}_r(X) := \min \left\{ d \in \mathbb{N} \,\middle|\, \begin{array}{l} \text{there exists a dominant rational} \\ \text{map } F \colon X \dashrightarrow \mathbb{P}^n \text{ of degree } d \end{array} \right\}.$$

Clearly, such a number is a birational invariant, and having $\mathrm{d}_r(X) = 1$ is equivalent to rationality. Moreover, since any dominant rational map from a curve to $\mathbb{P}^1$ can be resolved to a morphism, the notion of degree of irrationality does provide an extension of gonality to n-dimensional varieties.

We would like to recall that any dominant rational map $C \dashrightarrow C'$ between curves leads to the inequality $\mathrm{gon}(C) \geq \mathrm{gon}(C')$. On the other hand, the existence of a dominant rational map $X \dashrightarrow Y$ between varieties of dimension $n \geq 2$ does not work analogously on the degrees of irrationality. Indeed, there are counterexamples in the case of surfaces (cf. [22, 6]) and there are examples of non-rational threefolds that are unirational (see for instance [5, 11]).

Turning to consider the second symmetric product $C^{(2)}$ of a smooth complex projective curve C of genus g, we deal with the problem of computing its degree of irrationality. Clearly, there is a strong connection between the existence of a dominant rational map $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ and the genus of the curve C. For instance, rational and elliptic curves are such that the degree of irrationality of their second symmetric product is one and two respectively, whereas we shall see that $\mathrm{d}_r(C^{(2)}) \geq 3$ for any curve of genus $g \geq 2$.

Furthermore, the degree of irrationality of the second symmetric product seems to depend on the existence of linear series on the curve as well. Indeed, by using g_d^r's on C it is possible to construct rational dominant maps $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ leading to the following upper bound.

Proposition 1.1. *Let C be a smooth complex projective curve. Let δ_1 be the gonality of C, and for $m = 2, 3$ let δ_m be the minimum of the integers d such that C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^m$. Then*

$$\mathrm{d}_r(C^{(2)}) \leq \min \left\{ \delta_1^2 \,,\, \frac{\delta_2(\delta_2 - 1)}{2} \,,\, \frac{(\delta_3 - 1)(\delta_3 - 2)}{2} - g \right\}.$$

In the case of hyperelliptic curves of high genera the latter bound turns out to be an equality. Namely,

Theorem 1.2. *Let C be a smooth complex projective curve of genus $g \geq 2$ and assume that C is hyperelliptic. Then:*

   (i) *$3 \leq \mathrm{d}_r(C^{(2)}) \leq 4$ when either $g = 2$ or $g = 3$;*
   (ii) *$\mathrm{d}_r(C^{(2)}) = 4$ for any $g \geq 4$.*

On the other hand, when the curve is assumed to be non-hyperelliptic, the situation is more subtle and it is no longer true that the degree of irrationality of $C^{(2)}$ equals the square of the gonality of C for high enough genus. The main result we prove on generic curves is the following.

Theorem 1.3. *Let C be a smooth complex projective curve of genus $g \geq 4$ and assume that C is very general in the moduli space $\mathcal{M}_g$. Then $\mathrm{d}_r(C^{(2)}) \geq g - 1$.*

Without any assumption of generality on C, the latter inequality does not hold, but it is still possible to provide some estimations on the degree of irrationality of second symmetric products of non-hyperelliptic curves. The following result summarizes the lower bounds we achieve, and we list them by genus.

Theorem 1.4. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then the following hold:*

   (i) *if $g = 3, 4$, then $\mathrm{d}_r(C^{(2)}) \geq 3$;*
   (ii) *if $g = 5$, then $\mathrm{d}_r(C^{(2)}) \geq 4$;*
   (iii) *if $g = 6$, then $\mathrm{d}_r(C^{(2)}) \geq 5$;*
   (iv) *if $g \geq 7$, then $\mathrm{d}_r(C^{(2)}) \geq \max\{6, \mathrm{gon}(C)\}$.*

Another attempt to extend the notion of gonality to n-dimensional varieties is the following. Given an irreducible complex projective variety X, we define the number

$$\mathrm{d}_o(X) := \min\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a family } \mathcal{E} = \{E_t\}_{t \in T} \\ \text{covering } X \text{ whose generic member is} \\ \text{an irreducible } d\text{-gonal curve} \end{array} \right\}$$

and we may call it the *degree of gonality* of X. Hence $\mathrm{d}_o(X)$ is the minimum gonality of curves passing through the generic point of X. Notice that the generic member E_t is a possibly singular d-gonal curve, i.e. its normalization $\widetilde{E}_t$ admits a degree d covering $f_t : \widetilde{E}_t \longrightarrow \mathbb{P}^1$. The degree of gonality is a birational invariant, and $\mathrm{d}_o(X) = 1$ if and only if X is an uniruled variety. Moreover, $\mathrm{d}_o(C) = \mathrm{gon}(C)$ for any complex projective curve C.

Although this second extension of the notion of gonality appears less intuitive and more artificial than the degree of irrationality, the degree of gonality has a nice behavior with respect to dominance. Namely, if there exists a dominant rational map $X \dashrightarrow Y$ between two irreducible complex projective varieties of dimension n, then $\mathrm{d}_o(X) \geq \mathrm{d}_o(Y)$ as in the one-dimensional case.

Dealing with the problem of computing the degree of gonality of the second symmetric product $C^{(2)}$ of a smooth complex projective curve C, it is easy to check that $\mathrm{d}_o(C^{(2)}) = 1$ when the curve is either rational or elliptic and that $\mathrm{d}_o(C^{(2)}) = 2$ for any curve of genus two. Moreover, we prove the following.

Theorem 1.5. *Let C be a smooth complex projective curve of genus $g \geq 3$. For a positive integer d, let $\mathcal{E} = \{E_t\}_{t \in T}$ be a family of curves on $C^{(2)}$ parametrized over a smooth variety T such that the generic fiber E_t is an irreducible d-gonal curve, and for any point $P \in C^{(2)}$ there exists $t \in T$ such that $P \in E_t$. Then $d \geq \mathrm{gon}(C)$. Moreover, under the further assumption that $g \geq 6$ and $\mathrm{Aut}(C) = \{Id_C\}$, we have that equality holds if and only if E_t is isomorphic to C.*

In particular, we have $\mathrm{d}_o(C^{(2)}) \geq \mathrm{gon}(C)$. Furthermore, for any smooth curve C, its second symmetric product is covered by a family of copies of C. Hence $\mathrm{d}_o(C^{(2)}) \leq \mathrm{gon}(C)$, and the problem of computing the degree of gonality of second symmetric products of curves is now totally understood.

Theorem 1.6. *Let C be a smooth complex projective curve of genus $g \geq 3$. Then $\mathrm{d}_o(C^{(2)}) = \mathrm{gon}(C)$.*

At the end of this paper, we present further an application of Theorem 1.5 improving the bounds of [3] on the nef cone of the second symmetric product of a generic curve C of genus g. We would like to recall that the problem of describing the nef cone $Nef(C^{(2)})_{\mathbb{R}}$ in the Néron-Severi space $N^1(C^{(2)})_{\mathbb{R}}$ is reduced to estimate the slope $\tau(C)$ of one of the rays bounding the two-dimensional convex cone $Nef(C^{(2)})_{\mathbb{R}}$.

In [3, Theorem 1] are provided some bounds on $\tau(C)$ when the genus of C is $5 \leq g \leq 8$. The proof of such a result follows the argument of [19, Section 4] - which is based both on the main theorem of the latter paper and on the techniques introduced in [7] - and involves the gonality of moving curves on the second symmetric products.

By following the very same argument and by applying Theorem 1.5 to this setting, we achieve new bounds on the nef cone of $C^{(2)}$ when C has genus $6 \leq g \leq 8$. Namely,

Theorem 1.7. *Consider the rational numbers*

$$\tau_6 = \frac{32}{13}, \quad \tau_7 = \frac{77}{29} \quad and \quad \tau_8 = \frac{17}{6}.$$

Let C be a smooth complex projective curve of genus $6 \leq g \leq 8$ and assume that C is very general in the moduli space $\mathcal{M}_g$. Then $\tau(C) \leq \tau_g$.

In order to prove most of our results, the main technique is to use holomorphic differentials, following Mumford's method of induced differentials (cf. [18, Section 2]). In the spirit of [14], we rephrase our settings in terms of correspondences on the product $Y \times C^{(2)}$, where Y is an appropriate ruled surface. A general 0-cycle of such a correspondence $\Gamma \subset Y \times C^{(2)}$ is a Cayley-Bacharach scheme with respect to the canonical linear series $|K_{C^{(2)}}|$. That is, any holomorphic 2-form vanishing on all but one of the points of the 0-cycle vanishes in the remaining point as well. The latter property imposes strong conditions on the correspondence Γ, and the crucial point is to study the restrictions descending to the second symmetric product and then to the curve C.

Another important technique involved in the proofs is monodromy. In particular, we consider the generically finite dominant map $\pi_1 \colon \Gamma \longrightarrow Y$ projecting a correspondence Γ on the first factor, and we study the action of the monodromy group of π_1 on the generic fiber. Finally, an important role is played by Abel's theorem and some basic facts of Brill-Noether theory.

The plan of the paper is the following. Section 2 concerns preliminaries on symmetric products of curves and monodromy, whereas in Section 4 we develop the main techniques to manage our problems. In particular, given a smooth curve C of genus g and its k-fold symmetric product, with $2 \leq k \leq g-1$, we investigate how the existence of a correspondence on $C^{(k)}$ influences the geometry of the curve itself (see Theorem 4.3 and Corollary 4.4).

Dealing with this issue, we come across linear subspaces of $\mathbb{P}^n$ satisfying a condition of Cayley-Bacharach type. We spend the entire Section 3 analyzing them. They turn out to enjoy interesting properties, both on the dimension of their linear span and on their configuration in the projective space (cf. Theorem 3.3).

Section 5 and Section 6 are devoted to studying the degree of gonality and the degree of irrationality, respectively.

Finally, in the last section we deal with the ample cone on second symmetric products of curves and we prove Theorem 1.7.

Notation. We shall work throughout over the field $\mathbb{C}$ of complex numbers. Given a variety X, we say that a property holds for a *general* point $x \in X$ if it holds on an open non-empty subset of X. Moreover, we say that $x \in X$ is a *very general* - or *generic* - point if there exists a countable collection of proper subvarieties of X such that x is not contained in the union of those subvarieties. By *curve* we mean

a complete reduced algebraic curve over the field of complex numbers. When we speak of a *smooth* curve, we always implicitly assume it to be irreducible.

2. Preliminaries

2.1. **Definition and first properties.** Let C be a smooth complex projective curve of genus $g \geq 0$. For an integer $k \geq 1$, let $C^k := C \times \ldots \times C$ denote its k-fold ordinary product and let S_k be the k-th symmetric group. We define the *k-fold symmetric product* of C as the quotient

$$C^{(k)} := \frac{C^k}{S_k}$$

under the action of S_k permuting the factors of C^k. Hence the quotient map $\pi \colon C^k \longrightarrow C^{(k)}$ sending $(p_1, \ldots, p_k) \in C^k$ to the point $p_1 + \ldots + p_k \in C^{(k)}$ has degree $k!$. The k-fold symmetric product is a smooth projective variety of dimension k (cf. [2, p. 18]), and it parametrizes the effective divisors on C of degree k or, equivalently, the unordered k-tuples of points of C.

2.2. **Linear series and subordinate loci.** Let d and r be some positive integers. As is customary, we denote by $W_d^r(C) \subset \mathrm{Pic}^d(C)$ the subvariety parametrizing the complete linear systems on C of degree d and dimension at least r. We recall that the dimension of $W_d^r(C)$ is bounded from below by the Brill-Noether number $\rho(g, r, d) := g - (r+1)(g - d + r)$, and if the curve C is very general in the moduli space $\mathcal{M}_g$, then $\dim W_d^r(C)$ equals $\rho(g, r, d)$.

Let $G_d^r(C)$ be the variety of linear series on C of degree d and dimension exactly r whose points are said g_d^r's. We note that the gonality of C is the minimum d such that C admits a g_d^1. Moreover, any complete g_d^r on C can be thought of as an element of $W_d^r(C)$.

Given a linear series $\mathcal{D} \in G_d^r(C)$, we define the locus of divisors on C subordinate to $\mathcal{D}$ as

$$(2.1) \qquad \Gamma_k(\mathcal{D}) := \left\{ P \in C^{(k)} \,|\, D - P \geq 0 \text{ for some } D \in \mathcal{D} \right\}.$$

We point out that the linear series $\mathcal{D}$ is not assumed to be base-point-free. Furthermore, the locus $\Gamma_k(\mathcal{D})$ is a subvariety of $C^{(k)}$, and if the dimension of $\mathcal{D}$ is $r = k - 1$, then $\Gamma_k(\mathcal{D})$ is a divisor.

