THE RING OF BOUNDED POLYNOMIALS ON A SEMI-ALGEBRAIC SET

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Abstract. Let \( V \) be a normal affine \( \mathbb{R} \)-variety, and let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \) which is Zariski dense in \( V \). We study the subring \( B_V(S) \) of \( \mathbb{R}[V] \) consisting of the polynomials that are bounded on \( S \). We introduce the notion of \( S \)-compatible completions of \( V \), and we prove the existence of such completions when \( \dim(V) \leq 2 \) or \( S = V(\mathbb{R}) \). An \( S \)-compatible completion \( X \) of \( V \) yields a ring isomorphism \( \mathcal{O}_X(U) \cong B_V(S) \) for some (concretely specified) open subvariety \( U \supseteq V \) of \( X \). We prove that \( B_V(S) \) is a finitely generated \( \mathbb{R} \)-algebra if \( \dim(V) \leq 2 \) and \( S \) is open, and we show that this result becomes false in general when \( \dim(V) \geq 3 \).

Introduction

The general question studied in this paper can be stated as follows: Let \( S \) be a semi-algebraic subset of \( \mathbb{R}^n \) (i.e., a subset described by polynomial inequalities). How can we describe (conceptually or explicitly) the ring of polynomials that are bounded on \( S \)?

To address this question we will work in the following setup. Let \( V \) be an affine variety defined over \( \mathbb{R} \), and let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \). We write \( \mathbb{R}[V] \) for the ring of real polynomial functions on \( V \) (the coordinate ring of \( V \)) and consider its subring
\[
B_V(S) = \{ f \in \mathbb{R}[V] : f|_S \text{ is bounded} \}.
\]

The first systematic study of \( B_V(S) \), in the case \( S = V(\mathbb{R}) \), was undertaken in 1996 by Becker and Powers [2]. Their results have been generalized substantially by Monnier [12] and Schweighofer [20]. The emphasis there is on iterating the \( B_V \) construction (which requires a more abstract definition via the real spectrum) and on relations to sums of squares, sums of higher powers, and certificates for positivity; see also Marshall [11]. Motivation came in part from earlier work on the so-called holomorphy ring in the theory of real rings and fields (see [1] and [2] and references therein). In particular, rings \( A \) were studied in which all elements of the form \( 1 + \sum a_i^2 \) with \( a_i \in A \) are invertible. Of course, this condition is hardly ever satisfied for \( A = \mathbb{R}[V] \).

A principal difficulty in studying \( B_V(S) \) is that these rings need not be of finite type over \( \mathbb{R} \). For example, this is so for elementary reasons when \( S \) is neither relatively compact nor Zariski dense in \( V \) (Corollary 5.8). We will show, however, that there exist other and more genuine examples as well (Corollary 5.14). In [2],

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we prove that the structure of the real spectrum and using sophisticated arguments from real algebra.

Our investigations go in a somewhat different direction. We seek to understand the structure of $B_V(S)$ in terms of the geometry of $S$, in particular by constructing algebraic compactifications of $V$ that are suitably adapted to $S$. To describe the main results of this paper, let us make the simplifying assumption that the affine variety $V$ is non-singular and connected. Using resolution of singularities, we construct a quasi-projective variety $U$ containing $V$ as an open dense subset such that $\mathcal{O}_U(U)$ restricts isomorphically onto $B_V(V(R))$ (Theorems 3.8 and 4.1). We also generalize this construction from $S = V(R)$ to more general semi-algebraic sets $S \subset V(R)$. However, we have been able to prove a satisfactory general result only for $\dim(V) \leq 2$ (Theorem 1.5). Combining this with a theorem of Zariski, we prove that the $\mathbb{R}$-algebra $B_V(S)$ is finitely generated when $\dim(V) \leq 2$ and the semi-algebraic set $S$ satisfies a weak regularity condition (Theorem 5.12).

Here is a brief survey of the contents of this paper. After introducing the necessary terminology, we fix a connected normal affine $\mathbb{R}$-variety $V$ and a closed semi-algebraic set $S \subset V(R)$, and we study morphisms $V \to W$ into affine $\mathbb{R}$-varieties $W$ which are bounded on $S$. From this we obtain a characterization of the field of fractions of $B_V(S)$ in geometric terms (Proposition 2.2). In particular, we show that $B_V(S)$ has full transcendence degree $\dim(V)$ over $\mathbb{R}$ if, and only if, there exists a non-constant $f \in B_V(S)$ such that the set $f^{-1}(c) \cap S$ is compact for almost all real numbers $c$ (Theorem 2.6).

In Section 3 we introduce the notion of compatible completions. Given a normal affine variety $V$ and a semi-algebraic set $S \subset V(R)$, a complete variety $X$ containing $V$ as an open dense set is said to be compatible with $S$ if $S$ touches every irreducible component of $X \setminus V$ in a Zariski dense subset, provided it touches that component at all. Removing those components from $X$ that are not touched by $S$, we obtain an open subvariety $U \subset X$ containing $V$, and we show $\mathcal{O}_U(U) \cong B_V(S)$ (Theorem 3.8). The existence of compatible completions is studied in Section 4. Relative to $S = V(R)$, every non-singular affine variety $V$ has a compatible completion (Theorem 4.1). Relative to more general semi-algebraic subsets of $V(R)$, we can prove such a result for $\dim(V) \leq 2$ (Theorem 4.1). Turning things around, we prove in Theorem 4.1 that for every normal real quasi-projective variety $U$ there is an open affine subset $V \subset U$ and a semi-algebraic set $S \subset V(R)$ with $\mathcal{O}_U(U) \cong B_V(S)$. In Section 5 we finally study finite generation of the $\mathbb{R}$-algebra $B_V(S)$. After showing that $B_V(S)$ is not Noetherian when $S$ is not Zariski dense in $V$, we prove finite generation of $B_V(S)$ for $\dim(V) \leq 2$ (Theorem 5.12), using Zariski’s theorem. We also show that this result becomes false when $\dim(V) \geq 3$.

1. Notation and conventions

1.1. By an $\mathbb{R}$-variety we mean a separated and reduced $\mathbb{R}$-scheme $V$ of finite type, not necessarily irreducible. The structural sheaf of $V$ is written $\mathcal{O}_V$. If $V$ is affine, we write $\mathbb{R}[V] := \mathcal{O}_V(V)$ for the coordinate ring of $V$. If $V$ is irreducible, then $\mathbb{R}(V)$ denotes the function field of $V$.

The set $V(R)$ of $\mathbb{R}$-rational points is endowed with the euclidean topology. The notion of semi-algebraic subsets of $V(R)$ is well known when $V$ is affine and is easily transferred to the general case, a subset $S \subset V(R)$ being semi-algebraic if and only if $S \cap U(R)$ is semi-algebraic in $U(R)$ for every open affine subset $U$ of $V$. 
An irreducible \( \mathbb{R} \)-variety \( V \) is said to be real if it has a non-singular \( \mathbb{R} \)-point, or equivalently, if the function field \( \mathbb{R}(V) \) is (formally) real. It is also equivalent that \( V(\mathbb{R}) \) is Zariski dense in \( V \).

1.2. Let \( f : V \to W \) be a morphism of \( \mathbb{R} \)-varieties, and let \( S \) be a subset of \( V(\mathbb{R}) \). We say that \( f \) is bounded on \( S \) if the closure of \( f(S) \) in \( W(\mathbb{R}) \) is compact.

In particular, this applies in the case \( W = \mathbb{A}^1 \), i.e., when \( f \) is a regular function on \( V \). The main object of this paper is to study the subring

\[
B_V(S) := \{ f \in \mathbb{R}[V] : f \text{ is bounded on } S \}
\]

of \( \mathbb{R}[V] \), for \( V \) an affine \( \mathbb{R} \)-variety. Here are some immediate observations:

**Lemma 1.3.** Let \( V \) be an affine \( \mathbb{R} \)-variety, and let \( S, S' \) be subsets of \( V(\mathbb{R}) \).

(a) \( B_V(\overline{S}) = B_V(S) \);
(b) \( B_V(S) = \mathbb{R}[V] \) if and only if \( \overline{S} \) is compact;
(c) \( B_V(S \cup S') = B_V(S) \cap B_V(S') \);
(d) if \( W \) is the (reduced) Zariski closure of \( S \) in \( V \) and \( I \) is the vanishing ideal of \( W \) in \( \mathbb{R}[V] \), then \( B_V(S) \) contains \( I \), and \( B_W(S) = B_V(S)/I \) as subrings of \( \mathbb{R}[V]/I = \mathbb{R}[W] \);
(e) the subring \( B_V(S) \) is relatively integrally closed in \( \mathbb{R}[V] \).

**Proof.** For (e), observe that a relation \( f^n + a_1 f^{n-1} + \cdots + a_n = 0 \) with \( f \in \mathbb{R}[V] \) and \( a_1, \ldots, a_n \in B_V(S) \) implies that \( |f| \leq 1 + \max_i |a_i| \) holds (pointwise) on \( V(\mathbb{R}) \), hence \( f \) is bounded on \( S \).

1.4. For the entire paper, our base field will be the field \( \mathbb{R} \) of real numbers. All results remain true when \( \mathbb{R} \) is replaced by an arbitrary real closed field \( R \) and bounded is replaced by bounded over \( R \), compact by semi-algebraically compact etc.

2. Fibres of bounded morphisms

Throughout this section we assume that \( V \) is an irreducible affine \( \mathbb{R} \)-variety and that \( S \) is a fixed semi-algebraic subset of \( V(\mathbb{R}) \).

2.1. Let \( \varphi : V \to W \) be a morphism of \( \mathbb{R} \)-varieties. Given \( y \in W(\mathbb{R}) \), we write \( S_y := \{ x \in S : \varphi(x) = y \} \) for the fibre of \( y \) in \( S \). Given \( f \in \mathbb{R}[V] \), we define

\[
\Omega_\varphi(f) := \{ y \in W(\mathbb{R}) : f \text{ is unbounded on } S_y \}.
\]

This is a semi-algebraic subset of \( W(\mathbb{R}) \) which is contained in \( \varphi(S) \).

