

## INTERPOLATING BLASCHKE PRODUCTS AND ANGULAR DERIVATIVES

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ABSTRACT. We show that to each inner function, there corresponds at least one interpolating Blaschke product whose angular derivatives have precisely the same behavior as the given inner function. We characterize the Blaschke products invertible in the closed algebra

$$H^\infty[\bar{b} : b \text{ has finite angular derivative everywhere}].$$

We study the most well-known example of a Blaschke product with infinite angular derivative everywhere and show that it is an interpolating Blaschke product. We conclude the paper with a method for constructing thin Blaschke products with infinite angular derivative everywhere.

### 1. INTRODUCTION

Let

$$B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z}$$

denote a Blaschke product with zeroes  $(z_j)$  in the open unit disk  $\mathbb{D}$ . Recall that  $B$  is said to be interpolating if

$$\inf_k \prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| = \inf_k (1 - |z_k|^2) |B'(z_k)| > \delta > 0$$

and thin if

$$\lim_k (1 - |z_k|^2) |B'(z_k)| = 1.$$

Blaschke products are important building blocks of functions in  $H^\infty$ , the space of bounded analytic functions on the open unit disk  $\mathbb{D}$ , as well as in the classical Hardy space  $H^p$  where  $p > 0$ . An interpolating Blaschke product will have distinct zeroes, while a thin Blaschke product may have finitely many zeroes that are repeated. In terms of behavior, the best-behaved Blaschke products are, of course, finite Blaschke products. Thin Blaschke products are among the best-behaved (infinite) Blaschke products; on the so-called Gleason parts in the maximal ideal space of  $H^\infty$ , for example, they are either identically one in modulus or, considering a certain function  $L_\varphi$  mapping the disk onto the Gleason part, a thin Blaschke product composed with  $L_\varphi$  is a Möbius transformation. In this sense, they are as close

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as an infinite Blaschke product can get to being a finite Blaschke product. Thin interpolating Blaschke products have been studied extensively in other contexts (see, for example, [8], [12], [13], [15], [17], [20]). In addition, the strong separation of the zeroes as well as the fact that the zero sequence is an interpolating sequence for  $H^\infty \cap VMO$  makes them particularly useful.

In [6], Cohn focused on functions that are orthogonal to invariant subspaces under the shift operator; that is, spaces of the form  $H^2 \ominus \varphi H^2$ , where  $\varphi$  is an inner function. Along the way, he proved the following result:

**Theorem 1** (W. Cohn). *Let  $B$  be an interpolating Blaschke product with zeroes  $(z_k)$  and let  $\varphi$  be an inner function. If  $\sup_k |\varphi(z_k)| < \varepsilon < 1$ , then there is a positive constant  $\gamma$  with*

$$|B'(e^{i\theta})| \leq \gamma |\varphi'(e^{i\theta})| \text{ a.e.}$$

In Section 3, we will adapt the proof of Cohn's result to more general situations, showing that given any inner function  $u$ , there is an interpolating Blaschke product that has the same angular derivative behavior as  $u$  at each point. As a consequence, we will show that if  $\mathcal{I}$  denotes the set of all Blaschke products with finite angular derivative at each point of the unit circle, and if  $A := H^\infty[\bar{b} : b \in \mathcal{I}]$  denotes the closed algebra generated by  $H^\infty$  and the conjugates of functions in  $\mathcal{I}$ , then an inner function  $u$  is invertible in  $A$  if and only if it has finite angular derivative at each point of the unit circle. One interesting consequence of our main result in Section 3 is its application to the well-known problem of Garnett and Jones [10, p. 430] of approximating Blaschke products by interpolating Blaschke products: If  $B$  is a Blaschke product with infinite angular derivative at a point  $\xi$  of the unit circle and if  $b$  is an interpolating Blaschke product with  $\|b - B\|_\infty < 1$ , then  $b$  must have infinite angular derivative at the point  $\xi$ .

In [18, p. 184], there is an example of a Blaschke product with infinite angular derivative at *every* point of the unit circle. This Blaschke product is clearly a very "bad" Blaschke product. It is, in a sense, similar to the Blaschke product with zeroes along the radius at the points  $(1 - 1/k^2)$ , because the zeroes of each have the same modulus. This latter Blaschke product has the property that the pseudohyperbolic distance between successive zeroes,

$$\rho(z_k, z_{k+1}) = \left| \frac{z_k - z_{k+1}}{1 - \bar{z}_k z_{k+1}} \right|,$$

tends to 0 and it is therefore not an interpolating Blaschke product. It also is continuous at every point except  $z = 1$  and therefore has finite angular derivative at every point except  $z = 1$ . We will show, in Section 4, that the change produced by looking at  $z_k = (1 - 1/k^2)e^{i\theta_k}$  for a clever choice of  $\theta_k$  changes the Blaschke product from one that is not interpolating to one that is interpolating and, as in [18], one with finite angular derivative a.e. to one with no angular derivative finite. In Section 5 we include a brief discussion of a geometric way to construct thin Blaschke products with infinite angular derivative at each point of the unit circle.

## 2. ANGULAR DERIVATIVES

Recall that an analytic function  $f$  in the unit ball of  $H^\infty$  is said to have an angular derivative at  $\xi \in \partial\mathbb{D}$  (where  $\partial\mathbb{D}$  denotes the unit circle) if there exists a point  $\eta \in \partial\mathbb{D}$  such that  $(f(z) - \eta)/(z - \xi)$  has a finite nontangential limit as  $z \rightarrow \xi$ . (See [7, p. 50].) Perhaps the most important result is due to Carathéodory. We

state it below for the reader’s convenience (see [2]). We write  $f(\xi)$  to denote the radial limit of  $f$  at the point  $\xi \in \partial\mathbb{D}$ .

**Theorem 2** (Carathéodory’s theorem). *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be analytic and let  $\xi \in \partial\mathbb{D}$ .*

- (1) *If  $f$  has angular derivative at  $\xi$ , then  $|f'(\xi)| = \lim_{r \rightarrow 1} (1 - |f(r\xi)|)/(1 - r)$  and  $f'(\xi) = \bar{\xi}f(\xi)|f'(\xi)|$ .*
- (2) *If  $f_n : \mathbb{D} \rightarrow \mathbb{D}$  and  $f_n \rightarrow f$  uniformly on compacta, then*

$$|f'(\xi)| \leq \lim_{n \rightarrow \infty} |f_n'(\xi)|.$$

The following version will also be useful, [7, p. 51].

**Theorem 3** (Julia-Carathéodory theorem, second version). *For  $f : \mathbb{D} \rightarrow \mathbb{D}$  analytic and  $\xi \in \partial\mathbb{D}$ , the following are equivalent:*

- (1)  *$d(\xi) := \liminf_{z \rightarrow \xi} (1 - |f(z)|)/(1 - |z|) < \infty$ , where the limit is taken as  $z$  approaches  $\xi$  unrestrictedly in  $\mathbb{D}$ .*
- (2)  *$f$  has finite angular derivative  $f'(\xi)$  at  $\xi$ .*
- (3) *Both  $f$  and  $f'$  have finite nontangential limits at  $\xi$  with  $\lim_{r \rightarrow 1} f(r\xi) = f(\xi) = \eta \in \partial\mathbb{D}$ .*

In [2, Theorem 3], the authors showed (among other things) that if  $u$  is an inner function with  $u' \in H^{1/2}$ , then  $u$  is a Blaschke product. In addition, they considered conditions on the zeroes of a Blaschke product  $B$  that imply that  $B'$  is in the class  $H^{1-\alpha}$  for  $0 < \alpha < 1/2$  and studied the behavior of the derivative of a Blaschke product  $B$  with zeroes tending to the point  $\xi = 1$  nontangentially. Cohn’s work continued the study of  $H^p$  derivatives that appeared in other papers, including [2] and [1]. We say more about this in the next section.