2.3. **Canonical divisor on $C^{(k)}$.** Let $\phi \colon C \longrightarrow \mathbb{P}^{g-1}$ be the canonical map of the smooth curve C of genus g. For $1 \leq k \leq g - 1$, let us consider the k-fold symmetric product $C^{(k)}$ and the Grassmannian variety $\mathbb{G}(k - 1, g - 1)$ parametrizing $(k - 1)$-dimensional planes in $\mathbb{P}^{g-1}$. As $\phi(C)$ is a non-degenerate curve of $\mathbb{P}^{g-1}$, by the General Position Theorem the *Gauss map*

$$(2.2) \qquad \mathcal{G}_k \colon C^{(k)} \dashrightarrow \mathbb{G}(k - 1, g - 1),$$

sending a point $p_1 + \ldots + p_k \in C^{(k)}$ to the linear span of the $\phi(p_i)$'s in $\mathbb{P}^{g-1}$, is well defined.
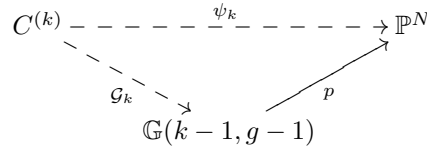
Let $|K_{C^{(k)}}|$ be the canonical linear system on $C^{(k)}$ and let

$$\psi_k \colon C^{(k)} \longrightarrow \mathbb{P}\left(H^{k,0}(C^{(k)})\right)$$

be the induced canonical map. Since

$$(2.3) \qquad H^{k,0}\left(C^{(k)}\right) \cong \bigwedge^{k} H^{1,0}(C)$$

(cf. [15]), we have the commutative diagram

$$
\begin{array}{ccc}
C^{(k)} & \xrightarrow{\;\;\psi_k\;\;} & \mathbb{P}^N \\
& \searrow^{\mathcal{G}_k} \quad \nearrow_{p} & \\
& \mathbb{G}(k-1, g-1) &
\end{array}
$$

where $N := \binom{g}{k} - 1$ and $p\colon \mathbb{G}(k-1, g-1) \longrightarrow \mathbb{P}^N$ is the Plücker embedding.

We recall that for any $L \in \mathbb{G}(g-k-1, g-1)$, the Schubert cycle $\sigma_1(L) := \{l \in \mathbb{G}(k-1, g-1) \mid l \cap L \neq \emptyset\}$ maps into a hyperplane section of $p\left(\mathbb{G}(k-1, g-1)\right) \subset \mathbb{P}^N$. Then it is possible to provide canonical divisors on the k-fold symmetric products of C as follows.

Lemma 2.1. *For any $L \in \mathbb{G}(g-k-1, g-1)$, let $\pi_L\colon \phi(C) \dashrightarrow \mathbb{P}^{k-1}$ be the projection from the $(g-k-1)$-plane L of the canonical image of C and let $\mathcal{D}_L$ be the associated linear series on C - not necessarily base-point-free - of degree $2g-2$ and dimension $k-1$. Then the effective divisor $\Gamma_k(\mathcal{D}_L)$ defined in (2.1) is a canonical divisor of $C^{(k)}$, that is, $\Gamma_k(\mathcal{D}_L) \in |K_{C^{(k)}}|$.*

In particular, a generic point $P = p_1 + \ldots + p_k \in C^{(k)}$ lies on the divisor $\Gamma_k(\mathcal{D}_L)$ if and only if the linear span of the $\phi(p_j)$'s in $\mathbb{P}^{g-1}$ is a point of the Schubert cycle $\sigma_1(L)$, that is, $\mathcal{G}_k(P)$ intersects L.

2.4. **Monodromy.** To conclude this section, we follow [10] to recall some basic facts on the monodromy of a generically finite dominant morphism $F\colon X \longrightarrow Y$ of degree d between irreducible complex algebraic varieties of the same dimension.

Let $U \subset Y$ be a suitable Zariski open subset of X such that the restriction $F^{-1}(U) \longrightarrow U$ is an unbranched covering of degree d. Given a generic point $y \in U$, by lifting loops at y to $F^{-1}(U)$, we may define the monodromy representation $\rho\colon \pi_1(U, y) \longrightarrow Aut\left(F^{-1}(y)\right) \cong S_d$ and we define the *monodromy group* $M(F)$ of F to be the image of the latter homomorphism.

Equivalently, let L be the normalization of the algebraic field extension $K(X)/K(Y)$ of degree d, and let $Gal(L/K(Y))$ be the *Galois group* of $L/K(Y)$, that is, the group of the automorphisms of the field L fixing every element of $K(Y)$. Then the monodromy group $M(F)$ and the Galois group $Gal(L/K(Y))$ are isomorphic (see [10, p. 689]). In particular, this implies that the monodromy group of F is independent of the choice of the Zariski open set U. Moreover, F should not necessarily be a morphism, but it suffices being a dominant rational map.

The simple fact we want to point out is that the action of $M(F)$ on the fiber $F^{-1}(y)$ is transitive because of the connectedness of X. Roughly speaking, this means that the points of the fiber over a generic point are indistinguishable. Namely, suppose that a point $x_i \in F^{-1}(y)$ enjoys some special property such that as we vary continuously the point y on a suitable open subset $U \subset Y$, that special property is preserved as we follow the correspondent point of the fiber. Then for any loop $\gamma \in \pi_1(U, y)$, we have that the ending point $x_j \in F^{-1}(y)$ of the unique lifting $\widetilde{\gamma}$ of γ starting from x_i must enjoy the same property. Hence the transitivity of the

action assures that there is no way to distinguish a point of the fiber over $y \in Y$ from another for enjoying a property as above.

3. Linear subspaces of $\mathbb{P}^n$ in special position

In this section we deal with sets of linear subspaces of the n-dimensional projective space satisfying a condition of Cayley-Bacharach type. In particular, we shall provide a bound on the dimension of their linear span in $\mathbb{P}^n$, and we shall present some examples proving the sharpness of the bound. The reasons for studying these particular linear spaces shall be clear in the next section, when we shall relate them to correspondences on symmetric products of curves.

To start we recall the following definition (cf. [9]).

Definition 3.1. Let $\mathcal{D}$ be a complete linear system on a projective variety X. We say that a 0-cycle $P_1 + \ldots + P_d \subset X^{(d)}$ *satisfies the Cayley-Bacharach condition with respect to* $\mathcal{D}$ if for every $i = 1, \ldots, d$ and for any effective divisor $D \in \mathcal{D}$ passing through $P_1, \ldots, \widehat{P_i}, \ldots, P_d$, we have $P_i \in D$ as well.

Let n and k be two integers with $n \geq k \geq 2$, and let $\mathbb{G}(k-1, n)$ be the Grassmann variety of $(k-1)$-planes in $\mathbb{P}^n$. For an integer $d \geq 2$, let us consider a set $\{l_1, \ldots, l_d\} \subset \mathbb{G}(k-1, n)$ and suppose that the associated 0-cycle $l_1 + \ldots + l_d$ satisfies the Cayley-Bacharach condition with respect to the complete linear series $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$.

We recall that for any $L \in \mathbb{G}(n-k, n)$, the Schubert cycle

$$\sigma_1(L) := \{l \in \mathbb{G}(k-1, n) | l \cap L \neq \emptyset\}$$

is an effective divisor of $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$. Thus the set $\{l_1, \ldots, l_d\}$ is such that for every $i = 1, \ldots, d$ and for any $L \in \mathbb{G}(n-k, n)$ with $l_1, \ldots, \widehat{l_i}, \ldots, l_d \in \sigma_1(L)$, we have $l_i \in \sigma_1(L)$ as well. Then it makes sense to give the following definition expressing a condition of Cayley-Bacharach type for $(k-1)$-dimensional linear subspaces of $\mathbb{P}^n$.

Definition 3.2. We say that the $(k-1)$-planes $l_1, \ldots, l_d \subset \mathbb{P}^n$ are in *special position with respect to* $(n-k)$-*planes* if for every $i = 1, \ldots, d$ and for any $(n-k)$-plane $L \subset \mathbb{P}^n$ intersecting $l_1, \ldots, \widehat{l_i}, \ldots, l_d$, we have $l_i \cap L \neq \emptyset$.

We note that the $(k-1)$-planes in the definition are not assumed to be distinct. In particular, it is immediate to check that two $(k-1)$-planes $l_1, l_2 \subset \mathbb{P}^n$ are in special position if and only if they coincide.

The main result of this section is the following.

Theorem 3.3. *Let* $2 \leq k, d \leq n$ *be some integers and suppose that the* $(k-1)$-*planes* $l_1, \ldots, l_d \subset \mathbb{P}^n$ *are in special position with respect to* $(n-k)$-*planes of* $\mathbb{P}^n$. *Then the dimension of their linear span* $S := Span(l_1, \ldots, l_d)$ *in* $\mathbb{P}^n$ *is* $s \leq \left[\frac{kd}{2}\right] - 1$.

In order to prove this result, let us state the following preliminary lemma.

Lemma 3.4. *Under the assumption of Theorem* 3.3, *suppose further that there exists a linear space* $R \subset \mathbb{P}^n$ *containing* $l_1, \ldots, \widehat{l_j}, \ldots, l_d$. *Then* $l_j \subset R$ *as well.*

Proof. Let r denote the dimension of R. If $r = n$ the statement is trivially true, and then let us assume $r < n$. As $k - 1 \leq r$ we have that $0 \leq r - k + 1 \leq n - k$, and we can consider an $(r - k + 1)$-plane $T \subset R$. Then T intersects each of the

$(k-1)$-planes $l_1, \ldots, \widehat{l_j}, \ldots, l_d$. Therefore, by special position property, any $(n-k)$-plane L containing T must intersect l_j, thus $l_j \cap T \neq \emptyset$. Therefore l_j meets every $(r-k+1)$-plane $T \subset R$, and hence $l_j \subset R$. □

Proof of Theorem 3.3. Let us fix $2 \leq k, d \leq n$. Notice that if $n \leq \left[\frac{kd}{2}\right] - 1$, the statement is trivially proved. Hence we assume hereafter that $n \geq \left[\frac{kd}{2}\right]$. We proceed by induction on the number d of $(k-1)$-planes.

Let $l_1, l_2 \subset \mathbb{P}^n$ be two $(k-1)$-dimensional planes in special position with respect to $(n-k)$-planes. Then we set $R := l_1$ and Lemma 3.4 implies $l_2 \subset R$. Hence $R = l_1 = l_2$ and $\left[\frac{kd}{2}\right] - 1 = k - 1 = \dim R$. Thus the statement is proved when $d = 2$.

By induction, suppose that the assertion holds for any $2 \leq h \leq d - 1$ and for any h-tuple of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^m$ in special position with respect to $(m-k)$-planes, with $m \geq h$.

Now, let $l_1, \ldots, l_d \subset \mathbb{P}^n$ be $(k-1)$-planes in special position with respect to $(n-k)$-planes.

We first consider the case where it is not possible to choose one of the l_i's such that it does not coincide with any of the others. In this situation, the number of distinct l_i's is at most $\left[\frac{d}{2}\right]$. Thus the dimension of their linear span in $\mathbb{P}^n$ is at most $k\left[\frac{d}{2}\right] - 1 \leq \left[\frac{kd}{2}\right] - 1$, as claimed.

Then we consider the $(k-1)$-plane l_1 and we suppose - without loss of generality - that it does not coincide with any of the others l_i's. Therefore it is possible to choose a point $p \in l_1$ such that $p \notin l_i$ for any $i = 2, \ldots, d$. Moreover, let $H \subset \mathbb{P}^n$ be a hyperplane not containing p and consider the projection

$$\pi_p : \quad \mathbb{P}^n - \{p\} \quad \longrightarrow \quad H \cong \mathbb{P}^{n-1},$$
$$q \quad \longmapsto \quad \overline{pq} \cap H.$$

For $2 \leq i \leq d$, let $\lambda_i := \pi_p(l_i) \subset H$ be the image of l_i on H. We claim that the $(k-1)$-planes $\lambda_2, \ldots, \lambda_d \subset H$ are in special position with respect to $(n-1-k)$-planes of $H \cong \mathbb{P}^{n-1}$. To see this fact, let $j \in \{2, \ldots, d\}$ and let $\Lambda \subset H$ be an $(n-1-k)$ plane intersecting $\lambda_2, \ldots, \widehat{\lambda_j}, \ldots, \lambda_d$. Since $p \in l_1$, it follows that the $(n-k)$-plane $L := Span(\Lambda, p) \subset \mathbb{P}^n$ intersects $l_1, \ldots, \widehat{l_j}, \ldots, l_d$. As they are in special position with respect to $(n-k)$-planes, we have that L intersects l_j as well. Then, given a point $q_j \in L \cap l_j$, we have that $\pi_p(q_j) \in \Lambda$. In particular, Λ meets λ_j at $\pi_p(q_j)$, and hence $\lambda_2, \ldots, \lambda_d \subset H$ are in special position with respect to $(n-1-k)$-planes of the hyperplane $H \cong \mathbb{P}^{n-1}$.