**Proposition 2.2.** Let \( B = B_V(S) \). For \( f \in \mathbb{R}[V] \), the following are equivalent:

(i) \( f \in \text{Quot}(B) \);
(ii) there exists a dominant morphism \( \varphi : V \to W \) of affine \( \mathbb{R} \)-varieties which is bounded on \( S \) such that \( \Omega_\varphi(f) \) is not Zariski dense in \( W \).

**Proof.** When \( V \) is an affine \( \mathbb{R} \)-variety and \( f \in \mathbb{R}[V] \), we write \( Z(f) := \{ x \in V(\mathbb{R}) : f(x) = 0 \} \) for the zero set of \( f \) in \( V(\mathbb{R}) \). More generally, if \( M \subset V(\mathbb{R}) \) is a semi-algebraic subset and \( f : M \to \mathbb{R} \) is a (semi-algebraic) map, we denote the zero set of \( f \) in \( M \) by \( Z(f) \).

Assume \( f \in \text{Quot}(B) \), so there is \( 0 \neq h \in B \) with \( fh \in B \). Let \( C \) be any finitely generated subalgebra of \( B \) containing \( h \) and \( fh \), write \( W = \text{Spec}(C) \), and let \( \varphi : V \to W \) be the morphism induced by the inclusion \( C \subset \mathbb{R}[V] \). Then \( \varphi \) is
bounded on $S$. Let $y \in \Omega_{\varphi}(f)$, so $f$ is unbounded on $S_y = S \cap \varphi^{-1}(y)$. On the other hand, $h$ and $fh$ lie in $C = \mathbb{R}[W]$, so they are constant on $S_y$. Together, this implies $h(y) = 0$. So $\Omega_{\varphi}(f)$ is contained in $Z(h)$, and is therefore not Zariski dense in $W$.

Conversely assume that there is a dominant morphism $\varphi : V \to W$ as in (ii). Via $\varphi$, we consider $\mathbb{R}[W]$ as a subring of $B_V(S) \subset \mathbb{R}[V]$. Write $\Omega := \Omega_{\varphi}(f)$. We first define a map $\tilde{f} : \varphi(S) \to \mathbb{R}$ by

$$\tilde{f}(y) := \begin{cases} \left(1 + \sup\{|f(x)| : x \in S_y\}\right)^{-1} & \text{if } S_y \neq \emptyset, \\ 1 & \text{if } S_y = \emptyset, \end{cases}$$

($y \in \varphi(S)$), where we put $\frac{1}{\infty} := 0$. The map $\tilde{f}$ has semi-algebraic graph, and $\tilde{f}^{-1}(0) = \Omega$. Let $D$ be the set of points in $\varphi(S)$ where $\tilde{f}$ fails to be continuous. Then $D$ is not Zariski dense in $W$. This can, for instance, be easily deduced from a cylindrical algebraic decomposition of the graph of $\tilde{f}$ (see [3], Theorem 2.3.1). Hence $D \cup \Omega$ is not Zariski dense in $W$ either.

By Lemma 2.3 below, there exists a continuous map $\tilde{h} : \varphi(S) \to R$ with semi-algebraic graph such that $|\tilde{h}| \leq \tilde{f}$ on $\varphi(S)$, and such that $Z(\tilde{h}) \subset D \cup \Omega$. From the definition of $f$ it is clear that $|f(x) \tilde{h}(\varphi(x))| \leq |f(x) \tilde{f}(\varphi(x))| < 1$ for every $x \in S$.

Since $Z(\tilde{h})$ is not Zariski dense in $W$, there exists $h \neq 0$ in $\mathbb{R}[W]$ with $Z(h) \subset Z(h)$. Since $\varphi(S)$ is compact, the Łojasiewicz inequality (see [3], Corollary 2.6.7) implies that there are $0 < c \in \mathbb{R}$ and a positive integer $N$ such that $|h^N| \leq c \cdot |\tilde{h}|$ on $\varphi(S)$. We conclude

$$|f(x) h(\varphi(x))^N| \leq c \cdot |f(x) \tilde{h}(\varphi(x))| < c$$

for all $x \in S$. This shows that $fh^n$ is bounded on $S$, and so $f = (fh^n)/h^n$ lies in Quot($B$).

The following easy fact was used in the last proof:

**Lemma 2.3.** Let $M \subset \mathbb{R}^n$ be a semi-algebraic subset, and let $f : M \to \mathbb{R}$ be a map with semi-algebraic graph and with $f \geq 0$ on $M$. Let $D$ be the set of points in $M$ where $f$ fails to be continuous. Then there exists a continuous semi-algebraic function $g : M \to \mathbb{R}$ such that $|g| \leq f$ on $M$ and such that $Z(g) \subset Z(f) \cup \overline{D}$.

**Proof.** Upon replacing $f$ by $\min\{f, 1\}$ we may assume $f \leq 1$ on $M$. The distance function $d_D : \mathbb{R}^n \to \mathbb{R}$, $d_D(x) = \inf\{|x - y| : y \in D\}$ is continuous with semi-algebraic graph, and it vanishes precisely on $\overline{D}$. The function

$$g(x) := f(x) \cdot \frac{d_D(x)}{1+d_D(x)} \quad (x \in M)$$

has the desired properties. Indeed, $f$ is continuous outside of $\overline{D}$, and hence so is $g$. Fix $x \in M \cap \overline{D}$. Then $g(x) = 0$, and for all $y \in M$ we have $d_D(y) \leq |y - x|$. This implies

$$0 \leq g(y) \leq \frac{|y - x|}{1 + |y - x|}$$

for all $y \in M$. In particular, $g$ is continuous in $x$. □
2.4. We fix an affine irreducible \( \mathbb{R} \)-variety \( V \) and a semi-algebraic set \( S \subset V(\mathbb{R}) \).

Given a morphism \( \varphi : V \to W \) of \( \mathbb{R} \)-varieties, we put

\[
\Omega_\varphi := \bigcup_{f \in \mathbb{R}[V]} \Omega_\varphi(f).
\]

If \( \mathbb{R}[V] \) is generated by \( x_1, \ldots, x_n \) as an \( \mathbb{R} \)-algebra, then \( \Omega_\varphi = \bigcup_{i=1}^n \Omega_\varphi(x_i) \). This shows that \( \Omega_\varphi \) is a semi-algebraic subset of \( W(\mathbb{R}) \). Clearly,

\[
\Omega_\varphi = \{ y \in W(\mathbb{R}) : \overline{\varphi(y)} \text{ is not compact} \}.
\]

Given an \( \mathbb{R} \)-algebra \( B \), we define the transcendence degree \( \text{trdeg}_B(B) \) of \( B \) as the maximum number of elements of \( B \) which are algebraically independent over \( \mathbb{R} \).

**Theorem 2.5.** Let \( S \subset V(\mathbb{R}) \) be a semi-algebraic set, and let \( B = B_V(S) \). The following conditions are equivalent:

(i) \( \text{Quot}(B) = \mathbb{R}(V) \);
(ii) \( \text{trdeg}_B(B) = \dim(V) \);
(iii) there is a birational morphism \( V \to W \) of affine varieties which is bounded on \( S \);
(iv) there is a dominant morphism \( \varphi : V \to W \) of affine varieties which is bounded on \( S \) such that \( \Omega_\varphi \) is not Zariski dense in \( W \);
(v) there is a non-constant \( f \in \mathbb{R}[V] \), bounded on \( S \), such that the set \( f^{-1}(c) \cap \overline{S} \) is compact for every \( 0 \neq c \in \mathbb{R} \).

Note that condition (v) implies that there exists \( \varphi : V \to W \) as in (iv) with \( W = \mathbb{A}^1 \).

**Proof.** (i) \( \Rightarrow \) (iii): Assume \( \text{Quot}(B) = \mathbb{R}(V) \). Choose a finitely generated \( \mathbb{R} \)-subalgebra \( C \) of \( B \) with \( \text{Quot}(C) = \mathbb{R}(V) \), and put \( W = \text{Spec}(C) \). The morphism \( \varphi : V \to W \) induced by the inclusion \( C \subset \mathbb{R}[V] \) satisfies condition (iii).

(iii) \( \Rightarrow \) (ii): (iii) implies \( \mathbb{R}[W] \to B \), which implies (ii).

(ii) \( \Rightarrow \) (i): The field extension \( \text{Quot}(B) \subset \mathbb{R}(V) \) is algebraic, by (ii). Therefore, given \( f \in \mathbb{R}[V] \), there exists \( 0 \neq t \in B \) such that \( tf \) is integral over \( B \). Since \( B \) is integrally closed in \( \mathbb{R}[V] \) (Lemma 1.3(c)), this shows \( tf \in B \), and hence \( f \in \text{Quot}(B) \).

(iii) \( \Rightarrow \) (v): For \( \varphi : V \to W \) as in (iii), let \( D \) be a proper closed subvariety of \( W \) such that the restriction \( \varphi^{-1}(W \setminus D) \to W \setminus D \) of \( \varphi \) is an isomorphism. Choose a non-constant function \( g \in \mathbb{R}[W] \) which vanishes on \( D \) and on the boundary of \( \varphi(S) \) in \( W(\mathbb{R}) \). Let \( 0 \neq c \in \mathbb{R} \). Then \( g^{-1}(c) \cap \varphi(S) = g^{-1}(c) \cap \overline{\varphi(S)} \) by the choice of \( g \), and this set is compact since \( \overline{\varphi(S)} \) is compact. Moreover, \( g^{-1}(c) \cap \varphi(S) \) is contained in \( (W \setminus D)(\mathbb{R}) \), and so the preimage

\[
\varphi^{-1}(g^{-1}(c) \cap \varphi(S)) = (g \circ \varphi)^{-1}(c) \cap \varphi^{-1}(\varphi(S))
\]

is compact as well. In particular, \( (g \circ \varphi)^{-1}(c) \cap \overline{S} \) is compact, and so it suffices to take \( f := g \circ \varphi \).

