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BIRTH-DEATH PROCESSES AND q-CONTINUED FRACTIONS

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Abstract. In the 1997 paper of Parthasarathy, Lenin, Schoutens, and Van Assche, the authors study a birth-death process related to the Rogers-Ramanujan continued fraction $r(q)$. We generalize their results to establish a correspondence between birth-death processes and a larger family of q-continued fractions. It turns out that many of these continued fractions, including $r(q)$, play important roles in number theory, specifically in the theory of modular forms and q-series. We draw upon the number-theoretic properties of modular forms to give identities between the transition probabilities of different birth-death processes.

1. Introduction

A *birth-death process* is a continuous-time Markov chain in which the states represent the size of a population. The transition from state n to state $n+1$ is called a "birth" and the transition from state n to state $n-1$ is called a "death" (see §2.1). Birth-death processes are an important object of study in probability, stochastic processes, and queuing theory, as they can be used to model population growth, epidemics, and queue length [18]. In [16], Parthasarathy et al. establish a correspondence between a particular birth-death model and the celebrated Rogers-Ramanujan continued fraction

$$r(q) := \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\ddots}}} = 1 - q + q^2 - q^4 + \dots \tag{1.1}$$

It is this correspondence between birth-death models and q-continued fractions that we examine more generally here. Additionally, we explore the effect of the deep number-theoretic and analytic properties of the Rogers-Ramanujan continued fraction on the corresponding birth-death process.

A birth-death process is determined by its birth and death rates. More precisely, for each state n in a birth-death process there is a birth rate λ_n and a death rate μ_n (for a formal definition of birth-death processes, see §2.1). Let $p_0(t)$ be the probability that the population has size 0 at time t, given that it has size 0 at time 0. Define

$$f_0(s) := \int_0^\infty e^{-st} p_0(t)dt. \tag{1.2}$$

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As we shall see in §2.1, it turns out that $f_0(0)$ has the following suggestive representation:

$$f_0(0) = \cfrac{1}{\lambda_0 - \cfrac{\lambda_0\mu_1}{\lambda_1+\mu_1 - \cfrac{\lambda_1\mu_2}{\lambda_2+\mu_2 - \cfrac{\lambda_2\mu_3}{\ddots}}}}. \tag{1.3}$$

Since we are interested in finding birth-death processes corresponding to certain q-continued fractions, we let $\{a_n\} := \{a_n(q)\}$ and $\{b_n\} := \{b_n(q)\}$ be sequences of rational functions of q which are positive for $q \in (0,1)$. We now define two more sequences of rational functions of q from the sequences $\{a_n\}$ and $\{b_n\}$. Let $\{\lambda_n\} := \{\lambda_n(q)\}$ and $\{\mu_n\} := \{\mu_n(q)\}$ be defined by $\lambda_0 := b_0$ and

$$\begin{cases} \lambda_{n-1}\,\mu_n := a_n, & n \geq 1, \\ \lambda_n + \mu_n := b_n, & n \geq 1. \end{cases} \tag{1.4}$$

Notice that $\{a_n\}$ (resp. $\{b_n\}$) will be the numerators (resp. denominators) of the continued fraction (1.3). By (1.4) we have

$$\lambda_n = b_n - \frac{a_n}{\lambda_{n-1}}, \quad n \geq 1. \tag{1.5}$$

Call a birth-death model *good* if its birth rates and death rates are positive. Observe that if $\lambda_n > 0$, then also $\mu_{n+1} > 0$ since $\lambda_n\mu_{n+1} = a_{n+1} > 0$. Therefore to show that a process is good it suffices to show that $\lambda_n > 0$ for $n \geq 0$. In §2.4 we establish conditions for a sequence of recursively defined birth rates $\{\lambda_n(q)\}$ to give rise to a good birth-death process. Specifically, with $\{a_n\}$ and $\{b_n\}$ as above we prove the following two theorems for $q \in (0,1)$ which provide conditions for verifying that a process is good.

Theorem 1.1. *Suppose that the following conditions are met.*
(a) There exists $k > 0 \in \mathbb{R}$ such that for all $q \in (0,1)$, we have $0 < b_n(q) < k$.
(b) There exists $C_q > 0 \in \mathbb{R}$, depending on q, such that

$$0 \leq \frac{d}{dq}\left(\sqrt{a_n(q)}\right) < C_q \quad \text{and} \quad 0 \leq -\frac{d}{dq}(b_n(q)) < C_q.$$

Then there exists $q_1 \in [0,1]$ such that $\lambda_n(q) > 0$ for all $n \geq 0$ if and only if $q \leq q_1$.

Remark. The main result of [16] is the special case of Theorem 1.1 where $b_n = 1$ and $a_n = q^n$, in which case $q_1 \approx 0.576$.

Theorem 1.2. *Let $\alpha > 0 \in \mathbb{R}$, and suppose that there exists $k > 0$ with the following properties.*
(a) $\lambda_k \geq \frac{1}{\alpha}\sqrt{a_{k+1}}$.
(b) For all $n \geq k$, we have $b_n - \alpha\sqrt{a_n} \geq \frac{1}{\alpha}\sqrt{a_{n+1}}$.
Then for all $n > k$, we have $\lambda_n \geq b_n - \alpha\sqrt{a_n} > 0$.

Theorem 1.2 will prove especially useful later in the examples.

A process is *recurrent* if it returns to some state with probability 1 and *transient* otherwise. It is *positive recurrent* if the expected time of return is finite. (For a more formal definition, see §2.5.) It turns out that recurrence of a birth-death process is related to the convergence of (1.3). In §2.5 we prove that a birth-death process satisfying the hypotheses of Theorem 1.1 is transient for $q < q_1$ and positive recurrent for $q = q_1$, and we show that a birth-death process satisfying the hypotheses of Theorem 1.2 for $q \in (0, 1)$ is transient on that range. We also examine the asymptotic behavior of the birth and death rates.

In §2.3 we analyze the continued fractions from a q-series perspective to compute formulas for polynomials associated with the birth-death processes. We then discuss a method of recovering transition probabilities such as $p_0(t)$ from the corresponding continued fraction. These results end up requiring important constructions in the theory of q-series. For example, the following result expresses the convergents in terms of the Pochhammer symbol and the q-binomial coefficient (see §2.3).

Corollary. *Let $A_n(s)/B_n(s)$ be the nth convergent of the continued fraction*

$$\frac{1}{s+1+\cfrac{aq}{s+1+bq+\cfrac{aq^2}{s+1+bq^2\dots}}}. \tag{1.6}$$

Then

$$\frac{A_n(s)}{B_n(s)} = \frac{q^{1/2}\displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(s+1)(-bq^{1/2}/(s+1);q)_{n-j}}{(-bq^{1/2}/(s+1);q)_j}}{\displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b/(s+1);q)_{n-j}}{(-b/(s+1);q)_j}}. \tag{1.7}$$

The above equation allows us to approximate the transition probabilities of a birth-death process from the corresponding continued fraction via (1.3).

Remark. The Rogers-Ramanujan continued fraction (1.1) is the case $a = 1$, $b = s = 0$ of the continued fraction (1.6).