By induction, the linear span $\Sigma := Span(\lambda_2, \ldots, \lambda_d) \subset H$ has dimension $\dim \Sigma \leq \left[\frac{k(d-1)}{2}\right] - 1$. Then the linear space $R := Span(\lambda_2, \ldots, \lambda_d, p) \subset \mathbb{P}^n$ has dimension $\dim R = \dim \Sigma + 1 \leq \left[\frac{k(d-1)}{2}\right] \leq \left[\frac{kd}{2}\right] - 1$ for any $k \geq 2$. Notice that R contains $l_2, \ldots, l_d$. Hence $l_1 \subset R$ as well by Lemma 3.4. Thus R contains the linear span in $\mathbb{P}^n$ of all the l_i's and the assertion follows. □

We would like to note that the assumption $k \geq 2$ in Theorem 3.3 is necessary. For instance, let $k = 1$ and consider three collinear points in $\mathbb{P}^n$. Clearly, they are in special position with respect to $(n-1)$-planes and $\left[\frac{kd}{2}\right] - 1 = 0$, but they span a line.

On the other hand, when $k = 2$ the theorem assures that if $\{l_1, \ldots, l_d\}$ is a set of d lines in special position with respect to $(n-2)$-planes of $\mathbb{P}^n$, then their linear span

has dimension lower than d. The following examples concern the configuration of such lines in $\mathbb{P}^{d-1}$ and show that the latter bound is sharp.

Example 3.5. Let us consider three distinct lines l_1, l_2, l_3 in $\mathbb{P}^n$. Then they are in special position with respect to $(n-2)$-planes if and only if they lie on a plane $\pi \subset \mathbb{P}^n$ and they meet at a point $p \in \pi$.

Indeed, suppose that l_1, l_2, l_3 are in special position with respect to $(n-2)$-planes. Therefore they must lie on a plane $\pi \subset \mathbb{P}^n$ by Theorem 3.3. Then consider the point $p = l_2 \cap l_3$ and let L be an $(n-2)$-plane such that $L \cap \pi = \{p\}$. Thus L must also intersect l_1 by special position property, and hence $p \in l_1$.

On the other hand, it is immediate to check that if three distinct lines of $\mathbb{P}^n$ lie on the same plane and meet at a point, then any $(n-2)$-plane intersecting two of them intersects the last one as well.

Example 3.6. Let $l_1, \ldots, l_4 \subset \mathbb{P}^n$ be four skew lines. Then they are in special position with respect to $(n-2)$-planes if and only if they lie on the same ruling of a quadric surface $Q \subset \mathbb{P}^3$. In particular, the l_i's span the whole $\mathbb{P}^3$.

If the l_i's enjoy special position property, they span a linear subspace $S \cong \mathbb{P}^3$ by Theorem 3.3. Let $Q \subset S$ be the quadric defined as the union of the lines intersecting l_1, l_2 and l_3. By special position property, any line $L \subset S$ intersecting l_1, l_2, l_3 must meet l_4, too. Hence $l_4 \subset Q$ as well, and it lies on the same ruling of the other l_i's.

To see the converse, it suffices to observe that any quadric surface $Q \subset \mathbb{P}^3$ is covered by two families of skew lines, $\mathcal{L}$ and $\mathcal{L}'$, such that any two lines $l \in \mathcal{L}$ and $l' \in \mathcal{L}'$ meet at a point (see e.g. [8, p. 478]).

Example 3.7. In general, if $l_1, \ldots, l_d \subset \mathbb{P}^{d-1}$ are skew lines lying on a non-degenerate surface $Q \subset \mathbb{P}^{d-1}$ of minimal degree, then they are in special position with respect to $(d-3)$-planes.

Under these assumptions, Q is a ruled surface of degree $d-2$ (cf. [8, p. 522]). If $L \subset \mathbb{P}^{d-1}$ is a $(d-3)$-plane intersecting $l_1, \ldots, l_{d-1}$, then $L \cap \Sigma$ is a curve C of degree $\deg C \le d-2$. In particular, C does not lie on the ruling of Q, and hence it must intersect l_d, too.

4. Correspondences with null trace on symmetric products of curves

In order to deal with correspondences with null trace on symmetric products of curves, we would like to recall the basic properties of Mumford's induced differentials (see [18, Section 2]) and their applications to the study of correspondences (cf. [14, Section 2]). Then we shall turn to symmetric products of curves and we shall prove the main result of this section, which gives a geometric interpretation of the existence correspondences with null trace on these varieties.

Let X and Y be two projective varieties of dimension n, with X smooth and Y integral.

Definition 4.1. A *correspondence of degree d on $Y \times X$* is a reduced n-dimensional variety $\Gamma \subset Y \times X$ such that the projections $\pi_1 \colon \Gamma \longrightarrow Y$, $\pi_2 \colon \Gamma \longrightarrow X$ are generically finite dominant morphisms and $\deg \pi_1 = d$. Moreover, if $\deg \pi_2 = d'$ we say that Γ is a *(d, d')-correspondence*.

So, let $\Gamma \subset Y \times X$ be a correspondence of degree d. Let $X^d = X \times \ldots \times X$ be the d-fold ordinary product of X and let $p_i \colon X^d \longrightarrow X$ be the i-th projection map,

with $i = 1, \ldots, d$. Let us consider the d-fold symmetric product $X^{(d)} = X^d/S_d$ of X together with the quotient map $\pi \colon X^d \longrightarrow X^{(d)}$. Then we define the set $U := \{y \in Y_{reg} \mid \dim \pi_1^{-1}(y) = 0\}$ and the morphism $\gamma \colon U \longrightarrow X^{(d)}$ given by $\gamma(y) := P_1 + \ldots + P_d$, where $\pi_1^{-1}(y) = \{(y, P_i) \mid i = 1, \ldots, d\}$.

By using Mumford's induced differentials, we want to define the trace map of γ. To this aim, we consider a holomorphic n-form $\omega \in H^{n,0}(X)$ and the $(n,0)$-form

$$\omega^{(d)} := \sum_{i=1}^d p_i^* \omega \in H^{n,0}(X^d),$$

which is invariant under the action of S_d. Thus for any smooth variety W and for any morphism $f \colon W \longrightarrow X^{(d)}$, there exists a canonically induced $(n,0)$-form ω_f on W (cf. [18, Section 2]). In particular, we define Mumford's *trace map* of γ as

$$\begin{array}{rccc} Tr_\gamma \colon & H^{n,0}(X) & \longrightarrow & H^{n,0}(U), \\ & \omega & \longmapsto & \omega_\gamma. \end{array}$$

Another way to define the trace map of γ is the following. Let us consider the sets $V := \{y \in U \mid \pi_1^{-1}(y) \text{ has } d \text{ distinct points}\}$ and

$$X_0^{(d)} := \pi \left(X^d - \bigcup_{i,j} \Delta_{i,j} \right),$$

where $\Delta_{i,j}$ is the (i,j)-diagonal of X^d, with $i, j = 1, \ldots, d$ and $i \neq j$. Moreover, let us define the map

$$\begin{array}{rccc} \delta_d \colon & H^{n,0}(X) & \longrightarrow & H^{n,0}(X_0^{(d)}), \\ & \omega & \longmapsto & \pi_*(\omega^{(d)}), \end{array}$$

i.e. $\omega^{(d)}$ is thought of as an $(n,0)$-form on $X_0^{(d)}$. Then $Im\, \gamma_{|V} \subset X_0^{(d)}$ and the Mumford's trace map of γ turns out to be $Tr_\gamma = \gamma_{|V}^* \circ \delta_d$ (cf. [14, Proposition 2.1]).

The following result shows that the property of having null trace imposes strong conditions on the correspondence $\Gamma \subset Y \times X$. We would like to note that in [14, Proposition 2.2] it is presented in the case of correspondences on surfaces, but it is still true when X and Y are n-dimensional varieties, and the proof follows the very same argument.

Proposition 4.2. *Let X and Y be two projective varieties of dimension n, with X smooth and Y integral. Let Γ be a correspondence of degree d on $Y \times X$ with null trace. Let $y \in Y_{reg}$ such that $\dim \pi_1^{-1}(y) = 0$ and let*

$$\pi_1^{-1}(y) = \{(y, P_i) \in \Gamma \mid i = 1, \ldots, d\}$$

be its fiber. Then the 0-cycle $P_1 + \ldots + P_d$ satisfies the Cayley-Bacharach condition with respect to the canonical linear series $|K_X|$. That is, for every $i = 1, \ldots, d$ and for any effective canonical divisor K_X containing $P_1, \ldots, \widehat{P_i}, \ldots, P_d$, we have $P_i \in K_X$.

Now we turn to symmetric products of curves and state the main result of this section. Let us consider a smooth projective curve C of genus g and let $C^{(k)}$ be its k-fold symmetric product, with $2 \leq k \leq g-1$. Let us denote by $\phi \colon C \longrightarrow \mathbb{P}^{g-1}$ the

map induced by the canonical linear series $|K_C|$ on C. The following result connects the existence of a correspondence with null trace on $C^{(k)}$ and the geometry of the canonical image of C.

Theorem 4.3. *Let C be a smooth projective curve of genus g and let Y be a projective integral variety of dimension $2 \leq k \leq g - 1$. Let Γ be a correspondence of degree $d \geq 2$ on $Y \times C^{(k)}$ with null trace. For a generic point $y \in Y_{reg}$, let $\pi_1^{-1}(y) = \{(y, P_i) \in \Gamma \,|\, i = 1, \ldots, d\}$ be its fiber, where $P_i = p_{i1} + \ldots + p_{ik} \in C^{(k)}$ for $i = 1, \ldots, d$. Then the linear span of all the $\phi(p_{ij})$'s in $\mathbb{P}^{g-1}$ has dimension*

$$s \leq \left[\frac{k\,d}{2}\right] - 1.$$

Proof. By Proposition 4.2 the 0-cycle $P_1 + \ldots + P_d$ satisfies the Cayley-Bacharach condition with respect to the canonical linear series $|K_{C^{(k)}}|$. For any $(g - k - 1)$-plane $L \subset \mathbb{P}^{g-1}$, Lemma 2.1 assures that the subordinate locus $\Gamma_k(\mathcal{D}_L)$ is an effective canonical divisor on $C^{(k)}$. Therefore for every $i = 1, \ldots, d$ and for any $L \in \mathbb{G}(g - k - 1, g - 1)$ with $P_1, \ldots, \widehat{P_i}, \ldots, P_d \in \Gamma_k(\mathcal{D}_L)$, we have $P_i \in \Gamma_k(\mathcal{D}_L)$ as well.

Now, let $\mathcal{G}_k \colon C^{(k)} \dashrightarrow \mathbb{G}(k - 1, g - 1)$ be the Gauss map sending a point $P = p_1 + \ldots + p_k$ to the linear span of the $\phi(p_j)$'s in $\mathbb{P}^{g-1}$. We recall that for any

$$L \in \mathbb{G}(g - k - 1, g - 1)$$

and for any $P \in C^{(k)}$, we have that $P \in \Gamma_k(\mathcal{D}_L)$ if and only if $\mathcal{G}_k(P)$ intersects L. Thus the 0-cycle $P_1 + \ldots + P_d$ is such that for every $i = 1, \ldots, d$ and for any $L \in \mathbb{G}(g - k - 1, g - 1)$ intersecting the $(k-1)$-planes $\mathcal{G}_k(P_1), \ldots, \widehat{\mathcal{G}_k(P_i)}, \ldots, \mathcal{G}_k(P_d)$, we have $\mathcal{G}_k(P_i) \cap L \neq \emptyset$.

In particular, the $(k - 1)$-planes $\mathcal{G}_k(P_1), \ldots, \mathcal{G}_k(P_d)$ are in special position with respect to $(g - k - 1)$-planes (cf. Definition 3.2). Therefore Theorem 3.3 assures that their linear span in $\mathbb{P}^{g-1}$ has dimension lower than $\left[\frac{kd}{2}\right]$. Since any $\mathcal{G}_k(P_i)$ is generated by $\phi(p_{i1}), \ldots, \phi(p_{ik})$, we conclude that the linear span of all the $\phi(p_{ij})$'s in $\mathbb{P}^{g-1}$ has dimension bounded by $\left[\frac{kd}{2}\right] - 1$, as claimed. $\qquad\square$

As we anticipated, Theorem 4.3 shall turn out to be very useful to deal with the degree of irrationality and the degree of gonality on second symmetric products of curves. In particular, when $k = 2$ the latter result assures that the $\phi(p_{ij})$'s span a linear subspace of $\mathbb{P}^{g-1}$ of dimension $s \leq d - 1$.