(v) \( \Rightarrow \) (iv) is trivial (we can take \( W = \mathbb{A}^1 \) and \( \varphi := f \)).

(iv) \( \Rightarrow \) (i): Let \( \varphi \) be as in (iv). Given any \( f \in \mathbb{R}[V] \), the set \( \Omega_\varphi(f) \) is not Zariski dense in \( W \). So Proposition 2.2 shows \( f \in \text{Quot}(B) \). \( \square \)
Under suitable conditions, we know a priori that the \( \mathbb{R} \)-algebra \( B \) is finitely generated (see Theorem 5.12 below). In these cases we can formulate Proposition 2.2 and Theorem 2.5 more succinctly:

**Corollary 2.6.** Assume that the \( \mathbb{R} \)-algebra \( B = B_V(S) \) is finitely generated, put \( W = \text{Spec}(B) \), and let \( \varphi : V \to W \) be the canonical morphism.

(a) \( f \in \mathbb{R}[V] \) lies in \( \text{Quot}(B) \) if and only if \( \Omega_{\varphi}(f) \) is not Zariski dense in \( W \).

(b) \( \varphi \) is birational if and only if \( \dim(W) = \dim(V) \), if and only if the set \( \Omega_{\varphi} \) is not Zariski dense in \( W(\mathbb{R}) \).

**Remark 2.7.** Even if \( B = B_V(S) \) fails to be finitely generated, we can characterize \( \text{trdeg}_B \) as the maximum dimension of an affine \( \mathbb{R} \)-variety \( W \) for which there exists a dominant morphism \( V \to W \) which is bounded on \( S \).

**Examples 2.8.**

1. Let \( V = \mathbb{A}^2 \) and consider the set \( S = \{(x, y) : 0 \leq x(x^2 + y^2) \leq 1\} \) in \( \mathbb{R}^2 \). Then \( y \notin B(S) \), but \( x, xy \in B(S) \), so \( y \in \text{Quot}(B) \). In fact, \( B = \mathbb{R}[x, xy, xy^2] \). If \( \varphi : V \to W = \text{Spec}(B) \) denotes the canonical map, then \( \Omega_{\varphi} = \Omega_{\varphi}(y) \) consists only of the origin in \( W \).

2. Let \( f, g \in \mathbb{R}[x, y] \) be two algebraically independent polynomials, and let \( S = \{ p \in \mathbb{R}^2 : |f(p)| \leq 1, |f(p)g(p)| \leq 1 \} \). Clearly, \( f, fg \in B(S) \), hence \( \text{trdeg}(B(S)) = 2 \). From Theorem 2.5 it follows that there exists a non-constant polynomial \( h \in B(S) \) such that all fibres \( h^{-1}(c) \cap S \) are compact for \( c \neq 0 \). Depending on the choice of \( f \) and \( g \), it may not be a priori clear how to find such \( h \). For a concrete example, take \( f = x^2y \) and \( g = y^2 \). Here, neither \( f \) nor \( fg \) have any compact fibres, but (for example \( h = xy \)) will do.

3. The same example also shows another phenomenon. If \( \text{trdeg}(B(S)) = 2 \), then there exists a birational morphism \( \varphi : \mathbb{A}^2 \to W \) of affine varieties that is bounded on \( S \). In many instances, the map \( \mathbb{A}^2 \to \mathbb{A}^2, (x, y) \mapsto ((f(x), f(ygw(y)) \) will not have this property, implying in particular that \( B(S) \) is strictly larger than \( \mathbb{R}[f, fg] \).

Finding a birational morphism \( \varphi \) as above can be seen as a first step towards determining \( B(S) \). (For \( f \) and \( g \) as in the previous example, it is easy to see that \( B(S) = \mathbb{R}[xy, x^2y, x^2y^3] \), while the map \( (x, y) \mapsto (f(x), f(y)g(y)) \) has generically degree four.) There does not seem to be any general procedure that will produce such \( \varphi \); see however Examples 3.7.3.

**2.9.** Throughout this paper, we shall assume for the most part that varieties are irreducible. Here are a few remarks on the reducible case. Let \( V \) be an affine \( \mathbb{R} \)-variety with irreducible components \( V_1, \ldots, V_r \), let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \), and write \( S_i := S \cap V_i(\mathbb{R}) \). The relation between \( B_V(S) \) and the rings \( B_{V_i}(S_i) (i = 1, \ldots, r) \) depends largely on the way the components \( V_i \) of \( V \) meet. Clearly, the restriction \( \mathbb{R}[V] \to \mathbb{R}[V_i] \) maps \( B_V(S) \) to \( B_{V_i}(S_i) (i = 1, \ldots, r) \). But \( B_V(S) \to B_{V_i}(S_i) \) need not be surjective, as the following example shows. Let \( V \) be the plane affine curve \( x(x^2 + y^2 - 1) = 0 \), which is the union of a circle \( V_1 \) and a line \( V_2 \), and take \( S = V(\mathbb{R}) \). Then \( B_V(S) \) consists of those polynomials which are constant on the line, and so the restriction map \( B_V(S) \to B_{V_i}(S_i) \) to the circle does not contain \( y|_{V_1} \) in its image.

Nevertheless, in the case when \( V \) is a curve, the relation between \( B_V(S) \) and the \( B_{V_i}(S_i) \) is understood fairly well; see [15] for details. Things become considerably more difficult in dimension at least two. For instance, while \( B_V(S) = \mathbb{R} \) implies \( B_{V_i}(S_i) = \mathbb{R} \) for all \( i \) when \( \dim(V) = 1 \) ([15], Lemma 1.12(5)), this is false in higher
dimensions. For an example, let $V$ be the union of two planes in 3-space, and let $S$ be the union of the first plane $V_1(\mathbb{R})$ with a strip $[0, 1] \times \mathbb{R}$ in the second plane, where the strip is transversal to the line $V_1 \cap V_2$. Then $B_V(S) = \mathbb{R} \neq B_{V_2}(S_2)$.

3. Bounded polynomials and completions of varieties

3.1. Let $X$ be a normal $\mathbb{R}$-variety. A prime divisor on $X$ will be a closed irreducible subset of codimension one in $X$. By a divisor on $X$ we always mean a Weil divisor, that is, an element of the free abelian group generated by the prime divisors on $X$. Linear equivalence of divisors is denoted $\sim$. If $Z$ is a prime divisor, the discrete valuation of $\mathbb{R}(X)$ associated with $Z$ will be denoted by $v_Z$. Thus $v_Z(f)$ is the vanishing order of $f$ along $Z$, for $f \in \mathbb{R}(X)^*$. Recall that the prime divisor $Z$ is said to be real if its residue field $\mathbb{R}(Z)$ can be ordered (cf. [11]).

Given a rational map $f: V \dashrightarrow W$ between irreducible varieties, we denote by $\text{dom}(f)$ the largest open subset of $V$ on which $f$ is defined.

3.2. Let $X$ be an $\mathbb{R}$-variety. Sometimes it will be convenient to work in the real spectrum $X_r$ of $X$. When $X$ is affine, $X_r = \text{Sper} \mathbb{R}[X]$ is the space of all orderings of the ring $\mathbb{R}[X]$ (see [3] §7). When $X$ is not necessarily affine, the topological space $X_r$ is defined by gluing the real spectra $(U_i)_r$ $(i = 1, \ldots, r)$ of an open affine cover $X = U_1 \cup \cdots \cup U_r$ (see [17], 0.4 for more details). Thus a point $\alpha \in X_r$ corresponds to a pair $(x_\alpha, P_\alpha)$ where $x_\alpha$ is a (scheme-theoretic) point of $X$ and $P_\alpha$ is an ordering of the residue class field of $X$ in $x_\alpha$. The support of $\alpha$ is $\text{supp}(\alpha) = \{x_\alpha\}$. In particular, $X(\mathbb{R})$ is a topological subspace of $X_r$ in the obvious way. Similarly, when $X$ is irreducible, the space $\text{Sper} \mathbb{R}(X)$ of all orderings of the function field of $X$ is identified with the subspace $\{\alpha \in X_r: \text{supp}(\alpha) = X\}$ of $X_r$.

The topological space $X_r$ is spectral, and hence there is a well-defined notion of constructible subsets of $X_r$ (see [17], 0.4). For every semi-algebraic subset $S$ of $X(\mathbb{R})$, there exists a unique constructible subset $\tilde{S}$ of $X_r$ such that $S = \tilde{S} \cap X(\mathbb{R})$. If $X$ is affine, then $\tilde{S}$ is the subset of $X_r$ defined by the same system of inequalities as $S$. It is well known that $\tilde{S}$ is open (resp. closed) in $X_r$ if and only if the same is true of $S$ in $X(\mathbb{R})$. For two points $\alpha, \beta \in X_r$, we say that $\alpha$ specializes to $\beta$, denoted $\alpha \rightsquigarrow \beta$, if $\beta$ is contained in the closure of $\alpha$.

Recall that a valuation $v$ of a field $K$ is called compatible with an ordering $\leq$ of $K$ if $0 < b \leq a$, for $a, b \in K^*$, implies $v(b) \geq v(a)$. The usefulness of the real spectrum in our context comes from the following lemma:

**Lemma and Definition 3.3.** Let $X$ be a normal $\mathbb{R}$-variety, let $Z$ be a prime divisor on $X$, and let $S$ be a semi-algebraic subset of $X(\mathbb{R})$. The following conditions are equivalent:

(i) $Z(\mathbb{R}) \cap (S \cap U(\mathbb{R}))$ is Zariski dense in $Z$, where $U = X \setminus Z$;

(ii) there is a specialization $\alpha \rightsquigarrow \beta$ in $X_r$ with $\alpha \in \tilde{S}$, $\text{supp}(\beta) = Z$ and $\alpha \neq \beta$;

(iii) the discrete valuation $v_Z$ of $\mathbb{R}(X)$ is compatible with some ordering in $\tilde{S} \cap \text{Sper} \mathbb{R}(X)$.

