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THE DERIVATIVE OF AN INCOHERENT EISENSTEIN SERIES

HUI XUE

Abstract. In this paper we study the derivative at the center of symmetry of an incoherent Eisenstein series which is associated to an imaginary quadratic field. We show that each nonconstant Fourier coefficient of the derivative can be expressed as the degree of certain zero cycles on a moduli scheme. This result is a generalization of the work by Kudla-Rapoport-Yang.

1. Introduction

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D < 0$. Let $(V, q) = (K, -\mathbf{N})$ be the two-dimensional quadratic space over $\mathbb{Q}$. More generally, although equivalently, we can consider a quadratic form $(K, -\kappa\mathbf{N})$, where $\kappa \in \mathbb{Q}^\times$ is such that $(-D, \kappa)_p = 1$ for all primes (finite or infinite) p and $(\cdot, \cdot)_p$ is the local Hilbert symbol at the prime p. This quadratic space is associated to the quaternion algebra $M_2(\mathbb{Q})$ in the sense that $M_2(\mathbb{Q})$, as a quadratic space, is isomorphic to the orthogonal direct sum $(K, \mathbf{N}) \oplus (K, -\mathbf{N})$.

In Section 2 we will define a (complete) incoherent Eisenstein series $E^*(g, s, \Phi)$ on $SL_2(\mathbb{A})$ attached to an incoherent collection $\mathcal{C}$ (in the sense of Kudla [6]; see Definition 2.1) of local quadratic spaces coming from (V, q). The Eisenstein series $E^*(g, s, \Phi)$ vanishes at the center of symmetry $s = 0$. Let $\tau = u + iv$ be in the upper half plane and write

$$\phi(\tau) = -\frac{d}{ds}(\sqrt{v}^{-1} E^*(g_\tau, s, \Phi))|_{s=0}$$

with $g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$. Then $\phi(\tau)$ is a nonholomorphic modular form of weight 1, whose Fourier expansion is written

$$\phi(\tau) = a_0(v) + \sum_{t<0} a_t(v) e^{2\pi i t\tau} + \sum_{n>0} a_t e^{2\pi i t\tau}.$$

In this paper we will study the Fourier coefficients of ϕ and show in Theorem 1 that for $t > 0$,

$$a_t = w_K^{-1} \deg(Z(t)). \tag{1.1}$$

Here $Z(t)$ is a 0-cycle on a moduli scheme M of dimension one, and w_K is half of the number of units in K. The negative Fourier coefficient $a_t(v)$ has an analogous interpretation; see Theorem 1 in Section 4 for the precise statement. The constant term $a_0(v)$ has some nice arithmetic meaning as well.

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The Eisenstein series $E^*(g, s, \Phi)$ in the simplest form, i.e. when D is prime, was already studied by Kudla-Rapoport-Yang in [7, 9]. Our result is thus a generalization of theirs to an arbitrary fundamental discriminant D. But our method differs from theirs and carries more flavor of that of Kudla [6]. The treatment in [9] gets rather complicated when D is not prime due to the complications involving the genus theory of K. In the present paper we avoid the use of genus theory by employing Siegel-Weil formulas associated to various *coherent* Eisenstein series. Each such coherent Eisenstein series, denoted by $E^*(g, s, \Phi^{(p)})$, arises from a global binary quadratic space $V^{(p)}$ for each prime p not split in K. The quadratic space $V^{(p)}$ is associated to $B^{(p)}$, the quaternion algebra of discriminant p. Actually $B^{(p)}$, as a quadratic space, is isomorphic to the orthogonal sum $(K, \mathbf{N}) \oplus V^{(p)}$. The Eisenstein series $E^*(g, s, \Phi^{(p)})$ differs from $E^*(g, s, \Phi)$ only at the prime p. We observe that for $t > 0$,

$$E_t^*(g_\tau, 0, \Phi) = -(2w_K)^{-1} h_K \sum_p f_p \ell_p \log(p) \cdot E_t^*(g_\tau, 0, \Phi^{(p)});$$

see Section 3 for the details. By applying the Siegel-Weil formula (2.11) it turns out that $E_t^*(g_\tau, 0, \Phi^{(p)})$ is related to the degree of $Z(t)$ over characteristic p in the expected fashion. Taking the sum over all nonsplit p then gives the desired identity (1.1).

In a series of papers [6, 8, 10], Kudla et al. proposed a far-reaching program on the derivatives of certain Eisenstein series, at the center of symmetry or not. The result in this paper may be viewed as a realization of Kudla's idea in the degenerate case. The method of this paper should apply to more general situations, for instance Eisenstein series arising from general CM fields, modulo some (nontrivial) analysis of 0-cycles on a more general moduli scheme.

Added in proof. After the paper was submitted the referee brought to our attention an unpublished manuscript cited in [1, Section 6]. In that manuscript Kudla-Yang also proves (1.1) for an arbitrary *odd* discriminant D. Their method still follows the one outlined in [9] and makes detailed analysis of genus theory. Actually they prove (1.1) for each genus class of K. As our main concern is on the sum of various genus classes, our treatment significantly simplifies the computation. In a subsequent paper, by slightly modifying the present method we are able to prove (1.1) for an arbitrary quadratic form $(K, -\kappa\mathbb{N})$, in which case Kudla-Yang's treatment seems very complicated if possible.

Notation. The letter G always denotes the group SL_2. The standard Borel subgroup of SL_2 is denoted by B. The following notation is used for elements in SL_2:

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The ring of adeles of $\mathbb{Q}$ is denoted by $\mathbb{A}$. We use $\omega(x)$ to denote the quadratic character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ associated to $K = \mathbb{Q}(\sqrt{-D})$. For an algebraic object L we write $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$ and write 1_L for the characteristic function of the set L. For $g \in G(\mathbb{A})$, let $g = n(b)m(a)k$ be its Iwasawa decomposition, where $k \in SO_2(\mathbb{R})SL_2(\widehat{\mathbb{Z}})$. Then the absolute value $|a(g)| = |a|_{\mathbb{A}}$ is well defined.

The additive character ψ of $\mathbb{A}/\mathbb{Q}$ is the standard one defined in Tate's thesis such that $\psi_\infty(x) = e^{2\pi i x}$. The letter γ_p denotes the Weil index of the local quadratic space $(K_p, \mathbf{N})$.

