

ON COXETER DIAGRAMS OF COMPLEX REFLECTION GROUPS

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ABSTRACT. We study Coxeter diagrams of some unitary reflection groups. Using solely the combinatorics of diagrams, we give a new proof of the classification of root lattices defined over $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$; there are only four such lattices, namely, the \mathcal{E} -lattices whose real forms are A_2 , D_4 , E_6 and E_8 . Next, we address the issue of characterizing the diagrams for unitary reflection groups, a question that was raised by Broué, Malle and Rouquier. To this end, we describe an algorithm which, given a unitary reflection group G , picks out a set of complex reflections. The algorithm is based on an analogy with Weyl groups. If G is a Weyl group, the algorithm immediately yields a set of simple roots. Experimentally, we observe that if G is primitive and G has a set of roots whose \mathbb{Z} -span is a discrete subset of the ambient vector space, then the algorithm selects a minimal generating set for G . The group G has a presentation on these generators such that if we forget that the generators have finite order, then we get a (Coxeter-like) presentation of the corresponding braid group. For some groups, such as G_{33} and G_{34} , new diagrams are obtained. For G_{34} , our new diagram extends to an “affine diagram” with $\mathbb{Z}/7\mathbb{Z}$ symmetry.

1. INTRODUCTION

1.1. Background on unitary reflection groups. Let G be a finite subgroup of the unitary group $U(n)$ generated by complex reflections, such that G acts irreducibly on \mathbb{C}^n . We shall simply say that G is a unitary reflection group. Let \mathcal{M} be the union of the fixed point sets of the complex reflections in G , and let $X_G = \mathbb{C}^n \setminus \mathcal{M}$. The fundamental group $\text{Braid}(G) := \pi_1(X_G/G)$ is called the (*generalized*) *braid group associated to G* . Let k be the minimum number of complex reflections needed to generate G . We say that G is *well generated* if $k = n$. The smallest subfield F of \mathbb{C} that contains all the complex character values of G is called the *field of definition* of G . We say that G is defined over F .

Unitary reflection groups were classified by Shephard and Todd in [13]. For a self-contained proof of the classification, which is similar in spirit to part of our work, see [7]. A convenient table of all these groups and their properties may be found in [6]. There is an infinite series, denoted by $G(de, e, r)$, and 34 others, denoted by G_4, G_5, \dots, G_{37} . Unitary reflection groups have many invariant theoretic properties that are similar to those of the orthogonal reflection groups. Most of these properties were initially established for the unitary reflection groups, via case-by-case verification through Shephard and Todd’s list. Recently, there has

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been a great deal of progress in trying to find unified and more conceptual proofs. (For example, see [4] and the references therein.) However, a coherent theory, like that of the classical Coxeter groups and Weyl groups, is still not in place, and many mysteries still remain. One of these mysteries involves the diagrams for unitary reflection groups.

Coxeter presentations of orthogonal reflection groups are encoded in their Coxeter-Dynkin diagrams. Similarly, for each unitary reflection group G , there is a diagram D_G , that encodes a presentation of G . (Such a D_G is given in [6] for all but six groups. For the remaining six groups, presentations of the corresponding braid group were conjectured in [5], in terms of certain diagrams. The proof of this conjecture was completed in [4].) Most of these diagrams were first introduced by Coxeter in [10]. The vertices of D_G correspond to complex reflections that form a minimal set of generators for G . Other than that, the definition of D_G is ad hoc and case by case. It is curious that even though these diagrams do not have any uniform definition, they contain a great deal of non-trivial information about the group G . We quote two sample results which state that the weak homotopy type of X_G/G and the invariant degrees of G can be recovered from the diagram D_G .

Theorem 1.2 ([4]). (a) *The universal cover of X_G/G is contractible.*

(b) *G has a minimal generating set of complex reflections, $\bar{\mathbf{b}} = \{\bar{b}_1, \dots, \bar{b}_k\}$, which can be lifted to a set of generators $\mathbf{b} = \{b_1, \dots, b_k\}$ of $\text{Braid}(G)$, with the following property: There is a set of positive homogeneous relations $R'_G(\mathbf{b})$ in the alphabet \mathbf{b} such that $\text{Braid}(G)$ and G have the following presentations:*

$$\text{Braid}(G) \simeq \langle \mathbf{b} | R'_G(\mathbf{b}) \rangle, \quad G \simeq \langle \bar{\mathbf{b}} | R'_G(\bar{\mathbf{b}}), \bar{b}_i^{n_i} = 1 \text{ for all } i \rangle,$$

where n_i is the order of \bar{b}_i in G . (These presentations are encoded by a diagram D_G with vertex set \mathbf{b} , the edges indicating the relations in $R'_G(\mathbf{b})$.)

Theorem 1.3 (Th. 5.5 of [12], [5], [4]). *Let G be a well generated unitary reflection group. The Coxeter number of G is defined to be the largest positive integer h such that $e^{2\pi i/h}$ is an eigenvalue of an element of G . Then the product of the generators of G corresponding to the vertices of D_G , in certain order, has eigenvalues $e^{2\pi i(d_j-1)/h}$, where d_j are the invariant degrees of G .*

If G is a Weyl group and V is the complexification of the standard representation of G , then both 1.2 (due to Briskorn, Saito, Deligne) and 1.3 (probably due to Borel, Chevalley, Steinberg) are classical. These results were first verified for most of the groups in Shephard and Todd's list by arguments split into many separate cases. Essentially classification-free proofs are now known, by recent work of David Bessis (see [4], where the long standing task of showing that X_G is $K(\pi, 1)$ and finding Coxeter-like presentation for $\text{Braid}(G)$ were completed). But we still do not know of a way to characterize the diagrams for unitary reflection groups.

In this article, we study the diagrams of a few unitary reflection groups. The main results are discussed below. They are motivated by analogies with Weyl groups.

1.4. Summary of results. Our approach is to view unitary reflection groups as sets of automorphisms of "complex lattices". Let $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$. The main examples of unitary reflection groups that we want to study act as automorphisms of a sequence of \mathcal{E} -lattices, namely, $A_2^\mathcal{E} \subseteq D_4^\mathcal{E} \subseteq E_6^\mathcal{E} \subseteq E_8^\mathcal{E}$. Our interest in these lattices stems from their importance in studying the complex hyperbolic reflection

group with Y_{555} diagram and its conjectured connection with the bimonster (see [1], [3]). In Section 2, we present a new proof of theorem 2.2 of [1], which states that $A_2^\mathcal{E}, D_4^\mathcal{E}, E_6^\mathcal{E}, E_8^\mathcal{E}$ are the only “ \mathcal{E} -root lattices”. Our proof is like the A - D - E classification of Euclidean root lattices and is similar in spirit to the arguments in [7] and [11]. It is purely a linear algebra argument that only uses the diagrams for the complex reflection groups. This proof should be viewed as an illustration of the usefulness of the “complex diagrams”.

In Section 3 we address the following question, that was raised in [6]: *How do we characterize the diagrams for the unitary reflection groups?* To this end, we describe an algorithm (in subsection 3.8 and Algorithm 3.9) which, given the group G , the integer k and a random vector in V , selects a set $\bar{\mathbf{a}} = \{\bar{a}_1, \dots, \bar{a}_k\}$ of reflections in G . Our algorithm is based on a generalization of a “Weyl vector”. We show that “Weyl vectors” exist for all unitary reflection groups (see Theorem 3.5). If G is a Weyl group, then one can easily check that $\bar{\mathbf{a}}$ is a set of simple roots of G .

If G is primitive and defined over an imaginary quadratic extension of \mathbb{Q} , then we experimentally observe that $\bar{\mathbf{a}}$ is a minimal set of generators of G . There exists a set of positive homogeneous relations $R_G(\mathbf{a})$ in the alphabet $\mathbf{a} = \{a_1, \dots, a_k\}$ such that *in every execution of the algorithm, the generators $\bar{\mathbf{a}}$ satisfy the relations $R_G(\bar{\mathbf{a}})$.*

We find that the reflections $\bar{\mathbf{a}}$ form Coxeter’s diagrams in the examples of our main interest, namely, the reflection groups related to the \mathcal{E} -root lattices. For some groups G , namely, $G_{12}, G_{29}, G_{31}, G_{33}$ and G_{34} , new diagrams are obtained. More precisely, the generators $\bar{\mathbf{a}}$ selected by Algorithm 3.9 do not satisfy the relations R'_G known from [6], [5]. In Section 4, we verify that *the group $\text{Braid}(G)$ has a presentation given by $\langle \mathbf{a} | R_G(\mathbf{a}) \rangle$, and G has a presentation given by $\langle \bar{\mathbf{a}} | R_G(\bar{\mathbf{a}}), \bar{a}_i^{n_i} = 1 \text{ for all } i \rangle$, that is, Theorem 1.2(b) holds for the new diagrams (see Proposition 4.6).*

We have verified that Theorem 1.3 also holds for the new diagrams for G_{29}, G_{33} and G_{34} . The other two groups G_{12} and G_{31} are not well generated.

Let $G \in \{G_{29}, G_{31}, G_{33}, G_{34}\}$. For these groups, the relations R_G that our generators satisfy are different from those previously known. We note that all the relations in R_G (i.e., those needed to present $\text{Braid}(G)$) are of the form $x_0 x_1 \cdots x_{m-1} = x_1 x_2 \cdots x_m$, for a set of generators $\{x_i : i \in \mathbb{Z}/k\mathbb{Z}\}$, which form a minimal cycle in the diagram. When $k = 2$, these are the Coxeter relations. For most G , the group $\text{Braid}(G)$ has a presentation consisting of only this kind of relations (see the table in [6]). Following Conway [8], we call these deflation relations. The deflation relations encountered in G_{33} and G_{34} are, moreover, all “cyclic” (see Lemmas 4.3 and 4.4). For G_{29}, G_{33} or G_{34} , a presentation of the corresponding braid group is obtained by taking one braid relation for each edge and one deflation relation for each minimal cycle in the diagram. This makes us wonder if the right notion of a diagram for these groups is the two-dimensional polyhedral complex obtained by attaching 2-cells to the minimal cycles in the graphs D_G , so that, finiteness of G translates into the vanishing of the first homology of the polyhedral complex.

Note that G_{33} and G_{34} are part of a few exceptional cases, in which, the diagrams known in the literature do not have some of the desirable properties. (For example, see question 2.28 in [6] and the remark following it.) So there seems to be some doubt whether the diagrams known in the literature for these examples are the “right ones”.

In Section 5 we describe affine diagrams for unitary reflection groups defined over \mathcal{E} . The affine diagrams are obtained from the unitary diagrams by adding an extra node. They encode presentations for the corresponding affine complex reflection groups. For each affine diagram, we describe a “balanced numbering” on its vertices, such as the $2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$ numbering on the affine E_8 diagram. The existence of a balanced numbering on a diagram implies that the corresponding reflection group is not finite. So an affine diagram cannot occur as a full subgraph of a diagram for a unitary reflection group. These facts, coupled with a combinatorial argument, complete the classification of \mathcal{E} -root lattices. The affine diagrams are often more symmetric compared to the unitary diagrams, as in the real case. For example, for G_{34} , (which is the reflection group of the Coxeter-Todd lattice $K_{12}^{\mathcal{E}}$), we get an affine diagram with rotational $\mathbb{Z}/7\mathbb{Z}$ symmetry.

The complex root systems that we use in computer experiments are described in Appendix A. Appendix B contains details of proofs of statements made in Section 4.

1.5. Shortcomings of Algorithm 3.9. If G is imprimitive or not defined over \mathbb{Q} or an imaginary quadratic extension of \mathbb{Q} , then our algorithm does not work in the sense that the set $\bar{\mathbf{a}}$ of reflections chosen by the algorithm usually does not form a minimal set of generators of G . Also, one knows from [6] and [5] that the braid groups of G_{12} and G_{24} can be presented using cyclic deflation relations, but we were unable to find such presentations of these braid groups on the generators selected by our algorithm. So while Algorithm 3.9 does not provide a definite characterization of the diagrams for the unitary reflection groups, the observations in the previous paragraphs seem to indicate that the diagrams have a geometric origin.

We finish this section by introducing some basic definitions and notations to be used.

1.6. Reflection groups and root systems. Let V be a complex vector space with a hermitian form (always assumed to be linear in the second variable). If $x \in V$, then $|x|^2 = \langle x, x \rangle$ is called the *norm* of x . Given a vector x of non-zero norm and a root of unity $u \neq 1$, let

$$\phi_x^u(y) = y - (1 - u)\langle x, y \rangle |x|^{-2}x.$$

The automorphism ϕ_x^u of the hermitian vector space V is called a *u -reflection* in x , or simply, a *complex reflection*. The hyperplane x^\perp (or its image in the projective space $\mathbb{P}(V)$), fixed by ϕ_x^u , is called the *mirror* of reflection. A *complex reflection group* G is a discrete subgroup of $\text{Aut}(V, \langle \cdot, \cdot \rangle)$, generated by complex reflections. A mirror of G is a hyperplane fixed by a reflection in G . A complex reflection (resp. complex reflection group) is called a *unitary reflection* (resp. *unitary reflection group*) if the hermitian form on V is positive definite. We shall omit the words “complex” or “unitary”, if they are clear from context. A unitary reflection group G acting on V is *reducible* (resp. *imprimitive*) if V is a direct sum $V = V_1 \oplus \cdots \oplus V_l$ such that $0 \neq V_1 \neq V$ and each V_j is fixed by G (resp. the collection of V_j is stabilized by G). Otherwise G is *irreducible* (resp. *primitive*). Unless otherwise stated, we always assume that G is irreducible.

Let G be a unitary reflection group acting on \mathbb{C}^k with the standard hermitian form. Let F be the field of definition of G . Let \mathcal{O} be the ring of integers in F . Let \mathcal{O}^* be the group of units of \mathcal{O} . A vector r in an \mathcal{O} -module K is *primitive* if $r = mr'$

with $m \in \mathcal{O}$ and $r' \in K$ implies that $m \in \mathcal{O}^*$. Let Φ be a set of primitive vectors in \mathcal{O}^k such that:

- Φ is stable under the action of G ,
- $\{r^\perp : r \in \Phi\}$ is equal to the set of mirrors of G , and
- given $r \in \Phi$, $ur \in \Phi$ if and only if u is a unit of \mathcal{O} .