In §3.1 we give several examples of famous number-theoretic q-continued fractions which correspond to good birth-death processes. One of them is

$$\frac{1}{1-\cfrac{q^3}{1+q^5-\cfrac{q^5}{\ddots}}} \tag{1.8}$$

whose power series expansion is essentially the classical Jacobi theta function

$$\vartheta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}. \tag{1.9}$$

Given a good birth-death model $\mathcal{B}$, denote by $f_{\mathcal{B}}(q)$ the normalized Laplace transform of $p_0(t)$ evaluated at $s = 0$ associated with $\mathcal{B}$ (in other words, the $f_0(s)$ as in equation (1.2) for the particular birth-death model $\mathcal{B}$). In the last section we prove some surprising relations between the transition probabilities $p_0(t)$ for different birth-death processes, including one for the birth-death process associated with $r(q)$, the Rogers-Ramanujan continued fraction (1.1). For example, let the birth-death process corresponding to (1.8) be $\mathcal{B}_3$; we will show that if $y > 0$, then we have

$$1 + 2e^{-2\pi y} f_{\mathcal{B}_3}\left(e^{-2\pi y}\right) = \sqrt{2/y}\left(1 + e^{-\frac{\pi}{2y}} f_{\mathcal{B}_3}\left(e^{-\frac{\pi}{2y}}\right)\right). \tag{1.10}$$

The above equation is a consequence of the fact that $\vartheta_3(q)$ essentially transforms to itself under the mapping $\tau \to -1/\tau$ (where $\tau \in \mathbb{H}$ with $q = e^{2\pi i \tau}$) on the upper half of the complex plane, which is in turn a consequence of the theory of modular forms. (Recall that a *modular form* is a type of meromorphic function that satisfies a special transformation property. For a formal definition, see [15].)

2. Theory of q-continued fractions and birth-death processes

Here we recall the formal definition of a birth-death process and important facts about orthogonal polynomials and q-continued fractions. We also prove Theorems 1.1 and 1.2 and study the asymptotic behavior of the birth-death processes.

2.1. **Birth-death processes.** Here we closely follow the treatment of the subject as presented in [18]. Given that a population has size m initially, let $p_{m,n}(t)$ be the probability that the population is at size n after time t. By definition, a birth-death process has the following *transition probabilities* for $m \geq 0$:

$$p_{m,m+1}(t) := \lambda_m t + o(t),$$
$$p_{m,m-1}(t) := \mu_m t + o(t),$$

where $\lim_{t\to 0} o(t)/t = 0$. We call λ_n the birth rates and μ_n the death rates of this process. These probabilities $p_{m,n}$ satisfy the *Kolmogorov forward equations for birth-death processes* [18]:

$$p'_{m,0}(t) = -\lambda_0 p_{m,0}(t) + \mu_1 p_{m,1}(t), \tag{2.1}$$

$$p'_{m,n}(t) = \lambda_{n-1} p_{m,n-1}(t) - (\lambda_n + \mu_n) p_{m,n}(t) + \mu_{n+1} p_{m,n+1}(t), \tag{2.2}$$

where the derivative is taken with respect to t. We are interested in the case where the initial population is zero, so we will denote $p_n(t) := p_{0,n}(t)$.

Following [11, p. 366], let $f_n(s)$ be the normalized Laplace transform of $p_n(t)$:

$$f_n(s) := (-1)^n \left(\prod_{i=1}^{n} \mu_i\right) \int_0^\infty e^{-st} p_n(t)dt,$$
$$f_0(s) := \int_0^\infty e^{-st} p_0(t)dt.$$

Taking the normalized Laplace transforms of (2.1) and (2.2) yields the recurrence relations

$$\frac{1}{f_0(s)} = s + \lambda_0 + \frac{f_1(s)}{f_0(s)}, \tag{2.3}$$

$$\frac{f_n(s)}{f_{n-1}(s)} = \frac{-\lambda_{n-1}\mu_n}{s + \lambda_n + \mu_n + \frac{f_{n+1}(s)}{f_n(s)}}. \tag{2.4}$$

These recurrence relations yield the continued fraction

$$f_0(s) = \frac{1}{s + b_0 -} \quad \frac{a_1}{s + b_1 -} \quad \frac{a_2}{s + b_2 +} \quad \cdots, \tag{2.5}$$

where $b_0 := \lambda_0$ and for $n \geq 1$ we have $a_n := \lambda_{n-1}\mu_n$ and $b_n := \mu_n + \lambda_n$. (Henceforth we adopt the notation for continued fractions as in (2.5).) We will be examining $f_0(0) = \int_0^\infty p_0(t)dt$, which is well defined for $q < q_1$ for some $q_1 \in (0, 1)$ to be specified in §2.5.

2.2. **Orthogonal polynomials.** Central to the study of birth-death processes and continued fractions are orthogonal polynomials. A sequence of polynomials is called *orthogonal* if there exists a distribution function ψ such that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\psi(x) = 0 \text{ for } m \neq n \text{ and}$$
$$\int_{-\infty}^{\infty} P_n^2(x)d\psi(x) \neq 0.$$

By Favard's Theorem [6, p. 21], polynomials defined by the following recurrence relation are orthogonal:

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n P_{n-1}(x), \tag{2.6}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, $b_n \in \mathbb{R}$ and $a_n > 0$.

In [13], Karlin and McGregor exhibit deep connections between birth-death processes and orthogonal polynomials, using the polynomials to compute approximations of the transition probabilities. Orthogonal polynomials will play a key role in our methods as well, as they have many useful and interesting properties. We now list some properties of orthogonal polynomials that will be used in later sections.

(F1) The zeros of P_n are real and distinct. The zeros of P_n and P_{n+1} interlace ([6, p. 28]).

(F2) The *spectrum* of ψ, defined by

$$S := \{x \; : \; \psi(x + \delta) - \psi(x - \delta) > 0 \text{ for all } \delta > 0\}$$

is countably infinite, and for each $s \in S$, every neighborhood of s contains a zero of infinitely many polynomials $P_n(x)$ ([6, p. 51, p. 60]).

(F3) The orthonormal polynomials corresponding to $P_n(x)$ are ([6, p. 23])

$$p_n(x) := \frac{P_n(x)}{(a_1 a_2 \cdots a_n)^{1/2}}$$

with a_i as in (2.6).

(F4) If $s \in S$, then ([6, p. 63])

$$0 < \psi(s+) - \psi(s-) \leq \left(\sum_{k=0}^{\infty} p_k^2(s) \right)^{-1},$$

where $\psi(s+) := \lim_{\delta \to 0^+} \psi(s + \delta)$ and $\psi(s-) := \lim_{\delta \to 0^-} \psi(s + \delta)$.

2.3. **Explicit formulas through generating functions.** In this section we analyze generating functions for sequences of orthogonal polynomials associated with birth-death probability models. We derive explicit formulas for the orthogonal polynomials (and consequently the birth and death rates) and the convergents of their corresponding continued fractions, which allows us to approximate the normalized Laplace transforms of the transition probabilities.

Let $\{\lambda_n\}$ be a sequence of real numbers satisfying $\lambda_0 = 1$ and $\lambda_{n-1}(1+bq^n-\lambda_n) = aq^n$ for $n \geq 1$. Define R_n by $R_0 = 1$ and $\lambda_n = R_{n+1}/R_n$. Then we have

$$R_{n+1} = (1 + bq^n)R_n - aq^n R_{n-1}.$$

Let $\{P_n(x)\}$ be a sequence of polynomials satisfying the conditions $P_{-1}(x) = 0$, $P_0(x) = 1$, and

$$P_{n+1}(x) = (x + bq^n)P_n(x) - aq^n P_{n-1}(x) \tag{2.7}$$

so that $P_n(1) = R_n$. Observe that the $P_n(x)$ are orthogonal polynomials with respect to some distribution ψ (see (2.6)).

