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SIGN CHANGES OF THE ERROR TERM IN WEYL'S LAW FOR HEISENBERG MANIFOLDS

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ABSTRACT. Let $R(T)$ be the error term in Weyl's law for the $(2l+1)$-dimensional Heisenberg manifold $(H_l/\Gamma, g_l)$. In this paper, several results on the sign changes and odd moments of $R(t)$ are proved. In particular, it is proved that for some sufficiently large constant c, $R(t)$ changes sign in the interval $[T, T+c\sqrt{T}]$ for all large T. Moreover, for a small constant c_1 there exist infinitely many subintervals in $[T, 2T]$ of length $c_1\sqrt{T}\log^{-5} T$ such that $\pm R(t) > c_1 t^{l-1/4}$ holds on each of these subintervals.

1. INTRODUCTION

Let (M, g) be a closed n-dimensional Riemannian manifold with metric g and Laplace-Beltrami operator Δ. Let $N(t)$ denote its spectral counting function, which is defined as the number of the eigenvalues of Δ not exceeding t. Hörmander [13] proved that Weyl's law

$$N(t) = \frac{vol(B_n)vol(M)}{(2\pi)^n} t^{n/2} + O(t^{(n-1)/2}) \tag{1.1}$$

holds, where $vol(B_n)$ is the volume of the n-dimensional unit ball.

Let

$$R(t) = N(t) - \frac{vol(B_n)vol(M)}{(2\pi)^n} t^{n/2}.$$

Hörmander's estimate (1.1) in general is sharp, as the well-known example of the sphere S^n with its canonical metric shows [13]. However, it is a very difficult problem to determine the optimal bound of $R(t)$ in any given manifold, which depends on the properties of the associated geodesic flow. Many improvements have been obtained for certain types of manifolds; see [1, 2, 3, 4, 7, 10, 14, 17, 20, 22, 25, 29, 30, 31].

1.1. **Weyl's law for $\mathbb{T}^2$: The Gauss circle problem.**

The simplest compact manifold with integrable geodesic flow is the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The exponential functions $e(mx + ny)(m, n \in \mathbb{Z})$ form a basis of eigenfunctions of the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$, which acts on functions on $\mathbb{T}^2$. The corresponding eigenvalues are $4\pi^2(m^2 + n^2), m, n \in \mathbb{Z}$. The spectral counting function

$$N_I(t) = \{\lambda_j \in Spec(\Delta) : \lambda_j \leq t\}$$

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is equal to the number of lattice points of $\mathbb{Z}^2$ inside a circle of radius $\sqrt{t}/2\pi$. The well-known Gauss circle problem is the study of the properties of the error term of the function $N_I(t)$.

In this case, formula (1.1) becomes

$$N_I(t) = \frac{t}{4\pi} + O(t^{1/2}), \tag{1.2}$$

which is the classical result of Gauss. Let $R_I(t)$ denote the error term in (1.2). Many authors improved the upper bound estimate of $R_I(t)$. The latest result is due to Huxley [14], which reads

$$R_I(t) \ll t^{131/416} \log^{26947/8320} t. \tag{1.3}$$

Hardy [11] conjectured that

$$R_I(t) \ll t^{1/4+\varepsilon}. \tag{1.4}$$

Cramér [5] proved that

$$\lim_{T\to\infty} T^{-3/2} \int_1^T |R_I(t)|^2 dt = C, \quad C = \frac{1}{6\pi^3} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}},$$

which is consistent with Hardy's conjecture. Here $r(n)$ denotes the number of ways in which n can be written as a sum of two squares.

Ivić [15] first used the large value technique to study the higher power moments of $R_I(t)$. He proved that the estimate

$$\int_1^T |R_I(t)|^A dt \ll T^{1+A/4+\varepsilon} \tag{1.5}$$

holds for each fixed $0 \le A \le 35/4$. The value of A for which (1.5) holds is closely related to the upper bound of $R_I(t)$. If we insert the estimate (1.3) into Ivić's machinery, we get that (1.5) holds for $0 \le A \le 262/27$.

The first author [28] studied the third and the fourth moments of $R_I(t)$. He proved the following two asymptotic formulas:

$$\int_1^T R_I^3(t) dt = c_3 T^{7/4} + O(T^{7/4-1/14+\varepsilon}), \tag{1.6}$$

$$\int_1^T R_I^4(t) dt = c_4 T^2 + O(T^{2-1/23+\varepsilon}), \tag{1.7}$$

where c_3 and c_4 are explicit constants.

In [30], the second author proved by a unified method that the asymptotic formula

$$\int_1^T R_I^k(t) dt = c_k T^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon}) \tag{1.8}$$

holds for $3 \le k \le 9$, where c_k and $\delta_k > 0$ are explicit constants.

1.2. **Weyl's law for $(2l+1)$-dimensional Heisenberg manifold.** Let $l \ge 1$ be a fixed integer and $(H_l/\Gamma, g)$ be a $(2l+1)$-dimensional Heisenberg manifold with a metric g. When $l = 1$, Petridis and Toth [25] proved that $R(t) = O(t^{5/6} \log t)$ for a special metric. Later in [4] this bound was improved to $O(t^{119/146+\varepsilon})$ for all left-invariant Heisenberg metrics. For $l > 1$ Khosravi and Petridis [20] proved that $R(t) = O(t^{l-7/41})$ holds for rational Heisenberg manifolds. In both [4] and [20] they first established a ψ-expression of $R(t)$ and then used the van der Corput method of exponential sums. Substituting Huxley's result of [14] into the arguments of [4] and [20], we can get that the estimate

$$R(t) = O(t^{l-77/416}(\log t)^{26947/8320}) \tag{1.9}$$

holds for all rational $(2l+1)$-dimensional Heisenberg manifolds.

It was conjectured that for rational Heisenberg manifolds, the pointwise estimate

$$R(t) \ll t^{l-1/4+\varepsilon} \tag{1.10}$$

holds, which was proposed in Petridis and Toth [25] for the case $l = 1$ and in Khosravi and Petridis [20] for the case $l > 1$. As an evidence of this conjecture, Petridis and Toth proved the following L^2 result:

$$\int_{I^3} \left| N(t;\vec{u}) - \frac{1}{6\pi^2} vol(M(\vec{u}))t^{3/2} \right|^2 d\vec{u} \le C_\delta t^{3/2+\delta}$$

for the 3-dimensional Heisenberg manifold H_1, where $N(t;\vec{u})$ is the counting function for H_1 with the metric

$$g(\vec{u}) = \begin{pmatrix} u_1^{-1} & 0 & 0 \\ 0 & u_2^{-1} & 0 \\ 0 & 0 & u_3^{-1} \end{pmatrix}$$

for any $\vec{u} = (u_1, u_2, u_3) \in I^3$, and $I = [1-\varepsilon, 1+\varepsilon]$. They also proved

$$\frac{1}{T}\int_T^{2T} \left| N(t) - \frac{1}{6\pi^2} vol(M)t^{3/2} \right| dt \gg T^{3/4}.$$

Now let $M = (H_l/\Gamma, g_l)$ be a $(2l+1)$-dimensional Heisenberg manifold with the metric

$$g_l := \begin{pmatrix} I_{2l\times 2l} & 0 \\ 0 & 2\pi \end{pmatrix},$$

where $I_{2l\times 2l}$ is the identity matrix.

