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THE TOPOLOGICAL BAUMGARTNER-HAJNAL THEOREM

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Abstract. Two new topological partition relations are proved. These are

$$\omega_1 \to (top\ \alpha+1)^2_k$$

and

$$\mathbb{R} \to (top\ \alpha+1)^2_k$$

for all $\alpha < \omega_1$ and all $k < \omega$. Here the prefix "*top*" means that the homogeneous set $\alpha+1$ is closed in the order topology. In particular, the latter relation says that if the pairs of real numbers are partitioned into a finite number of classes, there is a homogeneous (all pairs in the same class), well-ordered subset of arbitrarily large countable order type which is closed in the usual topology of the reals. These relations confirm conjectures of Richard Laver and William Weiss, respectively. They are a strengthening of the classical Baumgartner-Hajnal theorem.

1. Introduction

This paper proves two new results in the structural Ramsey theory of topological spaces. Ramsey theory is that branch of mathematics which, from an arbitrary structure, finds a homogeneous substructure. The structures involved may be linear orders, topologies, Banach algebras, etc. There are two types of problems. One may begin with a given structure and seek to determine its largest homogeneous substructure, or one can fix a substructure type and find the smallest structure which always contains a homogeneous substructure of the given type. In order to clarify the notion of homogeneous, we need some notation. If X is a set, define $[X]^n = \{s \subseteq X || s| = n\}$. If k is a cardinal number, we define

$$X \to (Y)^n_k$$

to mean the following: for every $\chi : [X]^n \to k$ there is $X_0 \subseteq X$ such that $X_0 \cong Y$ and $\chi \restriction [X_0]^n$ is constant, where $\cong$ denotes the appropriate isomorphism relation. In this notation, Ramsey (1930) proved the finite and infinite versions of the theorem which carries his name.

Ramsey's Theorem, Infinite version. $\forall n,\ k < \omega\ \omega \to (\omega)^n_k$.

Ramsey's Theorem, Finite version. $\forall n,\ k, l < \omega \exists m < \omega\ m \to (l)^n_k$.

For $n = 1$ we obtain the pigeon-hole principle as a special case. In the finite version to find the smallest value of m, even for $k = n = 2$, is an impossible task of calculation. The smallest m for $k = n = 2$ and $l = 5$ is unknown.

The theorems of Ramsey are about pure sets without additional structure. They can be generalized in many directions. The generalization to uncountable sets is

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called the partition calculus and has been extensively studied. The theory for ordered sets and topological spaces has many open problems. In this paper we prove the following two theorems:

Theorem 1. *For all* $\alpha < \omega_1$ *and all* $k < \omega$,

$$\omega_1 \to (top\,\alpha + 1)^2_k.$$

Theorem 2. *For all* $\alpha < \omega_1$ *and all* $k < \omega$,

$$\mathbb{R} \to (top\,\alpha + 1)^2_k.$$

Here ω_1 is the smallest uncountable ordinal. The prefix "*top*" denotes that the subset is homeomorphic to the ordinal $\alpha + 1$ with the order topology. In other words, it is a closed subset of the space concerned. Thus Theorem 1 can be stated as follows:

> for every $n < \omega$ and $\alpha < \omega_1$ if $\chi : [\omega_1]^2 \to n$, there is $X \subseteq \omega_1$ such that X is closed in the order topology on ω_1, and, the order type of X is $\alpha + 1$ and χ is constant on $[X]^2$.

The second theorem is similar; here we obtain a homogeneous set, closed in the ordinary topology of the reals and of order type $\alpha + 1$. Theorem 1 and Theorem 2 answer questions of Richard Laver and William Weiss, respectively. The classical theorem of Baumgartner-Hajnal shows, without the topological aspect, that if ϕ is any order type such that $\phi \to (\omega)^1_\omega$, then $\phi \to (\alpha)^2_k$.

These theorems belong to the first kind of problem, that of finding which homogeneous structures are contained in ω_1 and $\mathbb{R}$. ω_1 is the smallest ordinal such that $\omega_1 \to (\omega + 1)^2_k$, since if α is countable, then $\alpha \not\to (\omega + 1, \omega)^2$. This can be easily seen by an argument due to Sierpinski: Reorder α in order type ω and color a pair by 0 if the two orderings on α agree and by 1 if they disagree. Then there is no increasing $\omega + 1$-sequence in ω and no decreasing ω-sequence in α.

The Baumgartner-Hajnal theorem, although it is a theorem of ZFC, was originally proved by first assuming an additional axiom, Martin's Axiom (MA) and then showing via absoluteness that if the result is true under the additional hypotheses, then it is true under ZFC. Later, Fred Galvin found a proof entirely within ZFC, although his proof is more complicated. In this paper we shall utilize the original trick of Baumgartner-Hajnal. No one has yet succeeded in finding a proof of the topological Baumgartner-Hajnal theorem entirely within ZFC. We shall also need a well-known lemma from set theory, the Fodor regressive mapping theorem.

Definition. $C \subseteqq \omega_1$ is closed and unbounded (club) if

(1) $\forall X \subseteqq C,\ X$ is bounded $\to \sup(X) \in C$,
(2) $\forall \alpha < \omega_1,\ \exists \beta \in C\, \alpha < \beta$.

Definition. $S \subseteq \omega_1$ is stationary if S has a nonempty intersection with every club subset of ω_1.

Fodor's Theorem ([2])**.** *If* $S \subseteq \omega_1$ *is a stationary set and* $f : S \to: \omega_1$ *is such that for all* $\alpha \in S$, $f(\alpha) < \alpha$, *then there is a stationary* $S_0 \subseteq S$ *such that* f *is constant on* S.