### 3. INTERPOLATING BLASCHKE PRODUCTS AND ANGULAR DERIVATIVES

The theorem and proof below were motivated by a theorem and proof due to W. Cohn [6]. Our interest is in extending Cohn’s result to more general inner functions, and we are particularly interested in the applications of the theorem below, all of which appear in the various corollaries in this section. We wish to thank Stephen Gardiner for very helpful correspondence and for showing us a simplification of our proof of Proposition 4 below.

Recall that  $\mathbb{D}$  is the open unit disk,  $\partial\mathbb{D}$  is the unit circle, and  $H^\infty$  is the algebra of bounded analytic functions on the disk. A function  $u \in H^\infty$  such that  $\lim_{r \rightarrow 1} |u(re^{i\theta})| = 1$  a.e. is called an inner function. Every Blaschke product is an inner function, and every inner function can be written as the product of a Blaschke product and an inner function with no zeroes in the unit disk. In what follows, when  $f$  is analytic on  $\mathbb{D}$ , we use the notation  $(\log |f(z)|)^+ = \max(\log |f(z)|, 0)$ .

**Proposition 4.** *Let  $u$  and  $v$  be inner functions and let  $G_\alpha = \{z \in \mathbb{D} : |u(z)| > \alpha > 0\}$ . If there exists  $\gamma > 0$  so that*

$$\gamma \log |v| - \log |u| \leq 0$$

*on  $\partial G_\alpha \cap \mathbb{D}$ , then this inequality also holds inside  $G_\alpha$ .*

*Proof.* Define  $s(z) = (\gamma \log |v(z)| - \log |u(z)|)^+$  on  $G_\alpha$  and 0 elsewhere. Then  $s$  is subharmonic on  $G_\alpha$  and vanishes on  $\partial G_\alpha \cap \mathbb{D}$ , and it follows from the definition that the nonnegative function  $s$  is subharmonic on  $\mathbb{D}$ .

Let  $h$  be the least harmonic majorant of  $s$ . Since  $s$  is bounded above,  $h$  is bounded. By the Riesz decomposition,  $s = h - g$  on  $\mathbb{D}$ , where  $g$  is a Green potential, [10, p. 100, Exercise 20]. By Littlewood's theorem,  $g$  has radial limit 0 a.e. on  $\partial\mathbb{D}$ . Since the radial limit of  $s = 0$  a.e., the same is true of the harmonic function  $h$ . So  $h = 0$  on  $\mathbb{D}$ . Therefore,  $s$  is identically zero, completing the proof.  $\square$

**Theorem 5.** *Let  $u$  and  $v$  be inner functions. Suppose that  $|u(z_n)| \rightarrow 1$  when  $|v(z_n)| \rightarrow 1$ , for all sequences  $(z_n)$  with  $|z_n| \rightarrow 1$ . Then there exists a constant  $C$  such that*

$$(3.1) \quad \sup_{z \in \mathbb{D}} \frac{1 - |u(z)|}{1 - |v(z)|} \leq C.$$

*Proof.* Let  $0 < \alpha' < 1$ . Consider the set

$$G_{\alpha'} = \{z \in \mathbb{D} : |u(z)| > \alpha'\}.$$

Then  $G_{\alpha'}$  is open and bounded. Note that  $\log |u(z)|$  is harmonic and  $\log |v(z)|$  is subharmonic in  $G_{\alpha'}$ .

*Claim 1.* There exists  $\delta' < 1$  such that  $|v(z)| < \delta'$  on  $G_{\alpha'}^c = \{z \in \mathbb{D} : |u(z)| \leq \alpha' < 1\}$ .

*Proof of Claim 1.* If not, there would exist a sequence  $(z_n)$  in  $\mathbb{D}$  such that  $\lim |v(z_n)| = 1$  and  $\sup |u(z_n)| \leq \alpha'$ , contradicting our assumption.

*Claim.* There exists  $\gamma > 0$  such that

$$(3.2) \quad -\log |u(z)| \leq -\gamma \log |v(z)|$$

for  $z \in \partial G_{\alpha'} \cap \mathbb{D}$ .

Let  $z \in \partial G_{\alpha'} \cap \mathbb{D}$ . Then  $-\log |u(z)| \leq -\log \alpha'$ . On the other hand,  $z \in \partial G_{\alpha'}^c$ , so  $|v(z)| \leq \delta'$  and  $-\log |v(z)| \geq -\log \delta'$ . Therefore, for  $z \in \partial G_{\alpha'} \cap \mathbb{D}$  we have

$$-\log |u(z)| \leq -\gamma \log |v(z)|,$$

where  $\gamma = \log \alpha' / \log \delta' > 0$ .

By Proposition 4, equation (3.2) is also true in  $G_{\alpha'}$ .

Now

$$\lim_{x \rightarrow 1} \frac{-\log x}{1 - x} = 1,$$

so there exists  $r < 1$  such that for  $|u(z)| > r$  we know that there exists  $C_1 > 0$  such that

$$\frac{-\log |u(z)|}{1 - |u(z)|} > C_1.$$

Note that this inequality (with a possibly different constant) holds throughout  $G_{\alpha'}$ . Similarly, on  $\{z : |v(z)| > r\}$ , we have

$$\frac{-\log |v(z)|}{1 - |v(z)|} < C_2,$$

for some  $C_2 > 0$ . Therefore, on  $G_{\alpha'} \cap \{|v(z)| > r\}$ , there exists a constant  $C$  such that

$$1 - |u(z)| < C(1 - |v(z)|).$$

Since this inequality holds if  $|v(z)| \leq r$  (with a possible change in the constant), this holds throughout  $G_{\alpha'}$ . But our first claim shows that  $|v(z)| \leq \delta'$  on  $G_{\alpha'}^c$ , so

there exists a (possibly different constant  $C$ ) such that

$$1 - |u(z)| < C(1 - |v(z)|)$$

for all  $z \in \mathbb{D}$ , and we obtain inequality (3.1).  $\square$

In what follows, let  $M(H^\infty)$  denote the space of nonzero multiplicative linear functionals on  $H^\infty$ . Then we can identify the disk  $\mathbb{D}$  with a subset of the compact space  $M(H^\infty)$  via point evaluation. Given a collection of functions  $\mathcal{J} = \{f_\alpha \in L^\infty : \alpha \in \mathcal{I}\}$ , the closed subalgebra of  $L^\infty$  generated by  $H^\infty$  and  $\mathcal{J}$  is denoted by  $H^\infty[f_\alpha : \alpha \in \mathcal{I}]$ .

**Corollary 6.** *Let  $u$  and  $v$  be inner functions with  $\bar{u} \in H^\infty[\bar{v}]$ . If  $v$  has (finite) angular derivative at a point, then so does  $u$ . In particular, if  $H^\infty[\bar{v}] = H^\infty[\bar{u}]$ , then  $u$  and  $v$  have (finite) angular derivatives at the same points.*

*Proof.* Suppose that  $|v(z)| \rightarrow 1$  but  $|u(z)|$  does not tend to 1 as  $|z| \rightarrow 1$ . Then we can find  $x \in M(H^\infty) \setminus \mathbb{D}$  such that  $|x(v)| = 1$  and  $|x(u)| < 1$ . But then [10, p. 375]  $x \in M(H^\infty[\bar{v}])$ . Since  $u, \bar{u} \in H^\infty[\bar{v}]$  we would have  $1 = |x(u\bar{u})| = |x(u)|^2$ , a contradiction. Thus, our assumption is equivalent to the fact that  $|u(z)| \rightarrow 1$  when  $|v(z)| \rightarrow 1$ . By Theorem 5, for all  $z \in \mathbb{D}$  we have

$$\frac{1 - |u(z)|}{1 - |z|} \leq C \frac{1 - |v(z)|}{1 - |z|}.$$

Taking the limit inferior as  $z \rightarrow \xi \in \partial\mathbb{D}$  yields the result.  $\square$

It is a consequence of the well-known Chang-Marshall theorem ([5], [16]) that for each inner function  $u$ , there is an interpolating Blaschke product  $B$  such that

$$H^\infty[\bar{u}] = H^\infty[\bar{B}].$$

It also follows from this theorem that subalgebras of  $L^\infty$  containing  $H^\infty$  are determined by their maximal ideal spaces; that is, if  $M(A_1) = M(A_2)$ , where  $A_1, A_2$  are closed subalgebras of  $L^\infty$  with  $H^\infty \subseteq A_j$ , then  $A_1 = A_2$ .