Khosravi and Toth [21] proved that

$$\int_1^T R(t)^2 dt = C_{2,l}T^{2l+1/2} + O(T^{2l+1/4+\varepsilon}), \tag{1.11}$$

where $C_{2,l}$ is an explicit constant.

Khosravi [19] proved that the asymptotic formula

$$\int_1^T R^3(t)dt = C_{3,l}T^{3l+1/4} + O(T^{3l+3/14+\varepsilon}) \tag{1.12}$$

is true for some explicit constant $C_{3,l}$.

In [32] the second author proved that the asymptotic formula

$$\int_1^T R^k(t)dt = C_{k,l}T^{k(l-1/4)+1} + O(T^{k(l-1/4)+1-\eta_k+\varepsilon}) \tag{1.13}$$

holds true for any $3 \le k \le 9$, where $C_{k,l}$ and $\eta_k > 0$ are explicit constants.

Recently, Nowak [23, 24] proved that the estimate

$$\limsup_{t\to\infty} \frac{R(t)}{t^{l-1/4}\omega_l(t)} > 0$$

holds with

$$\omega_l(t) = \begin{cases} (\log t)^{1/4}, & \text{if } l \text{ is even,} \\ (\log_2 t \log_3 t)^{1/4}, & \text{if } l \text{ is odd,} \end{cases}$$

where $\log_r t = \log \log_{r-1} t, \log_1 t = \log t$.

Notation. For a real number t, let $[t]$ denote the integer part of t, $\{t\} = t - [t]$, $\|t\| = \min(\{t\}, 1-\{t\})$, $e(t) = e^{2\pi i t}$. ε always denotes a sufficiently small positive constant. $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote the set of real numbers, the set of integers, and the set of positive integers, respectively. $d(n)$ denotes the Dirichlet divisor function. Throughout this paper, $\mathcal{L}$ always denotes $\log T$.

2. Sign changes of $R(t)$

From now on, we always suppose that $R(t)$ denotes the error term in Weyl's law for the $(2l+1)$-dimensional Heisenberg manifold $(H_l/\Gamma, g_l)$.

In [12], Heath-Brown and the first author studied the sign changes of the error term $R_I(t)$. They proved that for a suitable constant $C > 0$, $R_I(t)$ changes sign on the interval $[T, T + C\sqrt{T}]$ for every sufficiently large T. Here the length $\sqrt{T}$ is almost best possible since they proved that in the interval $[T, 2T]$ there are many subintervals of length $\gg \sqrt{T} \log^{-5} T$ such that $R_I(t)$ does not change sign in any of these subintervals.

In this paper we shall show that similar results hold for $R(t)$. More precisely, we have the following theorems.

Theorem 1. *Let $c_1 > 0$ be a sufficiently small constant and $c_2 > 0$ be a sufficiently large constant. For any real-valued function $g(t)$ satisfying $|g(t)| \le c_1 t^{l-1/4}$, the function $R(t) + g(t)$ changes sign at least once in the interval $[T, T + c_2\sqrt{T}]$ for every sufficiently large T. In particular, there exist $t_1, t_2 \in [T, T + c_2\sqrt{T}]$ such that $R(t_1) \ge c_1 t_1^{l-1/4}$ and $R(t_2) \le -c_1 t_2^{l-1/4}$.*

Theorem 2. *There exist three positive absolute constants c_3, c_4, c_5 such that, for any large parameter T, there are at least $c_3\sqrt{T}\log^5 T$ disjoint subintervals of length $c_4\sqrt{T}\log^{-5} T$ in $[T, 2T]$ such that $\pm R(t) > c_5 t^{l-1/4}$ whenever t lies in any of these subintervals. We also have the estimate*

$$meas\{t \in [T, 2T] : \pm R(t) > c_5 t^{l-1/4}\} \gg T.$$

Remark 1. Our proof of Theorem 2 is a variant of the proof of Theorem 2 in Section 3 of [12]. However, our approach can prove that $R(t)$ (respectively $-R(t)$) has large values on long intervals of length $\gg \sqrt{T}\log^{-5} T$.

As an application of Theorem 2, we study the Ω-result of the error term in the asymptotic formula (1.13) for odd k. For any integer $k \ge 2$, define

$$\mathcal{F}_{k,l}(T) := \int_1^T R^k(t)dt - C_{k,l} T^{k(l-1/4)+1}.$$

We then have the following

Theorem 3. *The estimate*

$$\mathcal{F}_{k,l}(T) = \Omega(T^{k(l-1/4)+1/2} \log^{-5} T)$$

holds for any fixed odd integer $k \ge 3$.

Remark 2. The results of [32] show that (1.13) should be true for any integer $k \geq 3$. However, up to the present we can only prove it for $3 \leq k \leq 9$. Theorem 3 provides an Ω-result for any odd $k \geq 3$.

The corresponding result on $R_I(t)$ proved in [12] can be improved slightly via the same approach. We state it as the following theorem.

Theorem 4. *There exist three positive absolute constants c_6, c_7, c_8 such that, for any large parameter T, there are at least $c_6\sqrt{T}\log^3 T$ disjoint subintervals of length $c_7\sqrt{T}\log^{-3} T$ in $[T, 2T]$ such that $\pm R_I(t) > c_8 t^{1/4}$ whenever t lies in any of these subintervals. We also have the estimate*

$$meas\{t \in [T, 2T] : \pm R_I(t) > c_8 t^{1/4}\} \gg T.$$

Remark 3. By Theorem 4, the argument of Theorem 3 proves that the formula

$$\int_1^T R_I^k(t)dt = c_k T^{1+k/4} + \Omega(T^{(k+2)/4}\log^{-3} T)$$

holds for any odd integer $k \geq 3$.