In addition to Fodor's pressing-down lemma, we repeatedly use a very simple property of stationary sets.

Theorem. *If $S \subseteq \omega_1$ is a stationary set and $S = \bigcup_{i<\omega} S_i$ is a decomposition of S into countably many pieces, then there is $i < \omega$ such that S_i is stationary.*

To see how these sets are used, let us now demonstrate the following well-known result. The proof of this result is the basis for the present paper.

Theorem.

$$\omega_1 \to (top\,\omega + 1)^2_2.$$

Proof. Let $X : [\omega_1]^2 \to 2$ be given. For each $\alpha \in \omega_1$, try to choose a sequence $\{\alpha_n\}_{n<\omega}$ such that for all k, $\{\alpha_n\}_{n<k} \cup \{\alpha\}$ is 1-homogeneous and such that $\sup \alpha_n = \alpha$. To fulfill the last condition we first choose a sequence $\{\gamma_n\}$ whose limit is α and insist that $\gamma_n < \alpha_n$. If it is possible to choose such an infinite sequence for even one limit α, then we are done. Otherwise for each limit α choose a maximal sequence $\{\alpha_n\}_{n \leq m(\alpha)}$ where $m(\alpha) < \omega$. Now consider the mapping $\alpha \mapsto \alpha_{m(\alpha)}$. This is a regressive map, and by Fodor's Theorem there is an $S \subseteq \omega_1$ such that $\alpha_{m(\alpha)}$ is constant for all $\alpha \in S$; say that the constant value is δ. There is then a stationary subset of S such that $m(\alpha)$ is also constant, say l, on this set. Now, the first l values can also be made constant, as they are a finite subset of δ and are therefore only countably many values for the first parts. Passing again to a stationary subset we may assume that we have S stationary with a fixed sequence $\delta_0, \delta_1, \dots, \delta_l$ as a maximal sequence for all elements of S. Now we claim that S is homogeneous in color 1. Since if $\alpha < \beta$ are elements of S with color 0, then α can be used to continue the sequence for β, contradicting maximality. Thus S is 1-homogeneous. Every stationary set contains a closed $\omega + 1$ sequence. □

An argument such as the above is called a pressing down argument. To generalize this to arbitrary countable ordinal numbers we must take a slightly different approach. One cannot produce even a homogeneous $\omega + 2$ directly. We must first construct large closed sets with nice properties and then thin out these sets to get a homogeneous set, but with a smaller order type. The main property is what we call weakly reflective. These sets are constructed by pressing down over a sequence of elementary submodels. In order to move from one elementary submodel to another, we will need the color relation over these sets to be definable. This definability is achieved by looking at the Cantor-Bendixon ranking of the limit points. It is then possible to ensure that the color depends only on this ranking in a very simple (i.e. definable) fashion. This is described in section 3.

2. Fine structure of closed countable sets

This section is preparation for the next. We explain the structure of closed well-ordered subsets of $\mathbb{R}$ and ω_1. Also, we describe the combinatorial method we shall use to construct closed subsets of arbitrary order type.

Notation. If A is an ordered set, then $\mathrm{ot}(A)$ stands for the order type of A. If A is well ordered, then $\mathrm{ot}(A)$ is the unique ordinal number isomorphic to A.

Let A be a well-ordered subset of $\mathbb{R}$ or ω_1. This means well ordered in the usual orderings on $\mathbb{R}$ and ω_1. If the set A has a successor order type $\alpha + 1$ for some $\alpha < \omega_1$, we define A is closed to have the usual meaning that A contains all of its limit points. If however A has a limit order type, then A cannot be closed since $\sup(A)$ is a limit point of A not in A.

If $\overline{A} = A \cup \{\sup(A)\}$, then $\sup(A)$ is the only limit point of A not in A. In this case we have the following definition.

Definition. If A is a well-ordered set with limit order type, then A is internally closed if, for every $X \subseteq A$ such that X is bounded by an element of A (that is, there is $a \in A$ such that, for all $x \in X$, $x < a$), then $\overline{X} \subseteq A$.

For a closed or internally closed set A let $A' \subseteq A$ denote the set of all limit points of A, which are also elements of A. For $\alpha < \omega_1$ define the Cantor-Bendixon derivative by transfinite induction:

(1) $A^{(0)} = A$,
(2) $A^{(\alpha+1)} = (A^{(\alpha)})'$,
(3) $A^{(\lambda)} = \bigcap_{\beta<\lambda} A^{(\beta)}$.

We shall also use the Cantor normal form.

Theorem (Cantor)**.** *If α is an ordinal, then there is a unique sequence $\beta_0 \geq \ldots \geq \beta_l$ such that $\alpha = \omega^{\beta_0}+\cdots+\omega^{\beta_l}$. α is called indecomposable if $l = 0$ and decomposable if $l > 0$.*

To calculate order types of Cantor-Bendixon derivatives we have the following lemma.

Basic Lemma. *If $\gamma < \omega_1$, then*

$$\omega_1^{(\gamma)} = \{\mu | \mu = \omega^{\beta_0} + \cdots + \omega^{\beta_l}, l < \omega, \beta_0 \geq \ldots \geq \beta_l \geq \gamma\}.$$

Proof. By induction on γ. Assume that this statement is true for γ. The isolated points of $\omega_1^{(\gamma)}$ are then ordinals of the form $\omega^{\beta_0} + \cdots + \omega^{\gamma} \cdot n$ for $0 < n < \omega$. Thus the elements of $\omega_1^{(\gamma+1)}$ are $\omega^{\beta_0} + \cdots + \omega^{\beta_l}$, where $\beta_l \geq \gamma + 1$.