We now turn to a very simple proof of the existence of an interpolating Blaschke product with infinite angular derivative everywhere.

**Example 1** ([18, p. 184]). There is a Blaschke product  $u$  with infinite angular derivative at each point of the unit circle.

The example is remarkably simple: Let  $I_n$  be a sequence of arcs on the unit circle of length  $1/n$  and let  $\xi_n$  denote the center of  $I_n$ . Wrap these arcs around the circle; this will wrap around the circle infinitely many times. We place a zero at  $z_n := (1 - 1/n^2)\xi_n$ . Then  $(z_n)$  is obviously a Blaschke sequence and if we choose any point of the unit circle, say  $e^{i\theta}$ , then

$$\sum_{j: e^{i\theta} \in I_j} \frac{1 - |z_j|^2}{|e^{i\theta} - z_j|^2} = \infty,$$

and this implies that  $u$  has infinite angular derivative: using Frostman's theorem, one can show that if  $B$  is a Blaschke product with zero sequence  $(z_n)$  and  $e^{i\theta}$  is a point on  $\partial\mathbb{D}$ , then  $B$  has (finite) angular derivative at  $e^{i\theta}$  if and only if

$$\sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} < \infty.$$

(See [18, p. 184] for the computations and the proof of Frostman's theorem.) We will return to the Blaschke product  $u$  in a later section. For now, we need only to be aware of its existence.

**Corollary 7.** *There exists an interpolating Blaschke product with infinite angular derivative everywhere.*

*Proof.* Example 1 shows that there is a Blaschke product  $u$  that has infinite angular derivative at each point of the unit circle. The result follows immediately from the existence of the interpolating Blaschke product  $b$  such that  $H^\infty[\bar{u}] = H^\infty[\bar{b}]$  (see the comments preceding Example 1 above) and the previous corollary.  $\square$

If we consider this in the context of the important open question [10, p. 430] asking whether every Blaschke product can be approximated (in the uniform norm) by an interpolating Blaschke product, we note that if we were able to approximate to within 1 by an interpolating Blaschke product, the (global) behavior of the angular derivative must be the same. In what follows, we identify a function with its Gelfand transform and note that for each  $x \in M(H^\infty) \setminus \mathbb{D}$  there exists a unique measure  $\mu_x$  supported on the Shilov boundary,  $X$ , of  $M(H^\infty)$  such that

$$x(f) = \int_X f d\mu_x.$$

The support of  $\mu_x$  will be denoted by  $\text{supp } x$ .

**Corollary 8.** *Suppose that  $u$  and  $v$  are inner functions and  $\|u - v\|_\infty < 1$ . Then  $u$  has infinite angular derivative at each point if and only if  $v$  does.*

*Proof.* First we show that  $x \notin M(H^\infty[\bar{v}])$  implies  $x \notin M(H^\infty[\bar{u}])$ .

Our assumption implies that if  $x \in M(H^\infty) \setminus \mathbb{D}$  and  $|x(v)| < 1$ , then  $|x(u)| < 1$ ; the fact that  $|x(v)| < 1$  implies that  $v|_{\text{supp } x}$  is not invertible in  $M(H^\infty|_{\text{supp } x})$  (for if it were, we would have  $1 = |x(v \cdot \bar{v})| = |x(v)|^2$ ). Therefore [9, p. 39] there exists  $y$  with  $\text{supp } y \subseteq \text{supp } x$  and  $y(v) = 0$ . Thus  $|y(u)| < 1$ , so  $u$  cannot be constant on the support of  $y$  and, therefore, on the support of  $x$ . Consequently  $|x(u)| < 1$ , completing the proof that  $x \notin M(H^\infty[\bar{v}])$  implies that  $x \notin M(H^\infty[\bar{u}])$ . The same argument holds in reverse, so we see that  $M(H^\infty[\bar{u}]) = M(H^\infty[\bar{v}])$ . We know, from the Chang-Marshall theorem, that  $H^\infty[\bar{u}] = H^\infty[\bar{v}]$ . Now the result follows from our previous work.  $\square$

Our results can be applied most readily in situations in which inner functions tend to modulus 1. In this section, we discuss a typical application of our result above, motivated by the following theorem due to H. Hedenmalm, [13].

**Theorem 9** (Hedenmalm's theorem). *Let  $\mathcal{I} = \{b : b \text{ is thin}\}$ . If  $b_0$  is an inner function for which  $\bar{b}_0$  is in the algebra  $A := H^\infty[\bar{b} : b \in \mathcal{I}]$ , then  $b_0$  is a finite product of thin Blaschke products.*

We mention here that the results in this paper can be used to simplify Hedenmalm's proof; for example, it follows easily from Theorem 5 that  $b_0$  is either identically one in modulus or  $b_0 \circ L_\varphi$  is a finite Blaschke product, where  $L_\varphi$  is the Hoffman mapping of the open unit disk onto the Gleason part. The completion of the proof, however, is essentially the same as Hedenmalm's. Therefore we do not provide the details here; Hedenmalm's proof is already very clear, short, and readable.

The algebra we consider now is also a closed subalgebra of  $L^\infty$  generated by  $H^\infty$  and the conjugates of a certain class of Blaschke products. After establishing a useful lemma, we apply it to the following subalgebra of  $L^\infty$ : Let  $\mathcal{I}_1$  denote the family of Blaschke products with finite angular derivative at each point of the unit circle. Our main result (Theorem 5) allows us to give a description of the Blaschke products invertible in  $H^\infty[\overline{b} : b \in \mathcal{I}_1]$ .

In what follows, let  $Z(b)$  denote the set of zeroes of a Blaschke product in the maximal ideal space,  $M(H^\infty)$ . We begin with a lemma that is known, but we include a proof for completeness.

**Lemma 10.** *Let  $A$  be a closed subalgebra of  $L^\infty$  containing  $H^\infty$ . Let  $I$  be a nonempty index set and suppose that*

$$A = H^\infty[\overline{b_\alpha} : b_\alpha \text{ Blaschke}, \alpha \in I].$$

*If  $b_0$  is an inner function and  $\overline{b_0} \in A$ , then there are finitely many Blaschke products  $b_{\alpha_j}$ , where  $\alpha_j \in I$  and  $j = 1, \dots, n$  such that  $\overline{b_0} \in H^\infty[\overline{B}]$ , where  $B = \prod_{j=1}^n b_{\alpha_j}$ .*

*Proof.* Since  $b_0$  is invertible in  $A$ , we know that  $x \in M(H^\infty)$  and  $x(b_0) = 0$  implies that  $x \notin M(A)$ . By [10, p. 375],  $M(A) = \{x \in M(H^\infty) : |x(b_\alpha)| = 1 \text{ for all } \alpha \in I\}$ . So, if  $x \in Z(b_0) \setminus \mathbb{D}$ , then there exists  $\alpha_x \in I$  such that  $|x(b_{\alpha_x})| < 1$ . So

$$Z(b_0) \setminus \mathbb{D} \subseteq \bigcup_{x \in Z(b_0) \setminus \mathbb{D}} \{y \in M(H^\infty) : |b_{\alpha_x}(y)| < 1\}.$$

Compactness implies that there exist finitely many  $b_{\alpha_j}$ , for  $j = 1, \dots, n$  such that

$$Z(b_0) \setminus \mathbb{D} \subseteq \bigcup_{j=1}^n \{y \in M(H^\infty) : |b_{\alpha_j}(y)| < 1\}.$$

Therefore if  $x \in M(H^\infty) \setminus \mathbb{D}$ , then  $x(b_0) = 0$  implies  $|x(b_{\alpha_j})| < 1$  for some  $j$ . Let  $B = \prod_{j=1}^n b_{\alpha_j}$ .