If these conditions hold, we say that $Z$ and $S$ are compatible.

Note that every prime divisor which is compatible with $S$ has a real residue field.
Proof. Let \( T := \{ R \cap (S \cap U(R)) \} \), a closed semi-algebraic subset of \( \{ R \cap (S \cap U(R)) \} \). The set \( T \) is Zariski dense in \( S \) if and only if \( T \) contains a point with support \( Z \). The latter condition is equivalent to (ii), and (iii) is merely a reformulation of (ii). □

The following lemma is obvious (see [18], Lemma 0.2) and will be used frequently.

**Lemma 3.4.** Let \( V \) be a connected normal \( \mathbb{R} \)-variety, let \( S \subset V(\mathbb{R}) \) be a semi-algebraic set, and let \( Z \) be a prime divisor in \( V \) which is compatible with \( S \). Let \( f_1, \ldots, f_r \in \mathbb{R}(V) \) satisfy \( f_i \geq 0 \) on \( S \cap \text{dom}(f) \). Then
\[
v_Z\left(\sum_i f_i\right) = \min_i v_Z(f_i). \tag*{□}
\]

**Lemma 3.5.** Let \( V \) be a normal \( \mathbb{R} \)-variety, let \( S \subset V(\mathbb{R}) \) be a semi-algebraic set, and let \( Z \) be a prime divisor in \( V \) which is compatible with \( S \). If a rational function \( f \in \mathbb{R}(V)^* \) has a pole along \( Z \), then \( f \) is unbounded on \( S \cap \text{dom}(f) \).

**Proof.** The compatibility of \( Z \) with \( S \) means that there exists \( \alpha \in \bar{S} \cap \text{Sper} \mathbb{R}(V) \) which makes the discrete valuation ring \( \mathcal{O}_{V,Z} \) convex in \( \mathbb{R}(V) \). Assume that \( f \) is bounded on \( S' := S \cap \text{dom}(f) \). Since \( \bar{S}' \) has the same trace in \( \text{Sper} \mathbb{R}(V) \) as \( \bar{S} \), we have \( \alpha \in \bar{S}' \) as well. On the other hand, \( f \) bounded on \( S' \) implies that \( f \) lies in the \( \alpha \)-convex hull of \( \mathbb{R} \) in \( \mathbb{R}(V) \). In particular, \( f \in \mathcal{O}_{V,Z} \), which means that \( f \) does not have a pole along \( Z \). □

**Definition 3.6.** Let \( V \) be an irreducible \( \mathbb{R} \)-variety, and let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \). By a completion of \( V \) we mean an open dense embedding \( V \hookrightarrow X \) into a complete \( \mathbb{R} \)-variety. The completion \( X \) will be said to be compatible with \( S \) (or \( S \)-compatible) if for every irreducible component \( Z \) of \( X \setminus V \) the following conditions hold:

1. the local ring \( \mathcal{O}_{X,Z} \) is a discrete valuation ring;
2. the set \( Z(\mathbb{R}) \cap \bar{S} \) is either empty or Zariski dense in \( Z \).

(Here, of course, \( \bar{S} \) denotes the closure of \( S \) in \( X(\mathbb{R}) \).)

Note that (1) is saying that \( Z \) has codimension one in \( X \) and is not contained in the singular locus of \( X \). Condition (2) says (for normal \( X \)) that every irreducible component of \( X \setminus V \) is either compatible with \( S \) or disjoint from \( \bar{S} \) (cf. Lemma and Definition 3.3).

**Examples 3.7.**

1. Given a semi-algebraic subset \( S \) of \( \mathbb{R}^n \), the natural completion \( \mathbb{A}^n \subset \mathbb{P}^n \) of affine \( n \)-space is compatible with \( S \) if and only if \( S \) contains an open cone in \( \mathbb{R}^n \) (not necessarily centered at the origin).

2. The case of curves is simple: Given a (possibly singular) irreducible curve \( C \), there is a unique projective completion \( C \hookrightarrow X \) for which \( X_{\text{sing}} \subset C_{\text{sing}} \) (see [19], 4.6). The completion \( X \) is compatible with any semi-algebraic subset \( S \) of \( C(\mathbb{R}) \). The points of \( X \setminus C \) are called the points of \( C \) at infinity.

3. There are interesting classes of semi-algebraic sets \( S \) (in \( \mathbb{R}^n \), say) for which \( S \)-compatible completions (of \( V = \mathbb{A}^n \), in this case) can be constructed as toric varieties. For example, when \( S \) is defined by (finitely many) binomial inequalities \( ax^a < bx^b \), this always is the case. For such \( S \), the ring \( B(S) \) can be identified explicitly in terms of the defining inequalities, and \( B(S) \) is always finitely generated as an \( \mathbb{R} \)-algebra. We plan to expand on this remark in a future publication.
Our interest in compatible completions arises from the next result. It shows that such a completion calculates the ring of bounded polynomials.

**Theorem 3.8.** Let \( V \) be an affine normal \( \mathbb{R} \)-variety, let \( S \subset V(\mathbb{R}) \) be a semi-algebraic subset, and assume that the completion \( V \hookrightarrow X \) of \( V \) is compatible with \( S \). Let \( Y \) denote the union of those irreducible components \( Z \) of \( X \setminus V \) for which \( \overline{S} \cap Z(\mathbb{R}) = \emptyset \), and put \( U = X \setminus Y \). Then the inclusion \( V \subset U \) induces a ring isomorphism

\[
\mathcal{O}_X(U) \xrightarrow{\sim} B_V(S).
\]

**Proof.** Again, \( \overline{S} \) denotes the closure of \( S \) in \( X(\mathbb{R}) \). Since \( \overline{S} \) is compact and contained in \( U(\mathbb{R}) \), every element of \( \mathcal{O}_X(U) \) is bounded on \( S \). So the image of the restriction map \( \mathcal{O}_X(U) \to \mathbb{R}[V] \) is contained in \( B_V(S) \). It remains to show that every \( f \in B_V(S) \), considered as a rational function on \( U \), is regular on \( U \). Since \( U \) is normal, it suffices that \( v_Z(f) \geq 0 \) for every irreducible component \( Z \) of \( U \setminus V \). By the construction of \( U \), and since \( X \) is compatible with \( S \), the intersection \( \overline{S} \cap Z(\mathbb{R}) \) is Zariski dense in \( Z \), which means that the divisor \( Z \) is compatible with the set \( S \) (see Lemma and Definition 3.3). Since \( f \) is regular on \( V \) and bounded on \( S \), Lemma 3.3 implies that \( v_Z(f) \geq 0 \).

**3.9.** We illustrate the possible use of Theorem 3.8 by two examples. We use homogeneous coordinates \( (x_0 : x_1 : x_2) \) on \( \mathbb{P}^2 \), and we identify \( (x, y) \in \mathbb{A}^2 \) with \( (1 : x : y) \in \mathbb{P}^2 \). Let \( L = \{x_0 = 0\} \) be the line at infinity in \( \mathbb{P}^2 \).

Consider the strip \( S = \{(x, y): |x| \leq 1\} \) in \( \mathbb{R}^2 \). In \( \mathbb{P}^2(\mathbb{R}) \) we have \( \overline{S} \cap L(\mathbb{R}) = \{P\} \), where \( P = (0 : 0 : 1) \). To improve this, we consider the blowing-up \( \pi_1: X_1 \to \mathbb{P}^2 \) at \( P \). Then \( \mathbb{A}^2 \to X_1 \) is a completion of \( \mathbb{A}^2 \) for which \( X_1 \setminus \mathbb{A}^2 \) has two irreducible components, namely \( L' \) (the strict transform of \( L \)) and \( E_1 = \pi_1^{-1}(P) \). In \( X_1(\mathbb{R}) \) we have \( \overline{S} \cap L'(\mathbb{R}) = \emptyset \), and \( \overline{S} \cap E_1(\mathbb{R}) \) is Zariski dense in \( E_1 \). Therefore, the completion \( X_1 \) is compatible with \( S \), and by Theorem 3.8 we have \( B(S) = \mathcal{O}(U) \) for \( U := X_1 \setminus L' \). Clearly, \( \mathcal{O}(U) = \{f \in \mathbb{R}[x, y]: v_{E_1}(f) \geq 0\} \). Calculating the valuation \( v_{E_1} \), we find that

\[
v_{E_1} \left( \sum_{i,j} a_{ij}x^iy^j \right) = \min \{-j: a_{ij} \neq 0\}.
\]

So we get \( B(S) = \mathcal{O}(U) = \mathbb{R}[x] \).

**3.10.** Now, for another example, let \( T = \{(x, y) \in \mathbb{R}^2: |x| \leq 1, |xy| \leq 1\} \). As before we use the blowing-up \( \pi_1: X_1 \to \mathbb{P}^2 \) of the plane in \( P = (0 : 0 : 1) \). Again we have \( T \cap L'(\mathbb{R}) = \emptyset \) in \( X_1(\mathbb{R}) \), but this time \( T \cap E_1(\mathbb{R}) = \{P_1\} \) is a singleton. Therefore, we blow up \( X_1 \) in \( P_1 \) to get \( \pi_2: X_2 \to X_1 \), with exceptional fibre \( E_2 = \pi_2^{-1}(P_1) \). Then in \( X_2(\mathbb{R}) \) we find \( T \cap E_2'(\mathbb{R}) = \emptyset \), and \( T \cap E_2(\mathbb{R}) \) is Zariski dense in \( E_2 \). So \( X_2 \) is a completion of \( \mathbb{A}^2 \) which is compatible with \( T \), and \( B(T) = \mathcal{O}(W) \) for \( W := X_2 \setminus (L' \cup E_1') \), according to Theorem 3.8. We find \( \mathcal{O}(W) = \{f \in \mathbb{R}[x, y]: v_{E_2}(f) \geq 0\} \) and

\[
v_{E_2} \left( \sum_{i,j} a_{ij}x^iy^j \right) = \min \{i-j: a_{ij} \neq 0\},
\]

which shows \( B(T) = \mathcal{O}(W) = \mathbb{R}[x, xy] \).