2. Eisenstein series

To state the various Eisenstein series to be considered in this paper, we first recall some basics on the Weil representation of SL_2 associated to a binary quadratic space (V, q) over a local field F. Let $SO(V)$ be the group defined by

$$SO(V) = \{\sigma \in GL_F(V) : q(\sigma v) = q(v) \text{ for } v \in V\}.$$

Notice that if $(V, q) = (K, \Lambda \mathbf{N})$ for a quadratic algebra K over F, then $SO(V) = K^1 = \{x \in K \mid \mathbf{N}(x) = 1\}$. Given an additive character ψ, the Weil representation $r_V = r_{(V,q),\psi}$ of $SL_2(F)$ on the space $S(V)$ of Schwartz functions on V is defined by the following rules (see [5]):

$$\begin{aligned} r_V\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) f(x) &= \psi(aq(x)) f(x), \\ r_V\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) f(x) &= |a|\omega_V(a) f(ax), \\ r_V\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) f(x) &= \gamma_V \hat{f}(x), \end{aligned} \tag{2.1}$$

where γ_V, called the Weil index of V, is an eighth root of unity, ω_V is the quadratic character of $F^\times$ associated to the quadratic space V, which is defined by $\omega_V(x) = (x, -d_V)$, where d_V is the determinant of V and $(\cdot, \cdot)$ is the Hilbert symbol over F. The measure on V is taken to be self-dual with respect to the Fourier transform $\hat{f}(x) = \int_V f(y)\psi(q(x, y))dy$, where $q(x, y) = q(x + y) - q(x) - q(y)$. In particular, if $(V, q) = (K, \mathbf{N})$ and F is non-Archimedean, then the total measure of $\mathcal{O}_K$ is $|d_{K/F}|^{1/2}$, where $d_{K/F}$ is the relative discriminant of K/F. If F is Archimedean, that is $K = \mathbb{C}$, then the measure on K is given by $2dxdy$.

If r'_V is the Weil representation attached to $(V, \Lambda q)$, then

$$\omega' = \omega, \ \gamma' = \omega(\Lambda)\gamma, \ dx' = |\Lambda| dx.$$

The representation r_V extends to a representation, denoted by r_V again, of $SO(V) \times SL_2$ on the same space $S(V)$ by letting

$$r_V(h, g) f(x) = r_V(g) f(h^{-1}x).$$

The product of local Weil representations gives a global Weil representation of $SL_2(\mathbb{A}) \times SO(V_\mathbb{A})$ on the global Schwartz space $S(V_\mathbb{A})$.

Let

$$I(s, \omega) = \operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\omega |\cdot|^s)$$

be the global induced representation of $G(\mathbb{A})$. Each $\Phi(g, s) \in I(s, \omega)$ thus satisfies

$$\Phi(n(b)m(a)g, s) = \omega(a)|a|^{s+1}\Phi(g, s). \tag{2.2}$$

At $s = 0$, the induced representation $I(0, \omega)$ has the following decomposition into irreducible representations of $G(\mathbb{A})$:

$$I(0, \omega) = (\bigoplus_V \Pi(V)) \oplus (\bigoplus_{\mathcal{C}} \Pi(\mathcal{C})).$$

Here V runs over rational binary quadratic spaces with $\omega_V = \omega$, and $\mathcal{C}$ runs through incoherent collections (in the sense of Kudla [6]) of local binary quadratic spaces with $\omega_{\mathcal{C}} = \omega$. The subspace $\Pi(V)$ is the image of the equivariant map $\lambda_V : S(V_{\mathbb{A}}) \to I(0,\omega)$ defined by

$$\lambda_V(\bigotimes_p \varphi_p)(g) = \bigotimes_p ((r_{V,p}(g)\varphi_p)(0). \tag{2.3}$$

Similarly, $\Pi(\mathcal{C})$ is the image of the equivariant map $\lambda_{\mathcal{C}} : S(\mathcal{C}_{\mathbb{A}}) \to I(0,\omega)$ defined by

$$\lambda_{\mathcal{C}}(\bigotimes_p \varphi_p)(g) = \bigotimes_p ((r_{\mathcal{C}_p}(g)\varphi_p)(0). \tag{2.4}$$

Definition 2.1. Let $\mathcal{C} = \{\mathcal{C}_p\}$ be the incoherent collection of local quadratic spaces so that $\mathcal{C}_p = (V_p, q_p) = (K_p, -\mathbf{N})$ for each finite prime p, and $\mathcal{C}_\infty = (V_\infty, -q_\infty) \cong (\mathbb{C}, \mathbf{N})$.

Using the above intertwining map $\lambda_{\mathcal{C}}$ and the incoherent collection $\mathcal{C}$ we define an incoherent section $\Phi = \bigotimes_p \Phi_p = \lambda_{\mathcal{C}}(\bigotimes_p \varphi_p) \in I(0,\omega)$, where for each $p < \infty$, the Schwartz function φ_p is the characteristic function $1_{\mathcal{O}_{K,p}}$ of $\mathcal{O}_K \otimes \mathbb{Z}_p$, and $\varphi_\infty(z) = e^{-2\pi \mathbf{N}(z)}$ on $\mathbb{C} = V_\infty$.

For each prime p inert or ramified (including ∞) in K, we will also define a coherent section $\Phi^{(p)} \in I(0,\omega)$. First let $p < \infty$ be inert or ramified in K. Suppose (E,ι) is a fixed elliptic curve over $\overline{\mathbb{F}}_p$ with an embedding $\iota : \mathcal{O}_K \to \mathrm{End}(E)$. Thus E is supersingular and $\mathcal{O}^{(p)} = \mathrm{End}(E)$ is a maximal order in the quaternion algebra $B^{(p)}$ of discriminant p. The embedding ι of $\mathcal{O}_K$ into $\mathcal{O}^{(p)}$ induces an orthogonal decomposition (with respect to the reduced norm $\mathbf{N} = \mathbf{N}_B$)

$$B^{(p)} = K \oplus V^{(p)},$$

where

$$V^{(p)} = \{y \in B^{(p)} \,|\, ya = \overline{a}y \text{ for all } a \in K\} \tag{2.5}$$

and $(V^{(p)}, \mathbf{N})$ is isomorphic to $(K, \kappa\mathbf{N})$ for some $\kappa \in \mathbb{Q}$. We define

$$L^{(p)} = \mathcal{O}^{(p)} \cap V^{(p)}, \tag{2.6}$$

so $L^{(p)}$ is an $\mathcal{O}_K$-lattice in $V^{(p)}$. The Schwartz function $\varphi^{(p)} = \bigotimes_l \varphi_l^{(p)} \in S(V_{\mathbb{A}}^{(p)})$ is defined as follows. If $l < \infty$, then $\varphi_l^{(p)}$ is the characteristic function $1_{L_l^{(p)}}$ of the local lattice $L_l^{(p)}$. If $l = \infty$ and $(V_\infty^{(p)}, \mathbf{N}) \cong (\mathbb{C}, \Lambda\mathbf{N})$, then

$$\varphi_\infty^{(p)}(z) = e^{-2\pi|\Lambda|\mathbf{N}(z)}. \tag{2.7}$$

Now, let $p = \infty$ and let (E,ι) be an elliptic curve over $\mathbb{C}$ with CM by $\mathcal{O}_K$. The embedding $\mathcal{O}_K \to \mathrm{End}(E^a) = M_2(\mathbb{Z})$ induces an orthogonal decomposition $M_2(\mathbb{R}) = K \oplus V^{(\infty)}$. Here $\mathrm{End}^0(E^a)$ means the endomorphism ring of E as a two-dimensional real torus and

$$V^{(\infty)} = \{y \in B^{(\infty)} = M_2(\mathbb{R}) \,|\, ya = \overline{a}y \text{ for all } a \in K\}.$$

We write

$$L^{(\infty)} = \mathcal{O}^{(\infty)} \cap V^{(\infty)}, \tag{2.8}$$

so $L^{(\infty)}$ is an $\mathcal{O}_K$-lattice in $V^{(\infty)}$. For $l < \infty$ the local $\varphi_l^{(\infty)}$ is defined to be the characteristic function of $L_l^{(\infty)}$, and $\varphi_\infty^{(\infty)}(z) = e^{-2\pi|\Lambda|\mathbf{N}(z)}$ provided that $(V_\infty^{(\infty)}, \mathbf{N}) \cong (\mathbb{C}, \Lambda\mathbf{N})$.

