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SPECTRA FOR COMPACT QUANTUM GROUP COACTIONS AND CROSSED PRODUCTS

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Abstract. We present definitions of both the Connes spectrum and the strong Connes spectrum for actions of compact quantum groups on C^*-algebras and obtain necessary and sufficient conditions for a crossed product to be a prime or a simple C^*-algebra. Our results extend to the case of compact quantum actions, namely the results of Gootman, Lazar and Peligrad, which in turn, generalize results by Connes, Olesen, Pedersen and Kishimoto for abelian group actions. We prove in addition that the Connes spectra are closed under tensor products. These results are new for compact nonabelian groups as well.

1. Introduction

In his fundamental paper [4], Connes defines the invariant Γ called, in his name, the Connes spectrum, for abelian group actions on von Neumann algebras. Among other results, he obtained conditions for a crossed product to be a factor. Subsequently, Olesen and Pedersen [11] have defined the Connes spectrum for abelian group actions on C^*-algebras. They have extended the results of Connes to the case of crossed products of C^*-algebras by abelian group actions, obtaining conditions that such a crossed product be a prime C^*-algebra. However, the similar result for simple crossed products using the Olesen-Pedersen version of the Connes spectrum is false. In [9], Kishimoto has shown that the result is true for simple crossed products if his "strong Connes spectrum" is used instead of Olesen and Pedersen's Connes spectrum. Rieffel [15] and Landstad [10] have put the problem of finding a "good" definition of the Connes spectrum for compact, not necessarily abelian, group actions on C^*-algebras. In [10], Landstad remarks that a "good" definition of the Connes spectrum should lead to a result that generalizes the Olesen-Pedersen characterization of prime crossed products to the case of nonabelian compact group actions. Gootman, Lazar, and Peligrad [8] have defined the Connes spectrum and the strong Connes spectrum for compact, not necessarily abelian, group actions on C^*-algebras. In the case of abelian groups, these notions coincide with the previous ones. Moreover, in [8], Gootman, Lazar, and Peligrad prove the characterizations of the primeness and simplicity of crossed products using their definitions. In this paper we present definitions of both the Connes spectrum (Definition 3.1) and the strong Connes spectrum (Definition 4.1) in the case of compact quantum groups and prove the corresponding characterizations of primeness and simplicity of crossed products (Theorems 3.4 and 4.4). In addition, we prove that the Connes spectra are closed under tensor products (Propositions 5.4 and 5.6). This result is new for nonabelian compact groups as well. We will use the techniques developed by

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Woronowicz [16, 17], Boca [2], and the authors in [5, 7]. In addition, this paper contains new methods for the study of the hereditary C^*-subalgebras that are invariant under a compact quantum group coaction (Section 3) and for the proof of the key Lemma 2.2.

2. Preliminaries

Let $G = (A, \Delta)$ be a compact quantum group (see [17]) and let (B, G, δ) be a quantum dynamical system, where B is a C^*-algebra and δ is a coaction of A on B (see [2] or [14]). Denote by $\widehat{G}$ the set of all equivalence classes of irreducible representations of G ([16], section 4).

For each $\alpha \in \widehat{G}$, $u^\alpha = \sum_{i,j=1}^{d_\alpha} m_{ij}^\alpha \otimes u_{ij}^\alpha$ denotes a representative of each class. Then the linear space generated by $\{u_{ij}^\alpha | \alpha \in \widehat{G}, 1 \le i, j \le d_\alpha\}$ is a $*$-algebra $\mathcal{A}$, called the Woronowicz-Hopf algebra ([17], Section 5). For $\alpha \in \widehat{G}$ and $u^\alpha \in \alpha$ a unitary representative, denote $\overline{u^\alpha} = \sum_{i,j=1}^{d_\alpha} m_{ij}^\alpha \otimes {u_{ij}^\alpha}^*$. Then $\overline{u^\alpha}$ is a (not necessarily unitary) representation of G called the adjoint of u^α. We will denote by $\overline{\alpha}$ the equivalence class of $\overline{u^\alpha}$.

Set $\chi_\alpha = \sum_{i=1}^{d_\alpha} u_{ii}^\alpha$. We use F_α to denote the unique positive, invertible operator in $B(H_\alpha)$ that intertwines u^α with its double contragradient representation $(u^\alpha)^{cc}$ such that $tr(F_\alpha) = tr(F_\alpha^{-1})$. Set $M_\alpha = tr(F_\alpha)$ ([16], Theorem 5.4).

Since every positive matrix is unitarily equivalent to a diagonal matrix, we may assume that the matrices F_α are diagonal: $F_\alpha = diag\{f_1^\alpha, \ldots, f_{d_\alpha}^\alpha\}$. The formula $f_1^\alpha(u_{nm}^\alpha) = \delta_{nm} f_m$ defines a linear form on $\mathcal{A}$. The above assumption implies that all u_{ij}^α are mutually orthogonal in H_h and therefore

$$h({u_{ij}^\alpha}^* u_{mn}^\alpha) = \frac{1}{M_\alpha}\frac{1}{f_i^\alpha}\delta_{im}\delta_{jn} \quad \text{and} \tag{1}$$

$$h(u_{mk}^\alpha {u_{nl}^\alpha}^*) = \frac{f_l^\alpha}{M_\alpha}\delta_{mn}\delta_{lk}, \tag{2}$$

where h is the Haar state on G and δ_{rs} are the Kronecker δ's ([16], Theorem 5.7).

For each $\alpha \in \widehat{G}$, denote by B_α^δ the associated spectral subspace defined by (see [7] or [2]):

$$B_\alpha^\delta = \{P_\alpha(x) | \, x \in B\},$$

where $P_\alpha(x) = (\iota \otimes h_\alpha)(\delta(x))$ and $h_\alpha = M_\alpha h \cdot (\chi_\alpha * f_1^\alpha)^*$. Recall that for all $a, b \in A$, $h \cdot a(b) = h(ba)$ and for all linear functionals ξ on A, $a * \xi = (\xi \otimes \iota)(\Delta(a))$ (see [17], relation 1.14, or [5]). In particular, for $\alpha = \iota$, P_ι is the projection of B onto the fixed point algebra B^δ.

Let $c_{ij} = M_\alpha(u_{ij}^\alpha * f_1^\alpha)^*$. Note that, since F_α is a diagonal matrix we obtain $u_{ij}^\alpha * f_1 = f_i^\alpha u_{ij}^\alpha$ and hence $c_{ij} = M_\alpha f_i^\alpha {u_{ij}^\alpha}^*$.

Define the mapping $P_{ij}^\alpha : B \to B$ by

$$P_{ij}^\alpha(x) = (id \otimes h \cdot c_{ji}^\alpha)(\delta(x)),$$

for all $x \in B$. Note that $P_{ij}^\alpha P_{kl}^\alpha = \delta_{il} P_{kj}^\alpha$.

For each $\alpha \in \widehat{G}$, define

$$B_2^\delta(u^\alpha) = \{[P_{ij}^\alpha(x)]_{ij} | \, x \in B\} \subseteq B \otimes \mathcal{M}_{d_\alpha},$$

where $[P^{\alpha}_{ij}(x)]_{ij} = \sum_{i,j=1}^{d_\alpha} P^{\alpha}_{ij}(x) \otimes m^{\alpha}_{ij}$ with $\{m^{\alpha}_{ij} | 1 \leq i,j \leq d_\alpha\}$ the set of elementary matrices in the algebra $\mathcal{M}_{d_\alpha}$ of scalar matrices of order $d_\alpha \times d_\alpha$.

Notice that $B^{\delta}_2(u^\alpha)$ depends on the representative u^α, not only on the equivalence class $\alpha \in \widehat{G}$. However, for two equivalent representations u^α_1 and u^α_2, the corresponding B^δ_2 are spatially isomorphic.

The proofs of the following remarks are straightforward from the definitions.