Theorem 2.1. *With notation as above,*

$$\frac{1}{x^n}P_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2}/x^{2j} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b/x;q)_{n-j}}{(-b/x;q)_j}. \tag{2.8}$$

Above and throughout, $(\alpha;q)_n := \prod_{i=0}^{n-1}(1-\alpha q^i)$ is the q-Pochhammer symbol, the infinite extension of the q-Pochhammer symbol is $(\alpha;q)_\infty := \prod_{i=0}^{\infty}(1-\alpha q^k)$, and

$$\begin{bmatrix} a \\ b \end{bmatrix} = (q;q)_a / \left((q;q)_b (q;q)_{a-b}\right)$$

is the q-binomial coefficient.

Proof. We follow the approach of Ismail in [9]. First we construct the generating function $F(z) := \sum_{n=0}^{\infty} z^n P_n(x)$. Multiplying the recurrence relation (2.7) by z^{n+1} and summing from $n = 0$ to ∞ yields

$$F(z,q) = \frac{1}{1-xz}\left(1 + bz(1 - azq/b)F(qz)\right).$$

Iterating, and using $F(z,0) = 0$, we have

$$F(z,q) = \sum_{k=0}^{\infty} z^k b^k q^{k(k-1)/2} \frac{(azq/b;q)_k}{(xz;q)_{k+1}}. \tag{2.9}$$

Now we apply Gauss's q-binomial formula (see [12, p. 15])

$$(z;q)_k = \sum_{j=0}^{k} \frac{(q;q)_k}{(q;q)_j (q;q)_{k-j}} (-z)^j q^{j(j+1)/2} \tag{2.10}$$

and Heine's q-binomial formula [5, p. 8]

$$\frac{(\alpha z;q)_\infty}{(z;q)_\infty} = \sum_{i=0}^{\infty} \frac{(\alpha;q)_i}{(q;q)_i} z^i \tag{2.11}$$

to obtain

$$F(z,q)=\sum_{k=0}^{\infty} z^k b^k q^{k(k-1)/2} \sum_{i=0}^{\infty} \frac{(q^{k+1};q)_i}{(q;q)_i}(xz)^i \sum_{j=0}^{k} \frac{(q;q)_k(-az/b)^j}{(q;q)_j(q;q)_{k-j}} q^{j(j+1)/2}.$$

Collecting terms, switching the order of summation (see [20]), and substituting $n=i+j+k$, we have

$$F(z,q) = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{j(j+1)/2} a^j b^{-j} x^{n-j} \frac{(q;q)_{n-j}}{(q;q)_j} \sum_{k=j}^{n-j} x^{-k} b^k \frac{q^{k(k-1)/2}}{(q;q)_{k-j}(q;q)_{n-j-k}}.$$

Now equating coefficients of z^n and rewriting $(\alpha;q)_{n-j-k}$ as $\frac{(\alpha;q)_{n-j} q^{k(k+1)/2} q^{-k(n-j)}}{(q^{k+j-n}/\alpha;q)_k(-\alpha)^k}$ yields

$$P_n(x)=\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j q^{j(j+1)/2} a^j b^{-j} x^{n-j}}{(q;q)_j} \sum_{k=j}^{n-j} x^{-k} b^k q^{k(n-j)} \frac{(q^{j-n};q)_k}{(q;q)_{k-j}}.$$

Next, set $m=k-j$ and apply Heine's binomial formula (2.11) to obtain

$$P_n(x)=\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{nj-j(j-1)/2} a^j x^{n-2j} \frac{(q^{j-n};q)_j(-q^j b/x;q)_\infty}{(q;q)_j(-q^{n-j}b/x;q)_\infty}.$$

Equation (2.8) then follows after straightforward manipulations. □

Corollary 2.2. *With notation as above, we have*

$$R_n=\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b;q)_{n-j}}{(-b;q)_j}. \tag{2.12}$$

Remark. This affords us an explicit formula for λ_n.

Corollary 2.3. *If*

$$f(z)=(-bz;q)_\infty \sum_{j=0}^{\infty} \frac{(-1)^j a^j q^{j^2} z^{2j}}{(q;q)_j(-bz;q)_j}, \tag{2.13}$$

then

$$\lim_{n\to\infty} \frac{1}{x^n} P_n(x) = f(1/x). \tag{2.14}$$

Furthermore, let the positive elements of the support of the distribution function ψ with respect to which the polynomials $P_n(x)$ are orthogonal be $\{x_1, x_2, \dots\}$, where $x_1 > x_2 > x_3 > \dots > 0$ and let the zeros of (2.13) be $0 < z_1 < z_2 < z_3 \dots$. Then we have $z_k = 1/x_k$.

Proof. Equation (2.14) follows immediately by letting $n \to \infty$ in (2.8) and setting $z=1/x$. Let the zeros of $P_n(x)$ be $x_{n,k}, k=1,2,\dots,n$, where $x_{n,1} > x_{n,2} > x_{n,3} \dots x_{n,n}$. It is a well-known property of orthogonal polynomials (see §2.2 (F2)) that $\lim_{n\to\infty} x_{n,k} = x_k$. By (2.14), $\lim_{n\to\infty} f(1/x_{n,k}) = f(1/x_k) = 0$. □

Remark. If we set $b = 0, a = 1$, then the polynomials are given by $P_0 = 1$, $P_{-1} = 0$, and $P_{n+1}(x) = xP_n(x) - q^n P_{n-1}(x), n \geq 1$ and we recover (2.3) in [16]: $F(z;q) = \sum_{k=0}^{\infty}(-1)^k z^{2k} q^{k^2}/(q;q)_k$.

Corollary 2.4. *Let $A_n(s)/B_n(s)$ be the nth convergent of the continued fraction*

$$\frac{1}{s+1+}\,\frac{aq}{s+1+bq+}\,\frac{aq^2}{s+1+bq^2+}\cdots. \tag{2.15}$$

Then

$$\frac{A_n(s)}{B_n(s)} = \frac{q^{1/2}\displaystyle\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j a^j q^{j^2+j}\begin{bmatrix} n-j \\ j\end{bmatrix}\frac{(s+1)(-bq^{1/2}/(s+1);q)_{n-j}}{(-bq^{1/2}/(s+1);q)_j}}{\displaystyle\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j a^j q^{j^2}\begin{bmatrix} n-j \\ j\end{bmatrix}\frac{(-b/(s+1);q)_{n-j}}{(-b/(s+1);q)_j}}. \tag{2.16}$$

Moreover, as $n \to \infty$ we find

$$\begin{aligned}&\frac{1}{s+1+}\,\frac{aq}{s+1+bq+}\,\frac{aq^2}{s+1+bq^2+}\cdots\\ &\qquad = \frac{(-b/(s+1);q)_\infty q^{1/2}\displaystyle\sum_{j=0}^{\infty}\frac{a^j q^{j^2+j}}{(q;q)_j(-bq^{1/2}/(s+1);q)_j}}{(s+1)(-bq^{1/2}/(s+1);q)_\infty\displaystyle\sum_{j=0}^{\infty}\frac{a^j q^{j^2}}{(q;q)_j(-b/(s+1);q)_j}}.\end{aligned}$$

Remark. Ramanujan mentions the case $s = 0$ of (2.15) in his first letter to Hardy [3, p. 22].