For λ a limit the result is immediate from $\omega_1^{(\lambda)} = \bigcap_{\gamma<\lambda} \omega_1^{(\gamma)}$. □

Lemma. *The order type of $(\omega^{\omega^\alpha})^{(\gamma)}$ for $\gamma < \omega^\alpha$ is ω^{ω^α}.*

Proof.

$$(\omega^{\omega^\alpha})^{(\gamma)} = \{\mu | \mu = \omega^{\beta_0} + \cdots + \omega^{\beta_l}, l < \omega, \omega^\alpha > \beta_0 \geq \ldots \geq \beta_l \geq \gamma\}.$$

Now $\omega^{\beta_0} + \cdots + \omega^{\beta_l} = \omega^{\gamma} \cdot (\omega^{\beta_0-\gamma} + \cdots + \omega^{\beta_l-\gamma})$. Here $\beta_l - \gamma$ ranges from 0 to $\omega^\alpha - \gamma$. But since $\omega^\alpha - \gamma = \omega^\alpha$, we see that these two sets have the same order type. □

Lemma 1. $(\omega^\gamma)^{(\gamma)} = \emptyset$.

Proof. By the basic lemma,

$$(\omega^\gamma)^{(\gamma)} = \{\mu < \omega^\gamma | \mu = \omega^{\beta_0} + \cdots + \omega^{\beta_l}, l < \omega, \beta_0 \geq \ldots \geq \beta_l \geq \gamma\}.$$

But if $\mu < \omega^\gamma$, then $\beta_0 < \gamma$, so $(\omega^\gamma)^{(\gamma)} = \emptyset$. □

Lemma 2. *If $A^{(\gamma)} = \{max(A)\}$, then $ot(A - max(A))$ is indecomposable.*

Proof. Let $x = \max(A)$. If $\mathrm{ot}(A - \max(A))$ is decomposable, then there are δ and α such that $\mathrm{ot}(A - \max(A)) = \delta + \omega^\alpha$, where $\delta \geq \omega^\alpha$. Let D and E be closed sets such that $A = D + E$ and $\mathrm{ot}(D) = \delta + 1$ and $\mathrm{ot}(E) = \omega^\alpha + 1$. Since $\delta \geq \omega^\alpha$ we have $D^{(\gamma)} \neq \emptyset$. Thus $A^{(\gamma)}$ contains x but also other elements from D, contrary to our assumption. □

Lemma 3. *If A is a countable, internally closed, well-ordered set such that $ot(A)$ is indecomposable and γ is least such that $A^{(\gamma)} = \emptyset$, then the order type of A is ω^γ.*

Proof. If $\mathrm{ot}(A) < \omega^\gamma$, then there is $\delta < \gamma$ such that $\mathrm{ot}(A) < \omega^\delta$. Thus by Lemma 1 $A^{(\delta)} = \emptyset$; however γ is the least such ordinal. □

Lemma 4. *If A is closed or internally closed and $x, y \in A^{(\gamma)}$ are isolated points of $A^{(\gamma)}$ such that $x < y$ and $A^{(\gamma)} \cap (x, y) = \emptyset$, then $(x, y) \cap A$ has order type ω^γ.*

Proof. Let $B = [x, y] \cap A$; then B is closed and $B^{(\gamma)} = \{y\}$. Thus by Lemma 2, $\mathrm{ot}(B - \max(B))$ is indecomposable, and by Lemma 3 $\mathrm{ot}(B - \max(B)) = \omega^\gamma$. □

Lemma 5. *Let A be internally closed of order type ω^{ω^α}. Then $A - A^{(\gamma)}$ is the sum of ω^{ω^α} many internally closed sets of order type ω^γ.*

Proof. Let $\{x_i | i < \omega^{\omega^\alpha}\}$ be an enumeration of $A^{(\gamma)}$. Then by Lemma 4 $(x_i, x_{i+1}) \cap A$ has order type ω^γ. Thus $A - A^{(\gamma)} = \Sigma_{i<\omega^{\omega^\alpha}} (x_i, x_{i+1}) \cap A$. □

We shall write $A^{(\gamma)} - A^{(\gamma+1)} = \Sigma_i s_i$, where $s_i = (x_i, x_{i+1}) \cap A$. Using this notation we have the following definition.

Definition. A set $B \subseteq \Sigma_i s_i$, where each s_i is internally closed, is neat if for every s_i either $B \cap s_i = \emptyset$ or $B \cap s_i$ is internally closed of order type ω^γ, and moreover the set

$$\{i | B \cap s_i \neq \emptyset\}$$

has order type ω^{ω^α}.

Thus, in particular, every neat set is the sum of internally closed sets of order type ω^γ.