Such a set of vectors will be called a (*unitary*) *root system* for G , defined over \mathcal{O} . The group \mathcal{O}^* acts on Φ by multiplication. An orbit is called a *projective root*. A set of projective roots for G is denoted by Φ_* or $\Phi_*(G)$.

1.7. Lattices and their reflection groups. Let F be a number field. Let \mathcal{O} be the ring of integers of F . Assume that \mathcal{O} is a unique factorization domain. Fix an embedding of F in \mathbb{C} , and identify F and \mathcal{O} as subsets of \mathbb{C} via this embedding. Assume that \mathcal{O} forms a discrete set in \mathbb{C} . The examples that will be important to us are the integers, the Gaussian integers $\mathcal{G} = \mathbb{Z}[i]$ and the Eisenstein integers $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$. (We also briefly consider $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[(\sqrt{-7} + 1)/2]$.) To fix ideas, one may take $\mathcal{O} = \mathcal{E}$. In the next section we only work over this ring.

A *lattice* K , defined over \mathcal{O} , is a free \mathcal{O} -module of finite rank with an \mathcal{O} -valued hermitian form. Let $V = \mathbb{C} \otimes_{\mathcal{O}} K$ be the complex vector space underlying K . The *dual lattice* of K , denoted by K' , is the set of vectors $y \in V$ such that $\langle y, x \rangle \in \mathcal{O}$ for all $x \in K$.

A *root* of K is a primitive vector $r \in K$ of non-zero norm such that $\phi_r^u \in \text{Aut}(K)$ for some root of unity $u \neq 1$. The *reflection group* of K , denoted by $R(K)$, is the subgroup of $\text{Aut}(K)$ generated by reflections in the roots of K . The projective roots of K are in bijection with the mirrors of $R(K)$. If K is positive definite, then the roots of K form a unitary root system, denoted by Φ_K , for the unitary reflection group $R(K)$.

1.8. (Root) diagrams. Consider the permutation matrices acting on $k \times k$ hermitian matrices by conjugation. An orbit D of this action is called a (*root*) *diagram* or simply a diagram. (There is a closely related notion of Coxeter diagram defined in Section 4.) If $M = ((m_{ij}))$ is a representative of an orbit D , then we say that M is a *gram matrix* of D . Let $\Delta = \{r_1, \dots, r_k\}$ be a subset of a hermitian vector space V . Let $m_{ij} = \langle r_i, r_j \rangle$. The matrix $((m_{ij}))$ is called a *gram matrix* of Δ , and the corresponding diagram is denoted by $D(\Delta)$. Let Φ be a root system for a unitary reflection group G . If Δ is a subset of Φ such that $\{\phi_r^{u_r} : r \in \Delta\}$ is a minimal generating set for G (for some units u_r), then we say that $D(\Delta)$ is a *root diagram* for G .

Pictorially, a diagram D is conveniently represented by drawing a directed graph D with labeling of vertices and edges, as follows: Let $v(D) = \{x_1, \dots, x_k\}$ be the set of vertices of D . We remember the entry m_{ii} by labeling the vertex x_i with m_{ii} . We remember the entry m_{ij} by drawing a directed edge from x_j to x_i labeled with m_{ij} or, equivalently, by drawing a directed edge from x_i to x_j labeled with \bar{m}_{ij} (but not both).

Let D be a diagram with gram matrix $((m_{ij}))$. Assume that $m_{ij} \in \mathcal{O}$ for all i and j . Define $L(D)$ to be the \mathcal{O} -lattice generated by linearly independent vectors $\{x_1, \dots, x_k\}$ with $\langle x_i, x_j \rangle = m_{ij}$. Conversely, let L be an \mathcal{O} -lattice having a set of roots Δ which form a minimal spanning set for L as an \mathcal{O} -module. Then the diagram $D(\Delta)$ is called a *root diagram* or simply a diagram for L . Let D be a diagram for L . Then $L(D)$ surjects onto L preserving the hermitian form. If the

gram matrix of D is positive definite, then $L(D) \simeq L$. We shall usually denote the vertices of D and the corresponding vectors of L by the same symbol.

Two diagrams D and D' are *equivalent* if $L(D) \simeq L(D')$. In this case, we write $D \simeq D'$. Let $L = L(D)$, $v(D) = \{x_1, \dots, x_k\}$ and u_1, \dots, u_k be units. Then there is a diagram for L whose vertices correspond to the generators $\{u_1x_1, \dots, u_kx_k\}$. These two diagrams are equivalent. The only difference between them is in the edge labeling, which may differ by units.

2. THE EISENSTEIN ROOT LATTICES

Let $\omega = e^{2\pi i/3}$, $\theta = \omega - \bar{\omega}$, and $p = 1 - \bar{\omega}$. Let $\mathcal{E} = \mathbb{Z}[\omega]$. In this section we shall classify the \mathcal{E} -root lattices, which we define following Daniel Allcock (see [1]).

Definition 2.1. An *Eisenstein root lattice* or \mathcal{E} -root lattice is a positive definite \mathcal{E} -lattice K , generated by vectors of norm 3, such that $K \subseteq \theta K'$ (see [1]). A root lattice is *indecomposable* if it is not a direct sum of two proper non-zero root lattices.

Let D be a diagram with gram matrix $((m_{ij}))$. The following assumptions about D will remain in force for the rest of this section. We assume that $m_{ij} \in \mathcal{E}$ for all i and j . We assume that $m_{ii} = 3$ for all i . So we omit the labels on the vertices. If $m_{ij} = -p$, we omit the label on the edge going from j to i . If $m_{ij} = 0$, we omit the edge $\{i, j\}$. These conventions are adopted when we discuss connectedness of a diagram. Each \mathcal{E} -root lattice has at least one diagram. Any diagram for an indecomposable \mathcal{E} -root lattice is connected.

Remark 2.2. Let K be a positive definite \mathcal{E} -lattice satisfying $K \subseteq \theta K'$. The following observations are immediate: If $r \in K$ has norm 3, then r is a root of K . The order 3 reflections in r , denoted by $\phi_r = \phi_r^\omega$ and $\phi_r^{-1} = \phi_r^{\bar{\omega}}$, belong to the reflection group of K . Let x and y be two linearly independent vectors of K of norm 3. The gram matrix of $\{x, y\}$ must be positive definite, that is, $(|x|^2|y|^2 - |\langle x, y \rangle|^2) > 0$. Since $\langle x, y \rangle \in \theta\mathcal{E}$, one has $|\langle x, y \rangle| \leq \sqrt{3}$. So either $\langle x, y \rangle = 0$, which implies $\phi_x\phi_y = \phi_y\phi_x$, or $\langle x, y \rangle \in \theta\mathcal{E}^*$, which implies $\phi_x\phi_y\phi_x = \phi_y\phi_x\phi_y$.

Definition 2.3. Let $E_{2k}^\mathcal{E}$ be the \mathcal{E} -lattice having a basis x_1, \dots, x_k such that $|x_i|^2 = 3$ for $i = 1, \dots, k$, $\langle x_i, x_{i+1} \rangle = -p$ for $i = 1, \dots, k - 1$ and $\langle x_i, x_j \rangle = 0$ if $j > i + 1$. The lattice $E_{2k}^\mathcal{E}$ has a diagram $D_{E_{2k}^\mathcal{E}} = (x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_k)$. The \mathbb{Z} -modules underlying $E_2^\mathcal{E}$, $E_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$, with the bilinear form $\frac{2}{3} \text{Re}\langle x, y \rangle$, are the lattices A_2 , D_4 , E_6 and E_8 respectively (this can be easily checked by computing the discriminant and explicitly exhibiting the root systems A_2 , D_4 , E_6 and E_8 inside the real forms of these lattices).

We use the complex diagrams to give a new proof of the following theorem (Theorem 2.2 of [1]):

Theorem 2.4. *The only indecomposable Eisenstein root lattices are $E_{2i}^\mathcal{E}$ with $i = 1, 2, 3, 4$.*

The lattices $E_2^\mathcal{E}$, $E_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$ are positive definite. So if some Eisenstein lattice L has the root diagram of $E_{2j}^\mathcal{E}$, for some $1 \leq j \leq 4$, then $L \simeq E_{2j}^\mathcal{E}$. Thus it suffices to classify the equivalence classes of root diagrams of indecomposable \mathcal{E} -root lattices and show that there are only four classes. The proof of this classification, given below, is like the well-known classification of A - D - E root systems.

Definition 2.5. Let D be a connected root diagram for an \mathcal{E} -lattice L . A *balanced numbering* on D is a function from $v(D)$ to \mathcal{E} , denoted by $x \mapsto n_x$, such that $n_x = 1$ for some $x \in v(D)$ and

$$(1) \quad \sum_{x \in v(D)} n_x \langle a, x \rangle = 0$$

for each $a \in v(D)$. If $\{n_x : x \in v(D)\}$ is a balanced numbering on D , then the vector $y = \sum_{x \in v(D)} n_x \cdot x \in L$ is orthogonal to each $x \in v(D)$. So $|y|^2 = 0$. A diagram D is called an *affine diagram* if D admits a balanced numbering but no subdiagram of D admits one.

Figure 1 shows a few affine diagrams, each with a balanced numbering. The number shown next to a vertex x is n_x . Given $a \in v(D)$, suppose b_1, \dots, b_m are the vertices connected to a and c_j is the label on the directed edge going from b_j to a . Then equation (1) becomes $-3n_a = \sum_{j=1}^m c_j n_{b_j}$. This is easily verified.

We say that a connected diagram D is *indefinite* if D cannot appear as a full subgraph of a diagram of an Eisenstein root lattice. Otherwise, we say that D is *definite*.

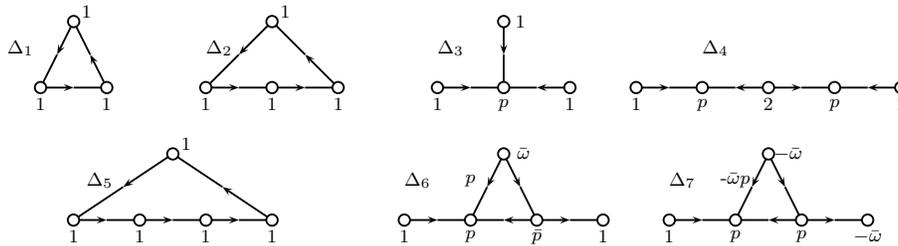


FIGURE 1. A few affine diagrams with balanced numbering. Choose any vertex v of Δ_i such that $\Delta_i \setminus \{v\}$ is connected. Then $\Delta_i \setminus \{v\}$ is a root diagram for $E_4^\mathcal{E}$ if $i = 1$, $E_6^\mathcal{E}$ if $i = 2, 3$ and $E_8^\mathcal{E}$ if $i = 4, 5, 6, 7$.

Lemma 2.6. Let D be a diagram that admits a numbering $\{n_x : x \in v(D)\}$ such that $y = \sum_{x \in v(D)} n_x \cdot x$ is a norm zero vector of $L(D)$ and $n_{x_j} = 1$ for some $x_j \in v(D)$. Then D is indefinite. In particular, if D admits a balanced numbering, then D is indefinite.

Proof. Suppose K is an \mathcal{E} -root lattice with a root diagram D_K . If $D \subseteq D_K$, then $y = \sum_{x \in v(D)} n_x \cdot x \in K$ has norm 0. Since K is positive definite, $y = 0$. If $n_{x_j} = 1$, then $x_j = -\sum_{x \neq x_j} n_x \cdot x$, contradicting the fact that $v(D_K)$ is a minimal generating set for K . \square

Definition 2.7. Let D be a diagram with vertices x_1, \dots, x_k and edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{k-1}, x_k\}$ and $\{x_k, x_1\}$, that is, a circuit of length k . Changing the vectors x_1, \dots, x_k by units if necessary, we may assume that $\langle x_i, x_{i+1} \rangle = -p$ for $1 \leq i \leq k-1$ and $\langle x_1, x_k \rangle = -up$ for some $u \in \mathcal{E}^*$. We denote this diagram by $\text{Circ}_{k,u}$.

Lemma 2.8. (a) Let $D = \text{Circ}_{3,-1}$ or $D = \text{Circ}_{3,\bar{\omega}}$. Then $D \simeq D_{E_6^\mathcal{E}}$.

(b) Let $D = \text{Circ}_{4,\bar{\omega}}$. Then $D \simeq D_{E_8^\mathcal{E}}$.

(c) Suppose $D = \text{Circ}_{k,u}$ with $k = 3, 4$ or 5 but D is not one of the three circuits considered in (a) and (b). Then D is indefinite.

(d) Suppose D is one of the diagrams given in Figure 1. Then D is indefinite.

Proof. (a) Let $v(D) = \{x_1, x_2, x_3\}$. Note that $\langle x_1, \bar{u}x_3 \rangle = \langle x_1, x_2 \rangle = -p$. If we take $x'_3 = x_2 - \bar{u}x_3$, then $\langle x_1, x'_3 \rangle = 0$. One checks that $|x'_3|^2 = 6 + 2\text{Re}(\bar{u}p) = 3$ and $\langle x_2, x'_3 \rangle = 3 + \bar{u}p \in p\mathcal{E}^*$, for $u = -1$ and $\bar{\omega}$. So D is equivalent to the diagram $D_{E_6^\mathcal{E}}$ formed by the roots x_1, x_2, x'_3 .

(b) Let $x'_4 = \bar{\omega}x_2 - px_3 - x_4$. One checks that x_1, x_2, x_3 and x'_4 form the diagram $D_{E_8^\mathcal{E}}$.