Now we claim that  $\overline{b_0} \in H^\infty[\overline{B}]$ . If not, there would exist  $y \in M(H^\infty[\overline{B}])$  with  $y(b_0) = 0$ . But then  $y \in Z(b_0) \setminus \mathbb{D}$ , while  $|y(B)| = 1$ . This is impossible, and we have established the claim.  $\square$

**Theorem 11.** *Let*

$$\mathcal{I}_1 = \{b : b \text{ Blaschke with finite angular derivative at each point of } \partial\mathbb{D}\}.$$

*If  $b_1$  is a Blaschke product with  $\overline{b_1}$  in the algebra  $A_1 := H^\infty[\overline{b} : b \in \mathcal{I}_1]$ , then  $b_1 \in \mathcal{I}_1$ .*

*Proof.* By Lemma 10, there is a finite product of Blaschke products in  $\mathcal{I}_1$  such that  $\overline{b_1} \in H^\infty[\overline{B_1}]$ . Now apply Corollary 6 to conclude that the fact that  $B_1$  has finite angular derivative [2, Corollary 1] at each point implies that  $b_1$  does as well.  $\square$

#### 4. INTERPOLATING BLASCHKE PRODUCTS AND INFINITE ANGULAR DERIVATIVE

We return to the example of a Blaschke product  $B_0$  with infinite angular derivative at each point of the unit circle as presented in [18, p. 184].

**Example 1.** There exists a Blaschke product,  $B_0$ , with infinite angular derivative at every point of the unit circle.

As is well known ([19], or [7, p. 132]), if  $\varphi$  is a self-map of the unit disk and the composition operator  $C_\varphi$  defined by  $C_\varphi(f) = f \circ \varphi$  is compact on  $H^2$ , then the angular derivative of  $\varphi$  exists at no point of the unit circle. Thus, since the fact that  $B_0$  is an inner function implies that the composition operator  $C_{B_0}$  is not compact on  $H^2$ , this example shows that the angular derivative condition is not sufficient to imply compactness in general. The desire to understand the abundance of interpolating Blaschke products and their angular derivative behavior is what motivated us to consider Cohn's work in a different context and to consider the applications to function algebras.

We now show that this Blaschke product is an interpolating Blaschke product. In what follows, we are in the situation described above; that is, we wrap intervals  $I_k$  of length  $1/k$  around the unit circle, placing a zero of the Blaschke product at  $z_k = (1 - 1/k^2)e^{i\theta_k}$  where  $e^{i\theta_k}$  is the center of the arc  $I_k$ . In the next section, we present a general way to construct thin Blaschke products with infinite angular derivative at each point of  $\partial\mathbb{D}$ .

We use the well-known equality below. Writing  $a_n = |a_n|e^{i\phi_n}$  we have

$$\begin{aligned} |1 - a_n|^2 &= 1 + |a_n|^2 - 2|a_n| \cos(\phi_n) \\ (4.1) \qquad &= 1 - 2|a_n| + |a_n|^2 + 2|a_n|(1 - \cos(\phi_n)) \\ &= (1 - |a_n|)^2 + 4|a_n|(\sin(\phi_n/2))^2. \end{aligned}$$

The next elementary proposition will be useful as well. Recall that the pseudo-hyperbolic distance between two points  $z, w \in \mathbb{D}$  is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|.$$

**Proposition 12.** *If  $0 < r < s < 1$ , then*

$$(4.2) \qquad \rho(r, re^{i\theta}) < \rho(r, se^{i\theta}).$$

*Proof.* Squaring both sides, the right hand side will be

$$\frac{(r - s)^2 + 4rs \sin^2(\theta/2)}{(1 - rs)^2 + 4rs \sin^2(\theta/2)}.$$

Now  $(1 - rs)^2 < (1 - r^2)^2$ , since we assume  $0 < r < s < 1$ . So

$$\frac{(r - s)^2 + 4rs \sin^2(\theta/2)}{(1 - rs)^2 + 4rs \sin^2(\theta/2)} > \frac{(r - s)^2 + 4rs \sin^2(\theta/2)}{(1 - r^2)^2 + 4rs \sin^2(\theta/2)}.$$

A computation shows that this is larger than

$$\frac{4r^2 \sin^2(\theta/2)}{(1 - r^2)^2 + 4r^2 \sin^2(\theta/2)} = \rho^2(r, re^{i\theta}).$$

□

These computations also show that  $\rho(r, se^{i\theta}) \geq \rho(r, s)$  in this case, a fact that can also be deduced from a geometric argument and will be used in Step 7 of the proof below (see also [10, p. 4]).

**Theorem 13.** *Let  $B$  be the Blaschke product with zeroes  $(z_n)$ , where  $z_n = (1 - 1/n^2)e^{i\theta_n}$  for  $e^{i\theta_n}$  the midpoint of the interval  $I_n$  and the intervals are chosen as in Example 1. Then  $B$  is an interpolating Blaschke product that does not have finite angular derivative at any point of  $\partial\mathbb{D}$ .*

*Proof.* We already know that the angular derivative does not exist at any point. It remains to check that the Blaschke product is interpolating.

In what follows, we rotate so that  $z_k$  is positive. Thus, since  $e^{i\theta_k}$  is the center of an interval of length  $1/k$ , the adjacent point  $z_{k+1} = (1 - 1/(k + 1)^2)e^{i\theta_{k+1}}$  has the property that  $e^{i\theta_{k+1}}$  is the midpoint of the subsequent arc of length  $1/(k + 1)$ . Therefore, the argument of  $z_{k+1}$  is

$$\theta_{k+1} = 1/(2k) + 1/(2k + 2) = (2k + 1)/(2k) \cdot 1/(k + 1) \geq 1/(k + 1).$$

Similarly, the argument of  $z_{k-1}$  is

$$\theta_{k-1} = -1/(2k) - 1/(2k - 2) < -1/k.$$

Now we define our terminology. Having fixed  $z_k$  and rotated so that it is positive, we “wrap” around the circle. Each time we pass  $\pi/2$ , there will be a last point before  $\pi/2$  called the *upper endpoint*. When we pass  $-\pi/2$ , after that, we enter the next *wrap*. The first wrap is the 1-wrap and we assume  $z_k$  is in the  $m$ -wrap.