For the error term $\Delta(x)$ in the divisor problem, the asymptotic formula (see [28] and [30])

$$\int_1^T \Delta^k(x)dx = C_k T^{k/4+1} + O(T^{k/4+1-\eta_k}) \tag{2.1}$$

holds for any integer $3 \leq k \leq 9$, where C_k and η_k are explicit constants. In [16], Ivić and the second author proved the estimate

$$\int_1^T \Delta^k(x)dx - C_k T^{k/4+1} = \Omega(G^{k+1}(T)\log^{-1} T)$$

for any $k \geq 2$, where

$$G(x) = (x\log x)^{1/4}(\log\log x)^{\frac{3}{4}(2^{4/3}-1)}(\log\log\log x)^{-5/8}$$

is the Ω-estimate of $\Delta(x)$ proved by Soundararajan [26]. In view of the work in [12], the proof of Theorem 3 implies that, for any odd integer $k \geq 3$, the estimate $\Omega(G^{k+1}(T)\log^{-1} T)$ can be substantially improved to $\Omega(T^{(k+2)/4}\log^{-5} T)$. A similar result also holds for $E(t)$, the error term in the mean square of the Riemann zeta-function $\zeta(s)$ over the critical line.

3. Background of Heisenberg manifolds and the analogue Voronoi formula for $R(2\pi x)$

In this section, we first review some background of Heisenberg manifolds. The reader can refer to [6], [9], [27] for more details.

3.1. **Heisenberg manifolds.** Suppose $x \in \mathbb{R}^l$ is a row vector and $y \in \mathbb{R}^l$ is a column vector. Define

$$\gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & I_l & y \\ 0 & 0 & 1 \end{pmatrix}, \quad X(x,y,t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The $(2l+1)$-dimensional Heisenberg group H_l is defined by

$$H_l = \{\gamma(x,y,t) : x, y \in \mathbb{R}^l, t \in \mathbb{R}\},$$

and its Lie algebra is

$$\mathfrak{H}_l = \{X(x,y,t) : x, y \in \mathbb{R}^l, t \in \mathbb{R}\}.$$

We say Γ is a uniform discrete subgroup of H_l if H_l/Γ is compact. A $(2l+1)$-dimensional Heisenberg manifold is a pair $(H_l/\Gamma, g)$ for which Γ is a uniform discrete subgroup of H_l and g is a left H_l-invariant metric.

For every l-tuple $r = (r_1, r_2, \cdots, r_l) \in \mathbb{N}^l$ such that $r_j | r_{j+1}$ $(j = 1, 2, \cdots, l-1)$, let $r\mathbb{Z}^l$ denote the l-tuple $x = (x_1, x_2, \cdots, x_l)$ with $x_j \in r_j\mathbb{Z}$. Define

$$\Gamma_r = \{\gamma(x,y,t) : x \in r\mathbb{Z}^l, y \in r\mathbb{Z}^l, t \in \mathbb{Z}\}.$$

It is clear that Γ_r is a uniform discrete subgroup of H_l. According to Theorem 2.4 of [9], the subgroup Γ_r classifies all the uniform discrete subgroups of H_l up to automorphisms. Thus (see [9], Corollary 2.5) given any Riemannian Heisenberg manifold $M = (H_l/\Gamma, g)$, there exists a unique l-tuple r as before and a left-invariant metric $\tilde{g}$ on H_l such that M is isometric to $(H_l/\Gamma, \tilde{g})$. So (see [9], 2.6(b)) we can replace the metric g by $\phi^* g$, where ϕ is an inner automorphism such that the direct sum split of the Lie algebra $\mathfrak{H}_l = \mathbb{R}^{2l} \oplus \mathfrak{Z}$ is orthogonal . Here $\mathfrak{Z}$ is the center of the Lie algebra and

$$\mathbb{R}^{2l} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}^l \right\}.$$

With respect to this orthogonal split of H_l the metric g has the form

$$\begin{pmatrix} h & 0 \\ 0 & g_{2l+1} \end{pmatrix},$$

where h is a positive-definite $2l \times 2l$ matrix and $g_{2l+1} > 0$ is a real number.

The volume of the Heisenberg manifold is given by

$$vol(H_l/\Gamma, g) = |\Gamma_r| \sqrt{det(g)}$$

with $|\Gamma_r| = r_1 r_2 \cdots r_l$ for $r = (r_1, r_2, \cdots, r_l)$.

3.2. **The spectrum of Heisenberg manifolds.** Let Σ be the spectrum of the Laplacian on $M = (H_l/\Gamma, g_l)$, where the eigenvalues are counted with multiplicities. According to [9] (p. 258), Σ can be divided into two parts, Σ_1 and Σ_2, where Σ_1 is the spectrum of $2l$-dimensional torus and Σ_2 contains all eigenvalues of the form

$$2\pi m^2 + 2\pi m(2n_1 + \cdots + 2n_l + l), m \in \mathbb{N}, n_j \in \mathbb{N} \cup \{0\},$$

each eigenvalue counted with the multiplicity $2m^l$.

3.3. **The Voronoi-type formula for** $R(2\pi x)$**.** In [32] the second author proved an analogue Voronoi formula for $R^*(x) := R(2\pi x)$. Suppose $T \geq 10$ is a large parameter, $\mathcal{L} = \log T$. Suppose $T \leq x \leq 2T, H \geq T$, and $J = [(\mathcal{L} - \log \mathcal{L})/2\log 2]$. Then we have

$$(3.1) \qquad R^*(x) = \frac{2^{2-l} x^{l-1/4}}{(l-1)!\pi} \sum_{1 \leq n \leq H^2(2^{2J+1}+1/2)} \frac{\tau_l(n; H, T)}{n^{3/4}} \cos\left(2\pi\sqrt{xn} - \frac{\pi}{4}\right) + O(T^{l-1/2}G(x) + T^{l-1/2}\mathcal{L}^2),$$

where

$$\tau_l(n;H,T): \quad = \sum_{\substack{n=h(2r-h),1\le h\le H\\ h\le r\le h(2^{2J+1}+1/2)}} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}}\left(1-\frac{h}{2r-h}\right)^{l-1}, \tag{3.2}$$

$$G(x) \quad = \sum_{m\le\sqrt{2T}} \min\left(1, \frac{1}{H\|\frac{x}{2m}-\frac{m}{2}+\frac{l}{2}\|}\right). \tag{3.3}$$

We note that if $n \le T\mathcal{L}^{-1}$, then

$$\tau_l(n;H,T) = \tau_l(n) := \sum_{\substack{n=h(2r-h)\\ h\le r}} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}}\left(1-\frac{h}{2r-h}\right)^{l-1}. \tag{3.4}$$

Remark 3.1. There is an error in the definition of $\tau_l(n)$ in [32], where the important condition $h \le r$ was omitted.