The following assertion is an immediate consequence of Theorem 4.3 connecting the existence of correspondences with null trace on $C^{(k)}$ and the existence of a complete linear series on C.

Corollary 4.4. *Under the assumption of Theorem 4.3, suppose in addition that C is non-hyperelliptic and that the number of distinct p_{ij}'s is $m > [\frac{kd}{2}]$. Then C possesses a complete g_m^r with $r \geq 1$.*

Proof. Let m be the number of distinct p_{ij}'s on C and let us denote by $q_1, \ldots, q_m$ these points. Consider the divisor $D = q_1 + \ldots + q_m$ of degree m on C. As the curve C is non-hyperelliptic, the canonical map ϕ is an embedding and the $\phi(q_t)$'s are all distinct. Hence their linear span in $\mathbb{P}^{g-1}$ has dimension lower than $[\frac{kd}{2}]$.

Therefore by the geometric version of Riemann-Roch theorem we have

$$\dim |D| = m - 1 - \dim \overline{\phi(D)} \geq m - \left[\frac{kd}{2}\right] \geq 1.$$

Thus $|D| = |q_1 + \ldots + q_m|$ is a complete g_m^r on C with $r \geq 1$. $\qquad\square$

Remark 4.5. In [9], Griffiths and Harris studied 0-cycles on an algebraic variety X satisfying the Cayley-Bacharach condition with respect to a complete linear system $|D|$. In particular, given such a 0-cycle $P_1 + \ldots + P_d$ and the rational map $\phi_{|D|} \colon X \dashrightarrow \mathbb{P}^r$, they present some results on the dimension of the linear span of the $\phi_{|D|}(P_i)$'s in $\mathbb{P}^r$ and, consequently, on the existence of linear series on X. We note that we start from an analogous situation with $X = C^{(k)}$, but the results of this section deal with the study of the geometry of the curve C and not with X.

To conclude this section, we would like to present two important examples of correspondences with null trace on the k-fold symmetric product, which shall be involved in the proofs of the following sections.

Example 4.6. For any dominant rational map $F \colon C^{(k)} \dashrightarrow \mathbb{P}^k$ of degree d, its graph

$$\Gamma := \overline{\left\{(y, P) \in \mathbb{P}^k \times C^{(k)} \mid F(P) = y\right\}}$$

is a $(d, 1)$-correspondence on $\mathbb{P}^k \times C^{(k)}$ with null trace. To see this fact, notice that the fiber $F^{-1}(y)$ over a generic point $y \in \mathbb{P}^k$ is given by d distinct points $P_1, \ldots, P_d \in C^{(k)}$. Hence Γ is a reduced variety and the projection $\pi_1 \colon \Gamma \longrightarrow \mathbb{P}^k$ is a generically finite dominant morphism of degree d. Moreover, we have that the map $\gamma \colon U \subset \mathbb{P}^k \longrightarrow \left(C^{(k)}\right)^{(d)}$ introduced at the beginning of this section is a rational map between smooth projective varieties. Therefore the indeterminacy locus can be resolved to a codimension 2 subvariety of $\mathbb{P}^k$. Being $H^{k,0}(\mathbb{P}^k)$ trivial, we then have that the trace map Tr_γ is null. Thus Γ is a $(d, 1)$-correspondence on $\mathbb{P}^k \times C^{(k)}$ with null trace. We note that this fact is still true for any smooth n-dimensional projective variety admitting a dominant rational map of degree d on $\mathbb{P}^n$.

Example 4.7. Let T be a $(k-1)$-dimensional smooth variety and let $\mathcal{E} = \{E_t\}_{t \in T}$ be a family of curves covering $C^{(k)}$ such that the generic member E_t is an irreducible d-gonal curve, i.e. the normalization of E_t has a base-point-free g_d^1. As in [14, proof of Corollary 1.7], we want to define a correspondence $\Gamma \subset Y \times C^{(k)}$ with null trace and degree d for an appropriate ruled variety Y.

To this aim, we think of the variety $C^{(k)}$ as it is embedded in some projective space and, for any $t \in T$, we denote by H_t the Hilbert scheme of curves on $C^{(k)}$ containing the point e_t representing E_t. We note that if t is generic, then $\dim H_t \geq k - 1$. Indeed, if $\dim H_t < k - 1$ for any t, then $\bigcup_{t \in T} H_t = \bigcup_{\deg \widetilde{E}_t, g(\widetilde{E}_t)} Hilb_{\deg \widetilde{E}_t, g(\widetilde{E}_t)}$ would be the union of countably many schemes of dimension $\leq k - 2$, and the curves parametrized over it would cover the k-dimensional variety $C^{(k)}$, a contradiction. Given a generic point $t \in T$, let $H \subset H_t$ be a $(k-1)$-dimensional subvariety containing e_t. We recall that the Hilbert scheme is a fine moduli space, so we can consider the universal family $\mathcal{U} := \{(e_\tau, P) \mid e_\tau \in H \text{ and } P \in E_\tau\}$ over H and - up to shrink H - the family $\widetilde{\mathcal{U}}$ of the normalized curves. Furthermore, by making a

base change

$$
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \widetilde{\mathcal{U}} \\
\rho \downarrow & & \downarrow pr_1 \\
B & \longrightarrow & H
\end{array}
$$

we can assume to have a map $\mu\colon \mathcal{F} \longrightarrow \mathbb{P}^1 \times B$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\ \mu\ } & \mathbb{P}^1 \times B \\
& \rho \searrow & \downarrow pr_2 \\
& & B
\end{array}
$$

is commutative, and for any $b \in B$ the restriction $\mu_b\colon \mathcal{F}_b \longrightarrow \mathbb{P}^1$ is the given g_d^1 on the curve $\mathcal{F}_b := \rho^{-1}(b)$. We note that any fiber of $\mathcal{F}$ is the normalization $\nu_b\colon \mathcal{F}_b \longrightarrow \mathcal{F}'_b$ of a curve lying on $C^{(k)}$. We then define the k-dimensional varieties $Y := \mathbb{P}^1 \times B$ and

$$
\Gamma := \overline{\left\{ ((z,b),P) \in Y \times C^{(k)} \mid P \in (\mathcal{F}'_b)_{reg} \text{ and } \mu_b \circ \nu_b^{-1}(P) = z \right\}}.
$$

We claim that $\Gamma \subset Y \times C^{(k)}$ is a correspondence of degree d with null trace. Since the $\mathcal{F}_b$'s are the normalizations of curves covering an open subset of $C^{(k)}$ and the map μ_b is a g_d^1, we have that both the projections $\pi_1\colon \Gamma \longrightarrow Y$ and $\pi_2\colon \Gamma \longrightarrow C^{(k)}$ are dominant morphisms, with $\deg \pi_1 = d$. Furthermore, the map π_2 is generically finite as well: if there exist infinitely many curves $\mathcal{F}'_b$ passing through the generic point $P \in C^{(k)}$, then B would be at least a k-dimensional variety. Finally, the space $H^{k,0}(Y)$ is trivial because Y is a ruled projective variety, and, by arguing as in the previous example, we deduce that the correspondence Γ has null trace.

5. Degree of gonality

Let C be a smooth complex projective curve of genus $g \geq 0$. In this section we deal with the gonality of moving curves on the second symmetric product $C^{(2)}$ and we compute the degree of gonality of this surface, which is the positive integer we defined to be

$$
\mathrm{d}_o(C^{(2)}) := \min \left\{ d \in \mathbb{N} \ \middle| \ \begin{array}{l} \text{there exists a family } \mathcal{E} = \{E_t\}_{t \in T} \\ \text{covering } C^{(2)} \text{ whose generic member is} \\ \text{an irreducible } d\text{-gonal curve} \end{array} \right\}.
$$

The degree of gonality of $C^{(2)}$ in cases of low genera is easily given. When C is a rational curve, then $C^{(2)} \cong \mathbb{P}^2$ and hence $\mathrm{d}_o(C^{(2)}) = 1$. On the other hand, if C is supposed to be an elliptic curve, then the fibers of the Abel map $C^{(2)} \longrightarrow J(C) \cong C$ are isomorphic to $\mathbb{P}^1$ and the second symmetric product of C is birational to $C \times \mathbb{P}^1$. Thus $\mathrm{d}_o(C^{(2)}) = \mathrm{d}_o(C \times \mathbb{P}^1) = 1$.

We note that for any $g \geq 0$, the surface $C^{(2)}$ is covered by the family $\mathcal{C} = \{C_p\}_{p \in C}$ of curves parametrized over C, where $C_p := C + p = \{p + q \mid q \in C\}$. Clearly, any C_p is isomorphic to C itself. Therefore the degree of gonality of the second symmetric product of a curve of genus $g \geq 0$ is such that

$$
(5.1) \qquad\qquad \mathrm{d}_o(C^{(2)}) \leq \mathrm{gon}(C).
$$

In particular, since the only rational curve lying on the second symmetric product of a hyperelliptic curve of genus $g = 2$ is the fiber of the g_2^1 via the Abel map $u\colon C^{(2)} \to J(C)$, we have that $\mathrm{d}_o(C^{(2)}) = 2$.

When the curve C has genus $g \geq 3$, Theorem 1.5 assures that the gonality of moving curves on $C^{(2)}$ must be greater than or equal to $\mathrm{gon}(C)$. Hence Theorem 1.6 follows straightforwardly from the latter result and inequality (5.1). We have then resolved the problem of computing the degree of irrationality on second symmetric products of curves, that is, $\mathrm{d}_o(C^{(2)}) = \mathrm{gon}(C)$ for any C of genus $g \geq 3$.

Remark 5.1. Let C be a smooth projective curve of genus $g \geq 0$ and let $C^{(k)}$ be its k-fold symmetric product, with $k \geq 3$.

When $k > g$, we have that $C^{(k)}$ is birational to $J(C) \times \mathbb{P}^{k-g}$ by Abel's theorem. Thus $C^{(k)}$ is covered by a family of rational curves, and hence the degree of gonality is $\mathrm{d}_o(C^{(k)}) = 1$.

On the other hand, when $k \leq g$, the k-fold symmetric product of C is covered by the family $\mathcal{C} = \{C_P\}_{P \in C^{(k-1)}}$ of curves parametrized over $C^{(k-1)}$, where $C_P := \{P + q \,|\, q \in C\}$. Since any such curve is isomorphic to C, we still have the inequality $\mathrm{d}_o(C^{(k)}) \leq \mathrm{gon}(C)$. In particular, it seems natural to conjecture that this bound is actually an equality, but the techniques we used above do not work to prove this fact.

Let us now prove Theorem 1.5. The argument of the proof is essentially based on Theorem 4.3 and Abel's theorem.

Proof of Theorem 1.5. Notice that if C is a hyperelliptic curve of genus $g \geq 3$, the only rational curve lying on $C^{(2)}$ is the fiber of the g_2^1 via the Abel map $u\colon C^{(2)} \to J(C)$. Therefore the gonality of the generic curve E_t must be $d \geq 2 = \mathrm{gon}(C)$, and the assertion follows. Then we assume hereafter that C is non-hyperelliptic. Moreover, aiming for a contradiction we suppose that $d < \mathrm{gon}(C)$ and we proceed by steps.

Step 1 (Correspondence on $C^{(2)}$). As $C^{(2)}$ is two-dimensional and $\mathcal{E}$ is a family of curves - up to restricting $\mathcal{E}$ to a subvariety of T - we can assume that T has dimension one. Following Example 4.7 we can then construct a family $\mathcal{F} \overset{\rho}{\longrightarrow} B$ of smooth d-gonal curves such that the diagram

$$\mathcal{F} \overset{\mu}{\longrightarrow} \mathbb{P}^1 \times B$$
$$\rho \searrow \quad \downarrow pr_2$$
$$B$$

is commutative, and for any $b \in B$ the restriction $\mu_b\colon \mathcal{F}_b \longrightarrow \mathbb{P}^1$ is the given g_d^1 on the curve $\mathcal{F}_b := \rho^{-1}(b)$. In particular, any such curve is the normalization $\nu_b\colon \mathcal{F}_b \longrightarrow \mathcal{F}_b'$ of a curve lying on $C^{(2)}$, which generically is one of the E_t's. Then we set $Y := \mathbb{P}^1 \times B$ and we have that the surface

$$\Gamma := \overline{\left\{ \big((z,b),P\big) \in Y \times C^{(k)} \mid P \in (\mathcal{F}_b')_{reg} \text{ and } \mu_b \circ \nu_b^{-1}(P) = z \right\}}$$

is a correspondence of degree d on $Y \times C^{(k)}$ with null trace.