Proof. We have the recursive relations for convergents of a continued fraction

$$\begin{cases} A_0(s) = 0 \\ A_1(s) = 1 \\ A_{n+1}(s) = (s+1+bq^n)A_n(s) - aq^n A_{n-1}(s), \quad n \geq 1\end{cases}$$

and

$$\begin{cases} B_{-1}(s) = 0 \\ B_0(s) = 1 \\ B_{n+1}(s) = (s+1+bq^n)B_n(s) - aq^n B_{n-1}(s), \quad n \geq 0.\end{cases}$$

Observe $A_n(s)$ and $B_n(s)$ satisfy similar recurrences to those for $P_n(s)$. In fact, it is straightforward to verify that $B_n(s) = P_n(s+1)$ and

$$A_n(s) = q^{(n-1)/2}P_{n-1}((s+1)/q^{1/2}).$$

The corollary follows by applying (2.8). □

As we saw in §2.1, the fraction in (2.15) is the normalized Laplace transform of $p_0(t)$ when $\lambda_n + \mu_n = bq^n$ and $\lambda_{n-1}\mu_n = aq^n$. We now outline a well-known method (see [16] for details) to compute the inverse Laplace transform of the convergents of the fraction, which allows us to approximate the transition probabilities $p_m(t)$. We know that $A_n(s)/B_n(s)$ is the nth convergent of (2.15). Since the polynomials $B_n(s)$ are orthogonal, their zeros are real and distinct. Using (2.16), we can then write $A_n(s)/B_n(s)$ as a sum of partial fractions and easily find the inverse Laplace transform using the formula

$$\mathcal{L}^{-1}\left(\frac{A_n(s)}{B_n(s)}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{st}\frac{A_n(s)}{B_n(s)}d\omega, \tag{2.17}$$

where $s = c + i\omega$, $c > 0$ ([11, p. 369]).

2.4. **Conditions for good birth-death processes.** In this section, we prove Theorems 1.1 and 1.2. Recall that we establish these theorems to give conditions under which a q-continued fraction gives rise to a good birth-death process.

Proof of Theorem 1.1. Since b_n is bounded above by k, we may write $b_n = k - \beta_n$, where $0 \le \beta_n < k$. Define R_n by $R_0 = 1$ and

$$\frac{R_{n+1}}{R_n} = \lambda_n, \quad n \ge 1.$$

By (1.5), we know $R_{n+1} = (k - \beta_n)R_n - a_n R_{n-1}$. Next, define the polynomials $P_n(x) = P_n(x, q)$ by $P_{-1}(x) = 0$, $P_0(x) = 1$, and

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - a_n P_{n-1}(x), \quad n \ge 0. \tag{2.18}$$

Notice that $P_n(k) = R_n$. We will study the zeros of the polynomials $P_n(x)$ in order to determine the signs of $P_n(k)$ and hence the signs of λ_n. By Favard's Theorem (2.6), the polynomials $P_i(x)$ are orthogonal with respect to some distribution ψ, and therefore all the roots of $P_n(x)$ are distinct and real (see §2.2 (F1)). Define $x_n = x_n(q)$ to be the largest zero of $P_n(x)$. Also, define

$$X(q) := \lim_{n\to\infty} x_n(q).$$

By orthogonality, all roots of the polynomials $P_i(x)$ are bounded and the zeros of $P_i(x)$ and $P_{i+1}(x)$ interlace (see §2.2 (F1)). Therefore, X exists and is in fact an element of the spectrum of ψ as it is a limit point of x_n (see §2.2 (F2)).

Next, notice that the zeros of P_n are the eigenvalues of the tridiagonal matrix

$$M_n(q) := \begin{pmatrix} \beta_0 & \sqrt{a_1} & 0 & \cdots & 0 \\ \sqrt{a_1} & \beta_1 & \sqrt{a_2} & \ddots & \vdots \\ 0 & \sqrt{a_2} & \beta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{a_{n-1}} \\ 0 & \cdots & 0 & \sqrt{a_{n-1}} & \beta_{n-1} \end{pmatrix}.$$

Let $v_n = (u_1, u_2, \cdots, u_n)$ be an eigenvector corresponding to the largest eigenvalue x_n. By the Hellmann-Feynman Theorem [10],

$$x_n'(q) = \frac{\langle M_n'(q)v_n, v_n\rangle}{\langle v_n, v_n\rangle},$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product, the derivatives are with respect to q, and $M_n'(q)$ is formed by taking the derivative of each individual entry. By the Perron-Frobenius Theorem for matrices with non-negative entries [4], $u_i \ge 0$ for $1 \le i \le n$. This implies that $x_n'(q) \ge 0$, so $X(q)$ is a non-decreasing function.

Now, we will show that $X(q)$ is a continuous function for $q \in (0,1)$. Fix a $q_0 \in (0,1)$. By assumption, there exists a $C = C_q$ such that $\frac{d}{dq}\sqrt{a_n(q_0)} < C$ and $\frac{d}{dq}\beta_n(q_0) = -\frac{d}{dq}b_n(q_0) < C$ for all n. Hence,

$$\begin{aligned} x_n'(q_0) &= \frac{\langle M_n'(q_0)v_n, v_n\rangle}{\langle v_n, v_n\rangle} \\ &\leq \frac{C(u_1^2 + 2u_1u_2 + u_2^2 + 2u_2u_3 + \cdots + 2u_{n-1}u_n + u_n^2)}{u_1^2 + u_2^2 + \cdots + u_n^2} \\ &\leq 3C. \end{aligned}$$

Using that $x_n'(q_0) \leq 3C$, it is not difficult to show that X is continuous at q_0.

Next we will prove that $X(q) \leq k$ if and only if $\lambda_n > 0$ for all n. First, if $X(q) \leq k$, then $P_n(k) > 0$ for all n, so $\lambda_n > 0$ for all n. Next, assume that $X(q) > k$. If $P_n(k) = 0$ for some n, then λ_n does not exist for all n. Otherwise $P_n(k) \neq 0$, so either $x_1(q) < k$ or $x_1(q) > k$. If $x_1 > k$, then $P_1(k) < 0$, so $\lambda_0 < 0$. If $x_1 < k$, then there exists an n such that $x_n(q) < k$ and $x_{n+1}(q) > k$, so $P_{n+1}(k) < 0$ and therefore $\lambda_n < 0$.

In the cases in which $X(q) < k$ for all $q \in (0,1)$ or $X(q) > k$ for all $q \in (0,1)$, define $q_1 := 1$ or $q_1 := 0$, respectively. Otherwise, define q_1 to be the largest q such that $X(q) = k$. By the previous paragraph and the properties of $X(q)$, this q_1 satisfies the theorem.

□

Proof of Theorem 1.2. We will induct on n. By assumption, we know that

$$\lambda_k \geq \frac{1}{\alpha}\sqrt{a_{k+1}}.$$

Therefore

$$\lambda_{k+1} = b_{k+1} - \frac{a_{k+1}}{\lambda_k} \geq b_{k+1} - \alpha\sqrt{a_{k+1}}.$$

This establishes the base case. For the inductive hypothesis, suppose that

$$\lambda_{n-1} \geq b_{n-1} - \sqrt{a_{n-1}}$$

for $n - 1 > k$. Then we have

$$\lambda_n = b_n - \frac{a_n}{\lambda_{n-1}} \geq b_n - \frac{a_n}{b_{n-1} - \alpha\sqrt{a_{n-1}}} \geq b_n - \alpha\frac{a_n}{\sqrt{a_n}} = b_n - \alpha\sqrt{a_n}.$$

By assumption, $b_n - \alpha\sqrt{a_n} \geq \sqrt{a_{n+1}}/\alpha$, which is greater than zero since $\alpha, a_n > 0$. □

2.5. **Asymptotic behavior of birth-death processes.** We will now examine the limiting behavior of the birth-death processes to determine whether they will be *recurrent* or *transient*. Let us first define these terms and others used in the statements of the theorems which follow.

Definition 2.5 ([2], p. 155, p. 158)**.** A process is *recurrent* if

$$\int_0^\infty p_n(t)dt = \infty,$$

where $p_n(t)$ is a transition probability as in §2.1. A process is *transient* if it is not recurrent. Moreover, a recurrent process is *positive recurrent* if for each state n, $\lim_{t\to\infty} p_n(t) > 0$ and *null recurrent* if for each state n, $\lim_{t\to\infty} p_n(t) = 0$.