Lemma 5.1. *Let $B \subseteq A - A^{(\omega^\gamma)}$ be neat, and let $x \in \overline{B} \cap A^{(\omega^\gamma)}$. Then there is $D \subseteq B$ such that D is internally closed of order type ω^{ω^γ} and $x = sup(D)$.*

Proof. Choose $\{x_n\}_{n<\omega}$ such that $x_n \in \overline{B} \cap A^{(\omega^\gamma)}$ is isolated in $A^{(\omega^\gamma)}$ and $\sup x_n = x$. Thus there are i_n such that $x_n = \sup(B \cap s_{i_n})$. Let $\omega^{\omega^\gamma} = \Sigma_n \omega^{\omega^{\gamma_n}}$. Now choose $D_n \subseteq B \cap s_{i_n}$ closed of order type $\omega^{\omega^{\gamma_n}} + 1$. Then $\bigcup_n D_n \cup \{x\}$ is closed of order type $\omega^{\omega^\gamma} + 1$. □

Lemma 6. *Let A be internally closed of order type ω^γ with $\gamma = \omega^{\delta_0} + \cdots + \omega^{\delta_{n-1}}$. Let $X \subseteq n$ be arbitrary and for $i \in X$, let*

$$B_i \subseteq A^{(\omega^{\delta_0} + \cdots + \omega^{\delta_i})} - A^{(\omega^{\delta_0} + \cdots + \omega^{\delta_{i+1}})}$$

be neat and such that if $i < j$, $i, j \in X$, then $B_j \subseteq \overline{B_i}$. Then there exists $D_i \subseteq B_i$ such that $D = \bigcup_{i \in X} D_i$ is internally closed, and $ot(D) = \omega^\mu$, where $\mu = \Sigma_{i \in X} \omega^{\delta_i}$.

Proof. Construct the sequence D_i for $i \in X$ by recursion from the largest to the smallest value of i. Let D_i and $j < i$, the next smallest member of X, be given. For each isolated point $x \in D_i$ there is, using Lemma 5, a $d_x \subseteq B_j$ such that d_x is internally closed of order type equal to $\omega^{\omega^{\gamma_j}}$. Let $D_j = \Sigma d_x$. Then $D = \bigcup_{i \in X} D_i$ is internally closed since the limit points of D_j are either elements of D_i itself or are the isolated points of D_i where i is the next element of X. One can see by induction that if $\nu = \Sigma_{i \in X, i<h} \omega^{\delta_i}$, then $D^{(\nu)} \subseteq \bigcup_{h>i} D_i$. Therefore, $\mu = \Sigma_{i \in X} \omega^{\delta_i}$ is the smallest ordinal such that $D^{(\mu)} = \emptyset$, and by Lemma 3 the order type of D is ω^μ. □

3. BH-sets and Martins axiom

We introduce some ideas from the original Baumgartner-Hajnal proof and extend them to closed sets. We use Martin's Axiom to guarantee the existence of sets with nice properties.

Notation. Fix a coloring $\chi : [\omega_1]^2 \to k$. For sets $A, B \subseteq \omega_1$ where for all $a \in A$ and $b \in B$, $a < b$, define $(A : B) = \{\{a, b\} | a \in A, b \in B\}$. Also $(A : x) = (A : \{x\})$.

Definition. A set $A \subseteq \omega_1$ is a BH-set if, for every $x > \sup A$, there is $A_0 \subseteq A$ a final segment of A such that $(A_0 : x)$ is monochromatic.

So for each x and BH-set A there is a color associated to a large enough monochromatic final segment $A_0 \subseteq A$. We call this the eventual color of A and x. We shall need the following two well-known consequences of Martin's Axiom. See [3] for a proof.

Proposition (MA)**.** *Martin's Axiom has the following consequences:*

(1) *If $\{A_\alpha | \alpha < \omega_1\}$ is a collection of subsets of ω such that the intersection of any finite number is infinite, then there is an infinite $B = \{b_0, b_1, \ldots\} \subseteq \omega$ such that for any $\alpha < \omega_1$ there is $n < \omega$ such that*
$$\{b_n, b_{n+1}, \ldots\} \subseteq A_\alpha.$$
(2) *If $\{f_\alpha | \alpha < \omega_1\}$ is a collection of functions from ω to ω, then there is a $g : \omega \to \omega$ such that for all $\alpha < \omega_1$ there is $n < \omega$ such that for all $m \geq n$, $f_\alpha(m) < g(m)$.*

Proposition 1. *If $A \subseteq \omega_1$ and $ot(A) = \omega$, then there is $B \subseteq A$ such that $ot(B) = \omega$ and B is a BH-set.*

Proof. We define sets $A_x \subseteq A$ by induction such that for all $x > \sup A$, $(A_x : x)$ is monochromatic, and any finite collection of sets A_x has infinite intersection. Define the sets by transfinite induction. Assume we have $\{A_y | y < x\}$ with the requisite property. For $i < k$ let $D_i = \{a \in A | \chi(\{a, x\}) = i\}$. Then one of the sets D_i must have the intersection property with the sets $\{A_y | y < x\}$, since otherwise there are a finite number of sets with finite intersection. Choose this set as A_x. Now by MA, there is $B \subseteq A$ such that for all $x > \sup A$ there is a final segment of B contained in A_x. This set B is a BH-set. □

Proposition (MA)**.** *Let $A = \Sigma_{i<\omega^\alpha} s_i$ be a sum of internally closed sets such that for all $n < \omega^\alpha$, s_i is a BH-set. Then there is a $B \subseteq A$ such that B is neat and B is a BH-set.*

Proof. By induction on α. Let $\{\alpha_n\}$ be such that $\omega^\alpha = \Sigma_n \omega^{\alpha_n}$. Let $A_m = \Sigma s_i$, where the summation runs over all i such that
$$\Sigma_{n=0}^{n=m-1} \omega^{\alpha_n} \leq i < \Sigma_{n=0}^{n=m} \omega^{\alpha_n}.$$
Thus A_m is the sum of ω^{α_n} many sets s_i. By induction there are neat BH-sets $B_m \subseteq A_m$. Since B_m is a BH-set there is a descending sequence of final segments $\{B_m(t)\}_{t<\omega}$ such that for every $x > \sup(A)$ there is t, where $(B_m(t) : x)$ is monochromatic. Let $f_x : \omega \to \omega$ be such that
$$(B_m(f_x(m)) : x) \text{ is monochromatic.}$$