As usual, let $\pi_1\colon \Gamma \longrightarrow Y$ be the restriction of the first projection map, and for a very general point $(z,b) \in Y$, let $\pi_1^{-1}(z,b) = \left\{ \big((z,b),P_i\big) \in Y \times C^{(2)} \,|\, i = 1, \ldots, d \right\}$

be its fiber, with $P_i = p_{2i-1} + p_{2i}$. Moreover, let $D = D_{(z,b)} \in Div(C)$ be the effective divisor given by

$$(5.2) \qquad D := p_1 + \ldots + p_{2d} = \sum_{j=1}^{m} n_j q_j$$

for some positive integers $n_j = \mathrm{mult}_{q_j}(D)$, where the q_j's are assumed to be distinct points of C.

Step 2 ($n_j = 1$ for all j). We suppose that $n_j = 1$ for any $1 \le j \le m$; that is, $m = 2d$, and the points defining D are all distinct. Let $\phi \colon C \longrightarrow \mathbb{P}^{g-1}$ be the canonical embedding of C and let $\overline{\phi(D)}$ be the linear span of the points $\phi(p_i)$ in $\mathbb{P}^{g-1}$. As Γ is a correspondence of degree d on $C^{(2)}$ with null trace, by Theorem 4.3 we have that $\dim \overline{\phi(D)} \le d - 1$.

If C is a non-hyperelliptic curve of genus $g = 3$, we have that $C^{(2)}$ is non-covered by a family of rational curves and $\mathrm{gon}(C) = 3$. Hence $d = 2$ and the curves of the family $\mathcal{F}$ are hyperelliptic. Moreover, the linear span $\overline{\phi(D)} \subset \mathbb{P}^2$ of the points $\phi(p_1), \ldots, \phi(p_4)$ is a line, that is, D is a canonical divisor on C. Let $\iota \colon C^{(2)} \longrightarrow C^{(2)}$ be the involution $p + q \longmapsto K_C - p - q$, sending a point to the residual of the canonical system. We note that the map ι_b induced on the generic curve $\mathcal{F}_b$ of the family $\mathcal{F}$ is the hyperelliptic involution. Thus the quotient surface $C^{(2)}/\langle \iota \rangle$ is covered by a family of rational curves, but this is impossible because the latter surface is of general type. To see this fact it suffices to observe that $C^{(2)}$ is a surface of general type and that the map $C^{(2)} \longrightarrow \mathbb{P}^{\binom{g}{2}-1}$ induced by the canonical linear system $|K_{C^{(2)}}|$ factors through ι, because the Gauss map $\mathcal{G}$ in (2.2) does as well.

On the other hand, let us assume $g \ge 4$. By the geometric version of Riemann-Roch theorem we have

$$\dim |D| = \deg D - 1 - \dim \overline{\phi(D)} \ge 2d - 1 - (d-1) = d = \frac{\deg D}{2}.$$

Therefore we have that either D is zero, D is a canonical divisor or C is hyperelliptic by Clifford's theorem (cf. [2, p. 107]). Notice that C is assumed to be non-hyperelliptic and $0 < d < \mathrm{gon}(C) \le \left[\frac{g+3}{2}\right]$. Hence $0 < \deg D = 2d < 2g - 2$ for any $g \ge 4$, and we have a contradiction. Thus the points $p_1, \ldots, p_{2d}$ cannot be distinct.

Step 3 ($n_j > 1$ for some j). Let us then assume that the points $p_1, \ldots, p_{2d}$ are not distinct, i.e. the integers n_j are not all equal to 1. For any $k = 1, \ldots, 2d$ let us consider the set $Q_k := \{q_j \in \mathrm{Supp} D \mid n_j = k\}$ of the points of D such that $\mathrm{mult}_{q_j}(D) = k$. Notice that the cardinality of any Q_k is at most $\left[\frac{2d}{k}\right]$. As the n_j's are not all equal to 1, there exist some $k > 1$ such that the corresponding sets Q_k are not empty. Let $h > 1$ be the minimum of these integers and - without loss of generality - suppose $Q_h = \{q_1, \ldots, q_s\}$, where $s \le \left[\frac{2d}{h}\right]$ is the cardinality of Q_h.

Since Y is connected, the fibers of π_1 over generic points of Y have the same configuration; i.e. the cardinality of any set Q_h is constant as we vary the point (z, b) on a suitable open set $U \subset Y$. Thus we may define a rational map $\xi \colon Y \dashrightarrow C^{(s)}$ sending a generic point $(z, b) \in Y = \mathbb{P}^1 \times B$ to the effective divisor $q_1 + \ldots + q_s \in C^{(s)}$. For a very general $b \in B$, let

$$\xi_b \colon \quad \mathbb{P}^1 \times \{b\} \quad \longrightarrow \quad C^{(s)},$$
$$(z, b) \quad \longmapsto \quad q_1 + \ldots + q_s$$

be the restriction of ξ to the rational curve $\mathbb{P}^1 \times \{b\} \subset Y$ and let us consider the composition with the Abel map

$$\mathbb{P}^1 \times \{b\} \xrightarrow{\xi_b} C^{(s)} \longrightarrow J(C) \,.$$

As $\mathbb{P}^1 \times \{b\}$ is a rational curve mapping into a Jacobian variety, the latter map is constant. Hence by Abel's theorem, either $|q_1 + \ldots + q_s|$ is a complete linear series of degree s and dimension at least 1 or ξ_b is a constant map. Being that $h > 1$ and $s \leq \left[\frac{2d}{h}\right]$, we have $s \leq d < \mathrm{gon}(C)$. Then $|q_1 + \ldots + q_s|$ cannot be such a linear series.

Therefore the map ξ_b must be constant. By the construction of ξ_b, this fact means that for any $z \in \mathbb{P}^1$ the divisor $D = D_{(z,b)}$ - defined in (5.2) by the fiber $\pi_1^{-1}(z,b)$ - must contain all the points $q_1, \ldots, q_s$ that are now fixed. We recall that $\pi_1^{-1}(z,b)$ is given by the points $((z,b), P_i) \in Y \times C^{(2)}$ such that $P_i \in \mathcal{F}'_b$ and $\mu_b \circ \nu_b^{-1}(P_i) = z$, where $P_i = p_{2i-1} + p_{2i}$. Hence one of the P_i's must lie on the curve $C + q_1$, one on $C + q_2$, and so on. As we vary z on $\mathbb{P}^1$, the P_i's must vary on $\mathcal{F}'_b$, but the latter condition must hold. It follows that the curve $\mathcal{F}'_b$ must have at least s irreducible components $\mathcal{F}'_{b1}, \ldots, \mathcal{F}'_{bs}$ such that $\mathcal{F}'_{bj} \subset C + q_j$ for $1 \leq j \leq s$. We recall that for the generic $b \in B$, the curve $\mathcal{F}'_b$ coincides with a generic element E_t of the family $\mathcal{E}$. Since both E_t and $C + q_j$ are irreducible curves, we deduce $s = 1$ and $E_t = C + q_1$. Then we get a contradiction because $C + q_1 \cong C$, and hence $d = \mathrm{gon}(E_t) = \mathrm{gon}(C)$. Thus we conclude that the gonality d of the generic E_t is $d \geq \mathrm{gon}(C)$.

Step 4 ($d = \mathrm{gon}(C)$). To conclude the proof of the statement, let us suppose that C is a curve of genus $g \geq 6$ with $Aut(C) = \{Id_C\}$ and $d = \mathrm{gon}(\widetilde{E}_t) = \mathrm{gon}(C)$. We want to prove that the generic E_t and C are isomorphic.

To this aim, let us consider the correspondence $\Gamma \subset Y \times C^{(2)}$ defined above, and for a generic point $(z,b) \in Y$, let $\pi_1^{-1}(z,b) = \left\{((z,b), P_i)\right\}$ be its fiber, with $P_i = p_{2i-1} + p_{2i}$. By arguing as in Step 1 we deduce that the p_i's cannot be distinct. If they were distinct, then $\dim |D| = \frac{\deg D}{2}$ and - by Clifford's theorem - we would have that either D is zero, D is a canonical divisor or C is hyperelliptic. We note that the assumption $Aut(C) = \{Id_C\}$ implies that C is non-hyperelliptic. Moreover, the degree of D is positive and

$$\deg D = 2d = 2\mathrm{gon}(C) \leq 2\left[\frac{g+3}{2}\right] < 2g - 2 \quad \text{for any } g \geq 6.$$

Hence the divisor D is neither zero nor canonical, and we have a contradiction.

Then we follow the argument of Step 3, and for the generic $b \in B$ we may define the map $\xi_b \colon \mathbb{P}^1 \times \{b\} \longrightarrow C^{(s)}$, with $s \leq d$.

If $s < d = \mathrm{gon}(C)$, the only possible choice is $s = 1$ because of the irreducibility of E_t. Hence the generic E_t and C turn out to be isomorphic.

On the other hand suppose that $s = d = \mathrm{gon}(C)$. Then $h = 2$, and the divisor $D = D_{(z,b)}$ in (5.2) has the form $D = 2(q_1 + \ldots + q_d)$. Since the point $(z,b) \in Y$ is generic and the projections $\pi_1 \colon \Gamma \longrightarrow Y$, $\pi_2 \colon \Gamma \longrightarrow C^{(2)}$ are generically finite dominant morphisms, we can assume that none of the P_i's lie on the diagonal curve $\Delta := \{p + p \mid p \in C\} \subset C^{(2)}$: indeed, if the fiber of π_1 over the generic point met the curve $\pi_2^{-1}(\Delta) \subset \Gamma$, such a curve would dominate the surface Y. Moreover, let

E_t be the generic element of the family $\mathcal{E}$ corresponding to the curve $\mathcal{F}'_b$. Thus - without loss of generality - the points $P_i \in E_t \subset C^{(2)}$ are given by

$$P_1 = q_1 + q_2, \; P_2 = q_2 + q_3, \ldots, \; P_{d-1} = q_{d-1} + q_d \text{ and } P_d = q_d + q_1.$$

Furthermore, since the map $\xi_b \colon \mathbb{P}^1 \times \{b\} \longrightarrow C^{(d)}$ is non-constant and $d = \mathrm{gon}(C)$, we have that $|q_1 + \ldots + q_d|$ is a base-point-free g^1_d on C. This implies that $(E_t \cdot C + q_j) = 2$ for any j. Indeed $(E_t \cdot C + q_j) \geq 2$ because two of the P_i's lie on $C + q_j$, and if there existed another point $p + q_j \in E_t$, then it would lie on the support of a divisor $D_{(\overline{z}, b)}$ of the g^1_d for some $\overline{z} \in \mathbb{P}^1$; thus q_j would be a base point. Hence $(E_t \cdot C + q) = 2$ for any $q \in C$ because the curves $C + q$ are all numerically equivalent. We then distinguish two cases.

Suppose that $d = 2n$ is even, and let us show that this situation cannot occur. So let us consider the permutation $\sigma \in S_d$ given by $\sigma(j) = j + n \mod d$ for any $1 \leq j \leq d$. Then σ induces an involution α_σ on $\{q_1, \ldots, q_d\}$, sending a point q_j to the point $q_{\sigma(j)}$. In other words, we can think of the q_j's as the vertices of a convex polygon whose sides correspond to the P_i's (e.g. the side P_1 is the one joining q_1 and q_2), and the involution above sends any q_j to the opposite vertex. In particular, this point of view shows that α_σ depends only on the configuration of the fiber over (z, b), and it can be defined independently from the choice of the indices of the q_j's. As $(E_t \cdot C + q) = 2$ for any $q \in C$ and the fiber of π_1 over (z, b) varies holomorphically as we vary $z \in \mathbb{P}^1$, we can extend the involution above to an automorphism $\alpha \colon C \longrightarrow C$. Since α is not the identity on C and $Aut(C) = \{Id_C\}$, we have a contradiction.

Finally, let us assume that $d = 2n + 1$ is odd and let us show that $E_t \cong C$. We can define a one-to-one map from $\{q_1, \ldots, q_d\}$ to the set of the P_i's by sending a point $q_j \in C$ to the point $q_{j+n} + q_{j+n+1} \in E_t$, where the indices are taken $\mod d$ (i.e. such a map sends any vertex of the polygon to the opposite side). By fixing $b \in B$ and varying $z \in \mathbb{P}^1$, we then have an isomorphism between C and E_t, as claimed.

Thus Theorem 1.5 is now proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Degree of irrationality

Let C be a smooth complex projective curve of genus $g \geq 0$. We want to study the degree of irrationality of the surface $C^{(2)}$,

$$\mathrm{d}_r(C^{(2)}) := \min \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a dominant rational} \\ \text{map } F \colon C^{(2)} \dashrightarrow \mathbb{P}^2 \text{ of degree } d \end{array} \right\},$$

in dependence both on the genus and on the gonality of C.