Remark 3.2. The term $T^{l-1/2}\mathcal{L}^2$ in (3.1) reads as $T^{l-1/2}\mathcal{L}^3$ in Proposition 6.1 of [32]. However, by a little more analysis in Section 6.2 of [32], we see that $T^{l-1/2}\mathcal{L}^3$ can be replaced by $T^{l-1/2}\mathcal{L}^2$.

4. Proof of Theorem 1

In this section we prove Theorem 1. We follow the approach of [12].

Let n_0 denote the smallest integer n such that $\tau_l(n) \neq 0$. From the definition of $\tau_l(n)$ it is easy to see that $n_0 = 1$ if $l = 1$ and $n_0 = 3$ if $l > 1$, and indeed

$$\tau_l(n_0) = \begin{cases} -1, & \text{if } l = 1, \\ e(l/2)3^{1/2-l}2^{l-1}, & \text{if } l > 1. \end{cases}$$

Suppose $|g(t)| \le c_1 t^{l-1/4}$. Let

$$R^{**}(t) = t^{-(2l-1/2)}(R(2\pi t^2) + g(2\pi t^2)), \quad t \ge 1, \tag{4.1}$$

and define

$$K_\zeta(u) := (1-|u|)(1+\zeta \sin 2\pi\alpha\sqrt{n_0}u), \quad u \ge 1, \tag{4.2}$$

where $\zeta = 1$ or -1 and $\alpha > 2$ is a large constant.

It is easy to see that Theorem 1 follows from Lemma 4.1 below.

Lemma 4.1. *Suppose $T \ge 10$ is a large parameter. Then for each $\sqrt{T} \le t \le \sqrt{2T}$, we have*

$$\begin{aligned}\int_{-1}^{1} R^{**}(t+\alpha u)K_\zeta(u)du = &-\frac{\zeta 2^{1-l}\tau_l(n_0)}{(l-1)!\pi n_0^{3/4}}\sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) \\ &+O(t^{-(2l-1/2)}\sup_{|u|\le 1}|g(2\pi(t+\alpha u)^2)| + t^{-1/2}\log^2 t).\end{aligned} \tag{4.3}$$

Proof. From (3.1) and the definition of n_0 we have

$$t^{-(2l-1/2)}R^*(t^2) = \frac{2^{2-l}}{(l-1)!\pi}\sum_{n_0\le n\le H^2(2^{2J+1}+1/2)}\frac{\tau_l(n;H,T)}{n^{3/4}} \times \cos\left(2\pi t\sqrt{n}-\frac{\pi}{4}\right)+O(t^{-1/2}G_1(t)+t^{-1/2}\log^2 t), \tag{4.4}$$

$$G_1(t) = \sum_{m\le\sqrt{2T}}\min\left(1,\frac{1}{H\|\frac{t^2}{2m}-\frac{m}{2}+\frac{l}{2}\|}\right). \tag{4.5}$$

We first estimate the integral $\int_{-1}^1 G_1(t+\alpha u)du$. It is well known that

$$\min(1,\frac{1}{H\|r\|})=\sum_{h=-\infty}^{\infty}a(h)e(hr) \tag{4.6}$$

with

$$a(0)\ll\frac{\log H}{H},\quad a(h)\ll\min(\frac{\log H}{H},\frac{H}{h^2}),\ h\ne 0.$$

Thus we have

$$\begin{aligned}\int_{-1}^1 G_1(t+\alpha u)du &= \sum_{h=-\infty}^{\infty}a(h)\sum_{m\le\sqrt{2T}}e\left(\frac{ht^2}{2m}-\frac{hm}{2}+\frac{hl}{2}\right)\\ &\quad\times\int_{-1}^1 e\left(\frac{2ht\alpha u+h\alpha^2u^2}{2m}\right)du\\ &\ll\sqrt{T}|a(0)|+\sum_{h=1}^{\infty}|a(h)|\sum_{m\le\sqrt{2T}}\frac{m}{ht\alpha}\\ &\ll\sqrt{T}H^{-1}\log^2 H,\end{aligned} \tag{4.7}$$

where the first derivative test was used.

Let

$$J_\zeta(\alpha,t,n):=\int_{-1}^1\cos(2\pi(t+\alpha u)\sqrt{n}-\frac{\pi}{4})K_\zeta(u)du.$$

Then we have

$$J_\zeta(\alpha,t,n)=J_1-J_2+J_3-J_4, \tag{4.8}$$

where

$$J_1=\cos(2\pi t\sqrt{n}-\frac{\pi}{4})\int_{-1}^1(1-|u|)\cos(2\pi\alpha u\sqrt{n})du,$$

$$J_2=\sin(2\pi t\sqrt{n}-\frac{\pi}{4})\int_{-1}^1(1-|u|)\sin(2\pi\alpha u\sqrt{n})du,$$

$$J_3=\zeta\cos(2\pi t\sqrt{n}-\frac{\pi}{4})\int_{-1}^1(1-|u|)\cos(2\pi\alpha u\sqrt{n})\sin(2\pi\alpha\sqrt{n_0}u)du,$$

$$J_4=\zeta\sin(2\pi t\sqrt{n}-\frac{\pi}{4})\int_{-1}^1(1-|u|)\sin(2\pi\alpha u\sqrt{n})\sin(2\pi\alpha\sqrt{n_0}u)du.$$

It is easy to see that $J_2=J_3=0$. By the first derivative test we get that

$$J_1\ll\alpha^{-1}n^{-1/2}. \tag{4.9}$$

For J_4 we have

$$J_4 = \frac{\zeta}{2}\sin(2\pi t\sqrt{n} - \frac{\pi}{4})\int_{-1}^{1}(1-|u|) \times \big(\cos(2\pi\alpha(\sqrt{n}-\sqrt{n_0})u) - \cos(2\pi\alpha(\sqrt{n}+\sqrt{n_0})u)\big)\, du.$$

So by the first derivative test again we get

$$J_4 = \begin{cases} -\frac{\zeta}{2}\sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}), & \text{if } n = n_0, \\ \ll \alpha^{-1}n^{-1/2}, & \text{if } n \neq n_0, \end{cases}$$

which combining (4.8) and (4.9) gives

$$J_\zeta(\alpha,t,n) = \begin{cases} -\frac{\zeta}{2}\sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}), & \text{if } n = n_0, \\ \ll \alpha^{-1}n^{-1/2}, & \text{if } n \neq n_0. \end{cases} \tag{4.10}$$