By MA there is $g : \omega \to \omega$ such that g eventually dominates each f_x. Define

$$\Theta(\{m, x\}) = \text{eventual color of } (B_m : x).$$

Let $M \subseteq \omega$ be a BH-set for Θ and define $B = \Sigma_{m \in M} B_m(g(m))$. Then B fulfills the requirements of the lemma. B is neat since each $B_m(g(m))$ is neat. To see that B is a BH-set, let $x > \sup(B)$. Then there is l_0 such that $\{m \in M | l_0 < m\}$ is homogeneous for $\Theta(\{-, x\})$ and such that $f_x(m) < g(m)$ for $l_0 < m$. Then $B_m(g(m)) \subseteqq B_m(f_x(m))$; thus $(B_m(g(m)) : x)$ is monochromatic in color $\Theta(\{m, x\})$. But this color is constant for $m \in M$ and $l_0 < m$. Thus $\Sigma_{m \in M, l_0 < m} B_m(g(m))$ is a final segment of B, monochromatic with respect to x. □

We now use BH-sets to construct internally closed BH-sets.

Lemma. *If A is an internally closed set of order type ω^{ω^α}, then there is $B \subseteq A$ such that B is internally closed of order type ω^{ω^α} and B is a BH-set.*

Proof. By induction on α. If $\alpha = 0$, then this is the consequence of Martin's Axiom stated above. If α is a limit, write $\omega^{\omega^\alpha} = \Sigma \omega^{\omega^{\alpha_n}}$ where the sequence $\{\alpha_n\}$ is increasing. Let $A = \Sigma A_n$ be a sum of internally closed sets such that $\text{ot}(A_n) = \omega^{\omega^{\alpha_n}}$. By induction there are $B_n \subseteq A_n$ such that B_n is an internally closed BH-set of order type $\omega^{\omega^{\alpha_n}}$. Now by the proposition on sums above with $\alpha = 1$, there is a neat BH-set $D = \Sigma D_n$, where $D_n \subseteq B_n$ is internally closed of order type $\omega^{\omega^{\alpha_n}}$. Let $C_n \subseteq D_n$ be a closed subset of order type $\omega^{\omega^{\alpha_{n-1}}}$. Then $C = \Sigma C_n$ is an internally closed BH-subset of A with order type ω^{ω^α}.

Let us now consider the case of a successor ordinal, and let $\alpha = \beta + 1$. Thus $\omega^{\omega^{\beta+1}} = \Sigma \omega^{\omega^\beta \cdot n}$. Let $A = \Sigma A_n$, where A_n is an internally closed set with $\text{ot}(A_n) = \omega^{\omega^\beta \cdot n}$. Define sets B_n, by recursion, with the following properties:

(1) $B_n \subseteq \overline{B_n}^{(\omega^\beta \cdot n)} - \overline{B_n}^{(\omega^\beta \cdot (n+1))}$.
(2) B_n is a BH-set.
(3) B_n is neat.

We begin the induction with B_i. We have that $A - A^{(\omega^\beta)} = \Sigma s_i$, where each s_i is an internally closed set of order type ω^{ω^β}. Thus by induction there is $t_i \subseteq s_i$ such that t_i is an internally closed BH-set. By the proposition on sums there is $B_0 \subseteq \Sigma t_i$ such that B is a neat BH-set.

Now given B_n, write $\overline{B_n}^{(\omega^\beta \cdot n)} - \overline{B_n}^{(\omega^\beta \cdot (n+1))}$ as a sum of Σs_i of internally closed sets of order type ω^{ω^β}. Now by the same procedure as above there is a neat BH-set $B_{n+1} \subseteq \Sigma s_i$. Let

$$\Theta(\{m, x\}) = \text{eventual color of } (B_m : x).$$

Let $M \subseteq \omega$ be a BH-set for Θ. Let $f_x : \omega \to \omega$ be such that

$$(B_m(f_x(m)) : x) \text{ is monochromatic,}$$

and let $g : \omega \to \omega$ eventually dominate each f_x.

Let $\{h_i\}$ be a sequence of numbers such that there are at least $i + 1$ elements of M greater than h_i and less than h_{i+1}.

Let $x_i = \min(B_{h_i}(g(h_i)))$. $\{x_m\}$ is cofinal in A since $x_n \notin A^{(\omega^\beta \cdot (m+1))}$. For each $i < \omega$ we make an application of Lemma 6 to the set $X = (h_i, h_{i+1}) \cap M$, and to the sets $C_m = (x_i, x_{i+1}) \cap B_m(g(m))$ for $m \in X$. These sets satisfy the hypothesis of Lemma 6, and so there are sets $D_m \subseteq C_m$ for $m \in M$ such that $\bigcup_{m \in M} D_m$ is internally closed and of order type $\omega^{\omega^{\beta\cdot l}}$ where $l = |X|$. Take $E_i \subseteq \bigcup_{m\in M} D_m$ to be closed with order type $\omega^{\omega^{\beta\cdot(l-1)}}$. Now

$$E_i \subseteq \bigcup\{B_m(g(m))|h_i < m < h_{i+1}, m \in M\}.$$

Finally let $B = \Sigma E_i$. Then B is internally closed. Also since $l \to \infty$ as $i \to \infty$, we have that

$$\mathrm{ot}(B) = \sup \omega^{\omega^{\beta\cdot l}} = \omega^{\omega^{\beta+1}} = \omega^{\omega^{\alpha}}.$$