(c) For $\nu = 1, \dots, 6$, let $(n_{1,\nu}, \dots, n_{5,\nu})$ be equal to $(1, -\omega, \bar{\omega}, \bar{\omega}, \bar{\omega})$, $(1, -\omega, \bar{\omega}, -1, -1)$, $(1, -\omega, \bar{\omega}, -1, \omega)$, $(1, -\omega, \bar{\omega}, -1, \omega)$, $(1, 1, 1, 1, 1)$ and $(1, -\omega, -\omega, -\omega, -\omega)$, respectively. One checks that the vector $y = \sum_{i=1}^k n_{i,\nu}x_i$ has norm zero in $L(\text{Circ}_{k,e^{2\pi i\nu/6}})$, for $k = 3, 4, 5$ and $\nu = 1, \dots, 6$, except for the three cases considered in parts (a) and (b). Part (c) now follows from Lemma 2.6. Part (d) also follows from Lemma 2.6. \square

Proof of Theorem 2.4. Let D be a root diagram for an indecomposable \mathcal{E} -root lattice K . We shall repeatedly use Lemma 2.8 in two ways. First, it implies that D cannot contain the diagrams mentioned in parts (c) and (d) of the lemma. Secondly, from the proof of Lemma 2.8, we observe the following:

If $\text{Circ}_{3,u}$ or $\text{Circ}_{4,u}$ is a subgraph of D , then we are in one of the cases considered in parts (a) or (b) of Lemma 2.8 and we can change one of the vertices to get an equivalent diagram, where one of the edges has been removed. However this may introduce new edges elsewhere in the graph.

Since K is indecomposable, D must be connected. Let $v(D) = \{x_1, \dots, x_k\}$. If $k = 1$ (resp. $k = 2$), then clearly $K \simeq E_2^\mathcal{E}$ (resp. $K \simeq E_4^\mathcal{E}$). If $k = 3$, then either $D \simeq D_{E_6^\mathcal{E}}$ or $D \simeq \text{Circ}_{3,u}$ with $u = -1$ or $\bar{\omega}$, which are again equivalent to $D_{E_6^\mathcal{E}}$.

Let $k = 4$. We may assume that the diagram D' formed by $\{x_1, x_2, x_3\}$ is $D_{E_6^\mathcal{E}}$. If possible, suppose D is not equivalent to $D_{E_8^\mathcal{E}}$. Also suppose that $\{x_2, x_4\}$ is an edge of D . Since D cannot be the affine diagram Δ_3 , either $\{x_1, x_2, x_4\}$ or $\{x_2, x_3, x_4\}$ is a circuit in D . Without loss of generality, suppose $\{x_1, x_2, x_4\}$ is a circuit. Then we can change x_4 by adding a multiple of x_1 to get an equivalent diagram where $\{x_2, x_4\}$ is not an edge. So $D \simeq E_8^\mathcal{E}$ or D is a circuit. In the latter case, $D \simeq \text{Circ}_{4,\bar{\omega}}$, since all other circuits of length 4 are indefinite, by Lemma 2.8(c). Lemma 2.8(b) implies that $\text{Circ}_{4,\bar{\omega}} \simeq E_8^\mathcal{E}$.

Let $k = 5$. We may assume that the diagram D' formed by $\{x_1, x_2, x_3, x_4\}$ is $D_{E_8^\mathcal{E}}$. Since D cannot be the affine diagram Δ_4 , there must be at least two edges joining x_5 with D' . Since the diagrams of the form $\text{Circ}_{5,u}$ are not definite, D must contain a circuit of length 3 or 4. If $\{x_1, x_5\}$ is an edge, it is part of a circuit of length 3 or 4 and, as before, we can remove it by shifting to an equivalent diagram. The remaining possibilities are shown in Figure 2.

(The arrows on edges are not important here, so they have been omitted.) In cases (iii) and (iv) of Figure 2, we may add a multiple of x_3 to x_5 and disconnect x_4 from x_5 (note that this does not introduce an edge between x_1 and x_5). So we are reduced to the first two cases. But (i) is affine (either Δ_6 or Δ_7) and (ii) contains the affine diagram Δ_3 (see Figure 1). \square

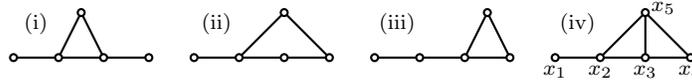


FIGURE 2. An undirected edge between x_i and x_j means $|\langle x_i, x_j \rangle| = |p|$.

3. AN ATTEMPT TO CHARACTERIZE THE DIAGRAMS FOR UNITARY REFLECTION GROUPS

In this section we want to address a question that was raised in [6]: *How do we characterize the diagrams for unitary reflection groups?* We maintain the definitions and notation introduced in subsections 1.6, 1.7 and 1.8. In subsection 3.8 and Algorithm 3.9, we shall describe an algorithm which, given a unitary reflection group G , picks out a set of reflections. As mentioned in the introduction, these reflections generate G (and lift to a set of generators for $\text{Braid}(G)$), if the field of definition of G is \mathbb{Q} or a quadratic imaginary extension of \mathbb{Q} . There are four fields to consider, namely, $\mathbb{Q}(\sqrt{-d})$, where $d = -1, -2, -3, -7$, and there are twelve groups to consider. Most of these can be viewed as sets of automorphisms of certain complex lattices defined over \mathcal{E} or \mathcal{G} , namely, $D_4^\mathcal{E}, E_6^\mathcal{E}, E_8^\mathcal{E}, K_{10}^\mathcal{E}, K_{12}^\mathcal{E}, D_4^\mathcal{G}$, and $E_8^\mathcal{G}$. For a description of $E_8^\mathcal{E}, E_8^\mathcal{G}$, and $K_{12}^\mathcal{E}$, see examples 11b, 13b and 10a, respectively, in chapter 7, section 8 of [9]. The lattice $K_{10}^\mathcal{E}$ is the orthogonal complement of any root in $K_{12}^\mathcal{E}$.

We take our cue from the fact that for a Weyl group, vertices of the Dynkin diagram correspond to the simple roots, which are the positive roots having minimal inner product with a Weyl vector. Our algorithm is based on a generalization of the notion of Weyl vector.

Definition 3.1. Let G be an irreducible unitary reflection group with a root system Φ defined over \mathcal{O} . Let $\Phi_* = \Phi/\mathcal{O}^*$ be the set of projective roots. Given a projective root r , let $o(r)$ denote the order of the subgroup of G generated by reflections in r . Let K be the \mathcal{O} -lattice spanned by Φ . Let V be the complex vector space underlying K , and let $\mathcal{M} = \bigcup_{r \in \Phi_*} r^\perp$ be the union of the mirrors. Define a function $\alpha : V \setminus \mathcal{M} \rightarrow V$ by

$$(2) \quad \alpha(w) = \sum_{r \in \Phi_*} o(r)^{-2} \frac{\langle r, w \rangle}{|\langle r, w \rangle|} \frac{r}{|r|}.$$

Remark 3.2. (1) Note that the quantity $o(r)^{-2} \frac{\langle r, w \rangle}{|\langle r, w \rangle|} \frac{r}{|r|}$ does not change if we change r by a scalar. So the function α is well defined and only depends on the reflection group G and not on the choice of the roots. The function α descends to a function $\bar{\alpha} : \mathbb{P}(V \setminus \mathcal{M}) \rightarrow \mathbb{P}(V)$. Also note that α is G -equivariant, that is, $\alpha(gw) = g\alpha(w)$ for all $g \in G$. So α induces a function from $\mathbb{P}(V \setminus \mathcal{M})/G$ to $\mathbb{P}(V)/G$.

- (2) If all the roots have the same norm, then the factor $|r|$ in the denominator is unnecessary and can be omitted from the definition of α .
- (3) The exponent -2 on $o(r)$ was found by experimenting with the example G_{26} , which has reflections of order two and three. We have included it to indicate one of the ways in which equation (2) may be modified to possibly

include other examples. If the stabilizer of each mirror has the same order, then the factor $o(r)^{-2}$ can be omitted from the definition of α . For example, if we consider reflection group of a Euclidean root lattice (resp. \mathcal{E} -root lattice), then $o(r)$ is always equal to 2 (resp. 3).

3.3. The case of Weyl groups. Let W be a Weyl group acting on a real vector space $V_{\mathbb{R}}$ via its standard representation. Let $\mathcal{M}_{\mathbb{R}}$ be the union of the mirrors of W . Then W can be viewed as a unitary reflection group acting on $V = V_{\mathbb{R}} \otimes \mathbb{C}$.

Claim. Let $w, w' \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$. If w and w' are in the same Weyl chamber, then $\alpha(w) = \alpha(w')$. Further, w and $\alpha(w)$ belong to the same Weyl chamber. So $\alpha(w)$ is a fixed point of α .

Proof. The statement is invariant upon scaling α by a positive factor, so we do the computation omitting the factor $o(r)^{-2} = 1/4$. Let $\Phi_+(w)$ be the set of roots having strictly positive inner product with w . Then $\Phi_+(w)$ can be chosen as a set of representatives for the projective roots, so $\alpha(w) = \sum_{r \in \Phi_+(w)} r/|r|$. Now, w and w' are in the same Weyl chamber if and only if $\Phi_+(w) = \Phi_+(w')$, so $\alpha(w) = \alpha(w')$.

To check that w and $\alpha(w)$ belong to the same Weyl chamber, first, let Φ be a root system of type A_n, D_n, E_6, E_7 or E_8 . Let $w \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$. Then $\rho = \frac{1}{2} \sum_{r \in \Phi_+(w)} r$ is a Weyl vector which belongs to the same Weyl chamber as w . Note that $\alpha(w) = \sqrt{2}\rho$ and $\alpha(\sqrt{2}\rho) = \sqrt{2}\rho$.

Now consider the non-simply laced case. Since α is G -equivariant, it is enough to check that w and $\alpha(w)$ are in the same Weyl chamber, for a single chamber. We show the calculation for B_n . The Weyl group of type C_n is isomorphic to the group of type B_n . Calculations for type G_2 and F_4 are only a little more complicated and will be omitted.

Let e_j be the j -th unit vector in \mathbb{Z}^n . The roots of B_n are $\{\pm e_j, \pm e_i \pm e_j : j < i\}$. Choose $w = (w_1, \dots, w_n)$ such that $0 < w_1 < \dots < w_n$. Then $\Phi_+(w) = \{e_j, e_i \pm e_j : j < i\}$. So

$$(3) \quad \rho^{(n)} = \alpha(w) = \sum_{r: \langle r, w \rangle > 0} r/|r| = (1, 1 + \sqrt{2}, 1 + 2\sqrt{2}, \dots, 1 + (n - 1)\sqrt{2}).$$

Observe that $\rho^{(n)}$ and w are in the same Weyl chamber. So $\alpha(\rho^{(n)}) = \alpha(w) = \rho^{(n)}$. □

Definition 3.4. Let Φ be a unitary root system defined over \mathcal{O} . In view of subsection 3.3, a fixed point of α will be called a *Weyl vector* when the root system is not defined over \mathbb{Z} . One may try to find a fixed point of α by iterating the function. Our method for selecting a set of generating reflections for G is based on this notion of Weyl vector. Before describing it, we show that Weyl vectors exist.

Theorem 3.5. *Let Φ be a root system for a unitary reflection group G . Then the function α , defined in (2), has a fixed point.*

For notational simplicity, let $\mu_r = |r|^{-1} \cdot o(r)^{-2}$, so that

$$(4) \quad \alpha(w) = \sum_{r \in \Phi_*} \mu_r |\langle r, w \rangle|^{-1} \langle r, w \rangle r.$$

The argument given below actually shows that, for any sequence of non-zero positive numbers μ_r , a function of the form (4) has a fixed point. We need a lemma, which converts the problem of finding a fixed point of α to a maximization problem.

Lemma 3.6. *Let Φ be a unitary root system, and let \mathcal{M} be the union of mirrors. Consider the function $S : \mathbb{P}(V) \rightarrow \mathbb{R}$, defined by*

$$(5) \quad S(w) = |w|^{-1} \langle \alpha(w), w \rangle = \sum_{r \in \Phi_*} \mu_r |w|^{-1} |\langle r, w \rangle|.$$

For $w \notin \mathcal{M}$, one has $\partial_w(S) = 0$ if and only if $\alpha(w) = |w|^{-1} S(w)w$. (Here ∂_w denotes the holomorphic derivative with respect to w .)

Proof. Let $n(w) = \langle \alpha(w), w \rangle$, so that $S(w) = n(w)/|w|$. We fix a basis for the vector space V and write $\langle r, w \rangle = r^* M w$, where r^* is the conjugate transpose of r and M is the matrix of the hermitian form. Differentiating, we get $\partial_w(|\langle r, w \rangle|^2) = \langle w, r \rangle r^* M$. So

$$\partial_w(\mu_r |\langle r, w \rangle|) = \mu_r (2|\langle r, w \rangle|)^{-1} \langle w, r \rangle r^* M.$$

Summing over Φ_* , one gets, $\partial_w n(w) = \frac{1}{2} \alpha(w)^* M$. Similarly we get $\partial_w(|w|) = \frac{1}{2|w|} w^* M$. It follows that

$$\partial_w(S(w)) = \partial_w\left(\frac{n(w)}{|w|}\right) = \frac{\frac{1}{2}|w|\alpha(w)^*M - \frac{1}{2|w|}n(w)w^*M}{|w|^2} = \frac{\alpha(w)^* - \frac{S(w)}{|w|}w^*}{2|w|}M.$$

Since M is an invertible matrix, the lemma follows. □

Proof of Theorem 3.5. We first prove the following claim:

Claim. If $w \in V$ lies on a mirror, then the function S cannot have a local maximum at w .

Fix a root $r_0 \in \Phi$ and $w \in r_0^\perp$. Assume $|w| = 1$. Take $w' = w + \epsilon \xi r_0$, where ξ is a complex root of unity and ϵ is a small positive real number so that ϵ^2 is negligible. We shall show that $S(w') > S(w)$, for suitable choice of ξ . Ignoring terms of order ϵ^2 , we have $|w'|^2 = |w|^2 + 2\operatorname{Re}(\epsilon \langle w, r_0 \rangle) = |w|^2 = 1$. Let $\Psi_0 = \Phi_* \cap w^\perp$ and $\Psi_1 = \Phi_* \setminus \Psi_0$. Then

$$S(w') = \sum_{r \in \Psi_0} \mu_r |\langle r, w' \rangle| + \sum_{r \in \Psi_1} \mu_r |\langle r, w' \rangle| = \epsilon \sum_{\Psi_0} \mu_r |\langle r, r_0 \rangle| + \sum_{\Psi_1} \mu_r |\langle r, w \rangle + \epsilon \xi \langle r, r_0 \rangle|.$$

Let $a = \langle r, w \rangle \neq 0$ and $b = \xi \langle r, r_0 \rangle$. Using the first order expansion of $(1+x)^{1/2}$, we have

$$|a + \epsilon b| = \sqrt{|a|^2 + 2\operatorname{Re}(\epsilon b \bar{a})} = |a| + \operatorname{Re}(\epsilon b \bar{a} |a|^{-1}).$$

It follows that

$$S(w') = S(w) + \epsilon \sum_{\Psi_0} \mu_r |\langle r, r_0 \rangle| + \epsilon \operatorname{Re}\left(\xi \sum_{\Psi_1} \mu_r \langle r, r_0 \rangle \overline{\langle r, w \rangle} |\langle r, w \rangle|^{-1}\right).$$

Note that the third term can be made non-negative by choosing ξ suitably, and the second term is positive since $r_0 \in \Psi_0$. This proves the claim.