Remark 2.1. (1) If u^α is an irreducible unitary representation of G, then

$$\delta(P^{\alpha}_{ij}(x)) = \sum_{k=1}^{d_\alpha} P^{\alpha}_{ik}(x) \otimes u^{\alpha}_{kj}.$$

(2) $B^\delta_\alpha = linspan\{P^\alpha_{ij}(x) |\, x \in B,\ i,j = 1,2,\ldots,d_\alpha\}$.

(3) For $x \in B$, let $X = [P^{\alpha}_{ij}(x)]_{ij} = \sum_{i,j=1}^{d_\alpha} P^{\alpha}_{ij}(x) \otimes m^{\alpha}_{ij}$. Then $X \in B^\delta_2(u^\alpha)$ and $\delta_{13}(X) = (X \otimes 1_A)(1_B \otimes u^\alpha)$, where 1_A is the unit of A and 1_B is the unit of the multiplier algebra of B. The leg numbering notation used here is the standard one ([1] and [16]). Also, $B^\delta_2(u^\alpha)$ is isomorphic as a Banach space to B^δ_α through the mapping $X \to \sum_{i=1}^{d_\alpha} P^{\alpha}_{ii}(x)$. Therefore:

$$B^\delta_2(u^\alpha) = \{X \in B \otimes \mathcal{M}_{d_\alpha} |\, \delta_{13}(X) = (X \otimes 1_A)(1_B \otimes u^\alpha)\}.$$

(4) Let $x \in B$ and fix $x_{i_0 j_0} = P^{\alpha}_{i_0 j_0}(x)$. Then $x_{i_0 j_0} \in B$ and $[P^{\alpha}_{ij}(x_{i_0 j_0})]_{ij} \in B^\delta_2(u^\alpha)$ is a matrix whose only nonzero row is the j_0-row and whose $j_0 j$-entry is given by $P^{\alpha}_{i_0 j}(x)$, for each $j = 1,2,\ldots,d_\alpha$. Furthermore,

$$B^\delta_2(u^\alpha) = linspan\{[P^{\alpha}_{ij}(x_{rs})]_{ij} |\, r,s = 1,2,...,d_\alpha\}.$$

With $\alpha \in \widehat{G}$ and v the right regular representation of G, we will use the following notation (see [17], relation 3.2 and [5]):

$$\mathcal{F}_v(a) = (id \otimes ha)(v^*).$$

In particular, for $a = (\chi_\alpha * f^\alpha_1)^*$ we denote

$$p_\alpha = \mathcal{F}_v((\chi_\alpha * f^\alpha_1)^*) = (id \otimes h_\alpha)(v^*).$$

Denote by $\widehat{A}$ the norm closure of the set of all operators of the form $\mathcal{F}_v(a)$, where $a \in A$.

Recall that the crossed product $B \times_\delta G$ is defined to be the C^*-algebra generated by all elements of the form $(\pi_u \times \pi_h)(\delta(b))(1 \otimes \mathcal{F}_v(a))$, where $a \in A$, $b \in B$, $\pi_u : B \to B(H_u)$ is the universal representation of the C^*-algebra B and $\pi_h : A \to B(H_h)$ is the GNS representation of A associated to the Haar state h.

Furthermore, if $\alpha_1, \alpha_2 \in \widehat{G}$, define

$$S_{\alpha_1,\alpha_2} = (1 \otimes p_{\overline{\alpha_1}})(B \times_\delta G)(1 \otimes p_{\overline{\alpha_2}}),$$
$$S_\alpha = S_{\alpha,\alpha}.$$

For properties of S_{α_1,α_2}, see [5], Lemma 3.1 and Proposition 3.2. It is straightforward to check that

$$S_{\alpha,\iota} = linspan\{(1 \otimes p_{\overline{\alpha}})\delta(b^*)(1 \otimes p_\iota) |\, b \in B_\alpha\}.$$

Note that, from the above definition, an element $(1\otimes p_\alpha)\delta(b)^*(1\otimes p_\iota)$ of $S_{\alpha,\iota}$ is an operator from $H_u\otimes\mathbb{C}\xi_h$ to $H_u\otimes p_\alpha H_h$. Since $p_\alpha H_h$ is a d_α^2-dimensional subspace of H_h (with basis $\left\{u_{ij}^{\alpha *}|1\le i,j\le d_\alpha\right\}$), every such operator can be represented as a $d_\alpha^2\times 1$ column matrix with entries in B.

If v is the right regular representation, then $ad(v)$ is a coaction of G on $B\times_\delta G$, defined by $ad(v)(x)=v_{23}(x\otimes 1)v_{23}^*$, for all $x\in B\times_\delta G$ ([5], Lemma 3.3). We consider the projection Q of $B\times_\delta G$ on $(B\times_\delta G)^{ad(v)}$, the C^*-subalgebra of fixed points for the coaction $ad(v)$:

$$Q(z)=(id_{B\times_\delta G}\otimes h)(ad(v)(z))=(id_{B\times_\delta G}\otimes h)(v_{23}(z\otimes 1)v_{23}^*).$$

In [5], Section 2.3 it is noticed that, if u^α is a representation of G on a Hilbert space H, the following is a coaction of G on $B\otimes K(H)$:

$$\delta_{u^\alpha}(b\otimes k)=u_{23}^\alpha\delta(b)_{13}(1\otimes k\otimes 1){u_{23}^\alpha}^*.$$

With $\mathcal{I}_\alpha=(B\times_\delta G)^{ad(v)}\cap S_\alpha$, and I_{d_α} the d_α-dimensional identity matrix, define the map $\Psi:(B\otimes\mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}\to\mathcal{I}_\alpha$ to be

$$\Psi(\Lambda)=[\lambda_{ij}\otimes I_{d_\alpha}]_{ij},$$

for each $\Lambda=[\lambda_{ij}]_{ij}\in(B\otimes\mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}$ (see [5], Section 4).

The following result is a generalization of [13], Lemma 2.10 to the case of compact quantum groups. The proof uses the matricial representation of the elements of $S_{\alpha,\iota}$ discussed above. We will use this result in Section 3.

Lemma 2.2. $Q(\overline{S_{\overline{\alpha},\iota}S_{\iota,\overline{\alpha}}})=\Psi(\overline{B_2^\delta(u^\alpha)^*B_2^\delta(u^\alpha)})$.

Proof. Let $b\in B$ and $b_{ij}=P_{ij}^\alpha(b)$. Then, if $\eta\in H_u$ and ξ_h is the cyclic vector in H_h, for $i_0,j_0=1,2,...,d_\alpha$ we have:

$$\begin{aligned}(1\otimes p_\alpha)\delta(b_{i_0j_0}^*)(1\otimes p_\iota)(\eta\otimes\xi_h)&=(1\otimes p_\alpha)(\sum_{l=1}^{d_\alpha}b_{i_0l}^*\otimes {u_{lj_0}^\alpha}^*)(1\otimes p_\iota)(\eta\otimes\xi_h)\\
&=\sum_{l=1}^{d_\alpha}(b_{i_0l}^*\otimes(p_\alpha {u_{lj_0}^\alpha}^* p_\iota))(\eta\otimes\xi_h)\\
&=\sum_{l=1}^{d_\alpha}(b_{i_0l}^*\otimes\mathcal{F}_v(a_\alpha)^*{u_{lj_0}^\alpha}^*\mathcal{F}_v(1)^*)(\eta\otimes\xi_h)\\
&=\sum_{l=1}^{d_\alpha}(b_{i_0l}^*\otimes(a_\alpha^*h*{u_{lj_0}^\alpha}^*))(\eta\otimes\xi_h)\\
&=\sum_{l,r,m,n}(b_{i_0l}^*\otimes M_\alpha f_1(u_{nm}^\alpha){u_{lr}^\alpha}^*h({u_{rj_0}^\alpha}^*u_{mn}^\alpha)(\eta\otimes\xi_h)\\
&=\sum_{l,r,m,n}M_\alpha f_1(u_{nm}^\alpha)h({u_{rj_0}^\alpha}^*u_{mn}^\alpha)(b_{i_0l}^*\otimes{u_{lr}^\alpha}^*)(\eta\otimes\xi_h)\\
&=\sum_{l,r,m,n}\delta_{j_0n}f_1(u_{nm}^\alpha)f_{-1}(u_{mr}^\alpha)(b_{i_0l}^*\otimes{u_{lr}^\alpha}^*)(\eta\otimes\xi_h)\\
&=\sum_{l=1}^{d_\alpha}b_{i_0l}^*\eta\otimes{u_{lj_0}^\alpha}^*\xi_h.\end{aligned}$$