To finish we note that B is a BH-set. For if $x > \sup(B)$, then there is an l_0 such that $\{m \in M|l_0 < m\}$ is homogenous for $\Theta(\{-,x\})$ and such that $f_x(m) < g(m)$ for $l_0 < m$. So when $l_0 < m$ we have that $(B_m(g(m)) : x)$ is monochromatic in color $\Theta(\{m,x\})$ and this color is constant for $m \in M$ and $l_0 < m$. Now there is i_0 such that for $i \geq i_0$ we have $l_0 < h_i$. For these i, $(E_i : x)$ is monochromatic in color in the stable color for $\Theta(\{-,x\})$. Thus $B_0 = \Sigma_{i_0<i}E_i$ is a homogeneous final segment of B. □

For closed sets of order type ω^{α} we can obtain the following.

Lemma on nice sets. *If A is a closed set of order type ω^{α} and if $\alpha = \omega^{\delta_0}+\ldots+\omega^{\delta_l}$, $\delta_0 \geq \ldots \geq \delta_l$, is the Cantor normal form for α, then there is a closed $B \subseteq A$ of order type ω^{α} and such that for $n < l$,*

$$B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_n})} - B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{n+1}})}$$

is a BH-set.

Proof. By induction on l. $A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{l-1}})}$ has order type $\omega^{\omega^{\delta_l}}$, so there is $B_0 \subseteq A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{l-1}})}$ which is a closed BH-set of order type $\omega^{\omega^{\delta_l}}$. Now each element of $B_0 - B_0'$ is the supremum of a closed set of order type $\omega^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{l-1}})}$. By induction the lemma holds for each of these sets and we are done. □

Note that for any x, the colors that these BH-sets eventually achieve with respect to x may be different. But it is a key point of the proof that there are only finitely many possibilities. This is a key point in the construction of weakly reflective sets, which is the subject of the next section.

4. Reflective sets

Definition. For closed subsets $A \subseteq \omega_1$ of order type $\omega^{\alpha}+1$ we define the notion of weakly reflective by transfinite induction on α. As the notion is only for ordinals $\alpha \geq 1$, we begin the induction with a special case and say that any singleton set A is weakly reflexive.

(1) $A = \bigcup_{n=0}^{\infty} A_n \cup \{y\}$, where for all $n < m$, $A_n < A_m$ and $y = \sup \bigcup A_n$,
(2) A_n is closed and has order type $\omega^{\alpha_n}+1$ for some α_n,
(3) for all $m < n$, $a \in A_m, b \in A_n$, $\chi(\{a,b\}) = \chi(\{a,y\})$,
(4) A_n is weakly reflective,
(5) if $\alpha_n = \omega^{\delta_0}+\cdots+\omega^{\delta_l}$ where $\delta_0 \geq \ldots \geq \delta_l$ is the Cantor normal form for α_n, then for all $i \leq l$

$$(A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_i})} - A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{i+1}})} : y)$$

is monochromatic.

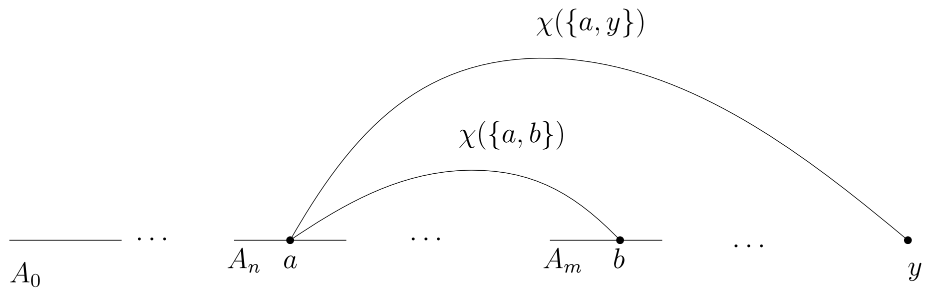

FIGURE 1. Weakly reflective set: $\chi(\{a,b\}) = \chi(\{a,y\})$

Definition. A set $A \subseteq \omega_1$ is strongly reflective if A fulfills the same conditions as above with conditions (4) and (5) changed to

(4. strong) A_n is strongly reflective.

(5. strong) $(\bigcup A_n : y)$ is monochromatic.

Weak Reflection Lemma. *There are weakly reflective subsets of ω_1 of arbitrary large countable order type.*

Proof. For each limit λ fix C_λ a set of order type ω with supremum λ. The nth element of C_λ is denoted by λ_n.

We prove by induction on α that every stationary set E contains a weakly reflective set of type $\Sigma(\omega^{\alpha_n}+1)+1$.

Fix a sequence $\langle N_\alpha : \alpha < \omega_1\rangle$ of elementary submodels of H_{ω_2} such that

(1) $\alpha < \beta$ implies $N_\alpha \prec N_\beta$,
(2) the sequence is continuous; that is, for λ a limit, $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$,
(3) $|N_\alpha| = \aleph_0$,
(4) $N_\alpha \in N_{\alpha+1}$,
(5) $\chi \in N_0$, $E \in N_0$.