Remark 2.2. The choice of φ_∞ or $\varphi_\infty^{(p)}$ is different from $(e^{-\pi|\Lambda|\mathbf{N}})$ in [9] due to the different normalizations of the measure on $\mathbb{C}$. In [9] the measure at infinity is given by $dxdy$ while ours is the self-dual one given by $2dxdy$.

Using the intertwining map (2.3), for each p not split in K, we define a *coherent* section

$$\Phi^{(p)} = \bigotimes_l \Phi_l^{(p)} = \lambda_{V^{(p)}}(\varphi^{(p)}) \in I(0, \omega).$$

These sections are related as follows.

Lemma 2.3. *If l is a (finite or infinite) prime different from p, then $\Phi_l^{(p)} = \Phi_l$.*

Proof. The case when $l = \infty$ is clear from the definitions, so we assume $l \neq p$ is finite. Suppose l is inert or split in K. As $B_l^{(p)}$ is split, we can choose $j \in \mathcal{O}_l^{(p),\times} - K_l^\times$ such that $j^2 = 1$ and $\mathcal{O}_l^{(p)} = \mathcal{O}_{K,l} \oplus \mathcal{O}_{K,l}j$; see [13] or [3]. Hence $L^{(p)} = V_l^{(p)} \cap \mathcal{O}_l^{(p)} = \mathcal{O}_{K,l}j$. The isomorphism $aj \mapsto a$ between $(K_l j, \mathbf{N})$ and $(K_l, \mathbf{N})$ sends $\varphi_l^{(p)}$ to φ_l, so $\Phi_l^{(p)} = \Phi_l$.

If l is ramified in K, then we can choose $j \in \mathcal{O}_l^{(p),\times} - K_l^\times$ such that $j^2 = 1$. The orthogonal decomposition $B_l^{(p)} = K_l \oplus K_l j$ gives $\mathcal{O}_l^{(p)} = \mathcal{O}_{K,l} + \mathcal{O}_{K,l}\varpi_l^{-d_l}(1+j)$ (see [3]), where ϖ_l is a uniformizer of $\mathcal{O}_{K,l}$ and $d_l = v_l(D)$ is the ramification index. Hence

$$\mathcal{O}_{K,l}j \subseteq L_l^{(p)} = V_l^{(p)} \cap \mathcal{O}_l^{(p)} \subset \varpi_l^{-d_l}\mathcal{O}_{K,l}j.$$

The first inclusion must be an equality because the norm on $L_l^{(p)}$ can only take integral values. This observation concludes that $\Phi_l^{(p)} = \Phi_l$ in this case. $\square$

Let $\Phi(g) \in \Pi(\mathcal{C}) \subset I(0, \omega)$ be as above, and let

$$\Phi(g, s) = \Phi(g)|a(g)|^s$$

be the standard extension to $I(s, \omega)$. Then the Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in B\backslash G} \Phi(\gamma g, s) \tag{2.9}$$

converges absolutely for $\mathrm{Re}(s) > 1$. The series $E(g, s, \Phi)$ has weight 1 at infinity [15], and it has an analytic continuation so that [6]

$$E(g, 0, \Phi) = 0.$$

Similarly, the Eisenstein series associated to $\Phi^{(p)}$ is defined by

$$E(g, s, \Phi^{(p)}) = \sum_{\gamma \in B\backslash G} \Phi^{(p)}(\gamma g, s), \tag{2.10}$$

where

$$\Phi^{(p)}(g, s) = \Phi^{(p)}(g)|a(g)|^s.$$

The theta kernel is defined by

$$\theta(h, g, \varphi^{(p)}) = \sum_{x \in V^{(p)}} r_{V^{(p)}}(h, g)(\varphi^{(p)})(x),$$

where $h \in SO(V_{\mathbb{A}}^{(p)}) = K_{\mathbb{A}}^1$ and $g \in G(\mathbb{A})$. Now we can recall the following Siegel-Weil formula; see Waldspurger [14, Prop. 31].

Proposition 2.4. *Suppose the measure of $K_{\mathbb{A}}^1/K^1$ is normalized so that its total measure is one. Then*

$$E(g, 0, \Phi^{(p)}) = 2 \int_{K^1 \backslash K_{\mathbb{A}}^1} \theta(h, g, \varphi^{(p)}) dh. \tag{2.11}$$

3. Whittaker integrals

We complete the Eisenstein series by a factor

$$E^*(g, s, \Phi) = D^{(s+1)/2} \Lambda(s+1, \omega) \cdot E(g, s, \Phi),$$

where $\Lambda(s, \omega) = \prod_{p \leq \infty} L(s, \omega_p)$ is the complete L-function of the character ω. In the range of absolute convergence $\mathrm{Re}(s) > 1$, the Eisenstein series has a Fourier expansion

$$E(g, s, \Phi) = \sum_{t \in \mathbb{Q}} E_t(g, s, \Phi)$$

with

$$E_t(g, s, \Phi) = \int_{\mathbb{Q} \backslash \mathbb{A}} E(n(b)g, s, \Phi) \psi(-tb) db. \tag{3.1}$$

It is easy to see that $E_t(g, s, \Phi) = 0$ if $t \notin \mathbb{Z}$ (see [9, Lemma 2.3]). So from now on we always assume that t is an integer.

For $t \neq 0$, unfolding (3.1) gives

$$E_t(g, s, \Phi) = \prod_p W_{t,p}(g_p, s, \Phi_p), \tag{3.2}$$

with

$$W_{t,p}(g_p, s, \Phi_p) = \int_{\mathbb{Q}_p} \Phi_p(wn(b)g_p, s) \psi_p(-tb) db \tag{3.3}$$

being called a local Whittaker integral. For $t = 0$, the constant term is given by

$$E_0(g, s, \Phi) = \Phi(g, s) + M(s)\Phi(g), \tag{3.4}$$

where $M(s)\Phi = \bigotimes_p M_p(s)\Phi_p$ is the intertwining map from $I(s, \omega)$ to $I(-s, \omega)$, and

$$M_p(s)\Phi_p(g) = W_{0,p}(g, s, \Phi_p). \tag{3.5}$$

The modified local Whittaker integral is defined as

$$W_{t,p}^*(g, s, \Phi_p) = L_p(s+1, \omega_p) \cdot W_{t,p}(g, s, \Phi_p). \tag{3.6}$$

All the above constructions and definitions also apply to $E(g, s, \Phi^{(p)})$ and will be assumed automatically.