From (4.4), (4.5), (4.7) and (4.10) we get (taking $H = T^2$)

$$\begin{aligned} &\int_{-1}^{1} R^{**}(t+\alpha u)K_\zeta(u)du \\ &= \frac{2^{2-l}}{(l-1)!\pi}\sum_{n_0\le n\le H^2(2^{2J+1}+1/2)}\frac{\tau_l(n;H,T)}{n^{3/4}}J_\zeta(\alpha,t,n) \\ &\quad + O(t^{-(2l-1/2)}\sup_{|u|\le 1}|g(2\pi(t+\alpha u)^2)| + T^{1/2}H^{-1}\mathcal{L}^2 + t^{-1/2}\log^2 t) \\ &= -\frac{\zeta 2^{1-l}\tau_l(n_0)}{(l-1)!\pi n_0^{3/4}}\sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) + \sum_{n_0+1\le n\le H^2(2^{2J+1}+1/2)}\frac{|\tau_l(n)|}{\alpha n^{5/4}} \\ &\quad + O(t^{-(2l-1/2)}\sup_{|u|\le 1}|g(2\pi(t+\alpha u)^2)| + t^{-1/2}\log^2 t) \\ &= -\frac{\zeta 2^{1-l}\tau_l(n_0)}{(l-1)!\pi n_0^{3/4}}\sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) \\ &\quad + O(t^{-(2l-1/2)}\sup_{|u|\le 1}|g(2\pi(t+\alpha u)^2)| + t^{-1/2}\log^2 t). \end{aligned} \tag{4.11}$$

□

5. The mean value of $R(t)$ in short intervals

Suppose $T \ge 10$ is a large parameter, $1 \le h \le \frac{1}{2}\sqrt{T}$. In this section we shall estimate the integral

$$I(T,h) = \int_1^T (R(x+h) - R(x))^2 dx,$$

which would play an important role in the proof of Theorem 2. This type of integral was studied for the error term in the mean square of $\zeta(1/2+it)$ by Good [8] and for the error term in the Dirichlet divisor problem by Jutila [18]. Our approach is based on Jutila [18], but with some modifications.

Without loss of generality, we shall estimate the integral

$$I^*(T,h) = \int_1^T (R^*(x+h) - R^*(x))^2 dx, \tag{5.1}$$

where $R^*(x)$ was defined in (3.1). We shall prove the following

Lemma 5.1. *The estimate*

$$I^*(T,h) \ll T^{2l} h \log^3 \frac{\sqrt{T}}{h} + T^{2l}\mathcal{L}^4 \tag{5.2}$$

holds uniformly for $1 \le h \le \frac{1}{2}\sqrt{T}$.

Remark. Lemma 5.1 is also true for $I(T,h)$.

Proof. Write

$$I^*(T,h) = \int_1 + \int_2, \tag{5.3}$$

where

$$\int_1 := \int_1^{100\max(h^2,T^{2/3})} (R^*(x+h) - R^*(x))^2 dx,$$
$$\int_2 := \int_{100\max(h^2,T^{2/3})}^{T} (R^*(x+h) - R^*(x))^2 dx.$$

From (1.11) we have

$$\int_1 \ll h^{2(2l+1/2)} + T^{\frac{2}{3}(2l+1/2)} \ll T^{2l} h. \tag{5.4}$$

In order to bound $\int_2$, we first estimate the integral

$$J(U,h) = \int_U^{2U} (R^*(x+h) - R^*(x))^2 dx, \quad 100\max(h^2,T^{2/3}) \le U \le T.$$

In (3.1) we use U in place of T and then take $H = U^{100}$, $J = [(\log U - \log\log U)/2\log 2]$. Let $z := \min(\varepsilon U h^{-1}, U \log^{-1} U)$. Define

$$R_1(x) := \frac{2^{2-l} x^{l-1/4}}{(l-1)!\pi} \sum_{1\le n\le z} \frac{\tau_l(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)$$
$$R_2(x) := \frac{2^{2-l} x^{l-1/4}}{(l-1)!\pi} \sum_{z\le n\le H^2(2^{2J+1}+1/2)} \frac{\tau_l(n;H,T)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right).$$

Then we have

$$R^*(x) = R_1(x) + R_2(x) + O(U^{l-1/2} G_2(x) + U^{l-1/2}\log^2 U), \tag{5.5}$$

where

$$G_2(x) := \sum_{m\le\sqrt{2U}} \min\left(1, \frac{1}{H\|\frac{x}{2m} - \frac{m}{2} + \frac{l}{2}\|}\right).$$

From (6.30) of [32] we have

$$\int_U^{2U} |R_2(x)|^2 dx \ll U^{2l+1/2} z^{-1/2} \log^3 z. \tag{5.6}$$

Lemma 6.1 of [32] implies that (trivially $G_2(x) \ll \sqrt{U}$)

$$\int_U^{2U} |U^{l-1/2} G_2(x)|^2 dx \ll U^{2l-1/2} \int_U^{2U} G_2(x) dx \ll U^{2l-99} \log H. \tag{5.7}$$

Let

$$M(x) = R_2(x) + O(U^{l-1/2}G_2(x) + U^{l-1/2}\log^2 U).$$

Then (5.6) and (5.7) implies

$$\int_U^{2U} |M(x)|^2 dx \ll U^{2l+1/2} z^{-1/2} \log^3 z + U^{2l}\log^4 U \\ \ll h^{1/2} U^{2l} \log^3 z + U^{2l} \log^4 U. \tag{5.8}$$

Now we estimate $\int_U^{2U} (R_1(x+h) - R_1(x))^2 dx$. From the definition of $R_1(x)$ we have

$$R_1(x+h) - R_1(x) = F_1(x) + F_2(x), \tag{5.9}$$

where

$$F_1(x) = \frac{2^{2-l}}{(l-1)!\pi}\left((x+h)^{l-1/4} - x^{l-1/4}\right)\sum_{1\le n\le z}\frac{\tau_l(n)}{n^{3/4}} \\ \times \cos\left(2\pi\sqrt{n(x+h)} - \frac{\pi}{4}\right) \ll hx^{-1}|R_1(x+h)|,$$

$$F_2(x) = \frac{2^{2-l}}{(l-1)!\pi} x^{l-1/4} \sum_{1\le n\le z} \frac{\tau_l(n)}{n^{3/4}} \\ \times \left(\cos\left(2\pi\sqrt{n(x+h)} - \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)\right).$$