**Step 1. An estimate on  $\rho(1 - 1/j^2, 1 - 1/k^2)$ .**

For  $j > k$ , we claim that

$$\rho(1 - 1/j^2, 1 - 1/k^2) \geq 1 - 2k^2/j^2.$$

*Proof of the claim.* For  $j > k$ ,

$$\rho(1 - 1/j^2, 1 - 1/k^2) = 1 - \frac{2/j^2 - 1/(j^2k^2)}{1/j^2 + 1/k^2 - 1/(j^2k^2)}.$$

Now  $(2 - 1/k^2)/j^2 < 2/j^2$  and  $(1 - 1/k^2)/j^2 + 1/k^2 \geq 1/k^2$ , so

$$\rho(1 - 1/j^2, 1 - 1/k^2) \geq 1 - 2k^2/j^2.$$

**Step 2. An estimate on  $\rho(r, re^{i\theta})$ .** Given  $z = r$  and  $w = re^{i\theta}$  we compute

$$(4.3) \quad \rho^2(z, w) = \frac{r^2|1 - e^{i\theta}|^2}{|1 - r^2e^{i\theta}|^2} = \frac{r^2(4 \sin^2(\theta/2))}{(1 - r^2)^2 + 4r^2 \sin^2(\theta/2)}.$$

We note that the function

$$\frac{4r^2x}{(1 - r^2)^2 + 4r^2x}$$

is an increasing function of  $x$ . Therefore, if  $\sin^2(\theta/2) > \sin^2(\alpha/2)$ , then

$$\rho^2(r, re^{i\theta}) \geq \rho^2(r, re^{i\alpha}).$$

**Step 3. An estimate on the tail of the product.** We estimate, for  $k \in \mathbb{Z}^+$  and  $M \in \mathbb{Z}^+$  with  $M > 3$ ,

$$\prod_{j \geq Mk^2} (1 - 2k^2/j^2).$$

Using the fact that  $-\log(1 - x) < x/(1 - x)$  for  $0 < x < 1$ :

$$- \sum_{j \geq Mk^2} \log \left( 1 - \frac{2k^2}{j^2} \right) \leq \sum_{j \geq Mk^2} \frac{3k^2}{j^2}.$$

Now

$$\sum_{j \geq Mk^2} \frac{3k^2}{j^2} \leq 3k^2 \int_{Mk^2-1}^{\infty} 1/x^2 dx \leq 3k^2/(Mk^2 - 1) \leq 6k^2/(Mk^2).$$

Thus, we have

$$\prod_{j \geq Mk^2} (1 - 2k^2/j^2) \geq e^{-6/M}.$$

Fix  $M > 3$  in what follows.

**Step 4. Estimating  $\rho(z_k, (1 - 1/k^2)e^{i\theta(k)})$ , where  $\theta(k) = 1/(k + 1)$  or  $\theta(k) = 1/k$ .** Recall that  $z_k = 1 - 1/k^2$ .

Using equation (4.1), we compute

$$\begin{aligned} &\rho^2(1 - 1/k^2, (1 - 1/k^2)e^{i\theta(k)}) \\ &= (1 - 1/k^2)^2 \frac{4 \sin^2(\theta(k)/2)}{(1 - (1 - 1/k^2)^2)^2 + 4(1 - 1/k^2)^2 \sin^2(\theta(k)/2)}. \end{aligned}$$

We know that  $\sin^2(\theta(k)/2) \geq \alpha/(k + 1)^2$  for some  $\alpha > 0$ , which is independent of  $k$ . So, using the fact that  $4ax/(b + 4ax)$  is increasing when  $a, b > 0$  (as in Step 2),

$$\begin{aligned} &\rho^2(1 - 1/k^2, (1 - 1/k^2)e^{i\theta(k)}) \\ &\geq (1 - 1/k^2)^2 \frac{\alpha/(k + 1)^2}{(1 - (1 - 1/k^2)^2)^2 + (1 - 1/k^2)^2 \alpha/(k + 1)^2}. \end{aligned}$$

We get an upper bound on the denominator:

$$\begin{aligned} &(1/k^4)(2 - 1/k^2)^2 + (1 - 1/k^2)^2(\alpha/(k + 1)^2) \\ &= ((k + 1)/k)^2 (\alpha/(k + 1)^2)(1/(\alpha k^2))(2 - 1/k^2)^2 + (1 - 1/k^2)^2(\alpha/(k + 1)^2). \end{aligned}$$

So the fraction we need to bound is

$$(1 - 1/k^2)^2 \frac{1}{(1 + 1/k)^2 \cdot 1/(\alpha k^2) \cdot (2 - 1/k^2)^2 + (1 - 1/k^2)^2}.$$

Now, the denominator is less than or equal to

$$\begin{aligned} &(1 + 2/k + 1/k^2)(4/(\alpha k^2)) + (1 - 2/k^2 + 1/k^4) \\ &\leq (1 + 1/k^2)^2 \cdot (1 + 9/(\alpha k^2)), \end{aligned}$$

for  $k > 4$ , say.

So

$$(4.4) \quad \rho^2(1 - 1/k^2, (1 - 1/k^2)e^{i\theta(k)}) \geq (1 - 1/k^2)^2 \left( \frac{1}{1 + 1/k^2} \right)^2 \left( \frac{1}{1 + 9/(\alpha k^2)} \right).$$

Note that since  $1/k \geq 1/(k + 1)$ , this estimate will also work when  $\theta(k) = 1/k$ .

**Step 5. Estimating  $\rho(z_j, z_k)$  when  $1 < j < k$  and  $z_j$  is in the  $m$ -wrap or  $z_j$  is in the left half-plane.**

If  $z_j$  is in the  $m$ -wrap, then choosing  $-\pi/2 < \arg(z_j) < 0$ , we have  $\arg(z_j) + 1/(j + 1) \leq 0$ . Thus  $-\pi/2 < \arg(z_j) \leq -1/(j + 1)$ .

If  $z_j$  is in the left half-plane, then  $\sin^2(\theta_j/2) \geq \sin^2(\pi/4) \geq 1/(j + 1)^2$  for all  $j$ . Therefore, in either case, we may use equation (4.2) and Step 2 to conclude that

$$\rho(z_k, z_j) \geq \rho((1 - 1/j^2), (1 - 1/j^2)e^{i/(j+1)}).$$

From equation (4.4), we have the estimate

$$\rho^2((1 - 1/j^2), (1 - 1/j^2)e^{i/(j+1)}) \geq (1 - 1/j^2)^2 \left( \frac{1}{1 + 1/j^2} \right)^2 \left( \frac{1}{1 + 9/(\alpha j^2)} \right),$$

and we note that this is greater than or equal to

$$(1 - 1/j^2)^4 \cdot (1 + 9/(\alpha j^2))^{-1}.$$

Now

$$\prod_{j=2}^{k-1} ((1 - 1/j^2)^4 \cdot (1 + 9/(\alpha j^2))^{-1}) \geq e^{-C_1(1-1/k)},$$

where  $C_1$  is a constant independent of  $k$ .

**Step 6. Estimating  $\rho(z_j, z_k)$  when  $j > k$  and  $e^{i\theta_j} \notin I_k$ .**

For these points we use equation (4.2) and the fact that  $1 - 1/j^2 > 1 - 1/k^2$ . Then when  $\sin^2(\theta_j/2) \geq \sin^2(\alpha(k)/2)$  we have

$$\rho(z_k, z_j) \geq \rho(z_k, (1 - 1/k^2)e^{i\theta_j}) \geq \rho(z_k, (1 - 1/k^2)e^{i\alpha(k)}),$$

since the pseudohyperbolic function is increasing in this case, as we saw above.

Note that for  $z_j$  in the first quadrant we have  $\theta_j \geq 1/(2k)$  and for  $z_j$  in the fourth quadrant we have  $\theta_j < 0$  and  $\theta_j < -1/(2k)$ . So for  $\alpha(k) = 1/(2k)$  we have

$$\rho(z_k, z_j) \geq \rho((1 - 1/k^2), (1 - 1/k^2)e^{i\alpha(k)}),$$

and we can compute this as in Steps 4 and/or 5. Note that there are at most  $Mk^2$  points here, because we have already handled the tail. Furthermore, this case includes the points in the left half-plane with  $k < j < Mk^2$ .

**Step 7. Estimating  $\rho(z_j, z_k)$  when  $j > k$  and  $e^{i\theta_j} \in I_k$ .**

In this case,  $z_j$  must be in a  $p$ -wrap with  $p > m$ . From equation (4.2), we know that  $\rho(r, re^{i\theta}) \leq \rho(r, se^{i\theta})$  if  $0 < r < s < 1$ . Let  $j_0$  be the index of the first point in the  $p$ -wrap and  $n_{m(p)}$  be the largest integer so that  $2^{n_{m(p)}}k < j_0$ . We compute an upper bound on the number of points we can have in the  $p$ -wrap with  $e^{i\theta_j} \in I_k$ .