When C is either a rational or an elliptic curve, the problem of determining the degree of irrationality of $C^{(2)}$ is totally understood. Namely, if C is rational, then $C^{(2)}$ is isomorphic to $\mathbb{P}^2$. Hence the second symmetric product is a rational surface and $\mathrm{d}_r(C^{(2)}) = 1$.

On the other hand, let us suppose that $g = 1$. By Abel's theorem $C^{(2)}$ is birational to the non-rational surface $C \times \mathbb{P}^1$. The curve C admits a double covering $f \colon C \longrightarrow \mathbb{P}^1$, and therefore we may define the degree two map

$$f \times Id_{\mathbb{P}^1} \colon C \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

Finally, being $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ birational surfaces, we conclude that

$$\mathrm{d}_r(C^{(2)}) = \mathrm{d}_r(C \times \mathbb{P}^1) = 2.$$

When C is a smooth complex projective curve of genus $g \geq 2$, the problem of computing the degree of irrationality of $C^{(2)}$ is still open. As a consequence of the main result in [1], we have the following result providing a lower bound.

Proposition 6.1. *Let $C^{(k)}$ be the k-fold symmetric product of a smooth curve C of genus $g \geq k \geq 2$. Then $\mathrm{d}_r(C^{(k)}) \geq k+1$.*

Proof. Let $F \colon C^{(k)} \dashrightarrow \mathbb{P}^k$ be a dominant rational map of degree d. By [1, Theorem 3.4], we have that $d\left(\mathrm{hl}(\mathbb{P}^k)+1\right) \geq \mathrm{hl}(C^{(k)})+1$, where $\mathrm{hl}(X)$ denotes the length of the graded algebra $H^{1,0}(X) \oplus \ldots \oplus H^{k,0}(X)$, that is, the maximum integer r such that there exist homogeneous elements $\omega_1, \ldots, \omega_r$ with $\omega_1 \wedge \ldots \wedge \omega_r \neq 0$. Since $\mathrm{hl}(\mathbb{P}^k) = 0$ and $\mathrm{hl}(C^{(k)}) = k$ by (2.3), we conclude that $d \geq k+1$, as claimed. $\square$

In particular, we have that $\mathrm{d}_r(C^{(2)}) \geq 3$ for any curve of genus $g \geq 2$. Moreover, this estimate turns out to be sharp when the curve has genus two. Indeed, Theorem 3.1 of [20] has presented an example of a genus two curve C' whose Jacobian satisfies $\mathrm{d}_r(J(C')) = 3$. As $C'^{(2)}$ maps birationally on $J(C')$ we have $\mathrm{d}_r(C'^{(2)}) = 3$ as well.

On the other hand, to provide upper bounds on $\mathrm{d}_r(C^{(2)})$ we have to present dominant rational maps $C^{(2)} \dashrightarrow \mathbb{P}^2$. In the examples below we exploit the existence of linear series on C in order to produce such maps. As a consequence, we achieve the upper bound on $\mathrm{d}_r(C^{(2)})$ stated in Proposition 1.1.

Example 6.2. Let $f \colon C \longrightarrow \mathbb{P}^1$ be a morphism of degree d. Then it is always possible to define the dominant morphism $F \colon C^{(2)} \longrightarrow \left(\mathbb{P}^1\right)^{(2)} \cong \mathbb{P}^2$ of degree d^2 given by $p + q \longmapsto f(p) + f(q)$.

Example 6.3. Suppose that C admits a birational mapping $f \colon C \longrightarrow \mathbb{P}^2$ onto a non-degenerate curve of degree d. Hence we may define a dominant rational map $F \colon C^{(2)} \dashrightarrow \mathbb{G}(1,2) \cong \mathbb{P}^2$ of degree $\binom{d}{2}$ by sending a point $p + q \in C^{(2)}$ to the line of $\mathbb{P}^2$ passing through $f(p)$ and $f(q)$.

Example 6.4. Let $f \colon C \longrightarrow \mathbb{P}^3$ be a birational map onto a non-degenerate curve of degree d. Consider a plane $H \subset \mathbb{P}^3$ and let $F \colon C^{(2)} \dashrightarrow H \cong \mathbb{P}^2$ be the dominant rational map sending a point $p + q \in C^{(2)}$ to the intersection of H with the line of $\mathbb{P}^3$ passing through $f(p)$ and $f(q)$. We note that the degree of F is $\frac{(d-1)(d-2)}{2} - g$. To see this fact, notice that the degree of F is the number of bisecant lines to $f(C)$ passing through a general point $y \in H$ and consider the projection $\pi_y \colon f(C) \longrightarrow \mathbb{P}^2$. As the number of such bisecant lines equals the number of nodes of the image $C' := (\pi_y \circ f)(C)$ and C' is a curve of degree d on $\mathbb{P}^2$, we conclude that $\deg F = p_a(C') - g(C') = \frac{(d-1)(d-2)}{2} - g$.

If C is assumed to be hyperelliptic, by Propositions 1.1 and 6.1 we have that $\mathrm{d}_r(C^{(2)})$ is either 3 or 4. We mentioned above an example of a hyperelliptic curve of genus two with $\mathrm{d}_r(C^{(2)}) = 3$. When the genus of C is $g \geq 4$, this is no longer possible, and Theorem 1.2 asserts that the degree of irrationality of hyperelliptic curves is exactly 4. In particular, the map on $\mathbb{P}^2$ reaching the minimum degree is the morphism described in Example 6.2.

When the curve C is non-hyperelliptic the situation is more subtle and we are not able to compute the precise value of the degree of irrationality of $C^{(2)}$. However,

Theorem 1.4 provides several lower bounds in dependence on the genus of the curve. Notice that for any curve of genus $4 \leq g \leq 7$ we have $\mathrm{d}_r(C^{(2)}) \geq g - 1$. As the following examples show, this fails to be true for larger values of g and it seems to happen when C covers certain curves.

Example 6.5. For an integer $d \geq 2$, let C be a non-hyperelliptic curve of genus $g \geq 2d^2 + 2$ provided with a degree d covering $f\colon C \longrightarrow E$ on an elliptic curve E (a particular case of this setting is given by bielliptic curves of genus greater than 9). Then we can define the dominant morphism $C^{(2)} \longrightarrow E^{(2)}$ of degree d^2 sending the point $p + q \in C^{(2)}$ to $f(p) + f(q) \in E^{(2)}$. As we saw at the beginning of this section, $\mathrm{d}_r(E^{(2)}) = 2$ and there exists a dominant rational map $E^{(2)} \dashrightarrow \mathbb{P}^2$ of degree 2. Therefore we obtain by composition a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree $2d^2$. Thus $\mathrm{d}_r(C^{(2)}) \leq 2d^2 < g - 1$.

Example 6.6. Let C' be the genus two curve of [20, Theorem 3.1] we mentioned above and suppose that C is a non-hyperelliptic curve of genus $g \geq 3d^2 + 2$ admitting a degree d covering of C'. Since $\mathrm{d}_r(C'^{(2)}) = 3$ and $C^{(2)}$ admits a covering of degree d^2 of $C'^{(2)}$, it is immediate to check that $\mathrm{d}_r(C^{(2)}) \leq 3d^2 < g - 1$.

On the other hand, when C is assumed to be very general in the moduli space $\mathcal{M}_g$, we have $\mathrm{d}_r(C^{(2)}) \geq g - 1$ for any genus $g \geq 4$, as we stated in Theorem 1.3.

Remark 6.7. Let C be a very general curve of genus $g \geq 3$. The dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ of minimum degree we are able to construct is one of those we used to establish Proposition 1.1. As in the proposition, let δ_1 be the gonality of C and for any $m \geq 2$ let δ_m be the minimum of the integers d such that C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^m$. The value of δ_m can be easily computed using the Brill-Noether number and - except for finitely many genera - the map of minimum degree is the one using g_d^2's in Example 6.3. We note further that we can construct analogously a dominant rational map $C^{(k)} \dashrightarrow \mathbb{P}^k$ of degree $\binom{\delta_k}{k}$ by using g_d^k's on C. Then we do not expect the bound in Theorem 1.3 to be sharp, and we conjecture that - except for finitely many genera - the degree of irrationality of symmetric products of a generic curve C of genus g is $\mathrm{d}_r(C^{(k)}) = \binom{\delta_k}{k}$, for any $1 \leq k \leq g - 1$.

Now, in order to prove the main theorems on this topic, we fix some notation and we state three preliminary lemmas.

By $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ we hereafter denote a dominant rational map of minimal degree, that is $d := \deg F = \mathrm{d}_r(C^{(2)})$. Given a point $y \in \mathbb{P}^2$, we consider its fiber

$$(6.1) \qquad F^{-1}(y) = \{p_1 + p_2, \ldots, p_{2d-1} + p_{2d}\} \subset C^{(2)}$$

and we define the *divisor $D_y \in Div(C)$ associated to y* as

$$(6.2) \qquad D_y := p_1 + p_2 + \ldots + p_{2d-1} + p_{2d}.$$

Then, by a simple monodromy argument we have the following.

Lemma 6.8. *There exists an integer $1 \leq a \leq d$ such that for a generic point $y \in \mathbb{P}^2$, we have $\mathrm{mult}_{p_j}(D_y) = a$ for any $j = 1, \ldots, 2d$.*

In particular, the divisor D_y defined above has the form $D_y = a(q_1 + q_2 + \ldots + q_m)$, where $m = \frac{2d}{a}$ and the q_j's are distinct points of C.

Proof. Let $G\colon C \times C \dashrightarrow \mathbb{P}^2$ be the dominant rational map of degree $2d$ defined as $G(p, q) := F(p + q) \in \mathbb{P}^2$. Given a generic point $y \in \mathbb{P}^2$, let

$$G^{-1}(y) = \{(p_1, p_2), (p_2, p_1), \ldots, (p_{2d-1}, p_{2d}), (p_{2d}, p_{2d-1})\} \subset C \times C$$

be its fiber. Then the divisor $D_y := p_1 + \ldots + p_{2d}$ is uniquely determined by the fiber $G^{-1}(y)$. Moreover, if m is the number of distinct points of $\{p_1, \ldots, p_{2d}\}$ and we denote by $q_1, \ldots, q_m$ these points, the divisor associated to y has the form $D_y = \sum_{j=1}^{m} a_j\, q_j$, for some positive integers $a_j := \mathrm{mult}_{q_j}(D_y)$. Therefore we have to prove that $a_1 = \ldots = a_m$.

As $C \times C$ is a connected surface, the action of the monodromy group $M\,(G) \subset S_{2d}$ of G is transitive. Hence it is not possible to distinguish any point of the fiber $G^{-1}(y)$ from another. Then for any $(r, s), (v, w) \in G^{-1}(y)$ we have that $\mathrm{mult}_r(D_y) = \mathrm{mult}_v(D_y)$ and $\mathrm{mult}_s(D_y) = \mathrm{mult}_w(D_y)$. In particular, we cannot distinguish the points (r, s) and (s, r), hence $\mathrm{mult}_r(D_y) = \mathrm{mult}_s(D_y)$. Thus the divisor D_y must have the same multiplicity at any p_i, i.e. there exists an integer $1 \leq a \leq 2d$ such that $a = \mathrm{mult}_{p_i}(D_y)$ for any $i = 1, \ldots, 2d$. Furthermore, a must divide $2d$ and the number of distinct p_i's is $m = \frac{2d}{a}$. Finally, being y generic on $\mathbb{P}^2$, we have that the number of distinct p_i's is at least 2. Hence $m \geq 2$ and $a \leq d$. $\square$

The second lemma is a consequence of Abel's theorem.

Lemma 6.9. *With the notation above, for a generic point $y \in \mathbb{P}^2$ with associate divisor $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, we have that the linear series $|q_1 + q_2 + \ldots + q_m|$ is a complete g_m^r on C with $r \geq 2$. Moreover, the integer a is lower than $d = \deg F$.*

Proof. Thanks to the previous lemma we are able to define the rational map $\xi\colon \mathbb{P}^2 \dashrightarrow C^{(m)}$, sending a generic point $y \in \mathbb{P}^2$ to the effective divisor $q_1 + q_2 + \ldots + q_m \in C^{(m)}$. As the image of $y \in \mathbb{P}^2$ depends on its fiber via the rational dominant map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$, we have that ξ is non-constant. Consider the resolution $\widetilde{\xi}\colon R \longrightarrow C^{(m)}$ of ξ and the composition with the Abel-Jacobi map $R \xrightarrow{\widetilde{\xi}} C^{(m)} \xrightarrow{u} J(C)$, where R is a rational surface. By the universal property of the Albanese morphism, the latter map factors through the Albanese variety $\mathrm{Alb}(R)$ of the rational surface R, which is 0-dimensional. Hence the composition $u \circ \widetilde{\xi}$ is a constant map. Being ξ non-constant, by Abel's theorem it follows that for all the generic points $y \in \mathbb{P}^2$, the divisors D_y are all linearly equivalent. Furthermore, as y vary on a surface, we deduce that the complete linear series $|q_1 + q_2 + \ldots + q_m|$ has dimension $r \geq 2$. To conclude, we recall that $1 \leq a \leq d$. If a were equal to d, then $m = 2$ and the linear series $|q_1 + q_2|$ would have degree 2 and dimension 2. Hence $a < d$. $\square$

Finally, the third lemma is an immediate consequence of Theorem 4.3 on correspondences with null trace on symmetric products of curves.