For the mean square of $F_1(x)$ we have

$$\int_U^{2U} |F_1(x)|^2 dx \ll h^2 U^{-2} U^{2l+1/2} \ll hU^{2l}. \tag{5.10}$$

We write

$$F_2^2(x) = F_{21}(x) + F_{22}(x), \tag{5.11}$$

where

$$F_{21}(x) = \frac{2^{4-2l}}{(l-1)!^2\pi^2} x^{2l-1/2} \sum_{1\le n\le z} \frac{\tau_l^2(n)}{n^{3/2}} \\ \times \left(\cos\left(2\pi\sqrt{n(x+h)} - \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)\right)^2,$$

$$F_{22}(x) = \frac{2^{4-2l}}{(l-1)!^2\pi^2} x^{2l-1/2} \sum_{1\le m\ne n\le z} \frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \\ \times \left(\cos\left(2\pi\sqrt{m(x+h)} - \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{mx} - \frac{\pi}{4}\right)\right) \\ \times \left(\cos\left(2\pi\sqrt{n(x+h)} - \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)\right).$$

By writing

$$\cos\left(2\pi\sqrt{n(x+h)} - \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right) \\ = \sum_{j=0}^{1} (-1)^{j+1} \cos\left(2\pi\sqrt{n(x+jh)} - \frac{\pi}{4}\right)$$

we get

$$F_{22}(x) = \frac{2^{4-2l}x^{2l-1/2}}{(l-1)!^2\pi^2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}\sum_{1\le m\ne n\le z}\frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \times \cos\left(2\pi\sqrt{m(x+j_1h)}-\frac{\pi}{4}\right)\times\cos\left(2\pi\sqrt{n(x+j_2h)}-\frac{\pi}{4}\right).$$

By the elementary formula

$$\cos a\cos b = \frac{\cos(a-b)+\cos(a+b)}{2}$$

we have

$$F_{22}(x) = F_{221}(x) + F_{222}(x), \tag{5.12}$$

where

$$F_{221}(x) = \frac{2^{3-2l}x^{2l-1/2}}{(l-1)!^2\pi^2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}\sum_{1\le m\ne n\le z}\frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \times\cos\left(2\pi\sqrt{m(x+j_1h)}-2\pi\sqrt{n(x+j_2h)}\right),$$

$$F_{222}(x) = \frac{2^{3-2l}x^{2l-1/2}}{(l-1)!^2\pi^2}\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}\sum_{1\le m\ne n\le z}\frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \times\sin\left(2\pi\sqrt{m(x+j_1h)}+2\pi\sqrt{n(x+j_2h)}\right).$$

Let

$$g_\pm(x) = 2\pi\sqrt{m(x+j_1h)} \pm 2\pi\sqrt{n(x+j_2h)}.$$

By the power series expansion

$$(1+t)^{1/2} = 1+\sum_{v=1}^{\infty}d_vt^v \quad (|t|\le 1/2) \tag{5.13}$$

we get that

$$g_\pm(x) = 2\pi\sqrt{x}(\sqrt{m}\pm\sqrt{n}) + 2\pi\sum_{v=1}^{\infty}\frac{d_vh^v}{x^{v-1/2}}(\sqrt{m}j_1^v\pm\sqrt{n}j_2^v),$$

which implies

$$|g'_\pm(x)| \gg x^{-1/2}|\sqrt{m}\pm\sqrt{n}| \ (m\ne n)$$

by noting that $m,n\le \varepsilon Uh^{-1}$. By the first derivative test we have

$$\begin{aligned}\int_U^{2U}F_{221}(x)dx &\ll U^{2l}\sum_{1\le m\ne n\le z}\frac{|\tau_l(m)\tau_l(n)|}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|} \\ &\ll U^{2l}\sum_{1\le m\ne n\le z}\frac{d(m)d(n)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|} \\ &\ll U^{2l}\log^4 z,\end{aligned} \tag{5.14}$$

where in the last step we have used the well-known estimate

$$\sum_{1\le m\ne n\le y}\frac{d(m)d(n)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|} \ll \log^4 y,\ \ y\ge 10.$$

We also have

$$\begin{aligned}(5.15)\qquad \int_U^{2U} F_{222}(x)dx &\ll U^{2l} \sum_{1\le m\ne n\le z} \frac{|\tau_l(m)\tau_l(n)|}{(mn)^{3/4}|\sqrt{m}+\sqrt{n}|}\\ &\ll U^{2l} \sum_{1\le m<n\le z} \frac{d(m)d(n)}{m^{3/4}n^{5/4}}\\ &\ll U^{2l}\log^3 z,\end{aligned}$$

by the well-known estimate $\sum_{n\le y} d(n) \ll y\log y$.

From (5.12), (5.14) and (5.15) we have

$$(5.16)\qquad \int_U^{2U} F_{22}(x)dx \ll U^{2l}\log^4 z.$$

By using the formulas

$$\cos u - \cos v = -2\sin(\frac{u+v}{2})\sin(\frac{u-v}{2})$$

and

$$\sin^2 u = (1-\cos 2u)/2$$

we have

$$\begin{aligned}(5.17)\qquad &\int_U^{2U} F_{21}(x)dx\\ &= \frac{2^{6-2l}}{(l-1)!^2\pi^2} \sum_{1\le n\le z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac12}\sin^2\left(\pi\sqrt{n(x+h)}+\pi\sqrt{nx}-\frac{\pi}{4}\right)\\ &\qquad\times\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)dx = S_1 - S_2,\end{aligned}$$

for instance, where

$$\begin{aligned}S_1 &= \frac{2^{5-2l}}{(l-1)!^2\pi^2} \sum_{1\le n\le z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac12}\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)dx,\\ S_2 &= \frac{2^{5-2l}}{(l-1)!^2\pi^2} \sum_{1\le n\le z} \frac{\tau_l^2(n)}{n^{3/2}}\\ &\times \int_U^{2U} x^{2l-\frac12}\sin\left(2\pi\sqrt{n(x+h)}+2\pi\sqrt{nx}\right)\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)dx.\end{aligned}$$

For each $n\le z$, let $L_n(t) = \int_U^t x^{2l-1/2}\sin\left(2\pi\sqrt{n(x+h)}+2\pi\sqrt{nx}\right)dx$. By the first derivative test

$$(5.18)\qquad L_n(t) \ll U^{2l}n^{-1/2}, U\le t\le 2U.$$

So by partial summation

$$\begin{aligned}&\int_U^{2U} x^{2l-\frac{1}{2}} \sin\left(2\pi\sqrt{n(x+h)}+2\pi\sqrt{nx}\right)\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)dx\\&=\int_U^{2U}\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)dL_n(x)\\&=L_n(2U)\sin^2\left(\pi\sqrt{n(2U+h)}-\pi\sqrt{2nU}\right)\\&-\int_U^{2U}L_n(x)\sin\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)\cos\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)\\&\qquad\times\left(\frac{\pi\sqrt{n}}{\sqrt{x+h}}-\frac{\pi\sqrt{n}}{\sqrt{x}}\right)dx\\&\ll U^{2l}n^{-1/2}+hU^{2l-1/2}.\end{aligned}$$