The *potential coefficients* are defined following the notation in [16]:

$$\pi_0 := 1, \tag{2.19}$$

$$\pi_n := \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = \frac{\lambda_0^2\lambda_1^2\cdots\lambda_{n-1}^2}{a_1a_2\cdots a_n} = \frac{P_n^2(k)}{a_1a_2\cdots a_n}, \tag{2.20}$$

with a_i as in (2.6) and where $k = 1$ if the continued fraction is of the second type.

As in [16], we will use Anderson's notation [2] to study the limiting behavior of the birth-death processes. Define the quantities

$$A := \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n}, \qquad B := \sum_{n=0}^{\infty} \pi_n.$$

The process corresponding to these potential coefficients is recurrent if $A = \infty$, and is positive recurrent if $A = \infty$ and $B < \infty$.

The following theorems will prove that for $q < q_1$ the birth-death processes are transient, and for $q = q_1 \in (0, 1)$ the processes are positive recurrent. Notice that this means that $f_0(0)$ is well defined because the integral $f_0(0) = \int_0^\infty p_0(t)dt$ is finite.

Theorem 2.6. *Let a_n, β_n, and k be defined as in Theorem* 1.1 *and its proof. Suppose for $q \in (0, q_1)$ that $0 < a_n \leq 1$. Moreover, suppose there exists some $N \in \mathbb{N}$ such that $\sup\{a_n\}_{n=N}^{\infty} = c$, where $0 < c < 1$ and $c^{\frac{1}{3}} \leq k - X(q)$. Then the birth-death process is transient. If we also have $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \beta_n = \beta$ for some $\beta \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \lambda_n = k - \beta \text{ and } \lim_{n\to\infty} \mu_n = 0.$$

Proof. Notice $P_n(k) = (k - x_{n,1})(k - x_{n,2})\cdots(k - x_{n,n})$, where the $x_{n,j}$ are the zeros of $P_n(x)$ such that $x_{n,i} < x_{n,j}$ for all $i < j$. Since the polynomials $P_n(x)$ are orthogonal polynomials, the zeros interlace (see §2.2 (F1)) and $(k - x_{n,j}) < (k - x_{n+1,j})$. Thus we have

$$P_{n+1}(k) > (k - x_{n+1,n+1})P_n(k) > (k - X(q))P_n(k) \geq c^{\frac{1}{3}}P_n(k). \tag{2.21}$$

Recall $X(q) = \lim_{n\to\infty} x_n(q)$, where x_n is the largest zero of P_n. Notice $X(q)$ is also the largest element in the spectrum of ψ, the distribution with respect to which the polynomials P_n are orthogonal (see §2.2 (F2)). Equation (2.21) implies

$$P_n(k) \geq (c^{\frac{1}{3}})^n \tag{2.22}$$

because $P_0(x) := 1$.

We will now show that B diverges. With the above inequality and the fact that $a_n \leq c$ for $n \geq N$, we have

$$B = 1 + \sum_{n=1}^{\infty} \frac{P_n^2(k)}{a_1a_2\cdots a_n} \geq 1 + \sum_{n=1}^{\infty} \frac{(c^{\frac{1}{3}})^{2n}}{a_1a_2\cdots a_n} \geq 1 + \sum_{n=1}^{N-1} c^{\frac{2n}{3}} + c^{N-1}\sum_{n=N}^{\infty}(c^{-\frac{1}{3}})^n,$$

which diverges because $c^{-\frac{1}{3}} > 1$. Thus $B = \infty$ and the process is not positive recurrent.

Next we will show that the process is not null-recurrent because

$$A := \sum_{n=0}^{\infty} 1/(\lambda_n\pi_n) < \infty.$$

From the same definition of π_n (2.20), and again using the fact that $a_n < c$ for all $n \geq N$ and $P_n(k) \geq (c^{\frac{1}{3}})^n$, we arrive at the following inequality:

$$A - \frac{1}{k-\beta_0} \leq \sum_{n=1}^{\infty} \frac{a_1 a_2 \cdots a_n}{P_n(k)P_{n+1}(k)} \leq \sum_{n=1}^{N-1} \frac{1}{(c^{\frac{1}{3}})^{2n+1}} + \sum_{n=N}^{\infty} \frac{c^{n-N+1}}{(c^{\frac{1}{3}})^{2n+1}}$$
$$= \frac{c^{\frac{2}{3}}(1-c^{\frac{2N}{3}})}{c^{\frac{2N}{3}}(c^{\frac{1}{3}}-c)} + \frac{c^{\frac{2}{3}(1-N)}}{1-c^{\frac{1}{3}}}.$$

Since this is finite, A is finite and the process therefore is not null-recurrent. Thus the process is transient.

We will now show that if $a_n \to 0$ and $\beta_n \to \beta$, then $\mu_n \to 0$ and $\lambda_n \to k - \beta$.

Because the zeros of $P_{n-1}(x)$ and $P_n(x)$ interlace, we find $P_n(k)/(k - x_{n,n}) \geq P_{n-1}(k)$. Recall that for $q < q_1$, we have $k > X(q) \geq x_{n,n}$, where $x_{n,n}$ is the largest zero of the nth polynomial $P_n(x)$, so the denominator $(k - x_{n,n}) > 0$. From the interlacing of the zeros of $P_n(x)$ and $P_{n+1}(x)$ (see §2.2 (F1)), the largest zeros of the polynomials P_n are increasing for a fixed q, the factors $(k-x_{n,n})$ are decreasing as n increases, and bounded below by $(k - X) > 0$. From the recursive definition of the polynomials P_n, and the fact that $\lambda_n = P_{n+1}(k)/P_n(k)$, we have

$$\lambda_n = \beta_n - a_n \frac{P_{n-1}(k)}{P_n(k)}.$$

We will now show that $\lim_{n\to\infty} \lambda_n$ is bounded above and below by $k-\beta$. Combining the facts deduced in the discussion above about λ_n, we have

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \left(\beta_n - a_n \frac{P_{n-1}(k)}{P_n(k)}\right) = k - \beta - \lim_{n\to\infty} a_n \frac{P_{n-1}(k)}{P_n(k)}.$$

But $a_n P_{n-1}(k)/P_n(k) \leq a_n/(k - x_{n,n}) \leq a_n/(k - X)$, so $\lim_{n\to\infty} \lambda_n \geq k - \beta - (\lim_{n\to\infty} a_n)/(k - X)$. Since $\lambda_n \leq \beta_n$ by definition for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \lambda_n = k - b$. This implies $\lim_{n\to\infty} \mu_n = 0$ because $\lambda_n + \mu_n = \beta_n$ for all n. Thus, when the numerators a_n converge to 0, the death rates converge to zero and the birth rates converge to $k - \beta$. □

Theorem 2.7. *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are defined as in Theorem* 1.2*, and also suppose that* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = b$*. Then*

$$\lim_{n\to\infty} \lambda_n = b, \quad \lim_{n\to\infty} \mu_n = 0,$$

and the process is transient.

Proof. By the definition of the λ_n, we have $b_n \geq \lambda_n$ for all n, and from Theorem 1.2 we have $\lambda_n \geq b_n - \alpha\sqrt{a_n}$ for sufficiently large n and some constant $\alpha > 0$ as in Theorem 1.2. Combining these two facts with the assumption that a_n and b_n converge, we have

$$b = \limsup_{n\to\infty} b_n \geq \limsup_{n\to\infty} \lambda_n \geq \liminf_{n\to\infty} \lambda_n \geq \liminf_{n\to\infty} (b_n - \alpha\sqrt{a_n}) = b,$$

and thus $\lim_{n\to\infty} \lambda_n = b$.