Consequently, the matrix of $(1 \otimes p_\alpha)\delta(b^*_{i_0j_0})(1 \otimes p_\iota)$ viewed as an operator from $H_u \otimes \mathbb{C}\xi_h$ to $H_u \otimes p_\alpha H_h$ is the $d_\alpha^2 \times 1$ column matrix whose entry $[(k-1)d_\alpha + j_0] \times 1$ is $b^*_{i_0k}$ and all the other entries are 0. Now let $c \in B$ and $c_{r_0s_0} = P^\alpha_{r_0s_0}(c)$. Then, similarly, $(1 \otimes p_\alpha)\delta(c_{r_0s_0})(1 \otimes p_\iota)$ can be represented by a $1 \times d_\alpha^2$ row matrix whose entry $1 \times [(k-1)d_\alpha + s_0]$ is c_{r_0k} and all the other entries are 0.

Therefore, the product $(1 \otimes p_\alpha)\delta(b^*_{i_0j_0})(1 \otimes p_\iota)\delta(c_{r_0s_0})(1 \otimes p_\alpha)$ is represented by a $d_\alpha^2 \times d_\alpha^2$ matrix X, partitioned into d_α^2 blocks X_{ij}, where each block X_{ij} has the entry j_0s_0 equal to $b^*_{i_0i}c_{r_0j}$ and the rest equal to 0, i.e. $X_{ij} = b^*_{i_0i}c_{r_0j} \otimes m_{j_0s_0}$.

Hence $X = \sum\limits_{i,j} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{j_0s_0}$.

On the other hand, by ([6], proof of Proposition 9), $v(p_\alpha \otimes 1) = \sum I_{d_\alpha} \otimes u^\alpha = \sum\limits_{p,q} I_{d_\alpha} \otimes m_{pq} \otimes u^\alpha_{pq}$. Hence:

$$\begin{aligned} v_{23}(X \otimes 1)v_{23}^* &= \sum_{i,j,p,q,k,l} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{pq}m_{j_0s_0}m_{kl} \otimes u^\alpha_{pq}{u^\alpha_{kl}}^* \\ &= \sum_{i,j,p,q,k,l} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes \delta_{qj_0}\delta_{s_0l}m_{pk} \otimes u^\alpha_{pq}{u^\alpha_{kl}}^* \\ &= \sum_{i,j,p,k} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{pk} \otimes u^\alpha_{pj_0}{u^\alpha_{ks_0}}^*. \end{aligned}$$

Applying $id \otimes h$ to the above expression and using Formula 2 above, we get:

$$\begin{aligned} Q(X) &= (\sum_{i,j,p,k} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{pk})h(u^\alpha_{pj_0}{u^\alpha_{ks_0}}^*) \\ &= \frac{1}{M_\alpha} f_1(u_{j_0s_0}) \sum_{i,j,p,k} \delta_{pk} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{pk} \\ &= \frac{1}{M_\alpha} f^\alpha_{j_0}\delta_{j_0s_0} \sum_{i,j,p} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes m_{pp} \\ &= \frac{1}{M_\alpha} f^\alpha_{j_0}\delta_{j_0s_0} \sum_{i,j} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes I_{d_\alpha}. \end{aligned}$$

Hence, if $j_0 = s_0$, we have:

$$Q(X) = c\sum_{i,j} b^*_{i_0i}c_{r_0j} \otimes m_{ij} \otimes I_{d_\alpha},$$

where $c = \frac{f^\alpha_{j_0}}{M_\alpha} > 0$.

But this is exactly $\Psi(M^*N)$, where $M \in B_2^\delta(u^\alpha)$ is the matrix whose j_0 row is $[cb_{i_0i}]$ and the other entries are 0 and $N \in B_2^\delta(u^\alpha)$ is the matrix whose $s_0 = j_0$ row is $[c_{r_0j}]$ and the other entries are 0. If $j_0 \neq s_0$, then $Q(X) = 0$ but, as can be easily checked, also $M^*N = 0$ and $\Psi(M^*N) = 0$. □

Now let (B, G, δ) be a quantum dynamical system. We say that a C^*-subalgebra $C \subset B$ is δ-invariant if the following two conditions hold:

(1) $\delta(C) \subseteq C \otimes A$,
(2) $\overline{\delta(C)(1 \otimes A)} = C \otimes A$.

In other words, C is called δ-invariant if the restriction of δ to C is a coaction. The set of all hereditary, δ-invariant C^*-subalgebras of B will be denoted by $\mathcal{H}^\delta(B)$.

A C^*-algebra B is called G-prime if the product of two nonzero δ-invariant ideals is nonzero.

A C^*-algebra B is called G-simple if B has no nontrivial δ-invariant two-sided ideals.

We will need the following remarks. Their proofs are straightforward.

Remark 2.3.
(1) $S_\iota = B^\delta \otimes 1$.
(2) Using the proof of Proposition 3.2 in [5], one can show that for $a_0, a_1 \in B^\delta$ and $\alpha \in \widehat{G}$, then $a_1 B^\delta_\alpha a_0 = (0)$ if and only if $(a_0 \otimes 1)S_{\alpha,\iota}(a_1 \otimes 1) = (0)$.
(3) If $C \in \mathcal{H}^\delta(B)$, then $C \times_\delta G$ is a hereditary subalgebra of $B \times_\delta G$.
(4) If $J \subset B^\delta$ is a two-sided ideal, then $D = \overline{JBJ} \in \mathcal{H}^\delta(B)$.

The next lemma and its corollary will be used in Section 4.

Lemma 2.4. *Let $\alpha \in \widehat{G}$. Then $S_\alpha = \overline{linspan\{S_{\alpha,\beta}S_{\beta,\alpha} | \beta \in \widehat{G}\}}^{\|\cdot\|}$.*

Proof. Since $\sum_{\beta \in \widehat{G}} p_\beta = 1$ in the strict topology of $\widehat{A}$ we have

$$(1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\alpha) = (1 \otimes p_\alpha)(B \times_\delta G)\sum_{\beta \in \widehat{G}}(1 \otimes p_\beta)(B \times_\delta G)(1 \otimes p_\alpha),$$

and the claim follows. ☐

Corollary 2.5. *Let $J \subset B^\delta$ be a two-sided ideal. Then $C = \overline{BJB} \in \mathcal{H}^\delta(B)$ and*

$$C^\delta \otimes 1 = \overline{linspan\{S_{\iota,\beta}(J \otimes 1)S_{\beta,\iota} | \beta \in \widehat{G}\}}^{\|\cdot\|}.$$

Proof. Clearly, $\delta(C) \subseteq C \otimes A$, since $\delta(J) = J \otimes 1$. The fact that $\delta(C)(1 \otimes A)$ is dense in $C \otimes A$ follows from the definition of C.

The equality $C^\delta \otimes 1 = \overline{linspan\{S_{\iota,\beta}(J \otimes 1)S_{\beta,\iota} | \beta \in \widehat{G}\}}^{\|\cdot\|}$ follows from Lemma 2.4 and Remark 2.3 (1). ☐

3. Connes spectrum and prime crossed products

A notion of a spectrum of an action δ of a compact group G on a C^*-algebra B was given in [8] by Gootman, Lazar, and Peligrad. They used the spectral subspaces $B^\delta_2(\alpha)$ to define the Arveson and Connes spectra and proved that the conjugate $\overline{\alpha}$ belongs to the Arveson spectrum $Sp(\delta)$ if and only if the closure of the ideal $S_{\alpha,\iota} * S_{\iota,\alpha}$ is essential in S_α ([8], Proposition 1.3). We are going to use this correspondence rather than the direct definition given in [8] to define the spectra for coactions of a compact quantum group on a C^*-algebra B.