We will now compute the local Whittaker integrals $W_{t,p}^*(g, s, \Phi)$ for $g = 1$ at $p < \infty$ and for

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} = n(u)m(v^{1/2}) \tag{3.7}$$

at $p = \infty$. Since our goal is modest we will confine ourselves to Whittaker integrals over $\mathbb{Q}_p$, although the results certainly hold over more general local fields.

Lemma 3.1. *Suppose $p < \infty$ is unramified in K. Then*

$$W^*_{t,p}(1, s, \Phi_p) = \sum_{r=0}^{v_p(t)} \left(\omega_p(p)p^{-s}\right)^r . \tag{3.8}$$

Proof. The Whittaker integral (3.3) is the sum of two integrals

$$W_{t,p}(1, s, \Phi_p) = \int_{\mathbb{Z}_p} \Phi_p(wn(b), s)\psi_p(-tb)db + \int_{\mathbb{Q}_p - \mathbb{Z}_p} \Phi_p(wn(b), s)\psi_p(-tb)db. \tag{3.9}$$

If $v_p(b) \geq 0$, then $wn(b) \in G(\mathbb{Z}_p)$ and the integrand in the first integral always equals 1; thus the first integral is 1 as $\mu(\mathbb{Z}_p) = 1$.

If $v_p(b) < 0$ we rewrite $wn(b)$ as

$$wn(b) = \begin{pmatrix} 1 & -b^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -b^{-1} & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^{-1} & 1 \end{pmatrix}. \tag{3.10}$$

Then by (2.2),

$$\Phi_p(wn(b), s) = w_p(-b)|b|^{-s-1}. \tag{3.11}$$

Therefore,

$$\begin{aligned} W_{t,p}(1, s, \Phi_p) &= 1 + \sum_{r=1}^{\infty} \omega_p(p)^r p^{-rs} \int_{\mathbb{Z}_p^\times} \psi_p(-bt/p^r)db \\ &= 1 + \sum_{r=1}^{v_p(t)} \mu(\mathbb{Z}_p^\times) \left(\omega_p(p)p^{-s}\right)^r \\ &\qquad + \omega_p(p)^{v_p(t)+1} p^{-(v_p(t)+1)s} \int_{\mathbb{Z}_p^\times} \psi_p(-bt/p^{v_p(t)+1})db \\ &= 1 + \sum_{r=1}^{v_p(t)} \frac{p-1}{p} \left(\omega_p(p)p^{-s}\right)^r - \omega_p(p)^{v_p(t)+1} p^{-(v_p(t)+1)s} p^{-1} \\ &= (1 - \omega_p(p)p^{-s-1}) \sum_{r=0}^{v_p(t)} (\omega_p(p)p^{-s})^r. \end{aligned}$$

Here we have used the fact that

$$\int_{\mathbb{Z}_p^\times} \psi_p(-bt/p^r)db = \begin{cases} \mu(\mathbb{Z}_p^\times) & \text{if } r < v_p(t) + 1, \\ -p^{-1} & \text{if } r = v_p(t) + 1, \\ 0 & \text{if } r > v_p(t) + 1. \end{cases} \tag{3.12}$$

The claim now follows from the definition of $W^*_{t,p}(1, s, \Phi)$. □

Corollary 3.2. *Assume $t \neq 0$. Then*
(1) If p is split in K, then $W^*_{t,p}(1, 0, \Phi_p) \neq 0$.
(2) If p is inert in K and $\omega_p(-t) = 1$, i.e. if $v_p(t)$ is even, then $W^*_{t,p}(1, 0, \Phi_p) = 1$.
(3) If p is inert in K and $\omega_p(-t) = -1$, i.e. if $v_p(t)$ is odd, then

$$W^{*,\prime}_{t,p}(1, 0, \Phi_p) = \frac{1}{2}(v_p(t) + 1)\log(p).$$

Corollary 3.3. *Suppose p is unramified in K.*
(1) If $t = 0$, i.e. $v_p(t) = \infty$, then

$$W^*_{0,p}(1, s, \Phi_p) = L(s, \omega_p).$$

(2) Let $M^*_p(s) = L(s+1, \omega_p) \cdot M_p(s)$. Then

$$M^*_p \Phi_p = L(s, \omega_p)\Phi_p(-s).$$

Lemma 3.4. *Suppose p is inert in K. Then*

$$W^*_{t,p}(1, s, \Phi^{(p)}_p) = -\frac{1 + 1/p}{1 + p^{-s-1}} + \sum_{r=0}^{v_p(t)} (-p^{-s})^r.$$

Proof. We will consider a slightly more general situation. Notice that, as quadratic spaces, $(V^{(p)}_p, \mathbf{N})$ is isomorphic to $(K_p, \Lambda_p \mathbf{N})$, where $\Lambda_p \in \mathbb{Z}_p$ has valuation 1. Under this isomorphism, the Schwartz function $\varphi^{(p)}_p$ is identified with the characteristic function $1_{\mathcal{O}_{K,p}}$ of the localization $\mathcal{O}_{K,p}$. Recall that (see [11, 15]) the value of $\Phi^{(p)}_p$ at $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Z}_p)$ is given by

$$\Phi^{(p)}_p(k) = \begin{cases} 1 & \text{if } v_p(c) \geq 1, \\ -\frac{1}{p} & \text{otherwise.} \end{cases} \tag{3.13}$$

Hence

$$\begin{aligned} W_{t,p}(1, s, \Phi^{(p)}_p) &= -\frac{1}{p} + \int_{\mathbb{Q}_p - \mathbb{Z}_p} \Phi^{(p)}_p(wn(b), s)\psi_b(-tb)db \\ &= -\frac{1}{p} - 1 + L_p(\omega, s+1)^{-1} \sum_{r=0}^{v_p(t)} (-p^{-s})^r, \end{aligned}$$

which implies the desired formula for $W^*_{t,p}(1, s, \Phi^{(p)}_p)$. □

Corollary 3.5. *Suppose p is unramified in K.*
(1) If $\omega_p(-t) = -1$, that is, if p is inert in K and $v_p(t)$ is odd, then

$$W^*_{t,p}(1, 0, \Phi^{(p)}_p) = -1.$$

(2) If $\omega_p(-t) = -1$, then

$$W^{*,\prime}_{t,p}(1, 0, \Phi_p) = -\frac{1}{2}(v_p(t) + 1)\log(p) \cdot W^*_{t,p}(1, 0, \Phi^{(p)}_p).$$

Lemma 3.6. *Suppose $v_p(D) = d_p \geq 1$ and suppose $V = (K_p, \Lambda_p \mathbf{N})$ is the quadratic space with $v_p(\Lambda_p) = 0$. Let $\Psi_p = r_V(1_{\mathcal{O}_{K,p}})(0)$. Then*

$$W^*_{t,p}(1, s, \Psi_p) = \gamma_p \omega_p(\Lambda_p) p^{-d_p/2}(1 + \omega_p(t\Lambda_p)p^{-s(v_p(t)+d_p)}). \tag{3.14}$$

Here γ_p is the Weil index of the quadratic space $(K_p, \mathbf{N})$.