Thus we get

$$S_2 \ll \sum_{1\le n\le z}\frac{\tau_l^2(n)}{n^{3/2}}(U^{2l}n^{-1/2}+hU^{2l-1/2}) \ll U^{2l}+hU^{2l-1/2} \ll U^{2l}. \tag{5.19}$$

By (5.13) we have

$$\pi\sqrt{n(x+h)}-\pi\sqrt{nx}=\frac{\pi h\sqrt{n}}{2\sqrt{x}}+O(\frac{h^2\sqrt{n}}{x^{3/2}}),$$

which implies that

$$\sin^2\left(\pi\sqrt{n(x+h)}-\pi\sqrt{nx}\right)=\sin^2\frac{\pi h\sqrt{n}}{2\sqrt{x}}+O(\frac{h^2\sqrt{n}}{x^{3/2}}).$$

Thus

$$\begin{aligned}S_1&=\frac{2^{5-2l}}{(l-1)!^2\pi^2}\sum_{1\le n\le z}\frac{\tau_l^2(n)}{n^{3/2}}\int_U^{2U}x^{2l-\frac{1}{2}}\sin^2\frac{\pi h\sqrt{n}}{2\sqrt{x}}dx\\&\quad+O\left(\sum_{1\le n\le z}\frac{|\tau_l^2(n)|}{n^{3/2}}\int_U^{2U}x^{2l-\frac{1}{2}}\frac{h^2\sqrt{n}}{x^{3/2}}dx\right)\\&=\frac{2^{5-2l}}{(l-1)!^2\pi^2}\sum_{1\le n\le z}\frac{\tau_l^2(n)}{n^{3/2}}\int_U^{2U}x^{2l-\frac{1}{2}}\sin^2\frac{\pi h\sqrt{n}}{2\sqrt{x}}dx\\&\quad+O\left(h^2U^{2l-1}\sum_{1\le n\le z}\frac{|\tau_l^2(n)|}{n}\right).\end{aligned}\tag{5.20}$$

Since $|\tau_l(n)|\le d(n)$, we have the estimate

$$\sum_{n\le y}\tau_l^2(n)\ll\sum_{n\le y}d^2(n)\ll y\log^3 y \quad (y\ge 2), \tag{5.21}$$

which immediately implies that

$$h^2U^{2l-1}\sum_{1\le n\le z}\frac{\tau_l^2(n)}{n}\ll h^2U^{2l-1}\log^4 z\ll hU^{2l-1/2}\log^4 U. \tag{5.22}$$

From (5.21) we can get

$$\begin{aligned}(5.23)\qquad &\sum_{1\le n\le z}\frac{\tau_l^2(n)}{n^{3/2}}\int_U^{2U}x^{2l-\frac12}\sin^2\frac{\pi h\sqrt{n}}{2\sqrt{x}}dx\\ &\ll \sum_{1\le n\le z}\frac{d^2(n)}{n^{3/2}}\int_U^{2U}x^{2l-\frac12}\min\left(1,\frac{h^2n}{x}\right)dx\\ &\ll h^2U^{2l-1/2}\sum_{1\le n\le U/h^2}\frac{d^2(n)}{n^{1/2}}+U^{2l+1/2}\sum_{U/h^2<n\le z}\frac{d^2(n)}{n^{3/2}}\\ &\ll hU^{2l}\log^3\frac{U}{h^2}\ll hU^{2l}\log^3\frac{\sqrt{U}}{h}.\end{aligned}$$

Combining (5.20), (5.22) and (5.23) we get

$$S_1\ll hU^{2l}\log^3\frac{\sqrt{U}}{h}, \tag{5.24}$$

which together with (5.17) gives

$$\int_U^{2U}F_{21}(x)dx\ll hU^{2l}\log^3\frac{\sqrt{U}}{h}. \tag{5.25}$$

From (5.9)-(5.11), (5.16) and (5.25) we get

$$\int_U^{2U}(R_1(x+h)-R_1(x))^2dx\ll hU^{2l}\log^3\frac{\sqrt{U}}{h}+U^{2l}\log^4 z. \tag{5.26}$$

Now combining (5.5), (5.8) and (5.26) we get

$$J(U,h)\ll hU^{2l}\log^3\frac{\sqrt{U}}{h}+U^{2l}\log^4 U,$$

which immediately implies that

$$\int_2\ll hT^{2l}\log^3\frac{\sqrt{T}}{h}+T^{2l}\log^4 T \tag{5.27}$$

via a splitting argument. Finally Lemma 5.1 follows from (5.3), (5.4) and (5.27). □

6. Proof of Theorem 2

In this section we shall prove Theorem 2. Our approach is a variant of the proof of Theorem 2 of [12].

Define

$$R_+(t)=\begin{cases}R(t), & \text{if } R(t)>0,\\ 0, & \text{otherwise}\end{cases}$$

and

$$R_-(t)=|R(t)|-R_+(t).$$

We first prove the following two lemmas.

Lemma 6.1. *The estimate*

$$\int_T^{2T}R_\pm^2(t)dt\gg T^{2l+1/2} \tag{6.1}$$

holds.

Proof. From (1.11) and (1.13) with $k = 4$, we get by Hölder's inequality that

$$T^{2l+1/2} \ll \int_T^{2T} R^2(t)dt \le \left(\int_T^{2T} |R(t)|dt\right)^{2/3} \left(\int_T^{2T} R^4(t)dt\right)^{1/3}$$
$$\le \left(\int_T^{2T} |R(t)|dt\right)^{2/3} T^{4l/3}.$$

Thus

$$\int_T^{2T} |R(t)|dt \gg T^{l+3/4}. \tag{6.2}$$

From (3.1) and Lemma 6.1 of [32], it is easy to verify that

$$\int_T^{2T} R(t)dt \ll T^{l+1/2}\mathcal{L}^2,$$

which implies

$$\int_T^{2T} R_\pm(t)dt \gg T^{l+3/4} \tag{6.3}$$

in view of (6.2). By (6.3) and Cauchy-Schwarz's inequality, it follows that

$$T^{l+3/4} \ll \left(\int_T^{2T} dt\right)^{1/2} \left(\int_T^{2T} R_\pm^2(t)dt\right)^{1/2}$$
$$\ll T^{1/2} \left(\int_T^{2T} R_\pm^2(t)dt\right)^{1/2},$$

which immediately implies Lemma 6.1. □

Lemma 6.2. *Suppose* $2 \le H \le \sqrt{T}$. *Then*

$$\int_T^{2T} \max_{h\le H}(R_\pm(t+h) - R_\pm(t))^2 dt \ll HT^{2l}\log^5 T.$$

Proof. It is easy to verify that

$$|R_\pm(t+h) - R_\pm(t)| \le |R(t+h) - R(t)|,$$

so it is sufficient to prove the estimate

$$I = \int_T^{2T} \max_{h\le H}(R(t+h) - R(t))^2 dt \ll HT^{2l}\log^5 T. \tag{6.4}$$