Definition 3.1.
(1) $Sp(\delta) = \{\alpha \in \widehat{G} | S_{\overline{\alpha},\iota}S_{\iota,\overline{\alpha}}$ is an essential ideal of $S_{\overline{\alpha}}\}$.
(2) $\Gamma(\delta) = \bigcap_{C \in \mathcal{H}^\delta(B)} Sp(\delta|_C)$.

The connection to the definition in [8] is made by the following lemma.

Lemma 3.2. *Let $\alpha \in \widehat{G}$. Then $\alpha \in Sp(\delta)$ if and only if $\overline{B^\delta_2(u^\alpha)^* B^\delta_2(u^\alpha)}$ is an essential ideal of $(B \otimes \mathcal{M}_{d_\alpha}(\mathbf{C}))^{\delta_{u^\alpha}}$.*

Proof. Let $\alpha \in Sp(\delta)$ and assume to the contrary that $\overline{B_2^\delta(u^\alpha)^* B_2^\delta(u^\alpha)}$ is not an essential ideal of $(B \otimes \mathcal{M}_{d_\alpha}(\mathbf{C}))^{\delta_{u^\alpha}}$.

Using Lemma 2.2, there exists a positive, nonzero element $c \in \mathcal{I}_{\overline{\alpha}}$, such that $c\mathcal{P}(S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}})c = 0$. Since, in particular, $c \in S_{\overline{\alpha}}$, then $\mathcal{P}(c(S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}})c) = 0$. The faithfulness of $\mathcal{P}$ now implies that $c(S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}})c = 0$, which is a contradiction with $\alpha \in Sp(\delta)$.

Conversely, assume that $\overline{B_2^\delta(u^\alpha)^* B_2^\delta(u^\alpha)}$ is an essential ideal of $(B\otimes\mathcal{M}_{d_\alpha}(\mathbf{C}))^{\delta_{u^\alpha}}$. By Lemma 2.2, $\mathcal{P}(S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}})$ is an essential ideal in $\mathcal{I}_{\overline{\alpha}}$. Using the same lemma,

$$\mathcal{P}(\overline{S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}}}) = \mathcal{I}_{\overline{\alpha}} \cap (\overline{S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}}}).$$

By Remark 3.5 in [5], $S_{\overline{\alpha}}$ is isomorphic to $I(\overline{\alpha})\otimes\mathcal{I}_{\overline{\alpha}}$, where $I(\overline{\alpha}) = \widehat{A}p_{\overline{\alpha}}$. It is easy to check that the image of $\mathcal{I}_{\overline{\alpha}} \subset S_{\overline{\alpha}}$ by this isomorphism is $\chi_{\overline{\alpha}}\otimes\{diag(x, x, ..., x)|x \in \mathcal{I}_{\overline{\alpha}}\}$, where $diag(x, x, ..., x)$ is the $d_\alpha \times d_\alpha$ matrix with all the diagonal elements equal to x and all the others equal to 0. Thus $\{diag(y, y, ..., y)|y \in \mathcal{I}_{\overline{\alpha}} \cap \overline{S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}}}\}$ is essential in $\{diag(x, x, ..., x)|x \in \mathcal{I}_{\overline{\alpha}}\}$. This implies that $\overline{S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}}}$ is essential in $S_{\overline{\alpha}}$. □

Proposition 3.3. *If B is G-prime and $\Gamma(\delta) = \widehat{G}$, then B^δ is prime.*

Proof. Assume, to the contrary, that B^δ is not prime. Then there exist two nonzero positive elements $a_0, a_1 \in B^\delta$ such that $a_1 B^\delta a_0 = (0)$. Since B is G-prime, $a_1 B a_0 \neq (0)$. On the other hand, since B is the closure of the linear span of its spectral subspaces $\{B_\alpha^\delta | \alpha \in \widehat{G}\}$, then there exists $\alpha_0 \in \widehat{G}$ such that

$$a_1 B_{\alpha_0}^\delta a_0 \neq (0). \tag{3}$$

Since $a_1 B^\delta a_0 = (0)$, Remark 2.3(1) above implies that

$$(1 \otimes p_\iota)((a_1 \otimes 1)(B \times_\delta G))(1 \otimes p_\alpha)(B \times_\delta G)(a_0 \otimes 1)(1 \otimes p_\iota) = (0),$$

that is,

$$(a_1 \otimes 1)S_{\iota,\overline{\alpha_0}} S_{\overline{\alpha_0},\iota}(a_0 \otimes 1) = (0).$$

Therefore, since $S_{\iota,\overline{\alpha_0}}(a_0^2 \otimes 1)S_{\overline{\alpha_0},\iota} \subset S_{\iota,\overline{\alpha_0}} S_{\overline{\alpha_0},\iota}$, then

$$(a_1 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0^2 \otimes 1)S_{\overline{\alpha_0},\iota}(a_0 \otimes 1) = (0). \tag{4}$$

Multiply the above equation to the left by $(a_0 \otimes 1)S_{\overline{\alpha_0},\iota}(a_1 \otimes 1)$ and to the right by $(a_0 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0 \otimes 1)$. We get:

$$(a_0 \otimes 1)S_{\overline{\alpha_0},\iota}(a_1^2 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0^2 \otimes 1)S_{\overline{\alpha_0},\iota}(a_0^2 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0 \otimes 1) = (0). \tag{5}$$

Regroup the terms on the left-hand side of equation (5) as:

$$[(a_0 \otimes 1)S_{\overline{\alpha_0},\iota}(a_1^2 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0 \otimes 1)][(a_0 \otimes 1)S_{\overline{\alpha_0},\iota}(a_0^2 \otimes 1)S_{\iota,\overline{\alpha_0}}(a_0 \otimes 1)] = (0). \tag{6}$$

Let $C = \overline{a_0 B a_0}$. Clearly, $C \in \mathcal{H}^\delta(B)$. The second factor on the left-hand side of equation (6) is just $S^c_{\overline{\alpha_0},\iota} S^c_{\iota,\overline{\alpha_0}}$, where $S^c_{\alpha,\beta}$ denotes the corresponding subspace of the crossed product $C \times_\delta G$.

Since $\Gamma(\delta) = \widehat{G}$, then $\overline{\alpha_0} \in \Gamma(\delta)$. Therefore $S^c_{\overline{\alpha_0},\iota} S^c_{\iota,\overline{\alpha_0}}$ is an essential ideal of $S^c_{\overline{\alpha_0}}$. Since the first factor in equation (6) is included in $S^c_{\overline{\alpha_0}}$, it follows that it equals (0). In particular,

$$(a_0 \otimes 1)S_{\overline{\alpha_0},\iota}(a_1 \otimes 1) = (0).$$

Using Remark 2.3(2), this means that $a_1 B_{\alpha_0}^\delta a_0 = (0)$, which is a contradiction to relation (3). □

Next we will prove the main result of this section. The result is a generalization of [8], Theorem 2.2.

Theorem 3.4. *The following are equivalent:*

(1) $B \times_\delta G$ *is prime.*
(2) B *is* G*-prime and* $\Gamma(\delta) = \widehat{G}$.

Proof. Assume that $B \times_\delta G$ is prime. Since for every δ-invariant ideal $J \subset B$, $J \times_\delta G$ is an ideal of $B \times_\delta G$, the fact that B is G-prime is immediate. Next we will show that $\Gamma(\delta) = \widehat{G}$.