Proof. The value of Ψ_p at $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Z}_p)$ is given by [11, 15]

$$\Psi_p(k) = \begin{cases} \omega_p(d) & \text{if } v_p(c) \geq d_p, \\ 0 & \text{if } 1 < v_p(c) < d_p, \\ \gamma_p p^{-d_p/2}\omega_p(-\Lambda_p c^{-1}) & \text{if } v_p(c) = 0. \end{cases}$$

By (3.10) and the above explicit value formula we get

$$\begin{aligned}
W^*_{t,p}(1,s,\Psi_p) &= \gamma_p p^{-d_p/2}\omega_p(\Lambda_p) + \int_{\mathbb{Q}_p-\mathbb{Z}_p} \Psi_p(wn(b),s)\psi_p(-tb)db \\
&= \gamma_p p^{-d_p/2}\omega_p(\Lambda_p) + \sum_{r\geq f} p^{-rs}\int_{\mathbb{Z}_p^\times} \omega_p(-b/p^r)\psi_p(-tb/p^r)db \\
&= \gamma_p p^{-d_p/2}\omega_p(\Lambda_p) + \omega_p(t)\sum_{r\geq f} p^{-rs}\int_{\mathbb{Z}_p^\times} \omega_p(tb/p^r)\psi_p(tb/p^r)db \\
&= \gamma_p p^{-d_p/2}\omega_p(\Lambda_p) + \omega_p(t)p^{-(v_p(t)+d_p)s}p^{-d_p/2}\epsilon(\omega_p,\psi_p) \\
&= \gamma_p p^{-d_p/2}\omega_p(\Lambda_p) + \gamma_p\omega_p(t)p^{-(v_p(t)+d_p)s}p^{-d_p/2} \\
&= \gamma_p\omega_p(\Lambda_p)p^{-d_p/2}(1+\omega_p(t\Lambda_p)p^{-s(v_p(t)+d_p)}),
\end{aligned}$$

where we have used the fact that

$$\int_{\mathbb{Z}_p^\times} \omega_p(tb/p^r)\psi_p(tb/p^r)db = \begin{cases} p^{-d_p/2}\epsilon(\omega_p,\psi_p) & \text{if } r = v_p(t)+d_p, \\ 0 & \text{otherwise,}\end{cases}$$

where $\epsilon(\omega_p,\psi_p) = \gamma_p$ is the local root number appearing in the functional equation for the zeta function of ω_p; see [11, Lemma 2.1.3]. □

Corollary 3.7. *Retain the notation in Lemma* 3.6.
(1) If $\omega_p(t\Lambda_p) = -1$, then $W^*_{t,p}(1,0,\Psi_p) = 0$ and

$$W^{*,\prime}_{t,p}(1,0,\Psi_p) = \gamma_p\omega_p(\Lambda_p)(v_p(t)+d_p)p^{-d_p/2}\log p.$$

(2) If $\omega_p(t\Lambda_p) = 1$, then

$$W^*_{t,p}(1,0,\Psi_p) = 2\gamma_p\omega_p(\Lambda_p)p^{-d_p/2}.$$

Applying this corollary to $\Psi_p = \Phi_p$ (with $\Lambda_p = -1$) and to $\Psi_p = \Phi_p^{(p)}$, respectively, we get the following.

Corollary 3.8. *Suppose $v_p(D) = d_p \geq 1$ and suppose $t \neq 0$. Then*
(1) If $\omega_p(-t) = 1$, then $W^*_{t,p}(1,0,\Phi_p)$ is nonzero.
(2) If $\omega_p(-t) = -1$, then $W^*_{t,p}(1,0,\Phi_p) = 0$ and

$$W^{*,\prime}_{t,p}(1,0,\Phi_p) = -\frac{1}{2}(v_p(t)+d_p)\log p\cdot W^*_{t,p}(1,0,\Phi_p^{(p)}). \tag{3.15}$$

Proof. Just notice that $V^{(p)} \cong (K_p,\Lambda_p)$ with $\omega_p(-\Lambda_p) = -1$ (see Section 2). □

The following corollary is a direct application of Lemma 3.6.

Corollary 3.9. *Suppose $v_p(D) \geq 1$. If $t = 0$, i.e. $v_p(t) = \infty$, then*

$$W^*_{0,p}(1,s,\Phi) = \gamma_p\omega_p(-1)p^{-d_p/2}$$

and

$$M^*_p(s)\Phi_p = \gamma_p\omega_p(-1)p^{-d_p/2}\Phi(-s).$$

Lemma 3.10. *Let $\tau = u + iv$ be in the upper half plane.*
(1) If $t > 0$, then $W^*_{t,\infty}(g_\tau, 0, \Phi_\infty) = 2i\sqrt{v}e^{2\pi i t\tau}$.
(2) If $t = 0$, then

$$W^*_{0,\infty}(g_\tau, s) = M^*_\infty(s)\Phi_\infty(g_\tau, s) = iv^{(1-s)/2}\pi^{-(s+1)}\Gamma\left(\frac{s+1}{2}\right),$$

and $W^*_{0,\infty}(g_\tau, 0, \Phi_\infty) = i\sqrt{v}$.
(3) If $t < 0$, then $W^*_{t,\infty}(g_\tau, 0, \Phi_\infty) = 0$, and

$$W^{*,\prime}_{t,\infty}(g_\tau, 0, \Phi_\infty) = i\sqrt{v}e^{2\pi i t\tau}\beta_1(4\pi|t|v).$$

Here for $t > 0$,

$$\beta_1(t) = \int_1^\infty u^{-1}e^{-ut}du.$$

Proof. This is Proposition 2.6 of [9]. □

Exactly the same proof gives the following result.

Lemma 3.11. *Let $\tau = u + iv$ in the upper half plane.*
(1) If $t < 0$, then $W^*_{t,\infty}(g_\tau, 0, \Phi^{(\infty)}_\infty) = -2i\sqrt{v}e^{2\pi i t\tau}$.
(2) If $t = 0$, then $W^*_{0,\infty}(g_\tau, 0, \Phi^{(\infty)}_\infty) = -i\sqrt{v}$.
(2) If $t > 0$, then $W^*_{t,\infty}(\tau, 0, \Phi^{(\infty)}_\infty) = 0$ and

$$W^{*,\prime}_{t,\infty}(\tau, 0, \Phi^{(\infty)}_\infty) = -i\sqrt{v}e^{2\pi i t\tau}\beta_1(4\pi|t|v).$$

Corollary 3.12. *If $t < 0$ and $\tau = u + iv$ is in the upper half plane, then*

$$W^{*,\prime}_{t,\infty}(g_\tau, 0, \Phi_\infty) = -\frac{1}{2}\beta_1(4\pi|t|v) \cdot W^*_{t,\infty}(g_\tau, 0, \Phi^{(\infty)}_\infty).$$

Remark 3.13. (1) From the above computations we observe the following dichotomy at each nonsplit p (finite or infinite): for each $t \neq 0$, exactly one of $W^*_{t,p}(1, 0, \Phi_p)$ and $W^*_{t,p}(1, 0, \Phi^{(p)}_p)$ equals zero.
(2) If $p < \infty$ is inert and ramified in K, then $W^*_{t,p}(1, 0, \Phi_p) = 0$ if and only if $\omega_p(-t) = -1$; if $p = \infty$, then $W^*_{t,\infty}(g_\tau, 0, \Phi_\infty) = 0$ if and only if $\omega_\infty(t) = -1$.
(3) For $t \neq 0$, an odd number of local Whittaker factors $W^*_{t,p}(1, 0, \Phi)$ (or $W^*_{t,\infty}(g_\tau, 0, \Phi_\infty)$) vanish because

$$\left(\prod_{p<\infty}\omega_p(-t)\right) \cdot \omega_\infty(t) = -1.$$

The constant term of the Fourier expansion of $E^{*,\prime}(g_\tau, 0, \Phi)$ is taken care of in the next proposition.