Since  $\sum_{l=j_0+1}^{m_0} 1/l \geq \int_{j_0+1}^{m_0} 1/x dx = \ln(m_0/(j_0 + 1))$ , we are out of  $I_k$  as soon as  $\ln(m_0/(j_0 + 1)) > 1/k$ , or the first integer for which  $m_0 > (j_0 + 1) \cdot e^{1/k}$ . Therefore, there are at most  $m_0 - j_0$  points, or the first integer greater than or equal to  $(j_0 + 1)(e^{1/k} - 1)$  points. Since  $j_0 < (2^{n_{m(p)}+1})k$  and  $e^{1/k} - 1 < 2/k$ , there are at most  $4(2^{n_{m(p)}})$  points.

For these points, from Step 1, the comments immediately following Proposition 12, and our work above, we have

$$\rho(z_j, z_k) \geq 1 - 2k^2/j^2 \geq (1 - 2/2^{2n_{m(p)}}).$$

Using our bound on the number of points, noting that  $n_{m(p)} > n_{m(p-1)}$  and setting  $p_0 = m + 1$ , we have a lower bound of

$$\prod_{p=m+1}^{\infty} (1 - 2/2^{2n_{m(p)}})^{2^{n_{m(p)}+2}} \geq e^{-1/2^{n_{m(p_0)}-4}}.$$

**Step 8.  $j < k$  and  $z_j$  in an  $n$ -wrap where  $n < m$ .** Let  $c = 1/9$ . Letting  $[r]$  denote the greatest integer less than or equal to  $r$ , we claim that the following is true.

**Proposition 14.** *The product for points in the  $n$ -wrap,  $n < m$ , is greater than or equal to*

$$e^{-48kc^{[2k/3]-1}} \rightarrow 1$$

as  $k \rightarrow \infty$ .

*Proof.* Recall that  $z_k$  is in the  $m$ -wrap. For this step, we consider  $z_j$  in the  $l$ -wrap with  $l < m$ . Note that  $z_j$  is in the right half-plane, because all our wraps are defined to lie in the right half-plane.

If  $z_{j_0}$  is the upper endpoint of the  $(j-1)$ -wrap, then to enter the  $j$ -wrap, we must have added an arc of length at least  $\pi$ . Therefore, since we add arcs of length  $1/j_0, \dots, 1/(j_0+l)$  we must have added at least  $\pi j_0$  points. Thus, if  $z_j$  is in the  $j$ -wrap,  $l \geq 3j_0$ . In particular,  $k \geq 3j_0$ . Now divide the right half-circle into angles of length  $1/k$  using sufficiently many angles to cover the right half-disk and keeping  $z_k > 0$  in the middle of an angle.

*Claim.* In the  $l$ -wrap, with  $l < m$ , there are at most two points in each angle of length  $1/k$ .

Suppose that  $z_j$  and  $z_{j+1}$  are both in the same angle in wrap  $l$ . Consider  $z_j$  in the lower half-plane: If  $\Im(z_j) < 0$ , then we have covered an arc of length at least  $1/(2j) + 1/(j+1)$  before we put down the third point. Now  $j < k$  so  $j+1 \leq k$ , and our sum is strictly greater than  $1/(j+1) \geq 1/k$  and we are out of the angle. This argument also handles the case  $\Im(z_j) > 0$ , so we have established the claim.

Note that there are at most  $4k$  such angles in the right half-plane. (While the angles may run into the left half-plane, this will add at most two angles and this will not affect the computation.) We consider two products, each with one point from each angle.

So consider one product. In each angle there is at most one point in each wrap. Therefore, if we have subsequent points within the angle,  $w_j$  and  $w_{j+1}$ , we know that  $w_j = z_r$  and  $w_{j+1} = z_s$  with  $s \geq 3r$ . So

$$\frac{1 - |w_{j+1}|}{1 - |w_j|} = r^2/s^2 \leq 1/9.$$

This holds for  $w_1, \dots, w_m$ , where  $w_m = z_{j_m}$  is in the same  $m$ -wrap as  $k$ . Here, we replace  $w_m$  by  $z_k$  and note that

$$k \geq 3j_{m-1},$$

so the estimate

$$\frac{1 - |z_k|}{1 - |w_{m-1}|} < 1/9$$

still holds. We use well-known computations ([14, p. 203, Corollary (Hayman; Newman)]) to conclude that for these  $z_j$  in the  $r$ th angle we have

$$\rho(z_j, z_k) \geq \left(1 - \frac{2c^{k-j}}{1 + c^{k-j}}\right),$$

where  $c = 1/9$ .

Since we have at most  $4k$  such angles, if we let  $\mathcal{I}$  denote the indices of the points in the  $n$ -wraps with  $n < m$ , we have that each point in such a wrap lies in one of our at most  $4k$  angles and we have just computed the pseudohyperbolic distance. Thus, as long as we choose one point from each of the  $4k$  angles we have

$$\prod_{j \in \mathcal{I}} \rho(z_j, z_k) \geq \left( \prod_{n \geq [2k/3]} (1 - 2c^n) \right)^{4k},$$

where  $[k]$  denotes the greatest integer less than or equal to  $k$ .

A computation shows that

$$\prod_{n \geq [2k/3]} (1 - 2c^n) \geq e^{-12c^{[2k/3]}}.$$

So, we have

$$\prod_{j \in \mathcal{I}} \rho(z_j, z_k) \geq e^{-48kc^{[2k/3]}} \rightarrow 1,$$

as  $k \rightarrow \infty$ . Recall that we had two such products, and this estimate holds for both of them.  $\square$

**Final Step.**

- (1) We bound the tail using Step 3 and make sure this is greater than  $e^{-6/M}$ . Thus,  $\prod_{j \geq Mk^2} \rho(z_j, z_k) > e^{-6/M}$ .

Now we consider the  $Mk^2 - 1$  points that are not in the tail.

- (2) For  $j > k$  and  $e^{i\theta_j} \in I_k$ , we know that  $z_j$  is in a  $p$ -wrap with  $p > m$ . We use Step 7 to conclude that the product is greater than

$$\prod_{m=1}^{\infty} (1 - 1/2^{2n_m - 1})^{((2^{n_m} + 2))},$$

where  $n_{m+1} > n_m$ . So, denoting this set of  $z_j$  by  $\mathcal{I}_1$  we have

$$\prod_{j \in \mathcal{I}_1} \rho(z_j, z_k) > e^{-C/2^{n_1}},$$

where  $C$  is a constant independent of  $k$ .

- (3) For  $j > k$  and  $e^{i\theta_j} \notin I_k$ , we use Step 6 to conclude that

$$\prod_{j=k}^{Mk^2-1} \rho(z_k, z_j) \geq \left( (1 - 1/k^2)^2 \left( \frac{1}{1 + 1/k^2} \right)^2 \left( \frac{1}{1 + 9/(\alpha'k^2)} \right) \right)^{Mk^2-1},$$

where  $\alpha'$  is a constant that depends on  $\alpha$  but not on  $k$ .

- (4) For  $j < k$  and  $z_j$  in the  $m$ -wrap or left half-plane, we use Step 5 to conclude that  $\prod_{j=1}^{k-1} \rho(z_j, z_k) \geq e^{-C_1(1-1/k)}$ .
- (5) For  $z_j$  in an  $n$ -wrap with  $n < m$ , we use Proposition 14 to bound the product below by a value tending to 1 as  $k \rightarrow \infty$ .

This bounds the product  $\prod_{j \neq k} \rho(z_j, z_k)$  below by a positive constant independent of  $k$ . So the Blaschke product is interpolating.  $\square$

5. THIN INTERPOLATING BLASCHKE PRODUCTS

As we learned from Daniel Suárez (private communication), the Blaschke product above is actually thin. This can be proved using the techniques above and modifying the pieces that do not tend to one using the Hayman-Newman estimates or, as in Suárez's proof, one can use the fact that the sequence is separated and an argument using Carleson squares. However, it seems more useful to describe a general procedure for obtaining such products, which we do below.