The function S is continuous on $\mathbb{P}(V)$, so it attains its global maximum, say at w_0 . The claim we just proved implies that $w_0 \notin \mathcal{M}$, so $\partial_{w_0}(S) = 0$. Let $w_1 = |w_0|^{-1} S(w_0)w_0$. Lemma 3.6 implies that $\alpha(w_1) = \alpha(w_0) = w_1$. □

Definition 3.7. Let $(\mathcal{O}, G, \Phi, K, V)$ be as in Definition 3.1. Given $w \in V$, let (r_1, \dots, r_N) be the projective roots of G arranged so that $d(r_1^\perp, w) \leq \dots \leq d(r_N^\perp, w)$, where d is the Fubini-Study metric on $\mathbb{P}(V)$. So

$$d(r^\perp, w) = \sin^{-1}(|\langle r, w \rangle|/|r||w|).$$

Let k be the minimum number of reflections needed to generate G . Define $\Delta(w) = \{r_1, \dots, r_k\}$. In other words, $\Delta(w)$ consists of k projective roots, whose mirrors are closest to w .

3.8. Method to obtain a set of “simple reflections”. We now describe a computational procedure in which the input is a unitary reflection group G (or equivalently, a projective root system Φ_* for G and the numbers $\{o(r) : r \in \Phi_*\}$) and the output is either the empty set or a non-empty set of projective roots, to be called the *simple roots*. The reflections in the simple roots are called *simple reflections*. A set of simple reflections form a *simple system*.

- (1) From the set $\{w : \alpha(w) = w\}$, choose w such that $S(w)$, defined in (5), is maximum.
- (2) Let $\Delta(w) = \{r_1, r_2, \dots, r_k\}$.
- (3) If r_1, \dots, r_k are linearly dependent, then return the empty set.¹
- (4) If r_1, \dots, r_k are linearly independent, then return $\Delta(w)$ (the simple roots).

In practice, for each group G to be studied, we execute the following algorithm many times.

3.9. Algorithm. Start with a random vector $w_0 \in V$. Generate a sequence w_n by $w_n = \alpha(w_{n-1})$. If the sequence stabilizes, then say that the algorithm converges and let $w = \lim w_n$. Note down $\Delta(w)$ and $S(w)$.

For computer calculation, we assume that w_n stabilizes if $|w_{n+1} - w_n|^2 / |w_n|^2$ becomes small, say less than 10^{-8} , and remains small and decreasing for many successive values of n . Note that if $\alpha(w) = w$, then $S(w) = |w|$. From the values of $S(w)$, the maximum, denoted by S_{\max}^G , can be found. (In all the examples that we have studied, $S(w)$ takes at most two values on the set of fixed points of α found experimentally, so finding the maximum is not difficult.) Each instance of the algorithm that produced a vector w with $S(w) = S_{\max}^G$ now yields a simple system $\Delta(w)$ provided that $\Delta(w)$ is a linearly independent set.

3.10. Observations for Weyl groups. First, suppose that Φ is a root system for a Weyl group W . We maintain the notation of subsection 3.3 and ignore the factor $o(r)^{-2} = \frac{1}{4}$ in the definition of α . Given $w \in V_{\mathbb{R}}$, let $\delta_w(r) = |\langle r, w \rangle| / |r|$. Arranging the roots according to increasing distance from w (according to the spherical metric or the Fubini-Study metric) is equivalent to arranging them according to increasing order of $\delta_w(r)$.

Claim. Let W be a Weyl group, $w_0 \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$ and $w = \alpha(w_0)$. Then Algorithm 3.9 converges in one iteration and yields a simple system $\Delta(w)$. For Weyl groups, the definition of a set of simple roots given in subsection 3.8 agrees with the classical notion. Further, for each simple root $r \in \Delta(w)$, we have $\delta_w(r) = 1$. So the simple mirrors are equidistant from w .

Proof. We saw in subsection 3.3 that $w = \alpha(w_0)$ is a fixed point of α , so $w = \alpha(w_0) = \alpha^2(w_0) = \dots$, that is, the algorithm converges in one iteration.

Suppose Φ is a root system of type A , D or E . Then $w = \alpha(w_0) = \sqrt{2}\rho$, where ρ is a Weyl vector. So $\Delta(w) = \Delta(\rho)$ consists of a set of simple roots and for each $r \in \Delta(w)$, one has $\langle r, \rho \rangle = 1$, so $\delta_w(r) = 1$.

¹If the group G is not well generated, then one should use obvious modifications. For example, step (3) should be rephrased as follows: If $\dim(\text{span}\{r_1, \dots, r_k\})$ is less than the rank of G , then return the empty set.

In type B_n , if w_0 is in the Weyl chamber containing the vector $\rho^{(n)}$ given in equation (3), then one has $w = \rho^{(n)} = \alpha(\rho^{(n)}) = \dots$. Observe that the function $\delta_w(r)$ attains its minimum for the simple roots $\Delta(w) = \{e_1, e_2 - e_1, \dots, e_n - e_{n-1}\}$ and $\delta(r) = 1$ for each $r \in \Delta(w)$. The claim was verified for G_2 and F_4 without difficulty. \square

3.11. Observations for complex root systems. In the following discussion, let G be one of the unitary reflection groups from the set

$$\{G_4, G_5, G_{25}, G_{26}, G_{32}, G_{33}, G_{34}\} \cup \{G_8, G_{29}, G_{31}\} \cup \{G_{12}\} \cup \{G_{24}\}.$$

The groups given in four subsets are defined over $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-7})$, respectively. In each case, a projective root system $\Phi_*(G)$ for G is chosen. These are described in Appendix A. For each of these groups G , we have run Algorithm 3.9 at least one thousand times and obtained many simple systems by method 3.8. The calculations were performed using a GP/PARI calculator. The main observations made from the computer experiments are the following:

The sequence $\{w_n\}$ stabilizes in each trial for each G mentioned above. The simple reflections generate G and satisfy the same set of relations R_G every time. In other words, the simple roots form the same “diagram” every time. For $G_4, G_5, G_8, G_{25}, G_{26}, G_{32}$, these are Coxeter’s diagrams, (see [6]). For G_{24} , the relations R_G are given in A.5 (this is the third presentation given in [5]). For G_{29}, G_{31}, G_{33} and G_{34} new diagrams are obtained. These diagrams are given in Figure 3, and corresponding presentations for the braid groups are given in subsection 4.5 and Proposition 4.6. Finally, for G_{12}^2 a presentation is given in A.4.

Further observations from the computer experiments are summarized below.

- (1) If G is not G_{29}, G_{31} or G_{32} , then for all w such that $\alpha(w) = w$, the function $S(w)$ attains the same value. So each trial of the algorithm yields a maxima w for S . For G_{29}, G_{31} and G_{32} , the function $S(w)$ attains two values on the fixed point set of α . In these three cases, the \mathbb{Z} -span of the roots form the E_8 lattice.
- (2) If $G \neq G_{33}$ and G is well generated, then in each trial of the algorithm, we find that the vectors $\{r_1, \dots, r_k\}$ are linearly independent. So each choice of a maxima w for the function S yields a set of simple roots $\Delta(w)$. For $G_{33} = R(K_{10}^E)$, in most of the trials, we find that $\{r_1, \dots, r_5\}$ are linearly dependent. So most trials do not yield a set of simple roots $\Delta(w)$. In an experiment with 5000 trials, only 297 yielded simple systems.
- (3) Let $G \in \{G_{33}, G_{34}, G_{29}\}$. These are the three well generated groups for which the diagrams obtained by method 3.8 are different from the ones in the literature. Let g_1, \dots, g_k be a set of simple reflections of G obtained by method 3.8. For these three groups, we have verified the following result (almost a restatement of 1.2):

Fix a permutation π so that the order of the product $p = g_{\pi_1} \dots g_{\pi_k}$ is maximum (over all permutations). Then, the order of p is equal to the

²In the notation of Section 4 the Coxeter diagram for G_{12} is a triangle with each edge marked with ∞ and for G_{24} it is a triangle with two double edges and one single edge. We have not drawn these. We should remark that in these two examples, we could not find presentations, on our generators, consisting of only cyclic homogeneous relations, though such presentations exist (see [6], [5]).

Coxeter number of G (denoted by h) and the eigenvalues of either p or p^{-1} are $e^{2\pi i(d_1-1)/h}, \dots, e^{2\pi i(d_k-1)/h}$, where d_1, \dots, d_k are the invariant degrees of G .

The invariant degrees of G_{29} , G_{33} and G_{34} are $(4, 8, 12, 20)$, $(4, 6, 10, 12, 18)$, and $(6, 12, 18, 24, 30, 42)$, respectively. In all three cases, h is equal to the maximum degree.

- (4) Start with $w_0 \in V$ and consider the sequence defined by $w_n = \alpha(w_{n-1})$. Roughly speaking, each iteration of the function α makes the vector w_n more symmetric with respect to the set of projective roots Φ_* . The function S measures this symmetry. So the fixed points w of α are often the vectors that are most symmetrically located with respect to Φ_* .

Based on the above discussion, an alternative definition of a Weyl vector may be suggested, namely, a vector $w \in V$, such that $d(w, \mathcal{M})$ is maximum. (This was suggested by Daniel Allcock.) It seems harder to compute these vectors, so we have not experimented much with this alternative definition; however, we would like to remark that for some complex and quaternionic Lorentzian lattices, similar analogs of Weyl vectors and simple roots are useful. (One such example is studied in [3]; other examples are studied in [2]. In these examples of complex and quaternionic Lorentzian lattices, the simple roots are again defined as those whose mirrors are closest to the “Weyl vector”.)

- (5) The method 3.8 fails for the primitive unitary reflection groups that are not defined over \mathbb{Q} or an imaginary quadratic extension of \mathbb{Q} and for the imprimitive groups $G(de, e, n)$, except when they are defined over \mathbb{Q} , that is, for the cases $G(1, 1, n+1) \simeq A_n$, $G(2, 1, n) \simeq BC_n$ and $G(2, 2, n) \simeq D_n$. It fails in the sense that the set $\Delta(w)$ does not in general form a minimal set of generators for the group. This was found by experimenting with $G(de, e, n)$ for small values of (d, e, n) and also with G_6 and G_9 (see Appendix A.4). Although method 3.8 fails, the following observation holds for $G(de, e, n)$:

Let k be the minimum number of reflections needed to generate $G(de, e, n)$ ($k = n$ or $n + 1$). There exists a vector $w \in V$ such that, if $r_1^\perp, \dots, r_k^\perp$ are the mirrors closest to w , then reflections in $\{r_1, \dots, r_k\}$ generate $G(de, e, n)$.

For $G(de, e, n)$, one can take $w = \rho^{(n)}$ (the vector we obtained for the Weyl group B_n ; see equation (3)). It is easy to check that $\Delta(\rho^{(n)})$ forms the known diagrams for $G(de, e, n)$. We found $\rho^{(n)}$ by using an algorithm that tries to find a point in V whose distance from \mathcal{M} is at a local maximum. For small values of d, e and n , we find that this algorithm always converges to $\rho^{(n)}$.

4. THE BRAID GROUPS FOR $G_{29}, G_{31}, G_{33}, G_{34}$

4.1. For this section, let $N \in \{29, 31, 33, 34\}$. Algorithm 3.9, applied to G_N , yields new diagrams, denoted by D_N (see Figure 3). Before stating the main result, Proposition 4.6, we need some notation.

4.2. **Notation.** Let $\{x_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ be elements of a monoid M and let $\mathcal{C}_m \langle x_0, \dots, x_{k-1} \rangle$ denote the positive and homogeneous relation

$$(6) \quad x_0 x_1 \cdots x_{m-1} = x_1 x_2 \cdots x_m.$$

For example, $\mathcal{C}_2\langle a, b \rangle$ (resp. $\mathcal{C}_3\langle a, b \rangle$) says that a and b commutes (resp. braids); while $\mathcal{C}_5\langle a, b \rangle$ (resp. $\mathcal{C}_4\langle a, b, c \rangle$) stands for the relation $ababa = babab$ (resp. $abca = bcab$). If x, y are elements in a group, let $c_x(y) := xyx^{-1}$.

Given a diagram D (such as in Figure 3) let $\text{Cox}(D, \infty)$ be the group defined by generators and relations as follows: The generators of $\text{Cox}(D, \infty)$ correspond to the vertices of D . The relations are encoded by the edges of D ; i.e., k edges between vertices p and q encodes the relation $\mathcal{C}_{k+2}\langle p, q \rangle$. In particular, no edge between p and q indicates that p and q commute, while an edge marked with ∞ indicates that $\text{Cox}(D, \infty)$ has no defining relation involving only the generators p and q . Let $\text{Cox}(D, n)$ be the quotient of $\text{Cox}(D, \infty)$ obtained by imposing the relation $p^n = 1$ for each vertex p of D . Let $\text{Cox. Rel}(D)$ be the defining relations of $\text{Cox}(D, \infty)$.