For each $y \in E$ try to choose a sequence $\{(A_n, \gamma_n) | n < \omega\}$ such that

(1) A_n is a closed set of order type $\alpha_n + 1$,
(2) $A_n \subseteq E$,
(3) $y_n < minA_n$, where y_n is the nth element of C_y,
(4) $A_n \in N_{\gamma_n}$ and $\gamma_n < y$,
(5) $\forall m < n \forall a \in A_m \forall b \in A_n \chi(\{a,b\}) = \chi(\{a,y\})$,
(6) if $\alpha_n = \omega^{\delta_0} + \cdots + \omega^{\delta_l}$, where $\delta_0 \geq \ldots \geq \delta_l$ is the Cantor normal form for α_n, then for all $i \leq l$

$$(A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_i})} - A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{i+1}})} : y)$$

is monochromatic.

If the set of $y \in E$ for which such a sequence exists is stationary, then we are done. So without loss of generality let us assume that for each $y \in E$ the attempt to construct a sequence fails.

Now for each $y \in E$ choose finite sequence $\{(A_n, \gamma_n) | n \leq k(y)\}$ which satisfies conditions (1) through (6) but which cannot be extended. For each $y \in E$ let $f(y) = \gamma_{k(y)}$. Then $f \in H_{\omega_2}$ is a regressive function on E, and by Fodor's lemma there is $S_0 \subseteq E$ and $\gamma < \omega_1$ such that $f(y) = \gamma$ for all $y \in S_0$. The sequence for

y is in the model N_γ. Now since N_γ is countable there are only countably many possibilities for the sequence, and we may pass to a stationary subset $S \subseteq S_0$ such that each element of S has the same maximal sequence. Denote this sequence by $\{(A_0\gamma_0), \ldots, (A_n, \gamma_n)\}$. Now, because this sequence satisfies condition (6), for each $y \in S$ there are only finitely many ways the edges $(A_0 \cup \ldots \cup A_n : y)$ can be colored. We may assume therefore that each element y of S is colored in the same way as the set $A = A_0 \cup \ldots \cup A_n$. Further, there is a first order formula $\Psi(A, S)$ which defines the color relation of any $y \in S$ to A. $\Psi(A, S)$ says, in the formal language of set theory, that each $x \in S$ has color c_i with respect to each element of the Cantor-Bendixon derivative

$$A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_i})} - A^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{i+1}})},$$

for every i.

Thus

$$H_{\omega_2} \vDash \exists S(S \text{ stationary} \wedge S \subseteq E \wedge \Psi(A, S)).$$

Since $A \in N_{\gamma+1} \prec H_{\omega_2}$ we have

$$N_{\gamma+1} \vDash \exists S(S \text{ stationary} \wedge S \subseteq E \wedge \Psi(A, S)).$$

Let $S' \in N_{\gamma+1}$ satisfy the formula, so

$$N_{\gamma+1} \vDash (S' \text{ stationary} \wedge S' \subseteq E \wedge \Psi(A, S')).$$

We now argue in $N_{\gamma+1}$. By induction there is a weakly reflective set $C \in N_{\gamma+1}$ of order type $\omega^{\alpha_{n+1}} + 1$ and such that $C \subseteq S'$. Now there is $B \subseteq C$ a closed subset of order type $\omega^{\alpha_{n+1}} + 1$ which satisfies the conclusion of the lemma on nice sets. Further, B is a weakly reflective set, as it is a closed subset of C.

Thus we have $B \in N_{\gamma+1}$, and since $B \in S'$ each element of B is colored the same way as $A = A_0 \cup \ldots \cup A_n$, as each element of S is colored to A. Now choose $y \in S$ such that $\gamma + 1 < y$. We claim that the sequence with respect to y can be continued by adding an element of $N_{\gamma+1}$. B is not quite sufficient, since the levels are not homogeneous with respect to y. However the sets

$$B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_i})} - B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{i+1}})}$$

are BH-sets. Let

$$D_i \subseteq B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_i})} - B^{(\omega^{\delta_0}+\cdots+\omega^{\delta_{i+1}})}$$

be a final segment. Then by Lemma 6 with $X = n$ there is $E \subseteq \bigcup_{i<n} D_i$ such that E is internally closed of order type $\omega^{\alpha_{n+1}}$, and by adding the supremum to E we get a closed set A_{n+1} of order type $\omega^{\alpha_{n+1}} + 1$. Now there is a choice of the final segments D_i such that the D_i are homogeneous with respect to y. Since there are only finitely many i there is such a set $A_{n+1} \in N_{\gamma+1}$, and this set can be used to continue the sequence for y. This is a contradiction to the choice of y. □

5. Homogeneous sets

Now that we have weakly reflective sets of arbitrary countable order type, we can construct strongly reflective sets of an arbitrary countable order type.

Lemma. *For every $\alpha < \omega_1$ there is $\beta < \omega_1$ such that a weakly reflective set of order type $\alpha + 1$ has a strongly reflective subset of order type $\beta + 1$.*

Proof. Abbreviate that statement as weak$(\beta+1) \to$ strong$(\alpha+1)$. The proof is by induction α. Let $\alpha_0 = \Sigma\alpha_n + 1$. Let γ_n be such that

$$\text{weak}(\gamma_n + 1) \to \text{strong}(\alpha_n + 1).$$

Let β_n be such that $(\beta_n + 1) \to \text{top}(\gamma_n + 1)^1_k$; then $\beta_0 = \Sigma\beta_n + 1$ will do. For the existence of β_n see [4]. □

Lemma. *For every $\alpha < \omega_1$ there is $\beta < \omega_1$ such that if A is a strongly reflective set of order type $\beta + 1$, there is a $B \subseteq A$ such that B is closed, homogeneous and of order type $\alpha + 1$.*