Lemma 6.10. *Let C be a non-hyperelliptic curve of genus $g \geq 5$ and let $d = \deg F < g - 1$. Then for a generic point $y \in \mathbb{P}^2$, we have that the points $p_1, \ldots, p_{2d} \in C$ in (6.1) and (6.2) are not distinct, that is, $a \neq 1$.*

Proof. Let $D_y = p_1 + \ldots + p_{2d}$ be the divisor associate to a generic point $y \in \mathbb{P}^2$. By contradiction, suppose that $p_1, \ldots, p_{2d}$ are distinct points of C. Let us consider the graph of the rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$,

$$(6.3) \qquad\qquad \Gamma := \overline{\{(y, p + q) \in \mathbb{P}^2 \times C^{(2)} \,|\, F(p + q) = y\}},$$

which is a correspondence with null trace on $\mathbb{P}^2 \times C^{(2)}$ of degree $d = \deg F$ (cf. Example 4.6). Let $\phi \colon C \longrightarrow \mathbb{P}^{g-1}$ denote the canonical map of C and let $\overline{\phi(D)}$ be the linear span of the points $\phi(p_i)$ in $\mathbb{P}^{g-1}$. Then $\dim \overline{\phi(D)} \leq d - 1$ by Theorem 4.3. Thus by the geometric version of the Riemann-Roch theorem we have

$$\dim |D_y| = \deg D_y - 1 - \dim \overline{\phi(D_y)} \geq 2d - 1 - (d-1) = d = \frac{\deg D_y}{2}.$$

Therefore by Clifford's theorem we have that either C is hyperelliptic, D_y is zero or D_y is a canonical divisor. On one hand, the curve C is assumed to be non-hyperelliptic. On the other hand, we have $0 < d < g - 1$, and hence $0 < \deg D_y < 2g - 2$. Thus we have a contradiction and the assertion follows. $\qquad \square$

So, let us prove the main results on degree of irrationality of second symmetric products of curves.

Proof of Theorem 1.3. Let C be a very general curve of genus $g \geq 4$ and let us prove that $\mathrm{d}_r(C^{(2)}) \geq g - 1$. When the genus of C is 4, the assertion follows from Proposition 6.1. Then let us assume that $g \geq 5$ and let $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d = \mathrm{d}_r(C^{(2)})$. Aiming for a contradiction we assume $d < g - 1$.

Let $y \in \mathbb{P}^2$ be a generic point and let D_y be its associate divisor. Since C is a non-hyperelliptic curve and $d < g - 1$, the lemmas above assure that there exists an integer $1 < a < d$ such that $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, where $m = \frac{2d}{a}$ and the q_j's are distinct points of C.

We claim that $a \neq 2$. If a were equal to 2, we would have $m = d$, and arguing as in Step 4 of the proof of Theorem 1.5, we could assume that the fiber over the generic $y \in \mathbb{P}^2$ has the form $F^{-1}(y) = \{q_1 + q_2, q_2 + q_3, \ldots, q_d + q_1\}$. Hence, by fixing a generic point $q \in C$, we would have that for any $p \in C$, there would exist a unique point $p' \in C$ such that $F(p + q) = F(q + p')$. Thus we could define an automorphism $\alpha_q \colon C \longrightarrow C$, sending a point $p \in C$ to the unique point $\alpha_q(p) \in C$ such that $F(p + q) = F(q + \alpha_q(p))$. In particular, α_q would not be the identity map, and we would have a contradiction because the only automorphism of a very general curve is the trivial one.

Then we have $a \geq 3$. By Lemma 6.9, the linear series $|q_1 + q_2 + \ldots + q_m|$ is a complete g_m^r of C with $r \geq 2$. Therefore the variety $W_m^r(C)$ parametrizing complete linear series of degree m and dimension at least r is non-empty. We recall that when C is a very general curve, the dimension of $W_m^r(C)$ equals the Brill-Noether number $\rho(g, r, m) := g - (r+1)(g - m + r)$. In particular, $|q_1 + q_2 + \ldots + q_m| \in W_m^2(C)$, and hence $\rho(g, 2, m) \geq 0$. It follows that

$$m \geq \frac{2g + 6}{3}.$$

On the other hand, we have $a \geq 3$ and $d < g - 1$. Therefore

$$m = \frac{2d}{a} < \frac{2g - 2}{3},$$

and we get a contradiction. $\qquad \square$

Proof of Theorem 1.4. Thanks to Proposition 6.1, we have that $\mathrm{d}_r(C^{(2)}) \geq 3$ and assertion (i) follows.

As usual, let $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d = \mathrm{d}_r(C^{(2)})$, and for a generic point $y \in \mathbb{P}^2$, we consider the associated divisor D_y. By Lemmas 6.8 and 6.9 we have that $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, where $1 \leq a < d$ and the q_j's are distinct points of C. Then we proceed by steps.

Step 1 $(g \geq 5 \Rightarrow d \geq 4)$. We assume that C has genus $g \geq 5$ and prove that $\mathrm{d}_r(C^{(2)}) \geq 4$. By Proposition 6.1 we have to check that $d \neq 3$. Aiming for a contradiction, we suppose that $d = \deg F = 3$.

By Lemma 6.9, we have that $|q_1 + q_2 + \ldots + q_m|$ is a complete linear series on C of degree m and dimension $r \geq 2$. As C is non-hyperelliptic and $g \geq 5$, Martens' theorem assures that $\dim W_m^r(C) \leq m - 2r - 1$ (see [2, p. 191]). As the number of q_j's is $m = \frac{2d}{a} = \frac{6}{a}$, we have that $W_m^r(C)$ has non-negative dimension only if $a = 1$.

Since $g \geq 5$ and $d = 3$, we have that $d < g - 1$, and hence the integer a cannot be equal to 1 by Lemma 6.10. Therefore we have a contradiction. Thus $d \geq 4$, and assertion (ii) follows as a consequence.

Step 2 $(g \geq 6 \Rightarrow d \geq 5)$. We prove that $d_r(C^{(2)}) \geq 5$ for any non-hyperelliptic curve C of genus $g \geq 6$. By the previous step, it suffices to see that $C^{(2)}$ does not admit dominant rational maps on $\mathbb{P}^2$ of degree 4. By contradiction, let us assume $d = \deg F = 4$.

The argument is the very same as Step 1. Thanks to Lemma 6.9 and Martens' theorem, we deduce $0 \leq \dim W_m^r(C) \leq m - 2r - 1$ with $r \geq 2$ and $m = \frac{2d}{a}$. Since $d = 4$, it follows that $a = 1$, but this situation cannot occur by Lemma 6.10. Then we have a contradiction and assertion (iii) holds.

Step 3 $(g \geq 7 \Rightarrow d \geq \mathrm{gon}(C))$. Suppose that C has genus $g \geq 7$ and - by contradiction - assume $d < \mathrm{gon}(C)$. Since $\mathrm{gon}(C) \leq \left[\frac{g+3}{2}\right] < g - 1$, Lemma 6.10 guarantees that $a \geq 2$, and hence $m = \frac{2d}{a} \leq d < \mathrm{gon}(C)$. On the other hand, $|q_1 + q_2 + \ldots + q_m|$ is a complete linear series on C of degree m; thus $m \geq \mathrm{gon}(C)$. Then we have a contradiction.

Step 4 $(g \geq 7 \Rightarrow d \geq 6)$. To conclude, we assume that C has genus $g \geq 7$ and prove that $\mathrm{d}_r(C^{(2)}) \geq 6$. Thanks to Step 2, we have to show that the degree of irrationality of $C^{(2)}$ is different from 5. Again we argue by contradiction and we suppose $d = \deg F = 5$.

As above, the inequality $0 \leq \dim W_m^r(C) \leq m - 2r - 1$ holds, where $r \geq 2$ and $m = \frac{10}{a}$. In this situation, the only possibilities are $a = 1$ and $a = 2$. The integer a must differ from 1 by Lemma 6.10. So, let us suppose that $a = 2$. Then $m = 5$ and the above inequality implies $r = 2$. In particular, the linear series $|q_1 + \ldots + q_5|$ is a complete g_5^2 on C. As m is prime, the map $C \longrightarrow \mathbb{P}^2$ defined by the g_5^2 is birational onto a non-degenerate plane quintic whose arithmetic genus is 6. Hence $g \leq 6$, a contradiction.

Thus assertion (iv) follows from Steps 3 and 4. $\square$

Proof of Theorem 1.2. Let C be a hyperelliptic curve of genus $g \geq 4$ and let us prove that $\mathrm{d}_r(C^{(2)}) \geq 4$. From Propositions 1.1 and 6.1 we deduce $3 \leq \mathrm{d}_r(C^{(2)}) \leq 4$. Aiming for a contradiction, we suppose that there exists a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree $d = 3$.

Let $y \in \mathbb{P}^2$ be a generic point, with fiber $F^{-1}(y) = \{p_1 + p_2, p_3 + p_4, p_5 + p_6\}$ and associated divisor $D_y = p_1 + \ldots + p_6$. Let $G \colon C \times C \dashrightarrow \mathbb{P}^2$ be the map of degree 6 defined as $G(p, q) := F(p + q)$, and let $G^{-1}(y) = \{(p_1, p_2), (p_2, p_1), \ldots, (p_6, p_5)\}$ be its fiber over y. By arguing as in the proof of Lemma 6.8, we have that the monodromy groups $M(F) \subset S_3$ and $M(G) \subset S_6$ act transitively on $F^{-1}(y)$ and $G^{-1}(y)$ respectively, since $C^{(2)}$ and $C \times C$ are connected surfaces. It follows that there is no way to distinguish either the points of the fiber $F^{-1}(y)$ or those of $G^{-1}(y)$ by some property varying continuously as y varies on $\mathbb{P}^2$. Thus we cannot distinguish the p_i's as well by such a property.

Now, let $f \colon C \longrightarrow \mathbb{P}^1$ be g_2^1 on C and let $\iota \colon C \longrightarrow C$ be the induced hyperelliptic involution. We recall that the canonical map $\phi \colon C \longrightarrow \mathbb{P}^{g-1}$ is the composition of the double covering f and the Veronese map $\nu_{g-1} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{g-1}$ (see e.g. [16, Proposition 2.2, p. 204]). Moreover, the image $\phi(C) \subset \mathbb{P}^{g-1}$ is set theoretically the rational normal curve of degree $g - 1$ and the covering $\phi \colon C \longrightarrow \phi(C)$ has degree two. Then two distinct points $p, q \in C$ have the same image if and only if they are conjugated under the hyperelliptic involution. Since $y \in \mathbb{P}^2$ is generic, we can assume - without loss of generality - that p_1 and p_2 are not conjugate under the hyperelliptic involution, that is, $\phi(p_1) \neq \phi(p_2)$. As the points of the $F^{-1}(y)$ are indistinguishable, it follows that $\phi(p_3) \neq \phi(p_4)$ and $\phi(p_5) \neq \phi(p_6)$ as well.

Consider the correspondence $\Gamma := \{(y, p + q) \in \mathbb{P}^2 \times C^{(2)} \mid F(p + q) = y\}$ defined as the graph of F. Since $\Gamma \subset \mathbb{P}^2 \times C^{(2)}$ has null trace and degree 3 (cf. Example 4.6), Theorem 4.3 assures that the points $\phi(p_1), \ldots, \phi(p_6)$ lie on a plane $\pi \subset \mathbb{P}^{g-1}$. With $\phi(C)$ being a rational normal curve, we have that the $\phi(p_i)$'s consist of at most three distinct points.