Let \tilde{A}_n denote the affine Dynkin diagram of type A_n . (Picture it as a regular polygon with $(n+1)$ vertices.) Fix an automorphism ρ of \tilde{A}_n (hence of $\text{Cox}(\tilde{A}_n, \infty)$) that rotates the Dynkin diagram \tilde{A}_n by an angle $2\pi/n$. Let x be a vertex of \tilde{A}_n . We say that the relation $\mathcal{C}_m\langle x, \rho(x), \dots, \rho^n(x) \rangle$ is *cyclic* in $\text{Cox}(\tilde{A}_n, \infty)$ if the relation $\mathcal{C}_m\langle \rho(x), \dots, \rho^n(x), x \rangle$ holds in the quotient $\text{Cox}(\tilde{A}_n, \infty) / \langle \mathcal{C}_m\langle x, \rho(x), \dots, \rho^n(x) \rangle \rangle$ (so $\mathcal{C}_m\langle y, \rho(y), \dots, \rho^n(y) \rangle$ holds for each vertex $y \in \tilde{A}_n$).

The two lemmas stated below help us verify Proposition 4.6. But these might be of independent interest for studying groups satisfying relations of the form (6). The proofs are straightforward and given in B.2 and B.3. It is easy to write down more general statements than those stated below and give a uniform proof, at least for Lemma 4.4. To keep things simple, we have resisted this impulse to generalize.

Lemma 4.3. *Let $(x_0, \dots, x_n) = (x, \rho(x), \dots, \rho^n(x))$ be the vertices of \tilde{A}_n .*

- (a) $\mathcal{C}_n\langle x_0, \dots, x_n \rangle$ is cyclic in $\text{Cox}(\tilde{A}_n, \infty)$ for all n .
- (b) $\mathcal{C}_{n+2}\langle x_0, \dots, x_n \rangle$ (resp. $\mathcal{C}_{2n+2}\langle x_0, \dots, x_n \rangle$) is cyclic in $\text{Cox}(\tilde{A}_n, \infty)$ if and only if $x_n = x_2$ (resp. $x_{n-1} = x_1$). In particular $\mathcal{C}_4\langle x_0, x_1, x_2 \rangle$ and $\mathcal{C}_6\langle x_0, x_1, x_2 \rangle$ are cyclic in $\text{Cox}(\tilde{A}_2, \infty)$.
- (c) Let $n > 2$. Then $\mathcal{C}_{2n+3}\langle x_0, \dots, x_n \rangle$ is cyclic in $\text{Cox}(\tilde{A}_n, \infty)$ if and only if $x_{n-1} = x_2$. In particular $\mathcal{C}_9\langle x_0, x_1, x_2, x_3 \rangle$ is cyclic in $\text{Cox}(\tilde{A}_3, \infty)$.

Lemma 4.4.

- (a) Assume x braids with y in a group G . Then the following are equivalent:
 - (i) $\mathcal{C}_4\langle x, y, z \rangle$.
 - (ii) $c_x(y)$ commutes with z .
 - (iii) $c_y(z)$ commutes with x .

If x also braids with z , then (i), (ii), (iii) are equivalent to (iv): $c_z(x)$ commutes with y .

- (b) Assume x braids with y and z in a group G . Then the following are equivalent:
 - (i) $\mathcal{C}_6\langle x, y, z \rangle$.
 - (ii) $c_x(y)$ braids with z .
 - (iii) $c_y(z)$ braids with x .
 - (iv) $c_z(x)$ braids with y .
- (c) Suppose x, y, z, w are the Coxeter generators of $\text{Cox}(\tilde{A}_3, \infty)$. Then $\mathcal{C}_9\langle x, y, z, w \rangle$ holds if and only if $c_{xy}(z)$ braids with w .

4.5. The relations. Let $N \in \{29, 31, 33, 34\}$. The vertices of the diagram D_N and D'_N given in Figure 3 correspond to generators of $\text{Braid}(G_N)$. The edges indicate

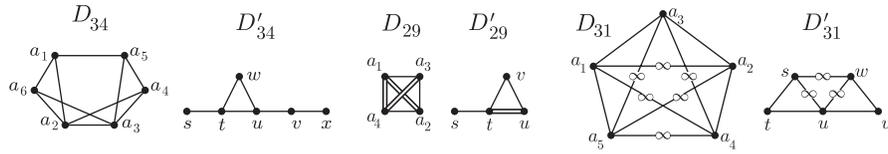


FIGURE 3. Let $D_{33} = D_{34} \setminus \{a_6\}$ and $D'_{33} = D'_{34} \setminus \{x\}$. The diagrams D_N and D'_N encode presentations for $\text{Braid}(G_N)$.

the Coxeter relations; however, more relations are needed to obtain a presentation of $\text{Braid}(G_N)$. These relations are given below. All of them are of the form (6).

$$\begin{aligned}
 E_{34} &= \{C_4\langle a_2, a_3, a_4 \rangle, C_4\langle a_3, a_4, a_5 \rangle, C_9\langle a_3, a_2, a_1, a_5 \rangle, \\
 &\quad C_4\langle a_2, a_1, a_6 \rangle, C_4\langle a_2, a_3, a_6 \rangle, C_9\langle a_1, a_5, a_3, a_6 \rangle\}, \\
 E'_{34} &= \{C_6\langle t, u, w \rangle\}, \\
 E_{29} &= \{C_4\langle a_4, a_2, a_1 \rangle, C_4\langle a_3, a_1, a_4 \rangle, C_6\langle a_3, a_1, a_2 \rangle\}, \\
 E'_{29} &= \{C_6\langle u, t, v \rangle\}, \\
 E_{31} &= \{C_4\langle a_4, a_2, a_1 \rangle, C_6\langle a_2, a_3, a_4 \rangle, C_3\langle a_3, a_5, a_4 \rangle, \\
 &\quad C_3\langle a_5, a_4, a_3 \rangle, C_4\langle a_1, a_5, a_2 \rangle, C_4\langle a_3, a_1, a_4 \rangle, C_4\langle a_4, a_2, a_5 \rangle\}, \\
 E'_{31} &= \{C_3\langle s, u, w \rangle, C_3\langle u, w, s \rangle\}.
 \end{aligned}$$

Let E_{33} (resp. E'_{33}) be the relations in E_{34} (resp. E'_{34}) that do not involve a_6 (resp. x). Let $R_N = \text{Cox. Rel}(D_N) \cup E_N$ and $R'_N = \text{Cox. Rel}(D'_N) \cup E'_N$. Let B_N (resp. B'_N) be the group generated by the vertices of the diagram D_N (resp. D'_N) satisfying the relations R_N (resp. R'_N). It was conjectured in [5] and proved in [4] that $\text{Braid}(G_N) \simeq B'_N$.

Proposition 4.6. *Assume the setup given in subsection 4.5. The presentations B_N and B'_N are equivalent. So B_N gives a presentation of $\text{Braid}(G_N)$. The quotient of B_N obtained by imposing the relations $a_i^2 = 1$, for all i , is isomorphic to G_N .*

Sketch of the proof. We define the maps $\varphi_N : B'_N \rightarrow B_N$ and $\psi_N : B_N \rightarrow B'_N$ on the generators. Let

$$\begin{aligned}
 \varphi_{29} &: (s, t, v, u) \mapsto (c_{(a_2 a_3 a_1)^{-1}}(a_4), a_1, a_2, a_3), \\
 \psi_{29} &: (a_1, a_2, a_3, a_4) \mapsto (t, v, u, c_{vut}(s)). \\
 \varphi_{29} &: (s, t, v, u) \mapsto (c_{(a_2 a_3 a_1)^{-1}}(a_4), a_1, a_2, a_3), \\
 \psi_{29} &: (a_1, a_2, a_3, a_4) \mapsto (t, v, u, c_{vut}(s)).
 \end{aligned}$$

$$\begin{aligned}
 \varphi_{31} &: (s, t, u, w, v) \mapsto (a_5, a_1, a_3, c_{a_3}^{-1}(a_4), c_{a_4}(a_2)), \\
 \psi_{31} &: (a_5, a_1, a_3, a_4, a_2) \mapsto (s, t, u, c_u(w), c_{uw^{-1}u^{-1}}(v)).
 \end{aligned}$$

$$\begin{aligned}
 \varphi_{34} &: (s, t, u, v, w) \mapsto (a_1, a_2, a_3, c_{a_3}(a_4), c_{a_2 a_1}(a_5), c_{a_4 a_2 a_1 a_5 a_3}(a_6)), \\
 \psi_{34} &: (a_1, a_2, a_3, a_4, a_5) \mapsto (s, t, u, c_{u^{-1}}(v), c_{(ts)^{-1}}(w), c_{(vuwt su)^{-1}}(x)).
 \end{aligned}$$

Let φ_{33} and ψ_{33} be the restrictions of φ_{34} and ψ_{34} , respectively.

To check that φ_N (resp. ψ_N) is a well defined group homomorphism, we have to verify that $(\varphi_N(s), \varphi_N(t), \dots)$ (resp. $(\psi_N(a_1), \psi_N(a_2), \dots)$) satisfy the relations R'_N (resp. R_N). This verification was done by hand using Lemmas 4.3 and 4.4. The details, given in B.5, B.6 and B.7, are rather tedious. It is easy to see that φ_N and ψ_N are mutual inverses, so $B_N \simeq B'_N$.

The generators a_i for B_N were found as follows. We first found generators \bar{a}_i of order 2 in G_N using Algorithm 3.9, and then we let a_i be a lift of \bar{a}_i ; that is, we chose R_N to be an appropriate subset of the relations satisfied by $\{\bar{a}_1, \bar{a}_2, \dots\}$ in G_N . So, from our construction, we know that there are reflections of order 2 in G_N that satisfy the relations R_N , that is, G_N is a quotient of $\bar{B}_N := B_N / \langle a_i^2 = 1 \text{ for all } i \rangle$. Using coset enumeration on the computer algebra system MAGMA, we verified that \bar{B}_N has the same order as G_N . \square

Remark 4.7. Here are some concluding remarks for this section.

- (1) The transformations, φ_N and ψ_N , given in the proof of 4.6 were found first by computing inside the reflection group G_N and then choosing appropriate lifts to $\text{Braid}(G)$.
- (2) It is interesting to note that all the relations encountered in the presentations of the braid groups of types G_{33} and G_{34} are cyclic relations (see Lemma 4.3) of the form $\mathcal{C}_n \langle x_1, \dots, x_m \rangle$ where x_1, \dots, x_m is a minimal cycle (actually a triangle or a square) inside the diagrams.
- (3) The relation $\mathcal{C}_9 \langle a_4, a_2, a_1, a_5 \rangle$ (resp. $\mathcal{C}_6 \langle a_2, a_3, a_4 \rangle$) hold as a consequence of the relations R_{33} (resp. R_{29}) (see B.8). So the content of Proposition 4.6 for $N = 29, 33$ and 34 can be succinctly stated as follows: The relations needed to present $\text{Braid}(G_N)$ are $\text{Cox. Rel}(D_N)$ and one relation of the form $\mathcal{C}_n \langle x_1, \dots, x_m \rangle$ for each minimal cycle x_1, \dots, x_m in D_N . Further, this n is the smallest integer for which the relation $\mathcal{C}_n \langle x_1, \dots, x_m \rangle$ holds in G_N . It might be interesting to find out all the unitary reflection groups for which a similar statement is true.
- (4) The situation with the non-well generated group G_{31} is not as nice. First of all, we were not able to verify the previous remark for $\text{Braid}(G_{31})$. Further, deleting the relations involving a_5 from R_{31} only yields a proper subset of R_{29} . The problem is that the relations $\mathcal{C}_4 \langle a_1, a_2 \rangle, \mathcal{C}_4 \langle a_1, a_4 \rangle, \mathcal{C}_4 \langle a_2, a_5 \rangle, \mathcal{C}_4 \langle a_3, a_4 \rangle, \mathcal{C}_4 \langle a_3, a_5 \rangle, \mathcal{C}_4 \langle a_4, a_5 \rangle$ hold in G_{31} , but we were not able to check whether these relations are implied by R_{31} . This would be equivalent to checking whether the relations $\mathcal{C}_4 \langle s, u \rangle, \mathcal{C}_4 \langle u, w \rangle, \mathcal{C}_4 \langle s, w \rangle$ are implied by R'_{31} .
- (5) Some relations of the form $\mathcal{C}_k \langle x_0, \dots, x_n \rangle$ were studied in [8]. Conway called them “deflation relations”, because they often deflate infinite Coxeter groups to finite groups. The groups studied in this section provide some examples of this. We mention two other examples: (i) The quotient of the affine Weyl group $\text{Cox}(\tilde{A}_n, 2)$ obtained by adding the cyclic (see Lemma 4.3(a)) relation $\mathcal{C}_n \langle x_0, \dots, x_n \rangle$ is the finite Weyl group $\text{Cox}(A_n, 2)$. (ii) There is a graph D with 26 vertices such that the quotient of the infinite Coxeter group $\text{Cox}(D, 2)$ obtained by adding the cyclic deflation relations $\mathcal{C}_{11} \langle x_0, \dots, x_{10} \rangle$ for each minimal cycle with 12 vertices in D is the wreath product of the monster simple group with $\mathbb{Z}/2\mathbb{Z}$ (see [8]).

5. THE AFFINE REFLECTION GROUPS

5.1. In this section we shall describe affine diagrams for the primitive unitary reflection groups defined over \mathcal{E} , except for G_5 . (Including G_5 would further complicate notation.) An affine diagram is obtained by adding an extra node to the corresponding “unitary diagram”. Each affine diagram admits a balanced numbering. A unitary diagram can be extended to an affine diagram in many ways. We have chosen one that makes the diagram more symmetric. The Weyl vector, that yielded the unitary diagram, is often fixed by the affine diagram automorphisms.

The discussion below and the lemma following it are direct analogs of the corresponding results for Euclidean root lattices. We have included a proof since we could not find a convenient reference.

Let Φ be a unitary root system defined over \mathcal{E} , for a unitary reflection group G . Let K be the \mathcal{E} -lattice spanned by Φ . Assume that the subgroup of $\text{Aut}(K)$ generated by reflections in Φ is equal to G . Let \mathcal{E}_0 be the one-dimensional free module over \mathcal{E} with zero hermitian form. Let $\tilde{K} = K \oplus \mathcal{E}_0$. Let us write the elements of \tilde{K} in the form (y, m) with $y \in K$ and $m \in \mathcal{E}$. Let \tilde{G} be the subgroup of $\text{Aut}(\tilde{K})$ generated by reflections in $\tilde{\Phi} = \{(x, m) : x \in \Phi, m \in \mathcal{E}\}$. Note that $\phi_{(x,m)}^\omega \in \tilde{G}$ if and only if $\phi_x^\omega \in G$. Consider the semi-direct product $K \rtimes G$, in which the product is defined by $(x, g) \cdot (y, h) = (x + gy, gh)$. The faithful action of $K \rtimes G$ on \tilde{K} via affine transformations is given by $(x, g)y = x + gy$.