Proof. Let $\text{ref}(\beta + 1) \to (\mu_0 + 1, \dots, \mu_k + 1)^2$ denote the statement that for every strongly reflective set A of order type $\beta+1$ there is an $i \le k$ and a closed subset $B \subseteq A$ of order type μ_i+1, homogeneous in color i. We show that for all $\mu_0, \dots, \mu_k < \omega_1$ there is a β such that $\text{ref}(\beta + 1) \to (\mu_0 + 1, \dots, \mu_k + 1)^2$. Proceed by induction on $\mu_0 + \mu_1 + \cdots + \mu_k$. Thus there is $\gamma < \omega_1$ such that for all $i < k$ and $\sigma < \mu_i$,

$$\text{ref}(\gamma + 1) \to (\mu_0 + 1, \dots, \mu_i + 1, \sigma + 1, \mu_i + 1, \dots, \mu_k + 1)^2.$$

Let $\beta = (\gamma + 1) \cdot \omega$. Let A be strongly reflective of order type $\beta + 1$. So $A = \Sigma A_n \cup \{y\}$, and let the color of the monochromatic set $(\bigcup A_n : y)$ be i. Let $\mu_i = \Sigma(\alpha_n + 1)$. Then apply the relation,

$$\text{ref}(\gamma + 1) \to (\mu_0 + 1, \dots, \mu_i + 1, \alpha_n + 1, \mu_i + 1, \dots, \mu_k + 1)^2$$

to the set A_n to get $B_n \subseteq A_n$. If for one n the color of b_n is not equal to i, then we are done. If every b_n is homogeneous in color i, then they have order type $\alpha_n + 1$. So $\bigcup B_n \cup \{y\}$ is closed and homogeneous in color i and is of order type $\Sigma(\alpha_n + 1) + 1 = \mu_i + 1$. □

Finally we show that the assumption of MA can be eliminated. Let $M \subseteq N$ be transitive models of set theory such that $\omega_1^N = \omega_1^M$ and $N \vDash MA$. For each limit $\lambda \in \omega_1$ let $C_\lambda \in M$ be a fixed omega sequence whose limit is λ. Then

$$N \vDash \omega_1 \to (\text{top}\,\alpha + 1)^2_k.$$

Fix α and let $g \in M$, $g : \alpha + 1 \to \omega$ be one to one and onto. Let P be the set of all functions $f : n \to \omega_1$ such that

(1) $f[n]$ is homogeneous,
(2) $f \circ g : n \to \omega_1$ is order preserving,
(3) if $\lambda \le \alpha$ is a limit ordinal, then $(f \circ g(\lambda))_n \le f \circ g(\lambda_n)$, where λ_n is the nth element of C_λ.

Define $f' \le f$ if $f \subseteq f'$. Thus in any model there is a homogeneous top$(\alpha+1)$ set if and only if P is not well-founded in that model. But well-foundedness is absolute. Thus P is not well-founded in N and therefore not well-founded in M, and so there is a homogeneous top$(\alpha + 1)$ in M.

6. The real numbers

In this section we describe the modifications that are required to prove that

$$\mathbb{R} \to (\text{top}\,\alpha + 1)^2_k.$$

Take a subset of $\mathbb{R}$ of size ω_1 and give it a second ordering of type ω_1. This set now has two orderings, the well ordering $<_{\omega_1}$ and the real ordering $<_{\mathbb{R}}$. We shall use the interval notation $(x, y) = \{w | x <_{\mathbb{R}} w <_{\mathbb{R}} y\}$ only in the sense of the real ordering. The addition that must be made is in the construction of a weakly reflective set. We shall claim by induction that for every stationary set E there is a weakly reflective subset such that the real and the well orderings agree on this weakly reflective set.

We shall need the following lemma.

Lemma. *If $x < y$ are such that (x, y) is stationary, then there is z such that (x, z) and (z, y) are stationary.*

Proof. If not, then there is an increasing sequence $\{x_n\}$ and a decreasing sequence $\{y_n\}$ such that both sequences have the same limit, and for all n, $(x, y) - (x_n, y_n)$ is nonstationary. Thus there are club sets C_n such that $C_n \cap (x, y) - (x_n, y_n) = \emptyset$. However, in this case $\bigcap C_n \cap (x, y)$ has at most one point, and so (x, y) is not stationary. A contradiction. $\square$

We now review the argument of the Weak Reflection Lemma. Choose $y \in$E and attempt to choose a sequence $\{A_n\}$ such that the conditions in the proof of the Weak Reflection Lemma hold and also that the real and the well-orderings agree on $\bigcup_{i<n} A_i \cup \{y\}$. If this is not possible, then as before we have a stationary set $S \subseteq E$ and a closed set $A \in N_\gamma$ such that A is maximal for all $y \in S$ and as such $y \in S$ has the same definable color relation to A. Note that for all $y \in S$, $\max(A) <_{\mathbb{R}} y$. Now choose $r_0 < r_1 < r_2$ rational numbers such that $(r_0, r_1) \cap S$ and $(r_1, r_2) \cap S$ are stationary. Then choose $B \in N_{\gamma+1}$ weakly reflective such that $B \subseteq (r_0, r_1) \cap E$ and B is larger than A in both orderings. Now choose $y \in (r_1, r_2) \cap S$. Finally choose $A_{n+1} \subseteq B$ homogeneous with respect to y, as in the Weak Reflection Lemma.

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