Let $C \in \mathcal{H}^\delta(B)$ and $\alpha \in \widehat{G}$. By Remark 2.3(3) above, $C \times_\delta G$ is a hereditary subalgebra of $B \times_\delta G$ and is therefore prime. Using [5], Corollary 4.9, $(C \otimes \mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}$ is prime and $C_2^\delta(\alpha) \neq (0)$ (since $C_\alpha^\delta \neq (0)$). Thus the ideal $\overline{C_2^\delta(\alpha)^* C_2^\delta(\alpha)}$ is essential in $(C \otimes \mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}$. Therefore $\alpha \in \Gamma(\delta)$ and so $\Gamma(\delta) = \widehat{G}$.

Conversely, assume that B is G-prime and $\Gamma(\delta) = \widehat{G}$.

For each $\alpha \in \widehat{G}$, the C^*-algebras $\overline{S_{\alpha,\iota} S_{\iota,\alpha}}$ and $\overline{S_{\iota,\alpha} S_{\alpha,\iota}}$ are strongly Morita equivalent ($S_{\alpha,\iota}$ being the imprimitivity bimodule). By Proposition 3.3, B^δ is prime and therefore, by Remark 2.3(1), S_ι is prime. Since S_ι is prime, so is the ideal $\overline{S_{\iota,\alpha} S_{\alpha,\iota}}$ and the Morita equivalent algebra $\overline{S_{\alpha,\iota} S_{\iota,\alpha}}$. By the definition of $\Gamma(\delta)$, $\overline{S_{\alpha,\iota} S_{\iota,\alpha}}$ is an essential ideal of S_α and thus S_α is prime also. The implication follows now from [5], Corollary 4.9. □

4. Strong Connes spectrum and simple crossed products

We begin by defining the strong Arveson and Connes spectra for compact quantum group coactions.

Definition 4.1.

(1) Strong Arveson spectrum $\tilde{Sp}(\delta) = \{\alpha \in \widehat{G} | \overline{S_{\overline{\alpha},\iota} S_{\iota,\overline{\alpha}}}^{\|\cdot\|} = S_{\overline{\alpha}}\}$.
(2) Strong Connes spectrum $\tilde{\Gamma}(\delta) = \bigcap_{c \in \mathcal{H}^\alpha(B)} \tilde{Sp}(\delta|_C)$.

Using similar arguments as in Lemma 3.2 we obtain the following result.

Lemma 4.2. *Let* $\alpha \in \widehat{G}$. *Then* $\alpha \in \tilde{Sp}(\delta)$ *if and only if* $\overline{B_2^\delta(u^\alpha)^* B_2^\delta(u^\alpha)} = (B \otimes \mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}$.

The following result makes a connection between the strong Connes spectrum and the simplicity of the fixed point algebra B^δ.

Proposition 4.3. *If* B *is* G*-simple and* $\tilde{\Gamma}(\delta) = \widehat{G}$*, then* B^δ *is simple.*

Proof. Let $J \subseteq B^\delta$ be a nonzero two-sided ideal. We will prove that $J = B^\delta$ and thus B^δ is simple. To this end we will show that $S_{\iota,\alpha}(J \otimes 1) S_{\alpha,\iota} \subseteq J \otimes 1$, for any $\alpha \in \widehat{G}$. The claim will then follow from Corollary 2.5.

Let $D = \overline{JBJ}$ and let $\alpha \in \widehat{G}$. Then $D \in \mathcal{H}^\delta(B)$. Since $\alpha \in \widehat{G} = \tilde{\Gamma}(\delta)$, we have

$$\overline{S_{\alpha,\iota}^D S_{\iota,\alpha}^D} = S_\alpha^D,$$

where $S_{\alpha,\iota}^D$ and $S_{\iota,\alpha}^D$ are the corresponding subspaces of $D \times_\delta G$.

By the definition of D, it obviously follows that $S_{\alpha,\iota}^D = \overline{(J \otimes 1) S_{\alpha,\iota} (J \otimes 1)}$ and $S_\alpha^D = \overline{(J \otimes 1) S_\alpha (J \otimes 1)}$.

Therefore,

$$\overline{(J\otimes 1)S_{\alpha,\iota}(J\otimes 1)S_{\iota,\alpha}(J\otimes 1)} = \overline{(J\otimes 1)S_{\alpha}(J\otimes 1)}. \tag{7}$$

Multiplying equation (7) to the left by $S_{\iota,\alpha}$ and to the right by $S_{\alpha,\iota}$ we get

$$\overline{S_{\iota,\alpha}(J\otimes 1)S_{\alpha,\iota}(J\otimes 1)S_{\iota,\alpha}(J\otimes 1)S_{\alpha,\iota}} = \overline{S_{\iota,\alpha}(J\otimes 1)S_{\alpha}(J\otimes 1)S_{\alpha,\iota}}. \tag{8}$$

By Remark 2.3(1), $S_{\iota,\alpha}S_{\alpha,\iota} \subseteq S_{\iota} = B^{\delta}\otimes 1$ and since $J \subseteq B^{\delta}$, the left-hand side of equation (8) is included in $J\otimes 1$. Therefore, the right-hand side of equation (8) is included in $J\otimes 1$:

$$S_{\iota,\alpha}(J\otimes 1)S_{\alpha}(J\otimes 1)S_{\alpha,\iota} \subseteq J\otimes 1. \tag{9}$$

Since $S_{\iota,\alpha}(J\otimes 1)S_{\alpha,\iota} \subseteq S_{\iota,\alpha}(J\otimes 1)S_{\alpha}(J\otimes 1)S_{\alpha,\iota}$, from equation (9) it follows that

$$S_{\iota,\alpha}(J\otimes 1)S_{\alpha,\iota} \subseteq J\otimes 1 \tag{10}$$

and we are done. □

We can now prove:

Theorem 4.4. *The following are equivalent:*

(1) $B \times_{\delta} G$ *is simple.*
(2) B *is G-simple and* $\tilde{\Gamma}(\delta) = \widehat{G}$.

Proof. First assume that $B\times_{\delta} G$ is simple. That B is G-simple follows easily since for every nontrivial ideal $J \in \mathcal{H}^{\delta}(B)$, $J\times_{\delta} G$ is a nontrivial ideal of $B\times_{\delta} G$.

Now let $\alpha \in \widehat{G}$ and $C \in \mathcal{H}^{\delta}(B)$. Then $C\times_{\delta} G$ is a hereditary subalgebra of $B\times_{\delta} G$ by Remark 2.3(3) and hence it is simple. By [5], Corollary 4.9, $S_{\alpha,\iota} \neq 0$ and S_{α} is simple. Hence $\overline{S_{\alpha,\iota}S_{\iota,\alpha}} = S_{\alpha}$ and $\alpha \in \tilde{\Gamma}(\delta)$.

Conversely, assume that B is G-simple and $\tilde{\Gamma}(\delta) = \widehat{G}$. By Proposition 4.3, B^{δ} is simple. Now, for every $\alpha \in \widehat{G} = \tilde{\Gamma}(\delta)$, the nonzero ideal $\overline{S_{\alpha,\iota}S_{\iota,\alpha}} \subseteq S_{\iota} = B^{\delta}\otimes 1$ is simple and so is the Morita equivalent algebra $\overline{S_{\alpha,\iota}S_{\iota,\alpha}} = S_{\alpha}$. The conclusion follows now from [5], Corollary 4.9. □

5. Spectra are closed under tensor products

In order to prove the results about the stability of the Connes spectrum and the strong Connes spectrum to tensor products, we need to make some notation. If $\alpha \in \widehat{G}$ and $\beta \in \widehat{G}$ and $u^{\alpha} \in \alpha$, $u^{\beta} \in \beta$, denote by $u^{\alpha}\odot u^{\beta} = \sum_{p,q,r,s} m^{\alpha}_{pq}\otimes m^{\beta}_{rs}\otimes u^{\alpha}_{pq}u^{\beta}_{rs}$ the Kronecker tensor product of u^{α} and u^{β}, which is a representation of A (see [17], Section 6). Then $u^{\alpha}\odot u^{\beta}$ is unitary if both $u^{\alpha}$and u^{β} are unitary. Moreover, $u^{\alpha}\odot u^{\beta}$ is equivalent to a direct sum of irreducible representations, $u^{\alpha}\odot u^{\beta} \cong \sum_i^{\oplus} u^{\rho_i}, \rho_i \in \widehat{G}$. The equivalence and $\rho_i \in \widehat{G}$ are unitary if both $u^{\alpha}$and u^{β} are unitary [17].