Since $b_n := \lambda_n + \mu_n$, we have $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty}(b_n - \lambda_n) = 0$. Thus $\lim_{n\to\infty} \mu_n = 0$.

Considering again the quantity A, we will now prove $A < \infty$, so the process described by these μ_n and λ_n is transient.

First, fix some $\epsilon > 0$. Since λ_n and μ_n converge, there exists some $N \in \mathbb{N}$ such that for $n \geq N$ we have $\mu_n < \epsilon$ and $\lambda_n < b + \epsilon$. Then

$$A - \sum_{n=0}^{N-1} \frac{\mu_1\mu_2\cdots\mu_n}{\lambda_0\lambda_1\cdots\lambda_n} = \frac{\mu_1\cdots\mu_{N-1}}{\lambda_0\cdots\lambda_{N-1}} \sum_{n=N}^{\infty} \frac{\mu_N\cdots\mu_n}{\lambda_N\cdots\lambda_n} \leq \frac{\mu_1\cdots\mu_{N-1}}{\lambda_0\cdots\lambda_{N-1}} \sum_{n=N}^{\infty} \frac{\epsilon^{n-N+1}}{(b+\epsilon)^{n-N+1}}$$

and so we have

$$A - \sum_{n=0}^{N-1} \frac{\mu_1\mu_2\cdots\mu_n}{\lambda_0\lambda_1\cdots\lambda_n} \leq \frac{\mu_1\cdots\mu_{N-1}}{\lambda_0\cdots\lambda_{N-1}}\frac{\epsilon}{b}.$$

Thus, since A is finite, the process described by these μ_n and λ_n is transient. □

Theorem 2.8. *Let a_n, b_n, β_n, and k be defined as in Theorem* 1.1 *and its proof. For $q = q_1 \in (0,1)$, if $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \beta_n = \beta$ for a_n and $b_n = k - \beta_n$ defined as in Theorem* 1.1*, the birth-death process is positive recurrent, and, if $\lim_{n\to\infty} \lambda_n$ exists,*

$$\lim_{n\to\infty} \lambda_n = 0 \textit{ and } \lim_{n\to\infty} \mu_n = k - \beta.$$

Proof. Consider the corresponding orthonormal polynomials $p_n(x) := P_n(x)/\sqrt{a_1a_2\cdots a_n}$ mentioned in §2.2 (F3). These polynomials satisfy the recurrence relation

$$\sqrt{a_{n+1}}p_{n+1}(x) = (x - \beta_n)p_n(x) - \sqrt{a_n}p_{n-1}(x)$$

with initial conditions $p_{-1} = 0$ and $p_0 = 1$. The p_n are orthonormal with respect to the distribution ψ, and $X(q_1) = k$ is in the spectrum of ψ, so by §2.2 (F4) we have

$$B = \sum_{n=0}^{\infty} \pi_n = 1 + \sum_{n=1}^{\infty} \frac{P_n^2(k)}{a_1a_2\cdots a_n} = \sum_{n=0}^{\infty} p_n^2(k) \leq \frac{1}{\psi(k+) - \psi(k-)} < \infty.$$

Since $\sum_{n=0}^{\infty} \pi_n$ converges, $\lim_{n\to\infty} \pi_n = 0$, so $1/\pi_n$ diverges, and thus $\sum_{n=0}^{\infty} 1/\pi_n = \infty$. We know $\lambda_n \leq b_n \leq k$, so

$$A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n} \geq \frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{\pi_n} = \infty.$$

Thus $A = \infty$ and $B < \infty$, so the process is positive recurrent.

We now use the fact that $B < \infty$ to determine the asymptotic behavior of the birth and death rates when $q = q_1$. We know that $B = 1 + \sum_{n=1}^{\infty} \lambda_0^2\lambda_1^2\cdots\lambda_{n-1}^2/(a_1a_2\cdots a_n)$ converges, so, by the ratio test for series convergence, for all $N \in \mathbb{N}$, there exists $n_N \geq N$ such that $\lambda_{n_N}^2/a_{n_N+1} \leq 1$. Thus, $0 < \lambda_{n_N}^2 \leq a_{n_N+1}$. By our assumption that $\lim_{n\to\infty} a_n = 0$, $\lim_{N\to\infty} \lambda_{n_N} = 0$. An infinite subsequence of $\{\lambda_n\}$ converges to 0, so if $\{\lambda_n\}$ converges, then $\lim_{n\to\infty}\lambda_n = 0$. Since $\lambda_n + \mu_n = b_n$, this implies $\lim_{n\to\infty} \mu_n = k - \beta$. □

3. Modular forms and special birth-death processes

In this section we first classify modular forms (see [15] for more on modular forms) satisfying relations which will turn out to have very interesting consequences for the transition probabilities of birth-death processes. We then examine modular continued fractions which correspond to special birth-death processes. Finally, we examine interesting relations between some special birth-death processes.

3.1. **Modular forms.** Here we establish definitions of *modularly symmetric* and *modularly reflexive* and illustrate the definition with two examples. Let f be a weight k modular form on a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, and let $\gamma = [a,b;c,d] \in GL_2(\mathbb{R})$. Then we define $\gamma(f) := (af+b)/(cf+d)$.

Definition 3.1. Suppose that f_1 and f_2 are weight k modular forms on a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$. Furthermore, suppose that there exists $\rho \in \mathbb{R}$, α and $l > 0$ in $\mathbb{Z}$, $\gamma \in GL_2(\mathbb{R})$, and $\chi : GL_2(\mathbb{R}) \to \mathbb{C}$ with $|\chi(\gamma)| = 1$ such that

$$f_1(-1/(\alpha\tau))^l = \rho\chi(\gamma)\tau^k\gamma(f_2(\tau)^l).$$

Then we say that $f_1(\tau)$ is *modularly symmetric* to $f_2(\tau)$. Moreover, if $f_1(\tau)$ is modularly symmetric to $f_2(\tau) = f_1(\tau)$, then we say that $f_1(\tau)$ is *modularly reflexive.*

Examples.
(1) If $f(\tau)$ is a modular form on $SL_2(\mathbb{Z})$, then $f(\tau)$ is modularly reflexive with $\alpha = 1$, $l = 1$, $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho = 1$, and $\chi(\gamma) = 1$.
(2) Slightly abusing notation, let $r(\tau) = q^{1/5}r(q)$, where $q = e^{2\pi i\tau}$. It is known (see §3.3) that

$$r\left(\frac{-1}{\tau}\right) = \frac{-\phi r(\tau) + 1}{r(\tau) + \phi},$$

where $\phi = (1+\sqrt{5})/2$ is the "golden ratio". Therefore, $r(q)$ is a weight 0 modularly reflexive form with $\alpha = 1$, $l = 1$, $\gamma = \begin{pmatrix} -\phi & 1 \\ 1 & \phi \end{pmatrix}$, $\rho = 1$, and $\chi(\gamma) = 1$.

As we shall see, birth-death processes corresponding to forms that are modularly symmetric satisfy very unexpected symmetry relations themselves (more precisely, the Laplace transforms of the transition probabilities evaluated at 0 satisfy interesting symmetry relations). The transformation $\tau \to -1/\tau$ on the positive imaginary axis maps $q = e^{2\pi i\tau}$ from the interval $(0,1)$ onto itself, and this is precisely the interval on which our birth-death models are good. In §3.3, we exhibit identities between the Laplace transforms of transition probabilities of different birth-death processes. Fundamentally, these identities exist because the modular forms associated to the birth-death processes are modularly symmetric.