In general, if we want to apply Theorem 3.8 to calculate \( B_V(S) \), we can start with some (normal) completion \( V \hookrightarrow X_0 \) of \( V \). By making suitable iterated blowing-ups
with centers over $X_0 \setminus V$, we try to “straighten out” the set $S$ at infinity more and more. When $V$ is a surface and the set $S$ is sufficiently regular at infinity, this procedure will always lead, after finitely many steps, to a completion of $V$ which is compatible with $S$; see Theorem 4.5 below.

4. Existence of compatible completions

In view of Theorem 3.8, we are now discussing the existence question for $S$-compatible completions. We start with the case $S = V(\mathbb{R})$.

**Theorem 4.1.** Every connected non-singular $\mathbb{R}$-variety $V$ has a completion $X$ which is compatible with the set $S = V(\mathbb{R})$ and such that the irreducible components of $X \setminus V$ are non-singular. If $V$ is quasi-projective, then $X$ can be chosen to be projective.

**Proof.** Start with any open dense embedding $V \hookrightarrow X$ into a complete $\mathbb{R}$-variety $X$ (which we can take projective if $V$ is quasi-projective). The singularities of $X$ are contained in $X \setminus V$, and by resolving them we get $X$ non-singular. Some irreducible components of $X \setminus V$ may have codimension $\geq 2$. This can be remedied by blowing up $X$ in these centers. So we can assume that every irreducible component of $X \setminus V$ has codimension one in $X$. Finally, by embedded resolution of singularities, we can achieve that the irreducible components of $X \setminus V$ are non-singular.

We claim that the completion $V \subset X$ is compatible with $V(\mathbb{R})$. To see this, fix an irreducible component $Z$ of $X \setminus V$ for which $Z(\mathbb{R}) \neq \emptyset$. Since $Z$ is non-singular, the function field $\mathbb{R}(Z)$ is real. By the Baer-Krull theorem, $\mathbb{R}(X)$ has an ordering $\alpha$ which is compatible with the discrete valuation $v_Z$. In particular, $Z$ is compatible with the set $V(\mathbb{R})$ (Lemma and Definition 3.3). $\square$

We now turn to compatible completions for more general semi-algebraic subsets $S$ of $V(\mathbb{R})$. We shall denote the interior of a set $M \subset V(\mathbb{R})$ by $\text{int}(M)$.

**Definition 4.2.** Let $V$ be an irreducible $\mathbb{R}$-variety, and let $S \subset V(\mathbb{R})$ be a semi-algebraic set.

(a) The set $S$ is said to be regular if $S \subset \text{int}(S \cap V_{\text{reg}}(\mathbb{R}))$.

(b) $S$ is called regular at infinity if $S = S_0 \cup S_1$, where $S_0, S_1$ are semi-algebraic sets with $\overline{S_0}$ compact and $S_1$ regular.

(c) $S$ is called Zariski dense at infinity if $S \cap (K \cap S)$ is Zariski dense in $V$ for every compact subset $K$ of $V(\mathbb{R})$.

**Remark 4.3.** If $S$ is regular at infinity and $\overline{S}$ is not compact, then $S$ is Zariski dense at infinity. Indeed, otherwise there would be a proper Zariski closed subset $Z$ of $V$ and a compact set $K \subset V(\mathbb{R})$ with $S \subset K \cup Z(\mathbb{R})$. This would imply $\text{int}(S \cap V_{\text{reg}}(\mathbb{R})) \subset K$, and by regularity at infinity we would get $\overline{S}$ compact, a contradiction.

We will need the following version of embedded resolution of singularities on a surface. By a curve on $X$ we mean a reduced effective divisor on $X$.

**Theorem 4.4.** Let $k$ be an infinite field, $X$ a normal quasi-projective surface over $k$, $C$ a curve in $X$, and $T$ a finite set of closed points in $C_{\text{sing}} \cap X_{\text{reg}}$. Then there
exists a birational morphism \( \varphi: \tilde{X} \to X \) of \( k \)-varieties with the following properties:

1. \( \varphi \) induces an isomorphism \( \tilde{X} \setminus \varphi^{-1}(T) \simeq X \setminus T \);
2. \( \tilde{X} \) is quasi-projective and \( \tilde{X}_{\text{sing}} = \varphi^{-1}(X_{\text{sing}}) \) (in particular, \( \tilde{X} \) is normal);
3. in all points of \( \varphi^{-1}(T) \), the divisor \( \varphi^{-1}(C) \) on \( \tilde{X} \) has normal crossings and non-singular components.

Proof. If \( X \) is non-singular and \( T = C_{\text{sing}} \), our statement becomes the usual one for embedded resolution of curves in surfaces (see for example [6], Thm. V.3.7, or [4], Sect. 3.5 for the case of an arbitrary infinite base field). This implies the above version, since \( T \) is contained in \( X_{\text{reg}} \). In more detail, let \( S = X_{\text{sing}} \cup (C_{\text{sing}} \setminus T) \) and put \( X_0 = X \setminus S \), a non-singular quasi-projective surface. Applying the usual embedded resolution of singularities to the divisor \( C \cap X_0 \) on \( X_0 \) and glueing it with the identity of \( X \setminus T \) yields the above version. \( \square \)

The following result proves the existence of compatible completions for surfaces, when the semi-algebraic set is regular at infinity (Definition 1.2):

**Theorem 4.5.** Let \( V \) be a connected normal quasi-projective surface over \( \mathbb{R} \), and let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \) that is regular at infinity. Then \( V \) has a projective completion which is compatible with \( S \). If \( V \) is non-singular, then the completion can be chosen to be non-singular as well.

Proof. We may assume that the function field \( \mathbb{R}(V) \) is real. For otherwise, \( \mathbb{R}(V) \) consists of finitely many singular points of \( V \). In that case, any normal completion of \( V \) will be compatible with any subset of \( \mathbb{R}(V) \).

We can also assume that \( S \) is closed in \( V(\mathbb{R}) \). Start with any completion \( V \hookrightarrow X_1 \) of \( V \). Since \( V \) is a normal surface, the singular set \( V_{\text{sing}} \) is finite. By resolving singularities of \( X_1 \setminus V_{\text{sing}} \) and gluing the resulting surface to \( V \), we can assume \( X_{1,\text{sing}} \subset V \). Let \( \overline{S} \) be the closure of \( S \) in \( X_1(\mathbb{R}) \), let \( \partial \overline{S} \) be its boundary in \( X_1(\mathbb{R}) \), and denote by \( C_1 \) the Zariski closure of \( \partial \overline{S} \) in \( X_1 \), a curve on \( X_1 \). Further write \( D_1 = X_1 \setminus \overline{S} \). Now apply the embedded resolution, as stated in Theorem 1.3 to the surface \( X_1 \), the curve \( C_1 \cup D_1 \) on \( X \), and the set \( T := (C_1 \cup D_1)_{\text{sing}} \cap D_1 \). We obtain a birational morphism \( \pi: X \to X_1 \) whose restriction \( \pi^{-1}(V) \to V \) is an isomorphism. Via \( \pi \) we can regard \( X \) as a completion of \( V \). Let \( D = X \setminus V = \pi^{-1}(D_1) \) and \( C = \pi^{-1}(C) \). The completion \( V \hookrightarrow X \) has the following properties:

1. \( X \) is projective and normal (non-singular if \( V \) is non-singular);
2. the irreducible components of the divisor \( C \cup D \) are non-singular in all points of \( D \), and if two components of \( C \cup D \) meet in a point of \( D \), then that point is a normal crossing singularity.

We show that \( V \hookrightarrow X \) is compatible with \( S \). Let \( Z \) be an irreducible component of \( D = X \setminus V \) for which the set \( \overline{S} \cap Z(\mathbb{R}) \) is non-empty. We have to show that this set is Zariski dense in \( Z \). Let \( x \in \overline{S} \cap Z(\mathbb{R}) \), and let \( u \in \mathcal{O}_{X,x} \) be a local equation for \( Z \). There can be at most one more irreducible component \( Z' \neq Z \) of \( C \cup D \) which passes through \( x \), by property (2). If such \( Z' \) exists, let \( v \in \mathcal{O}_{X,x} \) be a local equation for \( Z' \). Then \( u, v \) is a regular system of parameters for \( \mathcal{O}_{X,x} \). If no such \( Z' \) exists, we put \( Z' = Z \) and let \( v \in \mathcal{O}_{X,x} \) be any element for which \( u, v \) is a regular system of parameters for \( \mathcal{O}_{X,x} \).

Let \( U \) be an open neighbourhood of \( x \) in \( X(\mathbb{R}) \) such that \( U \cap (C \cup D)(\mathbb{R}) \subset (Z \cup Z')(\mathbb{R}) \). By shrinking \( U \), we can assume that \( u \) and \( v \) are regular functions on \( U \). Shrinking \( U \) further if necessary, there are precisely four connected components

...
of \{y \in U: (uv)(y) \neq 0\}. Since S is regular at infinity, it follows that \(\mathcal{S}\) contains at least one of these four components. In particular, \(\mathcal{S} \cap Z(\mathbb{R})\) contains a non-empty open subset of \(Z(\mathbb{R})\) and is therefore Zariski dense in \(Z\). This completes the proof. \(\square\)

Remarks 4.6.

1. We do not know whether Theorem 4.5 extends to \(\dim(V) > 2\). See however Theorem 4.11.

2. In Theorem 4.5, the regularity of \(S\) at infinity is not necessary in order for a compatible completion to exist. This is demonstrated by simple examples such as the following: Let \(V = k^2\) and consider the set \(S = \{(x, y) \in \mathbb{R}^2: xy^2 \geq 0\}\).