We describe, now, a simple geometric process for creating thin Blaschke products with infinite angular derivative at each point of the unit circle. Recall that the region

$$\{z \in \mathbb{D} : \frac{|1 - z|^2}{1 - |z|^2} < k\},$$

where  $k > 0$ , is called an orocycle at  $z = 1$ . All our orocycles will be rotations of one particular orocycle, and therefore will all have the same size. This is a disk in  $\mathbb{D}$  that is tangent to the unit circle at  $z = 1$ . Example 1 has no angular derivative at any point because there is an orocycle that, when rotated to *any* point  $e^{i\theta}$  (the orocycle at  $e^{i\theta}$ ) always contains infinitely many points, and it is easy to see from the definition of orocycle and from Frostman's theorem that this means that the angular derivative is infinite. This idea can be found in Cargo's work [4]. The construction below attempts to exploit the idea of placing infinitely many zeroes in every orocycle. The idea is very simple, but the proof that it works is not computationally simple. We will present the idea first, and then we will present the details.

Idea of proof: Choose a sequence of concentric circles centered at 0 (inductively) with radii  $r_k$  tending to 1 very quickly. This will need to be done with care, and we will show how to do it after we measure the number of points on each circle and the distances between them.

We will need to have  $\sum \sqrt{1 - r_k} < \infty$ . Choose an orocycle at the point  $z = 1$ ; that is, we choose points  $z \in \mathbb{D}$  satisfying  $\frac{1 - |z|^2}{|1 - z|^2} = 1/2$ . (It is only important that all our orocycles have the same size; the choice of  $1/2$  is not important.) Choose the first circle  $C_{r_1}$ . Now we choose points on the orocycles and on the circle  $\{z : |z| = r_1\}$  in the following manner: put the first orocycle  $\mathcal{O}_1$  down. Where that orocycle crosses the circle  $\{z : |z| = r_1\}$ , put the next orocycle passing through the point on the circle  $\{z : |z| = r_1\}$ , choosing the point with imaginary part positive in the first step. We choose our orocycles moving in the counterclockwise direction. Continue this process until you return to your starting point. (Note that at each stage, the circles will most likely not end where they start. We can omit a point, as long as we do not always omit the point in, approximately, the same place.) For the next stage, we will choose a circle  $C_{r_2} = \{z : |z| = r_2\}$ . We will put an orocycle  $\mathcal{O}_{1,2}$  (of the same size as the previous set) down. Draw a second orocycle  $\mathcal{O}_{2,2}$  of the same size that passes through the point of intersection of  $\mathcal{O}_{1,2}$  and the circle  $C_{r_2}$  with positive real part. Continue this process until you return, almost or exactly, to your starting point. Again the circles will most likely not end where they start, and we will have to omit some points again. We must make certain that all orocycles that were affected by omission in the previous step are not affected in this step, and this can be done by taking the first orocycle we put down in the appropriate spot. Repeat this process, choosing the radii as indicated below.

Note that we will choose the  $r_k$  so that  $\sum \sqrt{1 - r_k} < \infty$ , where  $r_k$  denotes the radius of the  $k$ -th circle.

**Step 1. Counting the number of points placed on the circles.** The distance between points along the orocycle can be measured by locating the points where  $|z| = r$  intersects the orocycle. We measure the distance between the two points on the orocycle based at the point  $z = 1$ . To do this, we solve

$$\frac{1 - r^2}{|1 - re^{i\theta}|^2} = 1/2.$$

Now,

$$(1 + 3r)(1 - r) = 4r(\sin(\theta/2))^2.$$

In particular, we may choose  $r$  close enough to 1 so that  $|\theta| < \pi/4$ . So, since

$$\lim_{\theta \rightarrow 0} \sin(\theta)/\theta = 1,$$

for the two points  $z_1$  and  $z_2$  that solve this equality (which will be “orocycle mates” on the circle we choose and on the orocycle for all points except possibly the pair of points involving an omitted point, in which case the distance between orocycle mates will be farther than our calculation) we have that the length of the arc joining  $z_1$  and  $z_2$ , denoted  $\text{arc}(z_1, z_2)$ , is

$$\text{arc}(z_1, z_2) = 2r|\theta| \asymp 2r|\sin(\theta)| \asymp \sqrt{1-r},$$

where, by  $c(r) \asymp d(r)$  we mean that there exist positive constants  $c_1$  and  $d_1$ , independent of  $r$  such that  $c_1(c(r)) \leq d(r) \leq c_2(c(r))$ .

To compute the distance between any of the two adjacent points on a particular circle, we may rotate so that the orocycle is based at the point  $z = 1$  and then use the computation we just completed.

Thus, there exist positive constants  $a_1$  and  $a_2$ , independent of  $r$ , such that we have  $N(r)$  points, where  $a_1r/\sqrt{1-r} \leq N(r) \leq a_2r/\sqrt{1-r}$ , on each circle of radius  $r$ .

We will choose the zeroes of the Blaschke product to be the points  $(z_k)$  that are the points of intersection of the orocycles with the circles of radii  $r_j$  and, as we said above, we choose them in such a way that the sums of the radii of the circles satisfy  $\sum_j \sqrt{1-r_j} < \infty$ .

**Step 2. Is  $(z_k)$  a Blaschke sequence?** Note that on the  $n$ -th circle we can think of (to within upper and lower bounds),  $r_n/\sqrt{1-r_n}$  points of modulus  $r_n$ . So

$$\sum_{k=1}^{\infty} (1 - |z_k|) = \sum_{n=1}^{\infty} (1 - r_n)(r_n/\sqrt{1-r_n}) < \infty,$$

by assumption. So it is a Blaschke sequence.

**Step 3. Is it a thin sequence?** We are going to choose our circles,  $C_j = \{z : |z| = r_j\}$ , so that the pseudohyperbolic distance between the circles is very close to one. This will allow us to choose our points. Using our estimate, we know that on  $C_j$  we have approximately  $r_j/\sqrt{1-r_j}$  points. Let  $z$  and  $w$  be “orocycle mates” chosen on an orocycle (where  $|z| = |w|$ ). Now  $\theta$  (in the previous step) denoted half the angle between consecutive points, so if we choose consecutive points and rotate so that the first is positive, we will have  $z = s$  and  $w = se^{2i\theta}$ , with  $0 < \theta < \pi/4$ . As above,

$$\rho^2(z, w) = \frac{4s^2 \sin^2(\theta)}{(1-s^2)^2 + 4s^2 \sin^2(\theta)}.$$

Now  $\sin^2(\theta/2) \leq \sin^2(\theta)$  for  $0 < \theta < \pi/4$ , and we have seen that  $\frac{x}{(1-s^2)^2+x}$  is an increasing function of  $x$ , so

$$\rho^2(z, w) \geq \frac{4s^2 \sin^2(\theta/2)}{(1-s^2)^2 + 4s^2 \sin^2(\theta/2)}.$$

But we know (from Step 1) that  $4s \sin^2(\theta/2) = (1+3s)(1-s)$ , so

$$\rho^2(z, w) \geq \frac{s(1+3s)(1-s)}{(1-s^2)^2 + s(1-s)(1+3s)}.$$

Since  $s \rightarrow 1$ , we have

$$\rho^2(z, w) \geq \frac{s(1 + 3s)}{(1 - s)(1 + s)^2 + s(1 + 3s)} \rightarrow 1.$$

Now, from our previous computation we know that there are roughly  $2\pi s/\sqrt{1 - s}$  such points. So, since

$$(5.1) \quad \left( \frac{s(1 + 3s)}{(1 - s)(1 + s)^2 + s(1 + 3s)} \right)^{2\pi s/\sqrt{1 - s}} \rightarrow 1$$

as  $s \rightarrow 1$ , our computation above shows that the product of the pseudohyperbolic distance of points on the same circle will tend to one and if we choose our sequence of radii tending to 1 very rapidly (as we do below), the pseudohyperbolic distance of points on circles with different radii will tend to 1 quickly as well. To see that this is thin, we do the following.