3.2. **Interesting q-continued fractions corresponding to birth-death processes.** Here we examine examples of modular continued fractions due to Ramanujan, Eisenstein, and Jacobi which correspond to birth-death probability models with positivity of the birth rates guaranteed by Theorems 1.1 and 1.2. Throughout, let λ_n and μ_n be the birth and death rates respectively as in equation (1.4).

In the examples in Table 1, define $\mu_0 := 0$, and $b_0 = \lambda_0 = 1$.

Note the relation between $f_{\mathcal{B}_3}(q)$ and the classical Jacobi theta function, a modular form (see [8, p. 290]):

$$\vartheta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + \frac{2q}{1-}\ \frac{q^3}{1+q^3-}\ \frac{q^5}{1+q^5-}\cdots = 1 + 2qf_{\mathcal{B}_3}(q).$$

TABLE 1

Example	a_n	b_n	$f_{\mathcal{B}_i}(q)$	**Reference**
$\mathcal{B}_1$	q^{2n-1}	$1+q^n$	$\frac{1}{1-}\,\frac{q}{1+q-}\,\frac{q^3}{1+q^2-}\,\frac{q^5}{1+q^3-\dots} = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}$	[3, p. 20]
$\mathcal{B}_2$	q^{2n-1}	$1+q^{2n}$	$\frac{1}{1-}\,\frac{q}{1+q^2-}\,\frac{q^3}{1+q^4-}\,\frac{q^5}{1+q^6-\dots} = \frac{(q^3;q^4)_\infty}{(q;q^4)_\infty}$	[3, p. 22]
$\mathcal{B}_3$	q^{2n+1}	$1+q^{2n+1}$	$\frac{1}{1-}\,\frac{q^3}{1+q^3-}\,\frac{q^5}{1+q^5-}\cdots$	
$\mathcal{B}_4$	q^{2n}	$1+q^{2n}$	$\frac{1}{1-}\,\frac{q^2}{1+q^2-}\,\frac{q^4}{1+q^4-}\cdots$	

Also note the relationship between $f_{\mathcal{B}_4}(q)$ and the second Jacobi theta function, a modular form [8, p. 290]:

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} = \frac{2q^{1/4}}{1-}\,\frac{q^2}{1+q^2-}\,\frac{q^4}{1+q^4-}\cdots = 2q^{1/4} f_{\mathcal{B}_4}(q). \tag{3.1}$$

The proofs that these examples constitute good birth-death processes are simple applications of Theorem 1.2. For example, for the first one, we observe that $\lambda_0 = 1$, $\lambda_1 = 1 \geq 1+q-\sqrt{q} = b_1-\sqrt{a_1}$. Furthermore, $a_n-b_n = 1+q^n-q^{\frac{2n-1}{2}} > q^n > q^{\frac{2n+1}{2}}$. By Theorem 1.2, we have $0 < \sqrt{a_{n-1}} < b_n - \sqrt{a_n} < \lambda_n < b_n$. Therefore, this birth-death process is good.

Before continuing with the next 5 examples, we introduce the notion of an "even part" of a continued fraction (see [11, p. 41]). Let

$$K(a_n/b_n) = \frac{a_0}{b_0+}\,\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots \tag{3.2}$$

be a continued fraction. Denote its nth convergent by C_n. Let $C_n^* = C_{2n}$. Then the *even part* of $K(a_n/b_n)$ is the continued fraction

$$K(a_n^*/b_n^*) = \frac{a_0^*}{b_0^*+}\,\frac{a_1^*}{b_1^*+}\,\frac{a_2^*}{b_2^*+}\cdots \tag{3.3}$$

with convergents C_n^*. Clearly if $K(a_n/b_n)$ converges, then $K(a_n^*/b_n^*)$ converges to the same value. The following examples all converge for $q < 1$. Furthermore, we have the equations

$$\begin{cases} a_0^* = a_0 b_1, \quad b_0^* = a_1 + b_0 b_1 \\ a_k^* = -a_{2k}a_{2k-1}b_{2k+1}/b_{2k}, \ k \geq 1 \\ b_k^* = a_{2k}b_{2k+1}/b_{2k-1} + a_{2k+1} + b_{2k}b_{2k+1}, \ k \geq 1. \end{cases} \tag{3.4}$$

3.3. **Symmetry between birth-death processes.** Here we give examples of modularly symmetric forms (see §3) corresponding to birth-death processes exhibited in §3.2.

Theorem 3.2. *Let $q = e^{2\pi i\tau}$, where $\mathrm{Im}(\tau) > 0$. The following are true:*
(1) $\vartheta(q)$ is a weight $1/2$ modularly reflexive form with $\alpha = 4$, $l = 1$, $\gamma = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $\rho = \frac{1}{\sqrt{2}}$, and $\chi(\gamma) = \sqrt{-i}$.

TABLE 2

Example	$b_0(=\lambda_0)$	a_n	b_n
$\mathcal{B}_5$	$1+q$	q^{4n-1}	$1+q^{2n}+q^{2n+1}$
$\mathcal{B}_6$	$1+q+q^2$	$(q^{2n}+q^{4n})(q^{2n-1}+q^{4n-2})$	$1+(q^{2n}+q^{4n})$ $+(q^{2n+1}+q^{4n+1})$
$\mathcal{B}_7$	$1+q$	$q^{2n-1}(q^n+q^{2n})$	$1+(q^n+q^{2n})+q^{2n+1}$
$\mathcal{B}_8$	$1+q+q^2$	$q^{4n}(q^{2n-1}+q^{4n-2})$	$1+q^{2n+1}+q^{4n}+q^{4n+2}$
$\mathcal{B}_9$	1	$\frac{q^{8n-4}(1-q^{4n+1})}{(1-q^{4n-1})}$	$1+q^n$

TABLE 3

Example	**Even part of**	$f_{\mathcal{B}_i}(q)$	**Reference**
$\mathcal{B}_5$	$\frac{q^{1/5}}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\cdots$	$\frac{(q;q^5)_\infty(q;q^4)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$	[7]
$\mathcal{B}_6$	$\frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1+}\cdots$	$\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}$	[3, p. 59]
$\mathcal{B}_7$	$\frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2+q}{1+}\ \frac{q^3}{1+}\ \frac{q^4+q^2}{1+}\cdots$	$\frac{(-1;q^2)_\infty}{2(-q;q^2)_\infty}$	[3, p. 23]
$\mathcal{B}_8$	$\frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^4}{1+}\ \frac{q^3+q^6}{1+}\ \frac{q^8}{1+}\cdots$	$\frac{(q;q^8)_\infty(q;q^7)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}$	[3, p. 23]
$\mathcal{B}_9$	$\vartheta_3(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{1}{1+}\ \frac{q}{1-q+}\ \frac{q^3}{1-q^3+}\ \frac{q^5}{1-q^5+}\cdots$	$(1-\vartheta_3(q))/(2q)$	[8, p. 282]

(2) $\vartheta_2(q)$ *is a weight* $1/2$ *modularly symmetric form to* $\vartheta_3(q)$ *with* $\alpha=4$, $l=1$, $\gamma=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\rho=\sqrt{2}$, *and* $\chi(\gamma)=\sqrt{i}$.

(3) $r(q)$ *is a weight* 0 *modularly reflexive form with* $\alpha=1$, $l=1$, $\gamma=\begin{pmatrix}-\phi&1\\1&\phi\end{pmatrix}$, *where* $\phi=\frac{1+\sqrt{5}}{2}$ *is the "golden ratio",* $\rho=1$, *and* $\chi(\gamma)=1$.