S is the union of the right half-plane with the \(x\)-axis. Clearly, \(\mathbb{R}^2\) is a completion of \(k^2\) which is compatible with \(S\), even though \(S\) is not regular at infinity. On the other hand, when \(S \subset V(\mathbb{R})\) is unbounded but contained in the union of a compact set and a proper subvariety of \(V\) (so \(S\) fails to be Zariski dense at infinity), there cannot be any completion of \(V\) compatible with \(S\) (see Proposition 3.7).

3. If \(C\) is a possibly singular (irreducible) affine curve, it is still true that \(C\) has a unique projective completion \(X\) which is non-singular in the added points (cf. also [19], Lemma 4.5). The completion \(X\) is compatible with every semi-algebraic subset \(S\) of \(C(\mathbb{R})\), and Theorem 3.8 holds in this case as well.

4. Let \(X\) be a non-singular connected projective surface over \(\mathbb{R}\), and let \(D\) be a curve on \(X\) with irreducible components \(Z_1, \ldots, Z_r\). Then it is sometimes possible to calculate the transcendence degree of the ring \(B = \mathcal{O}_X(X \setminus D)\) from the intersection matrix \(M_D = (Z_i \cdot Z_j)_{i,j=1,\ldots,r}\) of the divisor \(D\):

   1. If \(M_D\) is negative definite, then \(\text{trdeg}_B(B) = 0\), hence \(B = \mathbb{R}\).

   2. If \(M_D\) has a positive eigenvalue, then \(\text{trdeg}_B(B) = 2\).

If \(M_D\) is negative semi-definite and singular, there is no general statement about the transcendence degree of \(B\) (see [7], 8.3 for proofs).

For instance, consider \(T = \{(x, y) \in \mathbb{R}^2: |x| \leq 1, |xy| \leq 1\}\) as in [8]. There we found \(B(T) = \mathcal{O}(X_2 \setminus D)\) with \(D = L'' \cup E_1'\) (using the notation from [8]). The intersection matrix of \(D\) is

\[
M_D = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix},
\]

which has a positive eigenvalue. Thus we conclude \(\text{trdeg} B(T) = 2\) without actually calculating the ring \(B(T)\) as in [8]. In this example, the transcendence degree can also be read using Theorem 2.5 since the birational map \(k^2 \to k^2, (x, y) \mapsto (x, xy)\) is bounded on \(S\).

Let \(V\) be an affine normal \(\mathbb{R}\)-variety, and let \(S \subset V(\mathbb{R})\) be a semi-algebraic subset. We have seen that the ring \(B_V(S)\) is isomorphic to a ring \(\mathcal{O}_U(U)\) for some quasi-projective \(\mathbb{R}\)-variety \(U\), provided that \(V\) has a completion which is compatible with \(S\). We are now going to prove a converse (see Theorem 4.11 below). It roughly states, for every quasi-projective \(U\) which is real, that the ring \(\mathcal{O}_U(U)\) can be realized as \(B_V(S)\) for some open affine subset \(V\) of \(U\) and some semi-algebraic subset \(S\) of \(V(\mathbb{R})\).

Lemma 4.7. Let \(V\) be an irreducible affine \(\mathbb{R}\)-variety, and let \(X\) be a projective completion of \(V\) which is normal. Let \(Y_1, \ldots, Y_r\) be irreducible components of
X \setminus V which are real, and let Z_1, \ldots, Z_s be the remaining components (real or not) of X \setminus V. There exists a regular and basic closed semi-algebraic subset S of V(\mathbb{R}) that is compatible with Y_1, \ldots, Y_r and whose closure in X(\mathbb{R}) is disjoint from Z_1(\mathbb{R}), \ldots, Z_s(\mathbb{R}). In particular, the completion X of V is compatible with S.

**Proof.** Fix an index \( i \in \{1, \ldots, r\} \). Since X is normal and \( Y_i \) is real, we find a point \( p_i \in Y_i(\mathbb{R}) \) which is non-singular on X and on \( X \setminus V \). By this last fact, \( p_i \) has a generalisation \( \beta_i \) in Sper \( \mathbb{R}(Y_i) \). Since X is normal, \( \beta_i \) has a generalization \( \alpha_i \) in Sper \( \mathbb{R}(X) \). Every semi-algebraic subset \( S \) of \( X(\mathbb{R}) \) with \( \alpha_i \in S \) is compatible with \( Y_i \). There is an open affine subset \( U \) of \( X \) such that \( U(\mathbb{R}) = X(\mathbb{R}) \). Let \( x_1, \ldots, x_n \) be a system of generators for the \( \mathbb{R} \)-algebra \( \mathbb{R}[U] \). If \( c_i \in \mathbb{R} \) is positive and sufficiently small, the closed semi-algebraic set

\[ S_i := \left\{ p \in V(\mathbb{R}) : \sum_{k=1}^{n} (x_k(p) - x_k(p_i))^2 \leq c_i \right\} \]

will be contained in \( V_{reg}(\mathbb{R}) \) and regular, and moreover \( S_i \) will be compatible with \( Y_i \) and disjoint from \( Z_1(\mathbb{R}) \cup \cdots \cup Z_s(\mathbb{R}) \). This being done for \( i = 1, \ldots, r \), we may further assume that \( S_1, \ldots, S_r \) are pairwise disjoint by making \( c_1, \ldots, c_r \) even smaller if necessary.

We claim that \( S := S_1 \cup \cdots \cup S_r \) is a basic closed semi-algebraic subset of \( V(\mathbb{R}) \), which will complete the proof. For \( i = 1, \ldots, r \) put

\[ g_i := c_i - \sum_{k=1}^{n} (x_k - x_k(p_i))^2 \in \mathbb{R}[U], \]

and let \( V_0 \) be the maximal Zariski open subset of \( V \) for which \( g_i|_{V_0} \in \mathcal{O}_V(V_0) \) for \( i = 1, \ldots, r \). Since \( V \) is normal, the closed subset \( Z = V \setminus V_0 \) has pure codimension one in \( V \). Clearly, \( Z(\mathbb{R}) = \emptyset \) since \( U(\mathbb{R}) = X(\mathbb{R}) \) and the \( g_i \) are regular on \( U \). Choose \( h_1, \ldots, h_k \in \mathbb{R}[V] \) which generate the vanishing ideal of \( Z \) in \( \mathbb{R}[V] \), and put \( h := h_1^2 + \cdots + h_k^2 \). Then \( h \) has no real zeros on \( V \), and since \( V \) is normal, there exists \( N \geq 1 \) such that \( f_i := h^N g_i \in \mathbb{R}[V] \) for \( i = 1, \ldots, r \). Thus \( S_i = \{ p \in V(\mathbb{R}) : f_i(p) \geq 0 \} \) for every \( i \), and hence

\[ S_1 \cup \cdots \cup S_r = \{ p \in V(\mathbb{R}) : (-1)^{r+1} f_1(p) \cdots f_r(p) \geq 0 \}, \]

since the \( S_i \) are pairwise disjoint. \( \Box \)

An effective Weil divisor \( D \) on a normal \( \mathbb{R} \)-variety \( X \) will be called **totally real** if every irreducible component of \( D \) is a real variety. By \( \text{Cl}(X) \) we denote the class group of Weil divisors modulo linear equivalence on \( X \). We will use the following theorem:

**Theorem 4.8** (Roggero [16]). Let \( V \) be a normal affine real \( \mathbb{R} \)-variety of dimension at least 2. Then every divisor on \( V \) is linearly equivalent to a totally real effective divisor on \( V \). \( \Box \)

**Corollary 4.9.** Let \( X \) be a real normal projective \( \mathbb{R} \)-variety of dimension at least 2, and let \( H \) be an ample effective divisor on \( X \). For every divisor \( D \) on \( X \) there exists a totally real effective divisor \( D_0 \) on \( X \) such that \( D \sim D_0 + H_0 \), where \( H_0 \) is a linear combination of irreducible components of \( H \).
Lemma 4.10. Let $U$ be a connected normal quasi-projective $\mathbb{R}$-variety which is real. There exist real prime divisors $Y_1, \ldots, Y_r$ on $U$ such that the variety $U \setminus (Y_1 \cup \cdots \cup Y_r)$ is affine.

Proof. If $U$ is a curve, we can take $r = 1$ and $Y_1$ any real point on $U$. So we may assume $\dim(U) \geq 2$. Assume furthermore that $U$ is projective. Fix a non-singular point $p \in U(\mathbb{R})$ and an ample divisor $H$ on $U$, and let $L \subset |H|$ consist of those $D \in |H|$ that pass through $p$. The linear system $L$ has no fixed components, so by Bertini’s theorem (see for example Jouanolou [8], Corollary 6.11), the set of all $Z \in L$ for which $Z$ is irreducible and satisfies $Z_{\text{sing}} \subset U_{\text{sing}}$ has non-empty interior inside $L$. In particular, it is non-empty. Any such $Z$ is real since it contains $p$ as a regular real point. Moreover $U \setminus Z$ is affine since $Z$ is ample.

When $U$ is only quasi-projective, fix an open dense embedding $U \hookrightarrow X$ with $X$ normal and projective and such that $X \setminus U$ has pure codimension one in $X$ (the latter can be achieved by blowing up if necessary). Let $E$ be the divisor $X \setminus U$. By the projective case above we find an ample prime divisor $H$ on $X$ that is totally real. By Corollary 4.9 there exist $n \in \mathbb{Z}$ and a totally real effective divisor $D$ on $X$ such that $-E \sim D + nH$. For $m > \max\{0, n\}$ the divisor $E + D + mH$ is effective and ample. It follows that the irreducible components $Y_1, \ldots, Y_r$ of $(D + mH) \cap U$ are real and $U \setminus (Y_1 \cup \cdots \cup Y_r)$ is affine, as desired. \qed

Theorem 4.11. Let $U$ be a connected normal quasi-projective $\mathbb{R}$-variety which is real. There exists an open affine subset $V$ of $U$ and a regular and basic closed subset $S$ of $V(\mathbb{R})$ such that $\mathcal{O}(U) \cong \mathcal{O}(V(S))$.