Write

$$H = 2^\lambda b$$

such that $\lambda \in \mathbb{N}$ and $1 \le b < 2$. Similar to the argument of the proof of Lemma 2 of [12], we can deduce by using Lemma 5.1 that

$$\begin{aligned} I &\ll \lambda \sum_{\mu \le \lambda} \sum_{0 \le \nu < 2^\mu} \int_{T+\nu 2^{\lambda-\mu} b}^{2T+\nu 2^{\lambda-\mu} b} |R(t+2^{\lambda-\mu} b) - R(t)|^2 dt + T^{2l} \log^2 T \\ &\ll \lambda \sum_{\mu \le \lambda} \sum_{0 \le \nu < 2^\mu} (T^{2l} 2^{\lambda-\mu} b \log^3 T + T^{2l} \log^4 T) \\ &\ll \lambda \sum_{\mu \le \lambda} (T^{2l} 2^{\lambda} b \log^3 T + 2^\mu T^{2l} \log^4 T) \\ &\ll \lambda^2 H T^{2l} \log^3 T + \lambda H T^{2l} \log^4 T \\ &\ll H T^{2l} \log^5 T, \end{aligned}$$

namely (6.4) holds. □

Now we finish the proof of Theorem 2. For any function $P(t)$ and $Q(t)$, if

$$\omega(t) = P^2(t) - 4 \max_{h \le H} (P(t+h) - P(t))^2 - Q^2(t) > 0,$$

then

$$|P(t)| \ge 2 \max_{h \le H} |P(t+h) - P(t)|$$

and

$$|P(t)| \ge |Q(t)|.$$

The first inequality implies, for any $0 \le h \le H$,

$$P(t) - \frac{1}{2}|P(t)| \le P(t+h) \le P(t) + \frac{1}{2}|P(t)|,$$

and hence $P(t+h)$ has the same sign as $P(t)$. Moreover, by the second inequality above we get

$$|P(t+h)| \ge \frac{1}{2}|P(t)| \ge \frac{1}{2}|Q(t)|.$$

Now take $P(t) = R_\pm(t)$ and $Q(t) = \delta t^{l-1/4}$ for a sufficiently small $\delta > 0$. By Lemma 6.1 and Lemma 6.2 we get

$$\begin{aligned} \int_T^{2T} \omega(t) dt &\gg T^{2l+1/2} - O(H T^{2l} \log^5 T) - O(\delta^2 T^{2l+1/2}) \\ &\gg T^{2l+1/2} \end{aligned} \tag{6.5}$$

by taking $H = \delta \sqrt{T} \log^{-5} T$. Let $\mathscr{S} = \{t \in [T, 2T] : \omega(t) > 0\}$. By (6.5), the Cauchy-Schwarz inequality and (1.13) with $k = 4$ we get

$$\begin{aligned} T^{2l+1/2} &\ll \int_T^{2T} \omega(t) dt \le \int_{\mathscr{S}} \omega(t) dt \le \int_{\mathscr{S}} R_\pm^2(t) dt \\ &\le |\mathscr{S}|^{1/2} \left(\int_T^{2T} R^4(t) dt \right)^{1/2} \ll |\mathscr{S}|^{1/2} T^{2l}. \end{aligned}$$

Thus we get

$$|\mathscr{S}| \gg T.$$

This completes the proof of Theorem 2.

Remark for Theorem 4. The proof of Theorem 4 is the same except that we use $\log^3 T$ instead of $\log^5 T$. Here $\log^3 T$ appears since for $R_I(t)$ we can prove that the estimate

$$\int_1^T (R_I(t+h)-R_I(t))^2 dt \ll hT\log\frac{\sqrt{T}}{h} + T\mathcal{L}\log\mathcal{L} \tag{6.6}$$

holds for $1 \le h \le \sqrt{T}/2$, which implies that the $\log^5 T$ in Lemma 6.2 can be replaced by $\log^3 T$ if we have $R_I(t)$ in place of $R(t)$.

7. Proof of Theorem 3

In this section we prove Theorem 3. Suppose $k \ge 3$ is a fixed odd integer and T is a large parameter. Define

$$\delta = \begin{cases} -1, & \text{if } C_{k,l} \ge 0, \\ 1, & \text{if } C_{k,l} < 0, \end{cases}$$

where $C_{k,l}$ is defined in the formula (1.13).

By Theorem 2, let $t \in [T, 2T]$ such that

$$\delta R(u) > c_5 t^{l-1/4}, u \in [t, t+H],$$

with $H = c_4\sqrt{T}\log^{-5} T$. Then

$$\begin{aligned} c_5^k H t^{k(l-1/4)} &< \int_t^{t+H} \delta^k R^k(u)du = \delta^k \int_t^{t+H} R^k(u)du \\ &= C_{k,l}\delta^k \left((t+H)^{k(l-1/4)+1} - t^{k(l-1/4)+1}\right) \\ &\quad + \delta^k\left(\mathcal{F}_{k,l}(t+H) - \mathcal{F}_{k,l}(t)\right) \\ &= C_{k,l}\delta^k(k(l-1/4)+1)Ht^{k(l-1/4)} + O(H^2 t^{k(l-1/4)-1}) \\ &\quad + \delta^k\left(\mathcal{F}_{k,l}(t+H) - \mathcal{F}_{k,l}(t)\right). \end{aligned}$$

Hence we get

$$\delta^k\left(\mathcal{F}_{k,l}(t) - \mathcal{F}_{k,l}(t+H)\right) < B_{k,l}Ht^{k(l-1/4)}\left(1 + O(HT^{-1})\right), \tag{7.1}$$

where

$$B_{k,l} = C_{k,l}\delta^k(k(l-1/4)+1) - c_5^k \le -c_5^k < 0. \tag{7.2}$$

From (7.1) and (7.2) we have

$$|\mathcal{F}_{k,l}(t) - \mathcal{F}_{k,l}(t+H)| \gg Ht^{k(l-1/4)},$$

and Theorem 3 hence follows.

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