Proposition 3.14. *If $t = 0$, then $E^*_0(g_\tau, 0, \Phi) = 0$ and*

$$E^{*,\prime}_0(g_\tau, 0, \Phi) = w_K^{-1}h_K\sqrt{v}\left(\log(D) + \log(v) + 2\frac{\Lambda'(1,\omega)}{\Lambda(1,\omega)}\right),$$

where as usual $w_K = |\mathcal{O}^\times_K|/2$.

Proof. By Corollaries 3.3, 3.9 and equation (3.4) we get

(3.16)
$$\begin{aligned} E_0^*(g_\tau, s, \Phi) &= D^{(s+1)/2}\Lambda(s+1,\omega)\Phi(g_\tau, s) + D^{(s+1)/2}M^*(s)\Phi(g_\tau, s) \\ &= D^{(s+1)/2}v^{(s+1)/2}\Lambda(s+1,\omega) + iD^{s/2}v^{(1-s)/2}\Lambda(s,\omega)\prod_{p\mid D}\omega_p(-1)\gamma_p. \end{aligned}$$

As $\gamma_p = \epsilon(\omega_p, \psi_p)$, which is 1 for unramified p, and $\epsilon(\omega_\infty, \psi_\infty) = i$, we obtain

$$1 = \prod_{p\le\infty}\epsilon(\omega_p,\psi_p) = \epsilon(\omega_\infty,\psi_\infty)\prod_{p\mid D}\epsilon(\omega_p,\psi_p) = i\cdot\prod_{p\mid D}\gamma_p$$

and $\prod_{p\mid D}\gamma_p = -i$. So $\prod_{p\mid D}\omega_p(-1)\gamma_p = \omega_\infty(-1)\cdot(-i) = i$, which combined with (3.16) and the functional equation of $\Lambda(s,\omega)$ gives

$$\begin{aligned} E_0^*(g_\tau, s, \Phi) &= D^{(s+1)/2}v^{(s+1)/2}\Lambda(s+1,\omega) - D^{s/2}v^{(1-s)/2}\Lambda(s,\omega) \\ &= D^{(s+1)/2}v^{(s+1)/2}\Lambda(s+1,\omega) - D^{(1-s)/2}v^{(1-s)/2}\Lambda(1-s,\omega). \end{aligned}$$

Therefore $E_0^*(g_\tau, 0, \Phi) = 0$ and

$$\begin{aligned} E_0^{*,\prime}(g_\tau, 0, \Phi) &= 2\frac{d}{ds}\left(D^{(s+1)/2}v^{(s+1)/2}\Lambda(s+1,\omega)\right)|_{s=0} \\ &= v^{1/2}D^{1/2}\left(\log(D)\Lambda(1,\omega) + \log(v)\Lambda(1,\omega) + 2\Lambda'(1,\omega)\right), \end{aligned}$$

which combined with the following well-known class number formula

$$\Lambda(1,\omega) = \frac{h_K}{w_K\sqrt{D}} \tag{3.17}$$

completes the proof. □

We now summarize the above results to rewrite the nonconstant term Fourier coefficients in a more convenient form. To facilitate the description, for every nonsplit finite prime p, we let f_p be the residue degree of $K/\mathbb{Q}$ at p and define

$$\ell_p(t) = \begin{cases} \frac{v_p(t)+1}{2} & \text{if } p \text{ is inert in } K, \\ v_p(t) + d_p & \text{if } p \text{ is ramified in } K. \end{cases} \tag{3.18}$$

Proposition 3.15. *(1) If $t > 0$ is an integer, then*

$$E_t^{*,\prime}(g_\tau, 0, \Phi) = -(2w_K)^{-1}h_K\cdot\sum_p f_p\ell_p(t)\log(p)\cdot E_t(g_\tau, 0, \Phi^{(p)}),$$

where p ranges over finite primes that are not split in K.
(2) If $t < 0$ is an integer, then

$$E_t^{*,\prime}(g_\tau, 0, \Phi) = -(2w_K)^{-1}h_K\cdot\beta_1(4\pi|t|v)E_t(g_\tau, 0, \Phi^{(\infty)}).$$

Proof. We only give the proof of part (1). As $t > 0$ only local factors $W_{t,p}^*(1, 0, \Phi_p)$ at finite nonsplit p can vanish. We also know that $W_{t,p}^*(1, s, \Phi_p)$ is equal to 1 for almost all p. Thus we can take the product of local factors without worrying about the convergence. Now

$$
\begin{aligned}
E_t^{*,\prime}(g_\tau, 0, \Phi) &= \frac{d}{ds}\left(D^{(s+1)/2} W^*_{t,\infty}(g_\tau, s, \Phi_\infty) \prod_{p<\infty} W^*_{t,p}(1, s, \Phi_p) \right)|_{s=0} \\
&= D^{1/2} W^*_{t,\infty}(g_\tau, 0, \Phi_\infty) \cdot \left(\sum_{p,\,\text{nonsplit}} W^{*,\prime}_{t,p}(1, 0, \Phi_p) \prod_{l\neq p} W^*_{t,l}(1, 0, \Phi_l) \right) \\
&= D^{1/2} W^*_{t,\infty}(g_\tau, 0, \Phi_\infty) \\
&\quad \cdot \left(-\frac{1}{2} \sum_p f_p \ell_p \log(p) \cdot W^*_{t,p}(1, 0, \Phi_p^{(p)}) \prod_{l\neq p} W^*_{t,l}(1, 0, \Phi_l^{(p)}) \right) \\
&= -\frac{1}{2} \sum_p f_p \ell_p \log(p) \cdot E_t^*(g_\tau, 0, \Phi^{(p)}) \\
&= -\frac{1}{2} D^{1/2} \Lambda(1, \omega) \sum_p f_p \ell_p \log(p) \cdot E_t(g_\tau, 0, \Phi^{(p)}) \\
&= -(2w_K)^{-1} h_K \cdot \sum_p f_p \ell_p \log(p) \cdot E_t(g_\tau, 0, \Phi^{(p)}),
\end{aligned}
$$

where we have used Lemma 2.3 and Corollaries 3.5 and 3.8. □

The next proposition constitutes a key ingredient of our approach.