Proof. Let $U \hookrightarrow X$ be a normal projective completion of $U$, and let $Z_1, \ldots, Z_s$ be the irreducible components of $X \setminus U$. By Lemma 4.10 there exist further prime divisors $Y_1, \ldots, Y_r$ on $X$ which are real and such that the open subset $V := X \setminus (\bigcup_j Y_j \cup \bigcup_j Z_j)$ of $U$ is affine. By Lemma 4.7 we find a regular and basic closed semi-algebraic set $S$ in $V(\mathbb{R})$ which is compatible with $Y_1, \ldots, Y_r$ and which satisfies $S \cap Z_j(\mathbb{R}) = \emptyset$ for $j = 1, \ldots, s$. In particular, the completion $X$ of $V$ is compatible with $S$. Therefore $\mathcal{O}(U) \cong \mathcal{O}(V(S))$ by Theorem 3.5. \qed

Under certain conditions we know that the $\mathbb{R}$-algebra $B_V(S)$ is finitely generated; see, e.g., Theorem 3.12 below. Writing $W := \text{Spec} B_V(S)$ for the affine $\mathbb{R}$-variety defined by this ring, the canonical morphism $\varphi_S : V \to W$ is universal for morphisms that are bounded on $S$ from $V$ to affine varieties, i.e., every morphism $V \to W'$ into an affine $\mathbb{R}$-variety $W'$ which is bounded on $S$ factors uniquely through $\varphi_S$. Note that the variety $W$ is normal, provided $V$ is normal (Lemma 3.3(e)). We are wondering how to characterize all morphisms $V \to W$ of normal affine $\mathbb{R}$-varieties which are of the form $\varphi_S : V \to \text{Spec} B_V(S)$ for some semi-algebraic subset $S$ of $V(\mathbb{R})$ (with $B_V(S)$ finitely generated).

Although we do not know a general answer to this question, we can present the following sufficient condition for birational morphisms:

Proposition 4.12. Let $\varphi : V \to W$ be a birational morphism of connected normal real affine $\mathbb{R}$-varieties. Assume there is a totally real effective divisor $Z$ on $W$ such
that \( \varphi \) restricts to an isomorphism \( \varphi^{-1}(W \setminus Z) \cong W \setminus Z \). Then there exists a semi-algebraic set \( S \subset V(\mathbb{R}) \) such that \( \varphi^*\mathbb{R}[W] = B_V(S) \).

For the proof we need the following lemma:

**Lemma 4.13.** Let \( V \) be an affine normal \( \mathbb{R} \)-variety, and let \( Z \) be a given effective divisor on \( V \) which is totally real. There exists a compact semi-algebraic set \( S \) in \( V(\mathbb{R}) \) which is compatible with every irreducible component of \( Z \), and such that \( S \setminus (S \cap Z(\mathbb{R})) \) is dense in \( S \).

**Proof.** Let \( Z_1, \ldots, Z_r \) be the irreducible components of \( Z \), and fix \( i \in \{1, \ldots, r\} \). Since \( Z_i \) is real and not contained in \( V_{\text{sing}} \), we find a point \( p_i \in Z_i(\mathbb{R}) \) which is non-singular on \( Z_i \) and on \( V \). So \( p_i \) has a generalization \( \beta_i \) in \( \text{Sper} \mathbb{R}(Z_i) \), and in turn, \( \beta_i \) has a generalization \( \alpha_i \) in \( \text{Sper} \mathbb{R}(V) \). Every semi-algebraic subset \( S \) of \( V(\mathbb{R}) \) with \( \alpha_i \in S \) is compatible with \( Z_i \). If \( x_1, \ldots, x_n \) is a system of generators of the \( \mathbb{R} \)-algebra \( \mathbb{R}[V] \), we may therefore define \( S \) to be the closure of the set \( \{p \in V(\mathbb{R}) : p \notin Z(\mathbb{R}), x_1(p)^2 + \cdots + x_n(p)^2 < c\} \), where \( c \in \mathbb{R} \) is positive and sufficiently large. \( \square \)

**Proof of Proposition 4.12** By Lemma 4.13 there exists a compact basic closed semi-algebraic subset \( T \) of \( W(\mathbb{R}) \) which is compatible with every irreducible component of the divisor \( Z \), and such that \( T \setminus (T \cap Z(\mathbb{R})) \) is dense in \( T \). Let \( S := \varphi^{-1}(T) \subset V(\mathbb{R}) \). Since \( \varphi(S) \) contains \( T \setminus (T \cap Z(\mathbb{R})) \) by the hypothesis, we have \( \varphi(S) = T \). By compactness of \( T \) it is clear that \( \varphi^*(\mathbb{R}[W]) \subset B_V(S) \). We claim that this inclusion is an equality. Let \( f \in \mathbb{R}[V] \) with \( f \notin \varphi^*(\mathbb{R}[W]) \). Considering \( f \) as a rational function on \( W \), it follows that \( f \) has a pole along one of the irreducible components of \( Z \), since \( W \) is normal. Since \( T \) is compatible with that component, it follows from Lemma 3.5 that \( f \) is unbounded on \( T \cap \text{dom}_W(f) \). In particular, \( f \) is unbounded on \( T \setminus (T \cap Z(\mathbb{R})) \), and so \( f \) is unbounded on \( S \). \( \square \)

5. Finite generation of the ring of bounded polynomials

We first recall that \( B_C(S) \) is always finitely generated when \( C \) is a curve:

**Proposition 5.1.** Let \( C \) be an irreducible affine curve over \( \mathbb{R} \), possibly singular, and let \( S \subset C(\mathbb{R}) \) be a semi-algebraic subset. Assume that \( C \) is real.

(a) \( B_C(S) \) is finitely generated as an \( \mathbb{R} \)-algebra.

(b) If every point of \( C \) at infinity is real and lies in the closure of \( S \), then \( B_C(S) = \mathbb{R} \).

(c) Otherwise, \( B_C(S) \) has transcendence degree one over \( \mathbb{R} \).

**Proof.** Let \( C \to X \) be the completion of \( C \) for which \( X_{\text{sing}} \subset C_{\text{sing}} \) (Example 5.7.2), and let \( C' \subset X \) be the open subset which is the union of \( C \) and all \( \mathbb{R} \)-points of \( X \setminus C \) which lie in the closure of \( S \). Then \( \mathcal{O}_X(C') \to B_C(S) \). Indeed, even though Theorem 5.8 is not directly applicable here when \( C \) is singular, an inspection of the proof shows that it applies nevertheless. When \( C' \neq X \), then the curve \( C' \) is affine, and so \( B_C(S) \) is a finitely generated \( \mathbb{R} \)-algebra of transcendence degree one. When \( C' = X \) (that is the hypothesis of (b) is satisfied), then \( B_C(S) = \mathbb{R} \). \( \square \)

Before proceeding to our main result on finite generation, we study some consequences for the ring \( B_V(S) \) that arise from the existence of an \( S \)-compatible completion.
Lemma and Definition 5.2. Let \( V \) be an irreducible \( \mathbb{R} \)-variety, and let \( S \) be a semi-algebraic subset of \( V(\mathbb{R}) \). There is a unique smallest Zariski closed subset \( Z \) of \( V \) such that \( S \subset Z(\mathbb{R}) \cup K \) for some compact subset \( K \) of \( V(\mathbb{R}) \). We shall call \( Z \) the Zariski closure of \( S \) at infinity.

Proof. If Zariski closed subsets \( Z_1, Z_2 \) of \( V \) and compact subsets \( K_1, K_2 \) of \( V(\mathbb{R}) \) are given with \( S \subset Z_i(\mathbb{R}) \cup K_i \) \((i = 1, 2)\), then \( S \) is contained in \((Z_1(\mathbb{R}) \cap Z_2(\mathbb{R})) \cup (K_1 \cup K_2)\). This implies the assertion. \( \square \)

Clearly, \( S \) is Zariski dense at infinity (Definition 3.2) if and only if its Zariski closure at infinity is \( V \).

Proposition 5.3. Let \( V \) be an irreducible affine \( \mathbb{R} \)-variety, and let \( S \subset V(\mathbb{R}) \) be a semi-algebraic set. Let \( Z \) be the Zariski closure of \( S \) at infinity, and let \( I_Z \) be the full vanishing ideal of \( Z \) in \( \mathbb{R}[V] \). Then \( I_Z \) is equal to the conductor of \( B_V(S) \) in \( \mathbb{R}[V] \), that is, \( I_Z \) is the largest ideal of \( \mathbb{R}[V] \) which is contained in \( B_V(S) \).

Proof. Since \( S \subset Z(\mathbb{R}) \cup K \) for some compact set \( K \), it is clear that \( I_Z \subset B_V(S) \). Conversely, let \( b \in B_V(S) \) with \( b \notin I_Z \). The subset \( \{ t \in S \colon b(t) \neq 0 \} \) of \( V(\mathbb{R}) \) is unbounded. For if its closure \( K \) were compact, we would have \( S \subset K \cup \{ b = 0 \} \), contradicting the definition of \( Z \). Hence there exists a semi-algebraic curve \( \gamma \colon [0, \infty) \to S \), \( t \mapsto \gamma_t \) such that \( b(\gamma_t) \neq 0 \) for all \( t \geq 0 \) and such that \( \Gamma := \{ \gamma_t \colon t \geq 0 \} \) is unbounded in \( V(\mathbb{R}) \). This last condition means that there exists \( f \in \mathbb{R}[V] \) for which \( f(\gamma_t), t \geq 0 \), is unbounded. Since \( b \neq 0 \) on \( \Gamma \), it follows from Lemma 5.4 below that the product \( f^n b \) is unbounded on \( \Gamma \) for sufficiently large \( n \geq 1 \). So \( bf^n \notin B_V(S) \). \( \square \)

The following well-known fact was used in the last proof:

Lemma 5.4. Let \( f, g \colon [0, \infty) \to \mathbb{R} \) be two continuous semi-algebraic functions such that \( f \) is unbounded and \( g > 0 \) everywhere. Then \( f^n g \) is unbounded for sufficiently large \( n \geq 1 \). \( \square \)

Lemma 5.5. Let \( B \subset A \) be a ring extension, and let \( J \) be an ideal of \( A \) which is contained in \( B \). If \( J \) contains an element which is not a zero divisor of \( A \) and if \( J \) is finitely generated as an ideal of \( B \), then the extension \( B \subset A \) is integral.