Definition 5.1. Let $\Pi \subset \widehat{G}$ be a subset. We say that Π is closed under tensor products if for every $\alpha \in \Pi, \beta \in \Pi$ and $u^{\alpha} \in \alpha$, $u^{\beta} \in \beta$ it follows that every irreducible component of $u^{\alpha}\odot u^{\beta}$ belongs to Π.

If $X \in B_2^{\delta}(u^{\alpha})$ and $Y \in B_2^{\delta}(u^{\beta})$, we denote $X\odot Y = \sum_{l,k,i,j} X_{lk}Y_{ij}\otimes m^{\alpha}_{lk}\otimes m^{\beta}_{ij}$ (for the case of groups this notation was used in [12]). Standard calculations show that $X\odot Y \in B_2^{\delta}(u^{\alpha}\odot u^{\beta})$. Furthermore, $X\odot Y$ can be viewed as the matrix of order $d_\alpha d_\beta \times d_\alpha d_\beta$ partitioned in d_β^2 blocks of order $d_\alpha \times d_\alpha$ as follows: $X \odot Y = [X diag(Y_{ij})]$, where $diag(Y_{ij})$ is the $d_\alpha \times d_\alpha$ matrix with all the diagonal entries equal to Y_{ij} and all the others equal to 0.

Remark 5.2. If $u^\alpha \odot u^\beta \cong \sum_i^{\oplus} u^{\rho_i}, \rho_i \in \widehat{G}$, then:

(1) $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$ is spatially isomorphic to $\sum^{\oplus}(B \otimes \mathcal{M}_{d_{\rho_i}})^{\delta_{\rho_i}}$ ($*$-isomorphic if both $u^\alpha$and u^β are unitary) and
(2) $B_2^\delta(u^\alpha \odot u^\beta)$ is spatially isomorphic to $\sum^{\oplus} B_2^\delta(\rho_i)$.

The proof of the above remark follows immediately using a change of basis in $\mathcal{M}_{d_\alpha d_\beta}$.

First we prove

Lemma 5.3. *$\tilde{Sp}(\delta|_C)$ is closed under tensor products for every $C \in \mathcal{H}^\delta(B)$.*

Proof. We have to prove that if $\alpha, \beta \in \tilde{Sp}(\delta|C), C \in \mathcal{H}^\delta(B)$, then every irreducible component of $u^\alpha \odot u^\beta$ belongs to $\tilde{Sp}(\delta|C)$.

It is enough to prove the above claim for $C = B$. We first show that if $\alpha \in \tilde{Sp}(\delta)$ and $\beta \in \tilde{Sp}(\delta)$, then $B_2^\delta(u^\alpha \odot u^\beta)^* B_2^\delta(u^\alpha \odot u^\beta)$ is a dense ideal of $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$.

Indeed, by ([3], Theorem 2.1), $(B \otimes \mathcal{M}_{d_\alpha})^{\delta_\alpha}$ has an approximate identity $\{E_\lambda\}$ of the form $E_\lambda = \sum_1^{n_\lambda} (X_i^\lambda)^* X_i^\lambda$, $X_i^\lambda \in B_2^\delta(u^\alpha), i = 1, 2, ..., n_\lambda$. By ([7], Lemma 2.7), $\{E_\lambda\}$ is an approximate identity of $B \otimes \mathcal{M}_{d_\alpha}$. Hence $(Y_1 \odot I_{d_\alpha})^*(Y_2 \odot I_{d_\alpha}) \in B_2^\delta(u^\alpha \odot u^\beta)$, for all $Y_1, Y_2 \in B_2^\delta(u^\beta)$. Since $\beta \in \tilde{Sp}(\delta)$, $B_2^\delta(u^\beta)^* B_2^\delta(u^\beta)$ is a dense ideal of $(B \otimes \mathcal{M}_{d_\beta})^{\delta_\beta}$. Using an approximate identity of $(B \otimes \mathcal{M}_{d_\beta})^{\delta_\beta}$ of the form $F_\gamma = \sum_1^{m_\gamma} (Y_i^\gamma)^* Y_i^\gamma, Y_i^\gamma \in B_2^\delta(u^\beta)$, by the pattern we used above, it follows that $B_2^\delta(u^\alpha \odot u^\beta)^* B_2^\delta(u^\alpha \odot u^\beta)$ is a dense ideal of $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$.

On the other hand, since $u^\alpha \odot u^\beta$ is equivalent to a direct sum of irreducible representations, $u^\alpha \odot u^\beta \cong \sum_i^{\oplus} u^{\rho_i}, \rho_i \in \widehat{G}$, by Remark 5.2(1), $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$ is spatially $*$-isomorphic to $\sum^{\oplus}(B \otimes \mathcal{M}_{d_{\rho_i}})^{\delta_{\rho_i}}$. Thus, since by Remark 5.2(2), $B_2^\delta(u^\alpha \odot u^\beta)$ is spatially isomorphic to $\sum^{\oplus} B_2^\delta(\rho_i)$, it follows that $B_2^\delta(\rho_i)^* B_2^\delta(\rho_i)$ is dense in $(B \otimes \mathcal{M}_{d_{\rho_i}})^{\delta_{\rho_i}}$ for all i. Therefore $\rho_i \in \tilde{Sp}(\delta)$ for every i. Thus $\tilde{Sp}(\delta|_C)$ is closed under tensor products for every $C \in \mathcal{H}^\delta(B)$. □

Therefore:

Proposition 5.4. *$\tilde{\Gamma}(\delta)$ is closed under tensor products.*

Proof. This is obvious since $\tilde{\Gamma}(\delta) = \bigcap_{C \in \mathcal{H}^\alpha(B)} \tilde{Sp}(\delta|_C)$. □

Next we will prove that the Connes spectrum is closed under tensor products. As in the case of the strong Connes spectrum, first we will show that our Arveson spectrum $Sp(\delta|_C)$ is closed under tensor products for every $C \in \mathcal{H}^\delta(B)$.

Let $\alpha \in \widehat{G}$ and $\beta \in \widehat{G}$ and $u^\alpha \in \alpha$, $u^\beta \in \beta$. If u^α and u^β are unitary, then, as noticed above, $u^\alpha \odot u^\beta$ is a unitary representation. If u_1^α is a representation in the class α, not necessarily unitary, then there exists an invertible matrix $S \in \mathcal{M}_{d_\alpha}$ such that $u_1^\alpha = (S^{-1} \otimes 1) u^\alpha (S \otimes 1)$. Notice that $B_2^\delta(u_1^\alpha) = \{(1_B \otimes S^{-1}) X (1_B \otimes S) | X \in B_2^\delta(u^\alpha)\}$.

Lemma 5.5. *$Sp(\delta|_C)$ is closed under tensor products for every $C \in \mathcal{H}^\delta(B)$.*

Proof. We may assume that $C = B$. Let $\alpha, \beta \in Sp(\delta)$ and $u^\alpha \in \alpha, u^\beta \in \beta$ be unitary representatives of α and β. First we will show that

$$linspan\{(X \odot Y)^*(X \odot Y) | X \in B_2^\delta(u^\alpha), Y \in B_2^\delta(u^\beta)\}$$

is an essential ideal of $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$. It then follows immediately that each irreducible component of $u^\alpha \odot u^\beta$ belongs to $Sp(\delta)$.