**Choosing the radii.** Recall that  $N(r_j)$  denotes the number of points on the  $j$ -th chosen circle and that we calculated bounds on these numbers above.

Choose  $1 - \delta_n \rightarrow 1$  with  $\prod_{j=1}^\infty (1 - \delta_j) > 0$ . We begin the actual construction. We first choose a sequence of radii with  $\sum_{j=1}^\infty \sqrt{1 - r_j} < \infty$ . From this, we choose a subsequence as follows.

- (i) Choose the first circle,  $C_1$ , and put the first set of points down.
- (ii) Choose a radius  $w_2$  so that points on a circle  $C_r$  with  $r > w_2$  satisfy

$$(5.2) \quad \rho(z, C_1) \geq (1 - \delta_1)^{1/N(r_1)}$$

for  $z \in C_r$ . We are going to choose  $C_2$  with radius larger than  $w_2$  and satisfying one more condition we now describe. In particular we note that if  $z_k \in C_2$ , then

$$\prod_{z_j \in C_1} \rho(z_k, z_j) \geq (1 - \delta_1).$$

Note that the pseudohyperbolic distance between a point  $z \in C_s$  and a point  $w \in C_r$  satisfies ([10, p. 4])

$$\rho(z, w) \geq \rho(|z|, |w|) = \frac{r - s}{1 - rs},$$

where  $r > s$ . Recall that we have a bound on the number of points that we are putting down on a circle of radius  $r$ , and that bound is  $a/\sqrt{1 - r}$ , where  $a > 0$  is a constant independent of  $r$ . Since, for fixed  $s$ ,

$$\lim_{r \rightarrow 1^+} \left( \frac{r - s}{1 - rs} \right)^{a/\sqrt{1 - r}} = 1,$$

we may choose  $t_2$  so that if  $r > t_2$ , then

$$(5.3) \quad \left( \frac{r - r_1}{1 - rr_1} \right)^{a/\sqrt{1 - r}} > 1 - \delta_1.$$

Now choose  $C_2$  of radius  $r_2 > \max\{w_2, t_2\}$  so that both conditions (inequality (5.2) and inequality (5.3)) are satisfied and note that for  $z_l \in C_1$

and  $z_j \in C_2$  we have  $\rho(z_l, z_j) \geq \frac{r_2 - r_1}{1 - r_2 r_1}$ . Therefore,

$$\prod_{z_j \in C_2} \rho(z_l, z_j) \geq \left( \frac{r_2 - r_1}{1 - r_2 r_1} \right)^{a/\sqrt{1-r_2}} > (1 - \delta_1).$$

(iii) Choose  $C_3$  so that

$$\rho(z, C_1 \cup C_2) \geq (1 - \delta_2)^{1/(N(r_1)+N(r_2))}$$

for  $z \in C_3$ . So for  $z_k \in C_3$  we have

$$\prod_{z_j \in C_1 \cup C_2} \rho(z_k, z_j) \geq (1 - \delta_2).$$

As above, we may choose  $C_3$  so that, in addition to the conditions above for  $z_l \in C_1 \cup C_2$  we have

$$\prod_{z_j \in C_3} \rho(z_l, z_j) \geq \min \left\{ \left( \frac{r_3 - r_1}{1 - r_1 r_3} \right)^{a/\sqrt{1-r_3}}, \left( \frac{r_3 - r_2}{1 - r_3 r_2} \right)^{a/\sqrt{1-r_3}} \right\} > 1 - \delta_2.$$

Continue in this fashion.

The points will be enumerated as follows:  $z_1, \dots, z_{n_1} \in C_1, z_{n_1+1}, \dots, z_{n_2} \in C_2$ , and so on. To see that the corresponding Blaschke product is thin, choose  $\epsilon > 0$ . Suppose  $z_k \in C_{m_0}$ .

We compute distances in three steps:

**On the same circle.** For points  $z_j \in C_{m_0}$  with  $z_j \neq z_k$ , we know that  $\prod_{z_j \in C_{m_0}} \rho(z_k, z_j) \rightarrow 1$  as  $k \rightarrow \infty$  (since  $k \rightarrow \infty$  implies  $m_0 \rightarrow \infty$ ), by our estimates on the pseudohyperbolic distance between points on the same circle in Step 3 (see (5.1)).

**On previous circles.** We also have estimates for the pseudohyperbolic distance of points on previous circles,

$$\prod_{z_j \in C_m, m < m_0} \rho(z_k, z_j) \geq (1 - \delta_{m_0-1}).$$

**On future circles.** For  $z_j \in C_m$  with  $m > m_0$  we have

$$\prod_{z_j \in C_m, m > m_0} \rho(z_k, z_j) = \left( \prod_{z_j \in C_{m_0+1}} \rho(z_k, z_j) \right) \left( \prod_{z_j \in C_{m_0+2}} \rho(z_k, z_j) \right) \cdots.$$

By our estimates above,

$$\prod_{z_j \in C_m, m > m_0} \rho(z_k, z_j) \geq \prod_{l \geq m_0} (1 - \delta_l).$$

To complete the proof, given  $\epsilon > 0$ , choose  $m_0$  so  $\prod_{j \geq m_0} (1 - \delta_j) > 1 - \epsilon$ ,  $1 - \delta_{m_0-1} > 1 - \epsilon$ , and  $\prod_{z_j \in C_{m_0}, j \neq k} \rho(z_j, z_k) > 1 - \epsilon$ . Then  $\prod_{j \neq k} \rho(z_j, z_k) \geq (1 - \epsilon)^3$  for  $z_k \in C_{m_0}$ .

**Step 4. Is the angular derivative infinite everywhere?** Think of putting the orocycles down in steps.

As we noted above, it follows from Frostman’s theorem [18, p. 183] that if an orocycle at  $e^{i\theta}$  contains infinitely many points, then the angular derivative cannot be finite. For step one, note that if we have two consecutively chosen orocycles

$O_{\theta_1}$  and  $O_{\theta_2}$  corresponding to  $k = 2$  in the definition of an orocycle, then these determine  $n$  points on the circle  $C_1$ . Recall that  $O_{\theta_2}$  is drawn so that it passes through the point of intersection of  $O_{\theta_1}$  and  $C_1$  with positive real part and a point on the unit circle (which we call  $e^{i\theta_2}$ ). If we choose any other orocycle  $O_\theta$  centered at a point  $e^{i\theta}$ , where  $e^{i\theta}$  is between  $e^{i\theta_1}$  and  $e^{i\theta_2}$ , and of the same size as the other orocycles, then  $O_\theta$  will contain the point of intersection of  $O_{\theta_1}$  and  $O_{\theta_2}$ . For the next step, we choose our next circle  $C_2$  and draw our orocycles corresponding to  $k = 2$  as before. This time, however, we are choosing points on the circle  $C_2$ . Again, if an orocycle lies between two orocycles that we choose in step two, it will contain the point of intersection. Since this happens for each step,  $O_\theta$  must contain infinitely many points of the thin sequence. But, as in Example 1, if there are infinitely many points in an orocycle, then the angular derivative at the point cannot be finite. Therefore, the angular derivative is infinite at each point of the unit circle.

After this paper was completed, the authors learned that Akeroyd, Ghatage and Tjani have provided an example [3, Example 3.8] of a thin interpolating Blaschke product that is a subproduct of the Blaschke product constructed by Shapiro (and presented in Example 1 above) and has no angular derivative finite on the unit circle.

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