Proof. This is a special case of [5, Corollary 4.6]. \( \square \)

Lemma 5.6. Let \( Y \) be a connected normal \( \mathbb{R} \)-variety, and let \( V \subset Y \) be an affine open subset. If \( V \neq Y \), then \( \mathcal{O}_Y(Y) \) contains no non-zero ideal of \( \mathbb{R}[V] \).

Proof. The inclusion \( \mathcal{O}(Y) \subset \mathbb{R}[V] \) is proper. Indeed, otherwise there would be a morphism \( Y \to V \) which is the identity on \( V \), which is absurd if \( V \neq Y \). Assume \( b \in \mathcal{O}(Y) \) is such that \( bf \in \mathcal{O}(Y) \) for every \( f \in \mathbb{R}[V] \). Choose \( f \in \mathbb{R}[V] \) with \( f \notin \mathcal{O}(Y) \). Since \( Y \) is normal this means that there exists a prime divisor \( Z \) of \( Y \) with \( v_Z(f) \leq -1 \). Unless \( b = 0 \) we therefore get \( v_Z(bf^n) \leq -1 \) for large \( n > 0 \), and so \( bf^n \notin \mathcal{O}(Y) \). This proves the assertion. \( \square \)

Proposition 5.7. Let \( V \) be a connected normal affine \( \mathbb{R} \)-variety, and let \( S \) be an unbounded semi-algebraic subset of \( V(\mathbb{R}) \). Consider the following conditions:

(i) \( V \) has a completion which is compatible with \( S \);
(ii) the ring \( B_V(S) \) is Noetherian;
(iii) the conductor of $B_V(S)$ in $\mathbb{R}[V]$ is zero;
(iv) $S$ is Zariski dense in $V$ at infinity.

We have (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv).

**Proof.** (i) $\Rightarrow$ (iii): Let $V \hookrightarrow X$ be a completion which is compatible with $S$, let $Y$ be the union of the irreducible components $Z$ of $X \setminus Y$ with $Z(\mathbb{R}) = \emptyset$, and let $U = \mathcal{X} \setminus Y$. By Theorem 3.8, the inclusion $V \hookrightarrow U$ induces an isomorphism $\mathcal{O}(U) \to B_V(S)$. Since $S$ is unbounded we have $U \neq V$. By Lemma 5.6, $B_V(S)$ contains no non-zero ideal of $\mathbb{R}[V]$.  

(ii) $\Rightarrow$ (iii): We have $B \neq \mathbb{R}[V]$ since $S$ is unbounded. Let $J$ be an ideal of $\mathbb{R}[V]$ which is contained in $B_V(S)$. Since $B$ is Noetherian, Lemma 5.5 implies $J = (0)$. The equivalence of (iii) and (iv) follows from Proposition 5.3. □

**Corollary 5.8.** Let $V$ be a connected normal affine $\mathbb{R}$-variety, and let $S$ be an unbounded semi-algebraic subset of $V(\mathbb{R})$. If $S$ is not Zariski dense at infinity, then $B_V(S)$ is not Noetherian.

**Proof.** The vanishing ideal of the Zariski closure of $S$ at infinity is a proper non-zero ideal of $\mathbb{R}[V]$ that is contained in $B_V(S)$. Hence the conductor of $B_V(S)$ in $\mathbb{R}[V]$ is non-zero, and so $B_V(S)$ is not Noetherian by Proposition 5.7. □

**Remark 5.9.** Condition (i) in Proposition 5.7 implies $B_V(S) \cong \mathcal{O}_V(U)$ for some $\mathbb{R}$-variety $U$ (Theorem 3.8). In general, the ring $\mathcal{O}_V(U)$, and hence also $B_V(S)$, may fail to be Noetherian, see Subsection 5.13 below. Therefore, (i) does not imply (ii) in Proposition 5.7.

The implication (ii) $\Rightarrow$ (iv) in Proposition 5.7 can be strengthened as follows:

**Proposition 5.10.** Let $V$ be a normal affine $\mathbb{R}$-variety, and let $S$ be a semi-algebraic subset of $V(\mathbb{R})$ for which the ring $B_V(S)$ is Noetherian. Then for every semi-algebraic subset $T \subset S$ with $B_V(S) \neq B_V(T)$, the inequality

$$\text{trdeg}_R B_V(T) \leq \dim(S \setminus T)$$

holds.

Assume that $S$ is unbounded and (for simplicity) closed, and that $B_V(S)$ is Noetherian. Then conclusion (5.1) implies for any compact subset $T$ of $S$ that $S \setminus T$ is Zariski dense in $V(\mathbb{R})$. This means that $S$ is Zariski dense at infinity, and it shows that the implication (ii) $\Rightarrow$ (iv) of Proposition 5.7 is contained in Proposition 5.10.

**Proof.** Let $Z$ be the Zariski closure of $S \setminus T$ in $V$, and let $I = I_Z$ be the vanishing ideal of $Z$. Then $B(T)/I \cap B(T) \subset \mathbb{R}[V]/I$, and so

$$\text{trdeg}_R B(T)/I \cap B(T) \leq \text{trdeg}_R \mathbb{R}[V]/I = \dim Z = \dim(S \setminus T).$$

If (5.1) were false, we would conclude that $I \cap B(T) \neq (0)$. Now the ideal $I \cap B(T)$ of $B(T)$ is contained in $B(S)$. This contradicts Lemma 5.5 since $B(S)$ is integrally closed in $B(T)$ (even in $\mathbb{R}[V]$). □

We now consider algebraic surfaces. Theorem 5.12 below is one of our main results. It is based on the results of Sections 3 and 4 and on the following theorem.
Theorem 5.11 (Zariski). Let $W$ be a normal quasi-projective surface over a field $k$ of characteristic zero. Then the ring $\mathcal{O}_W(W)$ is finitely generated as a $k$-algebra.

See Zariski [22], remarks on Theorem 1 at the end of the article. An alternative proof in modern language was given in [9], based on ideas of Nagata.

Theorem 5.12. Let $V$ be a connected normal affine surface over $\mathbb{R}$. For every semi-algebraic subset $S$ of $V(\mathbb{R})$ which is regular at infinity, the $\mathbb{R}$-algebra $B_V(S)$ is finitely generated.

Proof. By Theorem 4.5, $V$ has a projective completion $V \hookrightarrow X$ which is compatible with $S$. If $Y$ is the union of all irreducible components of $X \setminus V$ without a real point, then $\mathcal{O}_X(X \setminus Y) \twoheadrightarrow B_V(S)$ by Theorem 5.11. This implies the assertion, since $\mathcal{O}_X(X \setminus Y)$ is finitely generated by Zariski's theorem, Theorem 5.11. □

5.13. It is folklore knowledge that Zariski’s theorem, Theorem 5.11, ceases to be true in dimensions larger than two. However, it seems not so easy to locate a reference for this fact in the published literature.

A construction of a quasi-projective non-singular threefold whose ring of global sections is not finitely generated was given by Vakil [21]: Take an elliptic curve $E$ over a field $k$, and let $L, L'$ be invertible sheaves on $E$ such that $\deg(L') > 0$, $\deg(L) = 0$ and $L$ is not torsion in $\text{Pic}(E)$. Let $W$ be the total space of the vector bundle $L \oplus L'$ on $E$. Then the ring $\mathcal{O}_W(W)$ is not Noetherian. By a variation of this construction, one can also obtain another example which is even quasi-affine (see [21]).

Using this construction for $k = \mathbb{R}$ in a case where the elliptic curve $E$ is real, we can apply Theorem 4.11 and conclude:

Corollary 5.14. There exists a non-singular affine $\mathbb{R}$-variety $V$ of dimension three and a regular and basic closed semi-algebraic subset $S$ of $V(\mathbb{R})$ for which the ring $B_V(S)$ is not Noetherian. □

5.15. From results of Kuroda [10], it is possible to deduce the existence of a rational, normal, quasi-affine threefold whose ring of regular functions is not finitely generated. This was pointed out to us by Sebastian Krug, thereby answering a question raised in an earlier version of this paper.

5.16. By Zariski’s theorem, an example as in 5.13 cannot exist in dimension two. However, there are examples of (non-normal) irreducible quasi-projective surfaces $W$ for which $\mathcal{O}_W(W)$ is not finitely generated. One such construction of a quasi-affine example is due to Nagata ([13], [14], see also [9] for a detailed account).

The construction of two-dimensional varieties whose ring of global sections is not finitely generated becomes much easier when we allow the variety to be reducible. Here is an example (again see [9] for details): In affine 3-space with coordinates $(x, y, z)$, let $U := V(xy)$ (the transversal union of two planes) and $L := V(y, z)$ (a line in one of the planes which intersects the other plane transversely). Then $W := U \setminus L$ is a reducible quasi-affine variety of dimension two for which the ring $\mathcal{O}_W(W)$ is not Noetherian.

To interpret this example in terms of bounded functions, let

$$V = U \setminus (U \cap V(z)) = W \setminus V(x, z),$$
an open affine subvariety of $W$, and write $L' = V(x, z)$. Let $S_0$ be a regular, compact and basic closed semi-algebraic subset of $U(\mathbb{R})$ which is disjoint from $L$ and compatible with $L'$. For example, $S_0 = \{(0, y, z) : (y - 2)^2 + z^2 \leq 1\}$ will do. Then $S = S_0 \cap V(\mathbb{R})$ is basic closed in $V(\mathbb{R})$, and $B_V(S) = \mathcal{O}_W(W)$ is not Noetherian.

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