Proposition 3.16. *Let $\tau = u + iv$ be in the upper half plane and $t \neq 0$. Then for each (finite or infinite) prime p nonsplit in K,*

$$
E_t(g_\tau, 0, \Phi^{(p)}) = \frac{2}{h_K} \sqrt{v} e^{2\pi i t \tau} \sum_{i=1}^{h_K} |\{y \in L_i^{(p)} | \mathbf{N}(y) = t\}|,
$$

where $L_1^{(p)} = L^{(p)}$ is the quadratic lattice defined in (2.6) and $L_i^{(p)}$, for $i = 2, \ldots, h_K$, is defined in the proof.

Proof. By the Siegel-Weil formula (2.11) and exchanging the order of integration we obtain

(3.19)

$$
\begin{aligned}
E_t(g_\tau, 0, \Phi^{(p)}) &= 2 \int_{K^1 \backslash K^1_{\mathbb{A}}} \int_{\mathbb{Q}\backslash\mathbb{A}} \theta(h, n(x) g_\tau, \varphi^{(p)}) \psi(-tx) dx dh \\
&= 2 \int_{K^1 \backslash K^1_{\mathbb{A}}} \left(\sum_{y \in V^{(p)}} \sum_{\mathbb{Q}\backslash\mathbb{A}} r_{V^{(p)}}(g_\tau) \varphi^{(p)}(h^{-1} y) \psi(x\mathbf{N}(y) - tx) dx \right) dh \\
&= 2 \int_{K^1 \backslash K^1_{\mathbb{A}}} \left(\sum_{y \in V^{(p)}(t)} r_{V^{(p)}}(g_\tau) \varphi^{(p)}(h^{-1} y) \right) dh \\
&= 2 \int_{K^1 \backslash K^1_{\mathbb{A}}} \left(\sum_{y \in V^{(p)}(t)} r_{V^{(p)}_\infty}(g_\tau) \varphi^{(p)}_\infty(h^{-1} y) \cdot 1_{\widehat{L}^{(p)}}(h^{-1} y) \right) dh,
\end{aligned}
$$

where $V^{(p)}(t) = \{y \in V^{(p)} \mid \mathbf{N}(y) = t\}$. The map $K^\times \to K^1$ defined by $z \mapsto z/\overline{z}$ induces an isomorphism

$$\mathrm{Cl}(K) = K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times / \widehat{\mathcal{O}}_K^\times \to K^1 K_\infty^1 \backslash K_{\mathbb{A}}^1 / U^1$$

with U^1 being the image of $\widehat{\mathcal{O}}_K^\times$. Let $\mathfrak{a}_i$, for $i = 1, \dots, h_K$, be representatives of $\mathrm{Cl}(K)$ with $\mathfrak{a}_1 = (1)$. Hence, under the above identification $h_i = \mathfrak{a}_i \overline{\mathfrak{a}}_i^{-1}$ becomes a set of representatives for $K^1 K_\infty^1 \backslash K_{\mathbb{A}}^1 / U^1$. As the total measure of $K^1 \backslash K_{\mathbb{A}}^1$ is one, the last integral becomes

$$E_t(g_\tau, 0, \Phi^{(p)}) = \frac{2}{h_K} \sqrt{v} e^{2\pi i t\tau} \sum_{i=1}^{h_K} \sum_{y \in V^{(p)}(t)} 1_{\widehat{L}^{(p)}}(h_i^{-1} y), \tag{3.20}$$

where we have used the fact that (by applying (2.1))

$$r_{V_\infty^{(p)}}(g_\tau) \varphi_\infty^{(p)}(h^{-1} y) = \sqrt{v} e^{2\pi i t\tau}$$

provided that $h^{-1} \in K_\infty^1$ and $y \in V^{(p)}(t)$. Let $L_i^{(p)} = V^{(p)} \cap \mathfrak{a}_i \overline{\mathfrak{a}}_i^{-1} \widehat{L}^{(p)}$ for $i = 1, \dots, h_K$. Then $1_{\widehat{L}^{(p)}}(h_i^{-1} y) = 1_{\widehat{L}_i^{(p)}}(y)$. Therefore we can rewrite (3.20) as

$$\begin{aligned} E_t(g_\tau, 0, \Phi^{(p)}) &= \frac{2}{h_K} \sqrt{v} e^{2\pi i t\tau} \sum_{i=1}^{h_K} \sum_{y \in V^{(p)}(t)} 1_{\widehat{L}_i^{(p)}}(y) \\ &= \frac{2}{h_K} \sqrt{v} e^{2\pi i t\tau} \sum_{i=1}^{h_K} |\{y \in L_i^{(p)} \mid \mathbf{N}(y) = t\}|, \end{aligned}$$

which concludes the proof. □

Combining Propositions 3.15 and 3.16 we obtain the following formulas about the nonconstant Fourier coefficients.

Proposition 3.17. *(1) If $t > 0$ is an integer, then*

$$E_t^{*,\prime}(g_\tau, 0, \Phi) = -w_K^{-1} \sqrt{v} e^{2\pi i t\tau} \cdot \sum_p f_p \ell_p \log(p) \sum_{i=1}^{h_K} |\{y \in L_i^{(p)} | \mathbf{N}(y) = t\}|,$$

where p ranges over primes which are not split in K.
(2) If $t < 0$ is an integer, then

$$E_t^{*,\prime}(g_\tau, 0, \Phi) = -w_K^{-1} \sqrt{v} e^{2\pi i t\tau} \cdot \beta_1(4\pi |t| v) \sum_{i=1}^{h_K} |\{y \in L_i^{(\infty)} | \mathbf{N}(y) = t\}|.$$

4. Special 0-cycles and conclusions

In this section we will first briefly describe a moduli scheme of dimension one and certain special 0-cycles on it. Readers should refer to [9] for more details. We consider the following moduli problem $\mathcal{M}$ over the category $(\mathrm{Sch}/\mathcal{O}_K)$. To each scheme S over $\mathcal{O}_K$, $\mathcal{M}(S)$ denotes the set of isomorphism classes of pairs (E, ι), where E is an elliptic curve over S and $\iota : \mathcal{O}_K \to \mathrm{End}_S(E)$ is a homomorphism such that the induced homomorphism $\mathrm{Lie}(\iota) : \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_S}(\mathrm{Lie}E) = \mathcal{O}_S$ coincides with the structure homomorphism of $\mathcal{O}_S$.

We quote the following well-known result on the functor $\mathcal{M}$.

Proposition 4.1. *(1) The coarse moduli scheme M of the functor $\mathcal{M}$ is given by*

$$M \cong Spec(\mathcal{O}_H),$$

where $\mathcal{O}_H$ is the ring of integers in the Hilbert class field H of K.
(2) Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime above p. Then there is a natural bijection

$$M(\overline{k(\mathfrak{p})}) \cong \mathrm{Cl}(K).$$

Here $\overline{k(\mathfrak{p})}$ is the algebraic closure of the residue field of $\mathfrak{p}$.

Definition 4.2. Let $(E, \iota) \in \mathcal{M}(S)$. A special endomorphism of (E, ι) is an element $y \in \mathrm{End}(E)$ such that

$$y \cdot \iota(a) = \iota(\overline{a}) \cdot y$$

for all $a \in \mathcal{O}_K$.