Lemma 5.2. *Given the setup above, assume that $|r|^2 = 3$ for all $r \in \Phi$ and $K' \supseteq p^{-1}K$.*

- (a) *The affine reflection group \tilde{G} is isomorphic to the semi-direct product $K \rtimes G$.*
- (b) *Let r_0 be a root of Φ such that the orbit Gr_0 spans K as a \mathbb{Z} -module. If $\phi_{r_1}^\omega, \dots, \phi_{r_k}^\omega$ generate G , then $\phi_{(r_1,0)}^\omega, \dots, \phi_{(r_k,0)}^\omega$, together with $\phi_{(r_0,1)}^\omega$ generate \tilde{G} .*
- (c) *If $\phi_{r_0}^\omega$ commutes (resp. braids) with $\phi_{r_j}^\omega$, then $\phi_{(r_0,1)}^\omega$ commutes (resp. braids) with $\phi_{(r_j,0)}^\omega$.*

Proof. (a) Identify K (resp. G) inside \tilde{K} (resp. \tilde{G}) via $y \mapsto (y, 0)$ (resp. $\phi_x^\omega \mapsto \phi_{(x,0)}^\omega$). Recall $p = 2 + \omega$. For $x \in K$, define

$$(7) \quad t_x(y, n) = (y, n - p^{-1}\langle x, y \rangle).$$

The automorphisms t_x of \tilde{K} are called translations. The subgroup of $\text{Aut}(\tilde{K})$ generated by translations is isomorphic to the additive group of K . For any root $(x, m) \in \tilde{\Phi}$, one has

$$\phi_{(x,m)}^\omega(y, n) = (\phi_x^\omega(y), n - mp^{-1}\langle x, y \rangle) = \phi_{(x,0)}^\omega \circ t_{\bar{m}x}(y, n).$$

Let us write $\phi_{(x,0)}^\omega = \phi_x^\omega$. From the above equation, one has, in particular,

$$(8) \quad t_x = (\phi_x^\omega)^{-1} \circ \phi_{(x,1)}^\omega.$$

From equation (8), it follows that $\phi_a^\omega t_x (\phi_a^\omega)^{-1} = t_{\phi_a^\omega(x)}$. So $(x, g) \mapsto t_x \circ g$ is an isomorphism from $K \rtimes G$ onto \tilde{G} .

(b) Let G_1 be the subgroup of \tilde{G} generated by $\phi_{r_1}^\omega, \dots, \phi_{r_k}^\omega$ and $\phi_{(r_0,1)}^\omega$. Then $G \subseteq G_1$. Let $x = gr_0$ be a root in the G -orbit of r_0 . Then $t_x = (\phi_x^\omega)^{-1} \phi_{(x,1)}^\omega = (\phi_x^\omega)^{-1} g \phi_{(r_0,1)}^\omega g^{-1} \in G_1$. Since the roots in the G -orbit of r_0 span K as a \mathbb{Z} -module, it follows that $t_x \in G_1$ for all $x \in K$. The translations, together with G , generate \tilde{G} . So $\tilde{G} = G_1$.

TABLE 1

G	$ \Phi_*(G) $	$\Phi_*(G)$	K
G_4	4	$(1, 1, \omega^j), (0, 0, p)$	$D_4^\mathcal{E}$
G_5	$4 + 4$	$\Phi_*(G_4)$ and $(p, p, 0), (1, 1, -2\omega^j)$	$D_4^\mathcal{E}$
G_{25}	12	$(\overline{p, 0, 0}), (1, \omega^j, \bar{\omega}^k)$	$E_6^\mathcal{E}$
G_{26}	$9 + 12$	$(\overline{1, -\omega^j, 0})$ and $\Phi_*(G_{25})$	
G_{32}	40	$(0, 1, \omega^j, \bar{\omega}^k), (1, \omega^j, \overrightarrow{-\bar{\omega}^k}, 0)$	$E_8^\mathcal{E}$

Part (c) follows from Remark 2.2 since $\{(r_0, 1), (r_j, 0)\}$ and $\{r_0, r_j\}$ have the same gram matrix. \square

5.3. Method to get an affine diagram. Lemma 5.2 applies to the root systems $\Phi(G_4), \Phi(G_{25})$ and $\Phi(G_{32})$ and the corresponding lattices $D_4^\mathcal{E}, E_6^\mathcal{E}$ and $E_8^\mathcal{E}$. A similar result holds for the root systems $\Phi(G_{33})$ and $\Phi(G_{34})$ and the corresponding lattices $K_{10}^\mathcal{E}$ and $K_{12}^\mathcal{E}$ if one replaces order three reflections by order two reflections and p by $-\omega$. In each of these cases, any root can be chosen as r_0 in part (b) of Lemma 5.2.

Further modifications are necessary for G_{26} . In this case $p^{-1}K$ is not a subset of K' but there is a sublattice $E_6^\mathcal{E} \subseteq K$ such that $p^{-1}E_6^\mathcal{E} \subseteq K'$. Accordingly, the translations t_x , given in (7), define automorphisms of K only for $x \in E_6^\mathcal{E}$. The conclusion in part (a) is that $\tilde{G} \simeq E_6^\mathcal{E} \rtimes G$. In part (b), any root of an order three reflection can be chosen as r_0 . The details are omitted.

Let $G \in \{G_4, G_{25}, G_{26}, G_{32}, G_{33}, G_{34}\}$. We take the diagram for G obtained by method 3.8 and extend it by adding an extra node corresponding to a suitable root, thus obtaining an affine diagram. These are shown in Figure 4. The extending node is joined with dotted lines. The vertices of an affine diagram of type G correspond to a minimal set of generators for the affine reflection group \tilde{G} . The edges indicate the Coxeter relations among the generators. Additional relations may be needed to obtain a presentation of \tilde{G} (like those given in subsection 4.5).

APPENDIX A. ROOT SYSTEMS FOR SOME UNITARY REFLECTION GROUPS

We describe a root system for each unitary reflection group considered in Section 3. We use the following notation. A set of coordinates marked with a line (resp. an arrow) above, means that these coordinates can be permuted (resp. cyclically permuted).

A.1. $G_4, G_5, G_{25}, G_{26}, G_{32}$. For each of these groups, a set of projective roots and the lattices spanned by these roots are given in Table 1. The groups G_5, G_{25} and G_{32} are reflection groups of the \mathcal{E} -root lattices $D_4^\mathcal{E}, E_6^\mathcal{E}$ and $E_8^\mathcal{E}$, respectively.

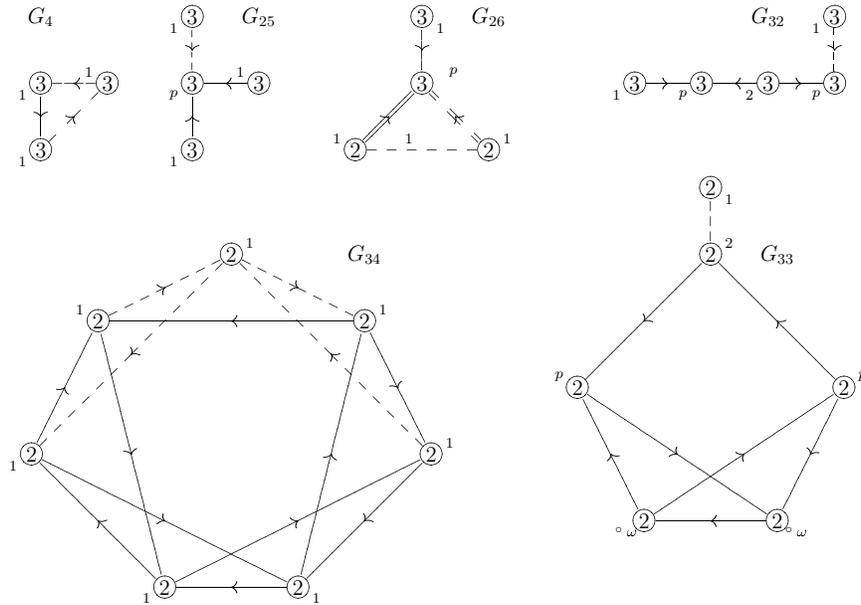


FIGURE 4. Affine diagrams with balanced numbering (shown next to the vertices). The number shown inside the vertex is the norm of the root as well as the order of a reflection in that root. An edge (resp. a double edge) between x and y implies the Coxeter relation $\phi_x\phi_y\phi_x = \phi_y\phi_x\phi_y$ (resp. $\phi_x\phi_y\phi_x\phi_y = \phi_y\phi_x\phi_y\phi_x$). An unmarked directed edge or double edge from y to x means that $\langle x, y \rangle = -p$ for $G_4, G_{25}, G_{26}, G_{32}$ and it means $\langle x, y \rangle = \omega$ for G_{33}, G_{34} .

A.2. $\mathbf{G}_{33}, \mathbf{G}_{34}$. These are reflection groups of the \mathcal{E} -lattices $K_{10}^{\mathcal{E}}$ and $K_{12}^{\mathcal{E}}$, respectively, where $K_{12}^{\mathcal{E}}$ is a complex form of the Coxeter–Todd lattice (see example 10a in chapter 7, section 8 of [9]) and $K_{10}^{\mathcal{E}}$ is the orthogonal complement of any vector of minimal norm in $K_{12}^{\mathcal{E}}$. The minimal norm vectors of $K_{10}^{\mathcal{E}}$ and $K_{12}^{\mathcal{E}}$ form root systems of type G_{33} and G_{34} , respectively.

A.3. $\mathbf{G}_8, \mathbf{G}_{29}, \mathbf{G}_{31}$. These are the primitive unitary reflection groups defined over $\mathcal{G} = \mathbb{Z}[i]$. Let $q = 1 + i$.

The reflection group of the two-dimensional \mathcal{G} -lattice $D_4^{\mathcal{G}} = \{(x, y) \in \mathcal{G}^2 : x + y \equiv 0 \pmod{q}\}$ is G_8 . The six projective roots are $(q, 0), (0, q)$ and $(1, i^j)$. The reflection group contains order four and order two reflections in these roots.

A set of projective roots for G_{29} can be chosen to be

$$\Phi_*(G_{29}) = \{(\overline{2}, 0, 0, 0), (1, \pm 1, i, i), (1, \pm 1, -i, -i), (\overline{q}, \pm q, 0, 0), (1, \pm 1, i, -i)\}.$$

There are a total of $4 + 6 + 6 + 12 + 12 = 40$ projective roots. They span the \mathcal{G} -lattice $E_8^{\mathcal{G}} = \{(x_1, \dots, x_4) \in \mathcal{G}^4 : x_i \equiv x_j \pmod{q}, \sum_i x_i \equiv 0 \pmod{2}\}$ whose real form is E_8 . The minimal norm vectors of $E_8^{\mathcal{G}}$ form a root system for G_{31} . The projective roots of G_{31} can be chosen to be

$$\Phi_*(G_{31}) = \{(1, \pm 1, \pm 1, \pm 1), (1, \overline{\mp 1}, \pm 1, \pm 1), (\overline{q}, \pm iq, 0, 0)\} \cup \Phi_*(G_{29}).$$

There are $2 + 6 + 12 + 40 = 60$ projective roots.

A.4. **G₁₂**. Following is a set of projective roots of G_{12} , defined over $\mathbb{Z}[\sqrt{-2}]$:

$$\Phi_*(G_{12}) = \{(\overline{2, 0}), \overline{(1, \pm 1 \pm \sqrt{-2})}, \overline{(\sqrt{-2}, \pm \sqrt{-2})}\}.$$

There are $2 + 8 + 2 = 12$ projective roots of norm 4 and G_{12} contains order two reflections in these.

Algorithm 3.9, applied to G_{12} , yields three generators s, t, v which satisfy the relations $\{stsv = tsvt, tsvs = vtsv\}$ in each trial. These relations are equivalent to the relations $\{stus = tust = ustu\}$ given in [6] using the substitution $u = c_s(v)$. So $\text{Braid}(G_{12})$ has a presentation given by $\langle s, t, v \mid stsv = tsvt, tsvs = vtsv \rangle$.

A.5. **G₂₄**. The following is a set of projective roots of G_{12} , defined over $\mathbb{Z}[(1 + \sqrt{-7})/2]$:

$$\Phi_*(G_{24}) = \{(\overline{2, 0, 0}), \overline{((1 - \sqrt{-7})/2, \pm(1 - \sqrt{-7})/2, 0)}, \overline{((1 + \sqrt{-7})/2, \pm 1, \pm 1)}\}.$$

The order two reflections in these $(3 + 6 + 12) = 21$ roots of norm 4 generate G_{24} .

The generators s, t, u chosen by Algorithm 3.9 satisfy the following relations:

$$sts = tst, \quad susu = usus, \quad utut = tutu, \quad (sut)^2s = u(sut)^2.$$

This happens to be the third presentation of $\text{Braid}(G_{24})$ given in [5]. (Note: the last relation above can be replaced by saying that u braids with sts^{-1} .)

A.6. **G₆, G₉**. Let $\zeta_m = e^{2\pi i/m}$. The projective roots of G_6 (defined over $\mathbb{Z}[\zeta_{12}]$) can be chosen to be

$$\Phi_*(G_6) = \{(\overline{2, 0}), (\pm\zeta_{12}q, q), (\pm\omega q, q)\} \cup \{(\pm 1, 1 + \zeta_{12}), (1 + \bar{\zeta}_{12}, \pm i)\}.$$

The group G_6 contains order two reflections in the six projective roots of norm 4 and order three reflections in the four projective roots of norm $(3 + \sqrt{3})$.