Let $Z \in (B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}, Z \geq 0$. Assume that $(X \odot Y)Z = 0$, for every $X \in B_2^\delta(u^\alpha), Y \in B_2^\delta(u^\beta)$. Let Z be partitioned into blocks as follows: $Z = \sum_{l,k=1}^{d_\beta} Z_{lk} \otimes m_{lk}^\beta$, where Z_{lk} are $d_\alpha \times d_\alpha$ matrices with entries in B. Since $(X \odot Y)Z = 0$, for every $X \in B_2^\delta(u^\alpha), Y \in B_2^\delta(u^\beta)$, it follows that $(X \odot Y)Z(I_{d\alpha} \odot Y^*) = 0$, for every such X, Y. In particular, if Y is as in Remark 2.1(4), that is, Y has only one nonzero row consisting of $y_1, y_2, ..., y_{d_\beta}$, we have

$$X \sum_{i,j=1}^{d_\beta} y_i Z_{ij} y_j^* = 0,$$

for every $X \in B_2^\delta(u^\alpha)$ and $Y \in B_2^\delta(u^\beta)$ as chosen, where the multiplication $y_i Z_{ij} y_j^*$ is the multiplication in B of y_i, y_j^* with each entry of Z_{ij}. First we prove the following:

$$\sum y_i Z_{ij} y_j^* \in (B \otimes \mathcal{M}_{d_\alpha})^{\delta_{u^\alpha}}, \tag{11}$$

for every $Y \in B_2^\delta(u^\beta)$ as chosen (i.e. with only one nonzero row).

In the following leg numbering notation, there are four places in the following order: $B, \mathcal{M}_{d_\alpha}, \mathcal{M}_{d_\beta}, A$.

Since $Z \in (B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$, we have

$$\delta_{14}(Z) = (1_B \otimes (u^\alpha \odot u^\beta)^*)(Z \otimes 1_A)(1_B \otimes (u^\alpha \odot u^\beta)). \tag{12}$$

By the definition of $u^\alpha \odot u^\beta$, we have

$$\begin{aligned} \delta_{14}(\sum Z_{ij} \otimes m_{ij}^\beta) = &(\sum 1_B \otimes m_{pq}^\alpha \otimes m_{rs}^\beta \otimes u_{sr}^{\beta *} u_{qp}^{\alpha *})(\sum Z_{ij} \otimes m_{ij}^\beta \otimes 1_A) \\ &\times (\sum 1_B \otimes m_{tu}^\alpha \otimes m_{vw}^\beta \otimes u_{pq}^\alpha u_{rs}^\beta). \end{aligned} \tag{13}$$

On the other hand, taking into account that $Y \in (B \otimes \mathcal{M}_{d_\beta})^{\delta_{u^\beta}}$, it follows that

$$\delta_{14}(Y_{13}) = (1_B \otimes (u^\beta)_{34}^*)(Y_{13} \otimes 1_A)(1_B \otimes (u^\beta)_{34}), \tag{14}$$

where $(u^\beta)_{34} = \sum 1_B \otimes I_{d_\alpha} \otimes m_{lk}^\beta \otimes u_{lk}^\beta$.

By combining Formulas 12 and 14 and taking into account that u^α and u^β are unitary, we get Formula 11. Therefore, since $\alpha \in Sp(\delta)$, it follows that $\sum y_i Z_{ij} y_j^* = 0$ for every such Y.

Let $u_1^\alpha \in \alpha$ be a not necessarily unitary representation, but such that $\overline{u_1^\alpha}$ is unitary. Then, since u^α and u_1^α are equivalent, there is an invertible matrix $S \in \mathcal{M}_{d_\alpha}$ such that $u_1^\alpha = (S^{-1} \otimes 1)u^\alpha(S \otimes 1)$. Notice that

$$B_2^\delta(u_1^\alpha) = \{(1_B \otimes S^{-1})X(1_B \otimes S) | X \in B_2^\delta(u^\alpha)\}.$$

Denote $V_{ij} = (1_B \otimes S^*) Z_{ij} (1_B \otimes S)$ for all $i, j = 1, 2, ..., d_\beta$. Thus, since $\sum y_i Z_{ij} y_j^* = 0$, it immediately follows that

$$\sum y_i V_{ij} y_j^* = 0,$$

for every Y as chosen.

In particular, $\sum y_i V_{ij}^{pq} y_j^* = 0$, for all $p, q = 1, 2, ..., d_\alpha$, where V_{ij}^{pq} is the entry pq of the $d_\alpha \times d_\alpha$ matrix V_{ij}. Hence, $\sum y_i (\sum_{p=1}^{d_\alpha} V_{ij}^{pp}) y_j^* = 0$. Let $d_{ij} = \sum_{p=1}^{d_\alpha} V_{ij}^{pp}$. Therefore, if $Y \in B_2^\delta(u^\beta)$ is as before and $D = \sum_{i,j=1}^{d_\beta} d_{ij} \otimes m_{ij}^\beta$, we have $YDY^* = 0$. By Remark 2.1(4), the matrices $Y \in B_2^\delta(u^\beta)$ that have only one nonzero row span $B_2^\delta(u^\beta)$ linearly. Therefore, $YDY^* = 0$ for every $Y \in B_2^\delta(u^\beta)$. Since $Z \geq 0$ it follows that $V \geq 0$ and so $D \geq 0$. Therefore $YD = 0$ for every $Y \in B_2^\delta(u^\beta)$. Notice that $V = \sum_{i,j} V_{ij} \otimes m_{ij}^\beta$ satisfies formula (12) with u^α replaced by u_1^α. This fact will be used in the proof of the next claim.

Claim. $D \in (B \otimes \mathcal{M}_{d_\beta})^{\delta_{u^\beta}}$.

The proof of the claim will be achieved in two steps:

Step 1. We prove that

$$d_{ij} \otimes 1_A = \sum_{p=1}^{d_\alpha} [(1_B \otimes u_1^\alpha)^* (V_{ij} \otimes 1_A)(1_B \otimes u_1^\alpha)]_{qq}, \tag{15}$$

where $[(1_B \otimes u_1^\alpha)^* (V_{ij} \otimes 1_A)(1_B \otimes u_1^\alpha)]_{qq}$ denotes the entry qq of the matrix $(1_B \otimes u_1^\alpha)^* (V_{ij} \otimes 1_A)(1_B \otimes u_1^\alpha)$, $q = 1, 2, ..., d_\alpha$. Tedious but straightforward calculations show that the right hand side of the above formula is

$$\begin{aligned}
&\sum_{q=1}^{d_\alpha} [(1_B \otimes u_1^\alpha)^* (V_{ij} \otimes 1_A)(1_B \otimes u_1^\alpha)]_{qq} \\
&= \sum_q [\sum_{p,n,r,s,l,k} V_{ij}^{pn} \otimes m_{rs}^\alpha m_{pn}^\alpha m_{lk}^\alpha \otimes (u_1^\alpha)_{sr}^* (u_1^\alpha)_{lk}]_{qq} \\
&= \sum_q [\sum_{p,n,r,s} V_{ij}^{pn} \otimes \delta_{sp} \delta_{nl} m_{rk}^\alpha \otimes (u_1^\alpha)_{sr}^* (u_1^\alpha)_{lk}]_{qq} \\
&= \sum_q \sum_{p,n,r,s} \delta_{qr} \delta_{qk} V_{ij}^{pn} \otimes (u_1^\alpha)_{pr}^* (u_1^\alpha)_{nk} \\
&= \sum_{p,n} V_{ij}^{pn} \otimes (\sum_q (u_1^\alpha)_{pq}^* (u_1^\alpha)_{nq}) = \sum_{p,n} V_{ij}^{pn} \otimes \delta_{pn} 1_A.
\end{aligned}$$

This last equality holds because we assume that $\overline{u_1^\alpha}$ is unitary. Therefore:

$$\sum_{q=1}^{d_\alpha} [(1_B \otimes u_1^\alpha)^* (V_{ij} \otimes 1_A)(1_B \otimes u_1^\alpha)]_{qq} = \sum_{p,n} V_{ij}^{pn} \otimes \delta_{pn} 1_A = \sum V_{ij}^{pp} \otimes 1_A = d_{ij} \otimes 1_A,$$

and Step 1 is proven.