Let p be a finite prime. The set of special endomorphisms of $(E, \iota) \in M(\overline{\mathbb{F}}_p)$ is denoted by $L^{(p)}(E, \iota)$. Of course $L^{(p)}(E, \iota)$ is nonempty if and only if p is not split in K.

Definition 4.3. For a positive integer t, let $\mathcal{Z}(t)$ denote the functor which to each scheme S over $\mathcal{O}_K$ associates the set of isomorphism classes of triples (E, ι, y), where $(E, \iota) \in \mathcal{M}(S)$ and $y \in L(E, \iota)$ such that

$$\mathbf{N}(y) = y \cdot y' = t.$$

Here y' is the main involution of $\mathrm{End}_S(E)$.

The coarse moduli scheme of $\mathcal{Z}(t)$ is denoted by $Z(t)$. It is easy to see that $Z(t)$ is Artinian and is located over finite characteristics p which are not split in K. The degree of $Z(t)$ is defined as in [12] to be

$$\deg(Z(t)) = \sum_{\xi \in Z(t)} \log(|\mathcal{O}_{Z(t),\xi}|), \tag{4.1}$$

where the sum is over the geometric points of $Z(t)$.

We quote the following well-known result about $\deg(Z(t))$, which is essentially due to Gross; see [9, Theorem 5.12] and [3].

Proposition 4.4. *The degree of $Z(t)$ is given by*

$$\deg(Z(t)) = \sum_p f_p \ell_p(t) \log(p) \cdot \sum_{i=1}^{h_K} |\{x \in L(E_i, \iota_i) \mid \mathbf{N}(x) = t\}|,$$

where p ranges over primes which are not split in K, (E_i, ι_i) for $i = 1, \ldots, h_K$ runs over points in $M(\overline{k(\mathfrak{p})})$, and f_p and ℓ_p are defined in (3.18).

To relate Proposition 4.4 with Proposition 3.17 we make the following observation. Let $(E_1, \iota_1) = (E, \iota) \in M(\overline{\mathbb{F}}_p)$ be the CM elliptic curve fixed in Section 2; thus $\mathcal{O}_K$ is optimally embedded in the maximal order $\mathcal{O}^{(p)} = \mathrm{End}(E_1)$ and $L(E, \iota) = L^{(p)}$. For $i = 1, \ldots, h_K$, each $\mathrm{End}(E_i)$ is a maximal order in $B^{(p)}$, in which $\mathcal{O}_K$ is optimally embedded. Thus by the Chevalley-Hasse-Noether theorem (see [4, Prop. 3.2] for a proof) there is an ideal $\mathfrak{a}_i$ such that

$$L(E_i, \iota_i) = \mathfrak{a}_i \overline{\mathfrak{a}}_i^{-1} L^{(p)}.$$

We may take $\mathfrak{a}_1 = (1), \mathfrak{a}_2, \ldots, \mathfrak{a}_{h_K}$ to be the same set of representatives as in Propositions 3.16 and 3.17. Thus we obtain the equality $L_i^{(p)} = L(E_i, \iota_i)$ for $i = 1, \ldots, h_K$, which implies

$$\sum_{i=1}^{h_K} |\{y \in L_i^{(p)} | \mathbf{N}(y) = t\}| = \sum_{i=1}^{h_K} |\{x \in L(E_i, \iota_i) \,|\, \mathbf{N}(x) = t\}|. \tag{4.2}$$

For a negative integer t and $\tau = u + iv$ in the upper half plane, we will define an Arakelov divisor concentrated at Archimedean points $M(\mathbb{C})$ of the scheme M. For a point (E, ι) of $M(\mathbb{C})$, let E^a be the underlying real torus. We define the space of special endomorphisms of (E^a, ι) to be

$$L^{(\infty)}(E, \iota) = \{y \in \mathrm{End}(E^a) \,|\, y \cdot \iota(a) = \iota(\overline{a}) \cdot y\}.$$

Notice that $\mathrm{End}(E^a) = M_2(\mathbb{Z})$. An analogous argument to the above shows that

$$\sum_{i=1}^{h_K} |\{y \in L_i^{(\infty)} | \mathbf{N}(y) = t\}| = \sum_{i=1}^{h_K} |\{x \in L^{(\infty)}(E_i, \iota_i) \,|\, \mathbf{N}(x) = t\}|. \tag{4.3}$$

The Arakelov divisor on M is defined by

$$Z(t, v) = \sum_{i=1}^{h_K} r_i(t, v) \tau_i, \tag{4.4}$$

where τ_i runs over Archimedean places of H and

$$r_i(t, v) = \beta_1(4\pi|t|v) \cdot |\{y \in L^{(\infty)}(E_i, \iota_i) \,|\, Q(y) = t\}|.$$

The degree of the Arakelov divisor $Z(t, v)$ is defined by [12]

$$\deg(Z(t, v)) = \sum_{i=1}^{h_K} r_i(t, v). \tag{4.5}$$

Now we can state the main result.

Theorem 1. *Let $\tau = u + iv$ be in the upper half plane and define*

$$\phi(\tau) = -\frac{d}{ds}(\sqrt{v}^{-1} E^*(g_\tau, s, \Phi))|_{s=0}.$$

The nonholomorphic modular form $\phi(\tau)$ has a Fourier expansion

$$\phi(\tau) = a_0(v) + \sum_{n>0} a_t e^{2\pi i t\tau} + \sum_{t<0} a_t(v) e^{2\pi i t\tau},$$

where for $t = 0$,

$$a_0(v) = w_K^{-1} h_K \left(\log(D) + \log(v) + 2\frac{\Lambda'(1, \omega)}{\Lambda(1, \omega)} \right),$$

for $t > 0$,

$$a_t = w_K^{-1} \deg(Z(t)),$$

and for $t < 0$,

$$a_t(v) = w_K^{-1} \deg(Z(t, v)).$$

Here $\deg(Z(t))$ *is defined in* (4.1) *and* $\deg(Z(t, v))$ *is defined in* (4.5).

Proof. The claim for $t = 0$ is proved in Proposition 3.14. The claim for $t > 0$ follows from Propositions 3.17, 4.4 and (4.2). The claim for $t < 0$ is obtained by combining Proposition 3.15 and (4.3). $\square$

To conclude the paper we follow [9] to recall a nice interpretation of the constant term $a_0(v)$. By a result of Colmez [2, p. 633], the Faltings height $h_{\mathrm{Fal}}(E)$ of the CM elliptic curve (E, ι) is given by

$$4h_{\mathrm{Fal}}(E) = -\log(D) - 2\log(2\pi) - 2\frac{L'(0,\omega)}{L(0,\omega)}.$$

Thus

$$a_0(v) = -w_K^{-1} h_K(\log(v) + 4h_{\mathrm{Fal}}(E) + C),$$

where the constant C is given by

$$C = \log(\pi) - \frac{\Gamma'(1/2)}{\Gamma(1/2)} + 2\log(2\pi).$$

By an observation in [10], the constant term $a_0(v)$ should also relate to the positive Fourier coefficients of the derivative of an Eisenstein series of weight $3/2$ at $s = 1/2$. Interested readers may consult [10] for more on this aspect.

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Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634

E-mail address: huixue@clemson.edu