The projective roots of G_9 (defined over $\mathbb{Z}[\zeta_8]$) can be chosen to be

$$\Phi_*(G_9) = \Phi_*(G_{12}) \cup \{(\pm 1, 1 + \sqrt{2}), (1 + \zeta_8, \pm i(1 + \zeta_8))\}.$$

There are twelve projective roots of norm 4 (those of G_{12}) and six of norm $(4 + 2\sqrt{2})$. The group G_9 contains order two reflections in all the roots and order four reflections in the roots of norm $(4 + 2\sqrt{2})$.

A.7. **G(de, e, n)**. Let $\zeta_m = e^{2\pi i/m}$, and let e_j be the j th unit vector in \mathbb{C}^n . The projective roots of $G(de, e, n)$ can be chosen to be $\{e_j, e_j - \zeta_{de}^t e_k : 1 \leq j < k \leq n, 1 \leq t \leq de\}$. For a detailed study of these groups, see [6].

APPENDIX B. PROOFS OF SOME STATEMENTS IN SECTION 4

B.1. **Notation**. We adopt the following notation. If x and y are elements in a group, we write $\bar{x} = x^{-1}$, $c_x(y) := xyx^{-1}$. We write $x \check{\wr} y$ (resp. $x \perp y$) as an abbreviation for “ x braids with y ” (resp. “ x commutes with y ”).

In the proof of Proposition 4.6, say, while working in the group B_{31} , instead of writing $\varphi(w) = c_{\bar{a}_3}(a_4)$, we simply write $w = c_{\bar{a}_3}(a_4)$. Similar abuse of notation is used consistently because it significantly simplifies writing.

B.2. Proof of Lemma 4.3.

Proof. Work in $\text{Cox}(\tilde{A}_n, \infty)$. Let $h_n = (x_0x_1 \cdots x_n)$ and $y_n = h_nx_0$.

Step 1. Observe that

$$(9) \quad y_nx_n = x_0 \cdots x_{n-1} x_nx_0x_n = x_0 \cdots x_{n-1} x_0x_nx_0 = x_0x_1x_0 x_2 \cdots x_n x_0 = x_1y_n.$$

Since $\rho^2(y_n) = x_1^{-1}\rho(y_n)x_2$, it follows that

$$(10) \quad \rho(y_n) = y_n \iff \rho^2(y_n) = x_1^{-1}y_nx_2 = y_nx_n^{-1}x_2.$$

Suppose $\mathcal{C}_{n+2}\langle x_0, \dots, x_n \rangle$ is cyclic. Then $\rho(y_n) = y_n$ implies $\rho^2(y_n) = y_n$, hence $x_n = x_2$ (from (10)). Conversely, if $n = 2$, then $\rho(y_n) = y_n$ implies $\rho^2(y_n) = y_n$ (by (10)). So $\mathcal{C}_{n+2}\langle x_0, \dots, x_n \rangle$ is cyclic.

Step 2. Using (9), we have

$$(11) \quad h_n^2x_{n-1} = y_n x_1 \cdots x_{n-2} x_nx_{n-1}x_n = y_nx_n(x_1 \cdots x_n) = x_1h_n^2.$$

It follows that

$$(12) \quad \rho(h_n^2) = h_n^2 \iff \rho^2(h_n^2) = x_1^{-1}h_n^2x_1 = h_n^2x_{n-1}^{-1}x_1.$$

As in Step 1 (but using (12) in place of (10)), we conclude that $\mathcal{C}_{2n+2}\langle x_0, \dots, x_n \rangle$ is cyclic if and only if $x_{n-1} = x_1$. This proves part (b).

(c) Let $z_n = h_n^2x_0$. From (11), it follows that

$$\rho(z_n) = z_n \iff \rho^2(z_n) = x_1^{-1}z_nx_2 = x_1^{-1}h_n^2x_0x_2 = h_n^2x_{n-1}^{-1}x_0x_2 = z_nx_{n-1}^{-1}x_2,$$

where the last equality is obtained by commuting x_{n-1} and x_0 , which holds since $n > 2$. Part (c) follows. \square

B.3. Proof of Lemma 4.4.

Proof. (a) $c_x(y) \perp z \iff zxy\bar{x} = xy\bar{x}z = \bar{y}xyz$ or equivalently $yzxy = xyzx$. Further $z \perp c_x(y) = c_{\bar{y}}(x) \iff c_y(z) \perp x$. If $x \not\wr z$, then $c_x(y) \perp z \iff y \perp c_{\bar{x}}(z) = c_z(x)$.

(b) Let $u = c_x(y)$. Then $c_u(z) = c_{xy\bar{x}}(z) = c_{xyz}(x)$ and $c_{\bar{z}}(u) = c_{\bar{z}x}(y) = c_{\bar{z}\bar{y}}(x)$. So

$$u \not\wr z \iff c_u(z) = c_{\bar{z}}(u) \iff c_{xyz}(x) = c_{\bar{z}\bar{y}}(x) \iff \mathcal{C}_6\langle x, y, z \rangle.$$

Since x braids with y and z , one has

$$x \not\wr c_y(z) \iff c_{\bar{y}}(x) \not\wr z \iff c_x(y) \not\wr z \iff y \not\wr c_{\bar{x}}(z) \iff y \not\wr c_z(x).$$

(c) Let $u = c_{xy}(z)$. Then $c_u(w) = c_{xyzy\bar{x}}(w) = c_{xyzy\bar{w}}(x) = c_{xyzw\bar{y}}(x) = c_{xyzwx}(y)$ and $c_{\bar{w}}(u) = c_{\bar{w}xy}(z) = c_{\bar{w}x\bar{z}}(y) = c_{\bar{w}\bar{z}x}(y) = c_{\bar{w}\bar{z}\bar{y}}(x)$. So

$$u \not\wr w \iff c_u(w) = c_{\bar{w}}(u) \iff c_{xyzwx}(y) = c_{\bar{w}\bar{z}\bar{y}}(x) \iff \mathcal{C}_9\langle x, y, z, w \rangle. \quad \square$$

Remark B.4. Note that Lemma 4.3 implies that the relations of the form $\mathcal{C}_m\langle x_0, \dots, x_n \rangle$ considered in Lemma 4.4 are cyclic if x, y, \dots satisfy the Coxeter relations of $\text{Cox}(\tilde{A}_n, \infty)$.

B.5. Proof of Proposition 4.6 for G_{33} and G_{34} .

Proof. (a) Work in the group generated by $a_1, a_2, a_3, a_4, a_5, s, t, u, v, w$ subject to the relations

$$(13) \quad s = a_1, \quad t = a_2, \quad u = a_3, \quad v = c_{a_3}(a_4), \quad w = c_{a_2 a_1}(a_5).$$

We find a sequence $R_{33,0} \subseteq R_{33,1} \subseteq \cdots \subseteq R_{33,10} = R_{33}$ such that $R_{33,j}$ is a set of relations in the alphabet a_1, a_2, \dots and $R_{33,j}$ is obtained from $R_{33,j-1}$ by adding a single relation $w_j(a_1, a_2, \dots)$. We find another sequence $R'_{33,0} \subseteq R'_{33,1} \subseteq \cdots \subseteq R'_{33,10} = R'_{33}$ such that $R'_{33,j}$ is a set of relations in the alphabet s, t, \dots and $R'_{33,j}$ is obtained from $R'_{33,j-1}$ by adding a single relation $w'_j(s, t, \dots)$.

For each successive $j \geq 1$, we assume $R_{33,j-1}$ (or equivalently $R'_{33,j-1}$) and show that the relation w_j is equivalent to the relation w'_j , hence $R_{33,j}$ is equivalent to $R'_{33,j}$. After ten such steps we find that R_{33} is equivalent to R'_{33} .

Let $R'_{33,0}$ (resp. $R_{33,0}$) be the relations in R'_{33} (resp. R_{33}) that only involve s, t, u (resp. a_1, a_2, a_3). Clearly $R'_{33,0}$ and $R_{33,0}$ are equivalent under (13). Assume $R'_{33,0}$, or equivalently $R_{33,0}$. The chain of equivalences below correspond to $j = 1, 2, 3, \dots$. In the following chain of equivalences it is understood that the leftmost relation is w_j and the rightmost one is w'_j . The implicit assumptions in the j th step are the relations $R_{33,j-1}$ and $R'_{33,j-1}$.

$$\begin{aligned} j = 1: & \quad a_4 \perp a_1 \iff c_{\bar{u}}(v) \perp s \iff v \perp s, \\ j = 2: & \quad a_4 \checkmark a_3 \iff c_{\bar{u}}(v) \checkmark u \iff v \checkmark u, \\ j = 3: & \quad \mathcal{C}_4\langle a_3, a_4, a_2 \rangle \iff c_{a_3}(a_4) \perp a_2 \iff v \perp t, \\ j = 4: & \quad a_4 \checkmark a_2 \iff c_{\bar{u}}(v) \checkmark t \iff v \checkmark c_u(t) = c_{\bar{t}}(u) \iff v \checkmark u. \end{aligned}$$

Note that $R'_{33,4}$ is the set of relations in R'_{33} not involving w and $R_{33,4}$ is the set of relations in R_{33} not involving a_5 . At this stage we have shown that $R'_{33,4}$ is equivalent to $R_{33,4}$. These are our implicit assumptions in the next step given below:

$$\begin{aligned} a_5 \perp a_2 & \iff c_{\bar{s}\bar{t}}(w) \perp t \iff w \perp c_{ts}(t) \iff w \perp s, \\ a_5 \checkmark a_1 & \iff c_{\bar{s}\bar{t}}(w) \checkmark s \iff w \checkmark c_t(s) = c_{\bar{s}}(t) \iff w \checkmark t, \\ a_5 \checkmark a_3 & \iff c_{\bar{s}\bar{t}}(w) \checkmark u \iff w \checkmark c_t(u) \iff \mathcal{C}_6\langle t, u, w \rangle, \\ \mathcal{C}_9\langle a_3, a_2, a_1, a_5 \rangle & \iff a_5 \checkmark c_{a_3 a_2}(a_1) \\ & \iff c_{\bar{s}\bar{t}}(w) \checkmark c_{ut}(s) = c_{u\bar{s}}(t) = c_{\bar{s}\bar{t}}(u) \iff w \checkmark u, \\ \mathcal{C}_4\langle a_3, a_4, a_5 \rangle & \iff a_5 \perp c_{a_3}(a_4) \iff c_{\bar{s}\bar{t}}(w) \perp v \iff w \perp v, \\ a_5 \checkmark a_4 & \iff c_{\bar{s}\bar{t}}(w) \checkmark c_{\bar{u}}(v) \iff c_{\bar{t}}(w) \checkmark c_v(u) \iff c_{\bar{t}}(w) \checkmark u \\ & \iff w \checkmark c_t(u) \\ & \iff \mathcal{C}_6\langle t, u, w \rangle. \end{aligned}$$

This proves Lemma 4.6 for G_{33} . □

Step 1. (b) Assume the relations R_{34} . Let s, t, u, v, w be as in (13). Let

$$(14) \quad x = c_{a_4 a_2 a_1 a_5 a_3}(a_6).$$

We have to check that $\{s, t, u, v, w, x\}$ satisfy R'_{34} . Part (a) implies that $\{s, t, u, v, w\}$ satisfy the relations in R'_{34} that does not involve x . The following observation is

helpful in checking the relations involving x . Let

$$\alpha = c_{a_5 a_3}(a_6), \quad \delta = \bar{a}_1 \bar{a}_2 \bar{a}_4, \quad \text{so that } x = c_{\bar{\delta}}(\alpha).$$

Claim. α commutes with a_2, a_3 and a_4 and braids with a_5 .

Proof of claim.

$$\begin{aligned} a_2 \perp c_{a_5 a_3}(a_6) &\iff a_2 \perp c_{a_3}(a_6) \iff \mathcal{C}_4 \langle a_2, a_3, a_6 \rangle, \\ c_{a_5 a_3}(a_6) \perp a_3 &\iff c_{a_3}(a_6) \perp c_{\bar{a}_5}(a_3) = c_{a_3}(a_5) \iff a_6 \perp a_5, \\ a_4 \perp c_{a_5 a_3}(a_6) &= c_{a_5 \bar{a}_6}(a_3) = c_{\bar{a}_6 a_5}(a_3) \iff a_4 \perp c_{a_5}(a_3) \iff \mathcal{C}_4 \langle a_3, a_4, a_5 \rangle, \\ a_5 \checkmark c_{a_5 a_3}(a_6) &\iff a_5 \checkmark c_{a_3}(a_6) = c_{\bar{a}_6}(a_3) \iff a_5 \checkmark a_3. \quad \square \end{aligned}$$

Now we can check the relations in R'_{34} involving x . Note that

$$\begin{aligned} c_{\delta}(a_1) &= c_{\bar{a}_1 \bar{a}_2}(a_1) = a_2, \quad c_{\delta}(a_2) = c_{\bar{a}_1}(a_4) = a_4, \\ c_{\delta}(a_3) &= c_{\bar{a}_1 \bar{a}_2 a_3}(a_4) = c_{\bar{a}_1 a_3}(a_4) = c_{a_3}(a_4). \end{aligned}$$

Conjugating by δ , we find that x commutes with $s = a_1$ (resp. $t = a_2$ or $u = a_3$) if and only if α commutes with $c_{\delta}(a_1) = a_2$ (resp. $c_{\delta}(a_2) = a_4$, or $c_{\delta}(a_3) = c_{a_3}(a_4)$).