(4) Let $u(\tau)=\sqrt{2}\,q^{1/8}\prod_{n=1}^{\infty}\frac{(1+q^{2n})}{(1+q^{2n+1})}$. *Then* $u(\tau)$ *is a weight* 0 *modularly reflexive form with* $\alpha=1/4$, $l=2$, $\gamma=\left(\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right)$, $\rho=1$, *and* $\chi(\gamma)=1$.

(5) Let $v(\tau)=q^{1/2}\prod_{n=0}^{\infty}\frac{(1-q^{8n+1})(1-q^{8n+6})}{(1-q^{8n+3})(1-q^{8n+5})}$. *Then* $v(\tau)$ *is a weight* 0 *modularly reflexive form with* $\alpha=8$, $l=2$, $\gamma=\begin{pmatrix}1&-\sigma^2\\\sigma^2&-1\end{pmatrix}$, *where* $\sigma=-1+\sqrt{2}$, $\rho=1$, *and* $\chi(\gamma)=1$.

Proof.
(1) This is well known. See, for example, Chapter 3 of Koblitz [14]. Alternatively, since

$$\vartheta(q)=\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}, \tag{3.5}$$

we can substitute $\tau=-1/\tau$ and, using the transformation law for the Dedekind η-function [15, p. 17], obtain

$$\frac{\eta(-1/(2\tau))^5}{\sqrt{-2i\tau}\,\eta(-1/\tau)^2\eta(-1/(4\tau))^2}=\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}. \tag{3.6}$$

(2) We have the following η-quotient representations [15, p. 17]:

$$\frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \quad \text{and} \quad \frac{\eta(4\tau)^2}{\eta(2\tau)} = \sum_{n=0}^{\infty} q^{(n+1/2)^2}. \tag{3.7}$$

Now make the transformation $\tau \to -1/\tau$ in $\eta(\tau)^2/\eta(2\tau)$.
(3) This identity dates back to Ramanujan (for a modern proof, see Proposition 2 in [7]):

$$r\left(\frac{-1}{\tau}\right) = \frac{-\phi r(\tau) + 1}{r(\tau) + \phi}, \tag{3.8}$$

where $\phi = (1+\sqrt{5})/2$ is the "golden ratio".
(4) This follows from the identity [7, p. 21]

$$u^2(-4/\tau) = \frac{1 - u^2(\tau)}{1 + u^2(\tau)}. \tag{3.9}$$

(5) This follows from the identity [7, p. 22]

$$v^2(-1/(8\tau)) = \frac{v^2(\tau) - \sigma^2}{\sigma^2 v^2(\tau) - 1}, \tag{3.10}$$

where $\sigma = -1 + \sqrt{2}$. $\square$

Corollary 3.3. *For $y > 0$, the following identities are true:*
(1)

$$1 + 2e^{-\pi(2y)} f_{\mathcal{B}_3}\left(e^{-\pi(2y)}\right) = \sqrt{2/y}\left(1 + e^{-\pi/(2y)} f_{\mathcal{B}_3}\left(e^{-\pi/(2y)}\right)\right).$$

(2)

$$-2e^{-\pi y/2} f_{\mathcal{B}_4}\left(e^{-\pi(2y)}\right) = \frac{1}{\sqrt{2y}} e^{\pi/(2y)} f_{\mathcal{B}_9}\left(e^{-\pi/(2y)}\right) - \frac{1}{2\sqrt{2y}}.$$

(3)

$$e^{2\pi/(5y)} f_{\mathcal{B}_5}\left(e^{-2\pi/y}\right) = \frac{-\phi\, e^{2\pi y} f_{\mathcal{B}_5}\left(e^{-2\pi y}\right) + 1}{e^{2\pi y} f_{\mathcal{B}_5}\left(e^{-2\pi y}\right) + \phi}.$$

(4)

$$\frac{e^{\pi y/2}}{2} f_{\mathcal{B}_7}^2\left(e^{-8\pi/y}\right) = \frac{1 - \frac{e^{\pi y/2}}{2} f_{\mathcal{B}_7}^2\left(e^{-2\pi y}\right)}{1 + \frac{e^{\pi y/2}}{2} f_{\mathcal{B}_5}^2\left(e^{-2\pi y}\right)}.$$

(5)

$$e^{\pi/(4y)} f_{\mathcal{B}_8}^2\left(e^{-\pi/(4y)}\right) = \frac{e^{2\pi y} f_{\mathcal{B}_8}^2\left(e^{-2\pi y}\right) - \sigma^2}{\sigma^2 e^{2\pi y} f_{\mathcal{B}_8}^2\left(e^{-2\pi y}\right) - 1}.$$

Proof.
(1) Recall that for the birth-death process $\mathcal{B}_3$ we have the equation

$$\vartheta(q) = 1 + 2q f_{\mathcal{B}_3}(q). \tag{3.11}$$

Set $\tau = iy$ in (3.6), and substitute the identity (3.11) in the resulting equation.
(2) Make the transformation $\tau \to -1/\tau$ in the $\mathcal{B}_9$ identity from Table 3, substitute this identity and (3.1) into (3.7), and then set $\tau = iy$.
(3) Substitute the $\mathcal{B}_5$ identity from Table 3 into (3.8) and then set $\tau = iy$.
(4) Substitute the $\mathcal{B}_7$ identity from Table 3 into (3.9) and set $\tau = iy$.
(5) Substitute the $\mathcal{B}_8$ identity from Table 3 into (3.10) and set $\tau = iy$. $\square$

Remark. The identities in Corollary 3.3 stemmed from the distinctive behavior of modular forms under the transformation $\tau \to -1/\tau$. There are further results which depend on algebraic equations satisfied by modular forms. For example, let $f(q) = f_{\mathcal{B}_5}(q)$. For $q = e^{2\pi i\tau}$ with Im $(\tau) > 0$, one can show the following (see Theorems 7.5.1, 7.5.3, and 7.5.4 in [5] and Entry 24 and Entry 25 in [3]):

(1)
$$\frac{1}{q} f(q)f(q^2)^2 = \frac{f(q^2) - f(q)^2}{f(q^2) + f(q)^2}.$$

(2)
$$\frac{3}{q} f(q)^2 f(q^3)^2 = \left(f(q^3) - f(q)^3\right)\left(1 + \frac{f(q)f(q^3)^2}{q^2}\right).$$

(3)
$$\begin{aligned}&\frac{5}{q^2}\; f(q)^2 f(q^4)^2 \left(\frac{f(q)f(q^4)}{q} - 1\right)^2 - \frac{f(q)f(q^4)}{q}\\ &= \left(\frac{f(q)^5}{q} + \frac{f(q^4)^5}{q^4}\right)\left(\frac{f(q)f(q^4)}{q} - 1\right) + \frac{f(q)^5 f(q^4)^5}{q^5}.\end{aligned}$$

(4)
$$\frac{f(q)^5}{f(q^5)} = \frac{1 - 2\frac{f(q^5)}{q} + 4\frac{f(q^5)^2}{q^2} - 3\frac{f(q^5)^3}{q^3} + \frac{f(q^5)^4}{q^4}}{1 + 3\frac{f(q^5)}{q} + 4\frac{f(q^5)^2}{q^2} + 2\frac{f(q^5)^3}{q^3} + \frac{f(q^5)^4}{q^4}}.$$

(5)
$$\begin{aligned}&\frac{1}{q^5} f(q)^5 f(q^2)^5 \left(f(q)^{10} f(q^2)^5 + f(q^2)^{10}\right) + f(q)^{10} - f(q^2)^5\\ &+ \frac{10}{q} f(q)^5 f(q^2)^5 \left(\frac{f(q)f(q^2)^5}{q^3} - \frac{f(q)^5}{q} + \frac{f(q^2)^5}{q^2} + 1\right) = 0.\end{aligned}$$

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