Step 2 (Proof of claim). We have to prove that

$$\delta_{13}(\sum_{i,j} d_{ij} \otimes m^{\beta}_{ij}) = (1_B \otimes u^{\beta})^*(\sum_{i,j} d_{ij} \otimes m^{\beta}_{ij} \otimes 1_A)(1_B \otimes u^{\beta}). \tag{16}$$

We will evaluate separately the right and left hand sides of formula (16) and show that they are the same. First, the right hand side:

$$\begin{aligned}(1_B \otimes u^{\beta})^*&(\sum_{i,j} d_{ij} \otimes m^{\beta}_{ij} \otimes 1_A)(1_B \otimes u^{\beta}) \\ &= \sum_{q,p,i,j,u,v} (1_B \otimes m^{\beta}_{qp} \otimes u^{\beta *}_{pq})(d_{ij} \otimes m^{\beta}_{ij} \otimes 1_A)(1_B \otimes m^{\beta}_{uv} \otimes u^{\beta}_{uv}) \\ &= \sum_{q,p,i,j,u,v} d_{ij} \otimes m^{\beta}_{qp} m^{\beta}_{ij} m^{\beta}_{uv} \otimes u^{\beta *}_{pq} u^{\beta}_{uv} \\ &= \sum_{q,p,i,j,u,v} d_{ij} \otimes \delta_{pi}\delta_{ju} m^{\beta}_{qv} \otimes u^{\beta *}_{pq} u^{\beta}_{uv} = \sum_{q,i,j,v} d_{ij} \otimes m^{\beta}_{qv} \otimes u^{\beta *}_{iq} u^{\beta}_{jv}.\end{aligned} \tag{17}$$

Next we will calculate the left hand side of formula (16). As noticed above, by multiplying formula (12) above by $1_B \otimes S^* \otimes I_{d_\beta} \otimes 1_A$ to the left and by $1_B \otimes S \otimes I_{d_\beta} \otimes 1_A$ to the right, and if we denote $V = \sum V_{ij} \otimes m^{\beta}_{ij}$, we get

$$\delta_{14}(V) = (1_B \otimes (u^{\alpha}_1 \odot u^{\beta})^*)(V \otimes 1_A)(1_B \otimes (u^{\alpha}_1 \odot u^{\beta})).$$

Therefore:

$$\begin{aligned}\delta_{14}(V) = &\sum_{r,s,k,l,i,j,p,q,t,u,v,w} (1_B \otimes m^{\alpha}_{rs} \otimes m^{\beta}_{kl} \otimes u^{\beta *}_{lk}(u^{\alpha}_1)^*_{sr})(V^{pq}_{ij} \otimes m^{\alpha}_{pq} \otimes m^{\beta}_{ij} \otimes 1_A) \\ &\times (1_B \otimes m^{\alpha}_{tu} \otimes m^{\beta}_{vw} \otimes (u^{\alpha}_1)_{tu} u^{\beta}_{vw}) \\ &= \sum V^{pq}_{ij} \otimes \delta_{sp}\delta_{qt} m^{\alpha}_{ru} \otimes \delta_{li}\delta_{jv} m^{\beta}_{kw} \otimes u^{\beta *}_{lk}(u^{\alpha}_1)^*_{sr}(u^{\alpha}_1)_{tu} u^{\beta}_{vw} \\ &= \sum V^{pq}_{ij} \otimes m^{\alpha}_{ru} \otimes m^{\beta}_{kw} \otimes u^{\beta *}_{ik}(u^{\alpha}_1)^*_{pr}(u^{\alpha}_1)_{qu} u^{\beta}_{jw}.\end{aligned}$$

Hence, if $k = i_0$ and $w = j_0$ we get

$$\delta_{14}(V_{i_0 j_0} \otimes m^{\beta}_{i_0 j_0}) = \sum_{p,q,r,u,i,j} V^{pq}_{ij} \otimes m^{\alpha}_{ru} \otimes m^{\beta}_{i_0 j_0} \otimes u^{\beta *}_{ii_0}(u^{\alpha}_1)^*_{pr}(u^{\alpha}_1)_{qu} u^{\beta}_{jj_0}$$

and if $r = u = l$,

$$\delta_{14}(V^{ll}_{i_0 j_0} \otimes m^{\alpha}_{ll} \otimes m^{\beta}_{i_0 j_0}) = \sum_{p,q,i,j} V^{pq}_{ij} \otimes m^{\alpha}_{ll} \otimes m^{\beta}_{i_0 j_0} \otimes u^{\beta *}_{ii_0}(u^{\alpha}_1)^*_{pl}(u^{\alpha}_1)_{ql} u^{\beta}_{jj_0}.$$

Therefore:

$$\delta_{13}(d_{i_0 j_0} \otimes m^{\beta}_{i_0 j_0}) = \sum_{p,q,i,j} V^{pq}_{ij} \otimes m^{\beta}_{i_0 j_0} \otimes u^{\beta *}_{ii_0}(\sum_{l=1}^{d_\alpha} (u^{\alpha}_1)^*_{pl}(u^{\alpha}_1)_{ql}) u^{\beta}_{jj_0}.$$

Since $\overline{u^{\alpha}_1}$ is a unitary representation, we have $\sum_{l=1}^{d_\alpha} (u^{\alpha}_1)^*_{pl}(u^{\alpha}_1)_{ql} = \delta_{pq}$, where, as usual, δ_{pq} is the Kronecker symbol. Hence:

$$\delta_{13}(d_{i_0 j_0} \otimes m^{\beta}_{i_0 j_0}) = \sum_{i,j} d_{ij} \otimes m^{\beta}_{i_0 j_0} \otimes u^{\beta *}_{ii_0} u^{\beta}_{jj_0}.$$

Thus:

$$\delta_{13}(D) = \sum_{i,j,i_0 j_0} d_{ij} \otimes m^{\beta}_{i_0 j_0} \otimes u^{\beta *}_{ii_0} u^{\beta}_{jj_0}. \tag{18}$$

Formulas (18) and (17) show that the claim is true.

Since $\beta \in Sp(\delta), D \in (B \otimes \mathcal{M}_{d_\beta})^{\delta_{u^\beta}}$ and $YD = 0$ for every $Y \in B_2^\delta(u^\beta)$, it follows that $D = 0$. This means in particular that all the diagonal entries of the matrix V are equal to 0. Since $V \geq 0$, it follows that $V = 0$ and thus $Z = 0$. Therefore $linspan\{(X \odot Y)^*(X \odot Y) | X \in B_2^\delta(u^\alpha), Y \in B_2^\delta(u^\beta)\}$ is an essential ideal of $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$, as claimed.

Now let $u^\alpha \odot u^\beta \cong \sum_i^\oplus u^{\rho_i}$, where the ρ_i are irreducible. Then, by Remark 5.2(1) above, it follows that $(B \otimes \mathcal{M}_{d_\alpha d_\beta})^{\delta_{u^\alpha \odot u^\beta}}$ is spatially $*$-isomorphic to $\sum^\oplus (B \otimes \mathcal{M}_{d_{\rho_i}})^{\delta_{\rho_i}}$. Thus, since $B_2^\delta(u^\alpha \odot u^\beta)$ is spatially isomorphic to $\sum^\oplus B_2^\delta(\rho_i)$ (Remark 5.2(2) above), it follows that $B_2^\delta(\rho_i)^* B_2^\delta(\rho_i)$ is an essential ideal of $(B \otimes \mathcal{M}_{d_{\rho_i}})^{\delta_{\rho_i}}$, for all i. Therefore $\rho_i \in Sp(\delta)$, for every i. Thus $Sp(\delta|_C)$ is closed under tensor products for every $C \in \mathcal{H}^\delta(B)$ and the lemma is proven. □

We can now state:

Proposition 5.6. *The Connes spectrum, $\Gamma(\delta)$, is closed under tensor products.*

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