Since x and a_3 commute, x braids with $v = c_{a_3}(a_4)$ if and only x braids with a_4 . Note that $c_{\delta}(a_4) = c_{\bar{a}_1 \bar{a}_2}(a_4) = c_{\bar{a}_1 a_4}(a_2) = c_{a_4 \bar{a}_1}(a_2) = c_{a_4 a_2}(a_1)$. It follows that

$$\begin{aligned} x \checkmark v &\iff x \checkmark a_4 \iff \alpha \checkmark c_{\delta}(a_4) = c_{a_4 a_2}(a_1) \iff \alpha = c_{a_5 a_3}(a_6) \checkmark a_1 \\ &\iff \mathcal{C}_9 \langle a_5, a_3, a_6, a_1 \rangle. \end{aligned}$$

Since x commutes with a_1 and a_2 , one has $x \perp w = c_{a_2 a_1}(a_5)$ if and only if $x \perp a_5$. Using the relations satisfied by α , one obtains $x = c_{a_4 a_2 a_1}(\alpha) = c_{a_4 a_2 \bar{\alpha}}(a_1) = c_{\bar{\alpha} a_4 a_2}(a_1)$, so $c_{\alpha}(x) = c_{a_4 a_2}(a_1)$. Also $c_{\alpha}(a_5) = c_{\bar{a}_5}(\alpha) = c_{a_3}(a_6) = c_{\bar{a}_6}(a_3)$. So

$$\begin{aligned} a_5 \perp x &\iff c_{\bar{a}_6}(a_3) \perp c_{a_4 a_2}(a_1) \iff c_{\bar{a}_4}(a_3) \perp c_{a_6}(c_{a_2}(a_1)) = c_{a_2}(a_1) \\ &\iff c_{\bar{a}_2}(c_{a_3}(a_4)) = c_{a_3}(a_4) \perp a_1. \end{aligned}$$

Step 2. Conversely, assume the relations R'_{34} . Let

$$a_1 = s, \quad a_2 = s, \quad a_3 = u, \quad a_4 = c_{\bar{u}}(v), \quad a_5 = c_{\bar{s} \bar{t}}(w), \quad a_6 = c_{\bar{u} \bar{s} \bar{t} \bar{w} \bar{u} \bar{v}}(x).$$

Let

$$\beta = c_{vu}(w), \quad \gamma = tsu\bar{x}.$$

It is useful to note the following relations:

$$(15) \quad \beta \text{ commutes with } s \text{ and } u, \text{ and } \beta \text{ braids with } c_t(s) \text{ and } c_t(u).$$

Proof of (15). Since s commutes with u, v and w , it commutes with β . Next, $\beta = c_{vu}(w) \perp u$ if and only if $w \perp c_{\bar{u} \bar{v}}(u) = v$. Now, β braids with $c_t(s) = c_{\bar{s}}(t)$ (resp. $c_t(u) = c_{\bar{u}}(t)$) if and only if β braids with t . Finally $\beta = c_{vu}(w) \checkmark t \iff c_u(w) \checkmark t \iff \mathcal{C}_6 \langle t, u, w \rangle$. \square

Note that

$$\beta = c_{v\bar{w}}(u) = c_{\bar{w}v}(u) = c_{\bar{w}\bar{u}}(v); \text{ so } x = c_{x\bar{u}\bar{s}\bar{t}\bar{w}\bar{u}}(v) = c_{\bar{\gamma}}(\beta).$$

Now we verify that a_1, \dots, a_6 satisfies the relations in R_{34} involving a_6 by reducing them to the relations in R'_{34} or those mentioned in (15).

$$\begin{aligned}
 a_6 \checkmark a_1 &\stackrel{c_\gamma}{\iff} \beta \checkmark c_{tsu\bar{x}}(s) = c_t(s), \\
 a_6 \checkmark a_2 &\stackrel{c_{s\gamma}}{\iff} c_s(\beta) \checkmark c_{stsu\bar{x}}(t) \iff \beta \checkmark c_{stsu}(t) = c_{st\bar{s}}(u) = c_{ts}(u) = c_t(u), \\
 a_6 \checkmark a_3 &\stackrel{c_\gamma}{\iff} \beta \checkmark c_{tsu\bar{x}}(u) = c_t(u), \\
 a_6 \perp a_4 &\iff x \perp c_{vuwstu} \bar{u}(v) = c_{vuwst}(v) = c_{vu}(v) = u, \\
 a_6 \perp a_5 &\stackrel{c_\gamma}{\iff} c_{\bar{w}\bar{u}}(v) \perp c_{tsu\bar{x}\bar{s}\bar{t}}(w) \\
 &= c_{t\bar{u}\bar{t}}(w) = c_{c_t(u)}(w) = c_{\bar{w}}(c_t(u)) = c_{\bar{w}\bar{u}}(t) \iff v \perp t, \\
 \mathcal{C}_4\langle a_2, a_3, a_6 \rangle &\iff a_6 \perp c_{a_2}(a_3) = c_t(u) \stackrel{c_\gamma}{\iff} \beta \perp c_{tsu\bar{x}t}(u) = c_{ts}(t) = s, \\
 \mathcal{C}_4\langle a_2, a_1, a_6 \rangle &\iff a_6 \perp c_{a_2}(a_1) = c_t(s) \stackrel{c_\gamma}{\iff} \beta \perp c_{tust}(s) = c_{tu}(t) = u, \\
 \mathcal{C}_9\langle a_5, a_3, a_6, a_1 \rangle &\iff c_{a_5 a_3}(a_6) \checkmark a_1 \iff a_6 \checkmark c_{\bar{a}_3 \bar{a}_5}(a_1) = c_{\bar{u}\bar{s}\bar{t}\bar{w}\bar{t}\bar{s}}(s) = c_{\bar{u}\bar{s}\bar{t}\bar{w}\bar{s}}(t) \\
 &\iff c_{\bar{u}\bar{v}}(x) \checkmark c_{\bar{s}}(t) \iff c_{xv}(u) \checkmark t \iff u \checkmark t.
 \end{aligned}$$

□

B.6. Proof of Proposition 4.6 for G_{29} .

Proof. Step 1. Define s, \dot{s}, t, v, u in B_{29} by

$$s = c_{a_4 a_2 a_3}(a_1), \quad \dot{s} = c_{\bar{a}_1 \bar{a}_3 \bar{a}_2}(a_4), \quad t = a_1, \quad v = a_2, \quad u = a_3.$$

One has

$$\begin{aligned}
 c_{\bar{a}_4 \bar{a}_1 \bar{a}_3 \bar{a}_2}(a_4) &= c_{a_3 \bar{a}_1 \bar{a}_3 \bar{a}_4 \bar{a}_1 \bar{a}_2}(a_4) \quad (\text{using } \mathcal{C}_4\langle a_3, a_1, a_4 \rangle \text{ in the form} \\
 &\quad \bar{a}_4 \bar{a}_1 \bar{a}_3 = a_3 \bar{a}_1 \bar{a}_3 \bar{a}_4 \bar{a}_1) \\
 &= c_{a_3 \bar{a}_1 \bar{a}_3}(a_2) \quad (\text{using } \mathcal{C}_4\langle a_4, a_1, a_2 \rangle \text{ in the form } c_{\bar{a}_4 \bar{a}_1 \bar{a}_2}(a_4) = a_2) \\
 &= c_{a_2 a_3}(a_1) \quad (\text{using } \mathcal{C}_6\langle a_3, a_1, a_2 \rangle \text{ in the form } a_2 \checkmark c_{a_3}(a_1)),
 \end{aligned}$$

that is,

$$s = \dot{s}.$$

Now we verify that s, t, v, u satisfy R'_{29} . Only the relations involving s require some work.

$$\begin{aligned}
 s \perp v &\iff c_{a_4 a_2 a_3}(a_1) \perp a_2 \iff c_{a_2 a_3}(a_1) \perp c_{\bar{a}_4}(a_2) = c_{a_2}(a_4) \\
 &\iff c_{a_3}(a_1) \perp a_4 \iff \mathcal{C}_4\langle a_3, a_1, a_4 \rangle, \\
 s \perp u &\iff c_{\bar{a}_1 \bar{a}_3 \bar{a}_2}(a_4) \perp a_3 \iff c_{\bar{a}_3 \bar{a}_2}(a_4) \perp c_{a_1}(a_3) = c_{\bar{a}_3}(a_1) \\
 &\iff c_{\bar{a}_2}(a_4) \perp a_1 \iff \mathcal{C}_4\langle a_4, a_2, a_1 \rangle, \\
 c_{\dot{s}}(a_1) &= c_{\bar{a}_1 \bar{a}_3 \bar{a}_2 a_4 a_2 a_3 a_1}(a_1) = c_{\bar{a}_1 \bar{a}_3 \bar{a}_2 a_4 a_2 a_3}(a_1) = c_{\bar{a}_1 \bar{a}_3 \bar{a}_2}(s) \\
 &= c_{\bar{a}_1}(s),
 \end{aligned}$$

where the last equality uses $s \perp a_2$ and $s \perp a_3$. So $s \checkmark t$.

Step 2. Conversely, define $a_1, a_2, a_3, a_4 \in B'_{29}$ by

$$a_1 = t, \quad a_2 = v, \quad a_3 = u, \quad \text{and} \quad a_4 = c_{vut}(s) = c_{vu\bar{s}}(t) = c_{\bar{s}vu}(t).$$

We have to check that a_1, a_2, a_3, a_4 satisfies R_{29} . Only the relations involving a_4 require some verification, which is done below. (We write $t \check{\frown}_4 v$ to denote the relation $\mathcal{C}_4\langle t, v \rangle$.)

$$\begin{aligned} a_4 \check{\frown} a_2 &\iff c_{\bar{s}vu}(t) \check{\frown} v \iff c_u(t) \check{\frown} v \iff \mathcal{C}_6\langle u, t, v \rangle, \\ a_4 \check{\frown}_4 a_3 &\iff c_{\bar{s}vu}(t) \check{\frown}_4 u \iff t \check{\frown}_4 c_{\bar{u}\bar{v}}(u) = v, \\ a_4 \check{\frown}_4 a_1 &\iff c_s(t) \check{\frown}_4 c_{vu}(t) = c_{c_u(\bar{t})}(v) \quad (\text{since } c_u(t) \check{\frown} v) \\ &\iff c_{\bar{t}}(s) \check{\frown}_4 c_{\bar{t}\bar{u}\bar{t}}(v) \iff s \check{\frown}_4 c_t(v) \iff c_s(t) = c_{\bar{t}}(s) \check{\frown}_4 v \iff t \check{\frown}_4 v. \end{aligned}$$

□

B.7. Proof of Proposition 4.6 for G_{31} .

Proof. Step 1. Assume a_1, \dots, a_5 satisfy R_{31} . Define s, t, u, v, w by

$$(16) \quad s = a_5, \quad t = a_1, \quad u = a_3, \quad w = c_{\bar{a}_3}(a_4), \quad v = c_{a_4}(a_2).$$

One has to check that s, t, \dots satisfy R'_{31} . The braid relations between s, t, u in R'_{31} are the same as the braid relations between a_5, a_1, a_3 in R_{31} . The rest of R'_{31} is verified below.

$$\begin{aligned} t \perp w &\iff 1 \perp c_{\bar{a}_3}(a_4) \iff c_{a_3}(a_1) \perp a_4 \iff \mathcal{C}_4\langle a_3, a_1, a_4 \rangle, \\ s \perp v &\iff a_5 \perp c_{a_4}(a_2) \iff \mathcal{C}_4\langle a_4, a_2, a_5 \rangle, \\ t \perp v &\iff a_1 \perp c_{a_4}(a_2) \iff \mathcal{C}_4\langle a_4, a_2, a_1 \rangle, \\ u \check{\frown} v &\iff a_3 \check{\frown} c_{a_4}(a_2) \iff \mathcal{C}_6\langle a_2, a_3, a_4 \rangle \quad (\text{see Lemma 4.4(b)}), \\ v \check{\frown} w &\iff c_{a_4}(a_2) \check{\frown} c_{\bar{a}_3}(a_4) \iff c_{a_3 a_4}(a_2) \check{\frown} a_4 \\ &\iff c_{c_{a_4}(\bar{a}_2)}(a_3) \check{\frown} a_4 \quad (\text{using } \mathcal{C}_6\langle a_2, a_3, a_4 \rangle) \\ &\iff a_3 \check{\frown} c_{a_4 a_2 \bar{a}_4}(a_4) = c_{a_4 a_2}(a_4) = a_2. \end{aligned}$$

Finally $suw = uws = wsu$ follows from $\mathcal{C}_3\langle a_5, a_4, a_3 \rangle$ and $\mathcal{C}_3\langle a_4, a_3, a_5 \rangle$ as shown below:

$$\begin{aligned} suw &= a_5 a_3 c_{\bar{a}_3}(a_4) = a_5 a_4 a_3, \quad uws = a_3 c_{\bar{a}_3}(a_4) a_5 = a_4 a_3 a_5, \\ wsu &= \bar{a}_3 a_4 a_3 a_5 a_3 = \bar{a}_3 a_3 a_5 a_4 a_3 = a_5 a_4 a_3. \end{aligned}$$

Step 2. Conversely, assume that s, t, u, v, w satisfy R'_{31} . Invert the equations in (16) to obtain

$$a_5 = s, \quad a_1 = t, \quad a_3 = u, \quad a_4 = c_u(w) = c_{\bar{s}}(w), \quad a_2 = c_{u\bar{w}\bar{u}}(v) = c_{\bar{s}\bar{w}}(v).$$

Again, the braid relations between a_5, a_1, a_3 are the same as the braid relations between s, t, u . The rest of R_{31} is verified below.

$$\begin{aligned} a_3 \check{\frown} a_2 &\iff u \check{\frown} c_{\bar{s}\bar{w}}(v) \iff c_{ws}(u) \check{\frown} v \iff u \check{\frown} v \\ &\quad (\text{since } wsu = uws \text{ implies } c_{ws}(u) = u), \\ a_4 \check{\frown} a_2 &\iff c_{\bar{s}}(w) \check{\frown} c_{\bar{s}\bar{w}}(v) \iff w \check{\frown} v. \end{aligned}$$

Now, since we know the braid relations between a_1, \dots, a_5 , Lemma 4.4 can be applied to conclude $\mathcal{C}_4\langle a_3, a_1, a_4 \rangle \iff c_{a_3}(a_1) \check{\frown} a_4$ etc. So the equivalences given in Step 1 show that $\mathcal{C}_4\langle a_3, a_1, a_4 \rangle, \mathcal{C}_4\langle a_4, a_2, a_5 \rangle, \mathcal{C}_4\langle a_4, a_2, a_1 \rangle$ and $\mathcal{C}_6\langle a_2, a_3, a_4 \rangle$ hold. One has

$$\mathcal{C}_4\langle a_1, a_5, a_2 \rangle \iff a_2 \perp c_{a_1}(a_5) \iff c_{\bar{s}\bar{w}}(v) \perp c_t(s) = c_{\bar{s}}(t) \iff c_{\bar{w}}(v) \perp t.$$

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