Suppose that they are exactly three. Then the $\phi(p_i)$'s are not collinear, because they lie on $\phi(C)$. Consider the lines

$$l_1 := \overline{\phi(p_1)\phi(p_2)}, \quad l_2 := \overline{\phi(p_3)\phi(p_4)}, \quad l_3 := \overline{\phi(p_5)\phi(p_6)} \subset \pi$$

and notice that each of them correspond to a point of $F^{-1}(y)$. Then we cannot distinguish them, and hence l_1, l_2 and l_3 are all distinct. Moreover, they must intersect at a same point $p \in \pi$ (cf. Example 3.5). Furthermore, each $\phi(p_i)$ has exactly two preimages on C because of the monodromy of G. Then we can assume - without loss of generality - that $p \neq \phi(p_1)$ and $\phi(p_1) = \phi(p_3)$. Thus l_1 and l_2 must coincide, a contradiction.

So, let us suppose that the $\phi(p_i)$'s consist of two distinct points. By Lemmas 6.8 and 6.9 there exists $a = 1, 2$ such that the divisor associated to y has the form $D_y = a(q_1 + \ldots + q_m)$, where $m = \frac{2d}{a}$ and the q_j's are distinct points of C. If $a = 2$ we have $m = 3$. Hence there are points q_1, q_2 mapping on $\phi(p_1)$ and q_3 on $\phi(p_2)$, but this situation cannot occur because we are distinguishing points. On the other hand, suppose that $a = 1$ and $m = 6$. As both $\phi(p_1)$ and $\phi(p_2)$ have two preimages on C, the q_j's must be at most four distinct points. Thus we have a contradiction, and the assertion of Theorem 1.2 holds. $\qquad\square$

7. Bounds on the ample cone of second symmetric products of curves

Let C be a smooth complex projective curve of genus g and let us assume that C is very general in the moduli space $\mathcal{M}_g$. In this section we apply Theorem 1.5 to the problem of describing the cone $Nef(C^{(2)})_{\mathbb{R}}$ of nef numerical equivalence classes

of $\mathbb{R}$-divisor, and we prove Theorem 1.7. To this aim we first recall some basic facts on this topic.

Given a point $p \in C$, we define the divisors on $C^{(2)}$ given by $C_p := \{p + q \mid q \in C\}$ and $\Delta := \{q + q \mid q \in C\}$. Let x and δ denote their numerical equivalence classes in the Néron-Severi group $N^1(C^{(2)})$. The vector space $N^1(C^{(2)})_{\mathbb{R}}$ of numerical classes of $\mathbb{R}$-divisors is spanned by the classes x and $\frac{\delta}{2}$ (cf. [2, p. 359]), where $x^2 = 1$, $\left(\frac{\delta}{2}\right)^2 = 1 - g$ and $\left(x \cdot \frac{\delta}{2}\right) = 1$. Then we deduce the formula governing the intersection on the Néron-Severi space, that is,

$$\left((a + b)x - b\frac{\delta}{2}\right) \cdot \left((c + d)x - d\frac{\delta}{2}\right) = ac - bdg.$$

Since $Nef(C^{(2)})_{\mathbb{R}}$ is a two-dimensional convex cone, it is completely determined by its two boundary rays. The first one is the dual ray of the diagonal via intersection pairing, and it is spanned by the class $(g - 1)x - \frac{\delta}{2}$. The other ray is spanned by the class $(\tau(C) + 1) x - \frac{\delta}{2}$, where $\tau(C)$ is the real number defined as

$$\tau(C) := \inf \left\{ t > 0 \,\middle|\, (t + 1)x - \frac{\delta}{2} \text{ is ample} \right\}.$$

Thus the problem of describing the nef cone $Nef(C^{(2)})_{\mathbb{R}}$ is reduced to compute $\tau(C)$. Clearly, as the self-intersection of an ample divisor is positive, it follows that $\tau(C) \geq \sqrt{g}$.

Remark 7.1. We note that when the genus of C is $g \leq 4$, the problem is totally understood (for details see [4, 13, 19]). On the other hand, there is an important conjecture - due to Kouvidakis - governing the case $g \geq 5$. It asserts that $\tau(C) = \sqrt{g}$, i.e. the nef cone is as large as possible. Such a conjecture has been proved in [4, 13] when the genus g is a perfect square, whereas the problem is still open in the other cases.

We recall further that when the genus of C is $g \geq 9$, Kouvidakis' conjecture is implied by Nagata's on the Seshadri constant at g generic points in $\mathbb{P}^2$, and this fact leads to several bounds on $\tau(C)$ (see for instance [19]). On the other hand, the best previously known bounds for generic curves of genus $5 \leq g \leq 8$ are those of [3, Theorem 1].

Moreover, we would like to note that the bounds of [3, Theorem 1] are $\tau_5 = \frac{9}{4}$, $\tau_6 = \frac{37}{15}$, $\tau_7 = \frac{189}{71}$ and $\tau_8 = \frac{54}{19}$. Since $\frac{32}{13} < \frac{37}{15}$, $\frac{77}{29} < \frac{189}{71}$ and $\frac{17}{6} < \frac{54}{19}$, we deduce that Theorem 1.7 does provide an improvement of the bounds on the ample cone of $C^{(2)}$ when $6 \leq g \leq 8$.

As we anticipated, the argument to prove Theorem 1.7 is the very same as [3, Theorem 1]. Then let us recall two preliminary results involved in the proof.

Given a smooth complex projective variety X and a nef class $L \in N^1(X)_{\mathbb{R}}$, we define the *Seshadri constant* of L at a point $y \in X$ as the real number

$$\epsilon(y; X, L) := \inf_E \frac{(L \cdot E)}{\text{mult}_y E},$$

where the infimum is taken over the irreducible curves E passing through y. The following holds (see [19, Theorem 1.2]).

Theorem 7.2. *Let D be a smooth curve of genus $g - 1$. Let a, b be two positive real numbers such that $\frac{a}{b} \geq \tau(D)$ and for a very general point $y \in D^{(2)}$*

$$\epsilon\left(y; D^{(2)}, (a+b)x - b\frac{\delta}{2}\right) \geq b.$$

If C is a very general curve of genus g, then $\tau(C) \leq \frac{a}{b}$.

Moreover, we need the following lemma (cf. [3, Lemma 3] and [12, Theorem A]).

Lemma 7.3. *Let X be a smooth complex projective surface. Let T be a smooth variety and consider a family $\{y_t \in E_t\}_{t \in T}$ consisting of a curve $E_t \subset Y$ through a very general point $y_t \in X$ such that $\mathrm{mult}_{y_t} E_t \geq m$ for any $t \in T$ and for some $m \geq 2$. If the central fiber E_0 is a reduced irreducible curve and the family is non-trivial, then*

$$E_0^2 \geq m(m-1) + \mathrm{gon}(E_0).$$

Proof of Theorem 1.7. Assume that $g = 6$ and let us prove that $\tau(C) \leq \frac{32}{13}$. To this aim, consider a very general curve D of genus $g(D) = g - 1 = 5$, together with its second symmetric product $D^{(2)}$. Let $a = 32$, $b = 13$ and consider the numerical equivalence class

$$(7.1) \qquad L := (a+b)x - b\frac{\delta}{2} \in N^1(D^{(2)}),$$

which is nef by [3, Theorem 1]. Thanks to Theorem 7.2 it suffices to prove that the Seshadri constant of L at a generic point $y \in D^{(2)}$ is greater than or equal to b, i.e. there is not a reduced irreducible curve $E \subset D^{(2)}$ passing through y such that $(L \cdot E)/\mathrm{mult}_y E < b$.

Let $\mathcal{F}$ be the set of pairs (F, z) such that $F \subset D^{(2)}$ is a reduced irreducible curve, $z \in F$ is a point and $(L \cdot F)/\mathrm{mult}_z F < 13$. Such a set consists of at most countably many algebraic families and y is generic on $D^{(2)}$, thus we have to show that each of these families is discrete (cf. [7, Section 2]).

We argue by contradiction and we assume that there exists a family $\mathcal{E} = \{(y_t \in E_t)\}_{t \in T}$ such that for any $t \in T$, the curve $E_t \subset D^{(2)}$ is reduced and irreducible, the point y_t is very general on $D^{(2)}$ and

$$(7.2) \qquad \frac{(L \cdot E_t)}{\mathrm{mult}_{y_t} E_t} < b = 13.$$

We claim that for any $t \in T$, we have

$$(7.3) \qquad (L \cdot E_t) \geq b.$$

Let $(n + \gamma)x - \gamma\frac{\delta}{2} \in N^1(D^{(2)})$ be the numerical equivalence class of E_t. Since the class x is ample, we have $(x \cdot E_t) = n > 0$. With $(L \cdot E_t) = an - b\gamma g$, we then have that (7.3) holds when $\gamma \leq 0$.

So, let us assume $\gamma > 0$. Since $\mathcal{E}$ is a family of curves covering $D^{(2)}$, we have $(L \cdot E_t) \geq 0$ (cf. [19, Lemma 2.2]). Furthermore, $D^{(2)}$ is a non-fibered surface, hence there are at most finitely many irreducible curves of zero self-intersection and numerical class $(n + \gamma)x - \gamma(\delta/2)$. Therefore we can assume $E_t^2 = n^2 - (g-1)\gamma^2 > 0$, that is, $n \geq \gamma\sqrt{g-1} + 1$. Notice that $a \geq b\sqrt{g-1}$; thus $(L \cdot E_t) = an - (g-1)b\gamma > b\sqrt{g-1}\left(\gamma\sqrt{g-1} + 1\right) - (g-1)b\gamma > b$ and the claim follows.

By (7.2) and (7.3) we deduce that $\mathrm{mult}_{y_t} E_t > (L \cdot E_t)/b \geq 1$ for any $t \in T$. As E_t is a reduced curve, we have that $\mathrm{mult}_z E_t = 1$ for any generic point $z \in E_t$.

Hence $\mathcal{E} = \{(y_t \in E_t)\}_{t \in T}$ is a non-trivial family. Without loss of generality, let us assume that the central fiber is such that

$$m := \mathrm{mult}_{x_0} E_0 \le \mathrm{mult}_{x_t} E_t$$

for any $t \in T$. Thanks to Lemma 7.3 we have that the curve E_0 has self-intersection $E_0^2 \ge m(m-1) + \mathrm{gon}(E_0)$. Furthermore, Theorem 1.5 assures that $\mathrm{gon}(E_0) \ge \mathrm{gon}(D)$, where $\mathrm{gon}(D) = \left\lceil \frac{g(D)+3}{2} \right\rceil$ because D is assumed to be very general in $\mathcal{M}_{g-1}$. Hence

$$(7.4) \qquad E_0^2 \ge m(m-1) + \left\lceil \frac{(g-1)+3}{2} \right\rceil = m(m-1) + 4.$$

Finally, inequality (7.2) leads to $(L \cdot E_0) \le bm - 1$. Thus by the Hodge Index Theorem we have

$$(7.5) \qquad m(m-1) + 4 \le E_0^2 \le \frac{(L \cdot E_0)^2}{L^2} \le \frac{(13m-1)^2}{179},$$

but this is impossible. Hence we get a contradiction, and we proved that $\tau(C) \le \frac{32}{13}$ for any generic curve of genus $g = 6$.

Now, let us assume that C has genus $g = 7$ and let D be a very general curve of genus $g(D) = g - 1 = 6$. In order to follow the above argument, we set $a = 77$ and $b = 29$. We just proved that $\tau(D) \le \frac{32}{13}$, hence the class L defined in (7.1) is still nef. Then we can argue as above and we have $E_0^2 \ge m(m-1) + \mathrm{gon}(E_0)$. We recall that $E_0 \subset D^{(2)}$ is a singular reduced irreducible curve lying on the second symmetric product of the genus six curve D. Therefore E_0 is not isomorphic to D, and $\mathrm{gon}(E_0) \ge \mathrm{gon}(D) + 1$ by Theorem 1.5. Then we obtain the analogue of inequality (7.4), that is, $E_0^2 \ge m(m-1) + 5$. Thus (7.5) becomes

$$m(m-1) + 5 \le E_0^2 \le \frac{(L \cdot E_0)^2}{L^2} \le \frac{(29m-1)^2}{883},$$

which is still impossible. Then we have that $\tau(C) \le \frac{77}{29}$ for any generic curve of genus $g = 7$.

Analogously, let $g = 8$ and consider a generic curve D of genus $g(D) = g - 1 = 7$. Since $\frac{17}{6} > \frac{77}{29} \ge \tau(D)$, we can argue as above by setting $a = 17$ and $b = 6$. Then we have $E_0^2 \ge m(m-1) + \mathrm{gon}(E_0)$, where $\mathrm{gon}(E_0) \ge \mathrm{gon}(D) + 1 = 6$. Therefore

$$m(m-1) + 6 \le E_0^2 \le \frac{(L \cdot E_0)^2}{L^2} \le \frac{(6m-1)^2}{37},$$

which still leads to a contradiction. Thus $\tau(C) \le \frac{17}{6}$ for any very general curve of genus $g = 8$, and the proof ends. $\qquad\square$

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