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REGULARITY OF MORREY COMMUTATORS

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ABSTRACT. This paper is devoted to presenting a new proof of boundedness of the commutator $bI_\alpha - I_\alpha b$ (in which I_α and b are regarded as the Riesz and multiplication operators) acting on the Morrey space $L^{p,\lambda}$ under $b \in \mathrm{BMO}$, and naturally, developing a regularity theory of commutators for Morrey-Sobolev spaces $I_\alpha(L^{p,\lambda})$ via a completely original iteration of I_α. Even in the special case of $I_\alpha(L^p)$, this is a new theory.

1. INTRODUCTION

This beginning section outlines our current work and collects the fundamental notation that will be used throughout the paper.

1.1. **Overview.** This article is two-fold: first to give an alternative argument (based on the new dual principle for Morrey spaces in [10]) for the following implication (cf. [19], [13], [14], [22]):

$$b \in \mathrm{BMO} \Longrightarrow [b, I_\alpha] : L^{p,\lambda} \to L^{\frac{p\lambda}{\lambda-\alpha p},\lambda} \text{ being bounded under } \lambda > \alpha p,$$

thereby establishing the corresponding property for some constant $c > 0$:

$$b \in \mathrm{BMO} \Longrightarrow \sup_{\|f\|_{L^{p,\lambda}} \le 1} \int_{B(0,1)} \exp(c|[b, I_\alpha]f(x)|)\, dx < \infty \text{ under } \lambda = \alpha p;$$

second, to develop, with much more effort on relating the traces of Morrey potentials $I_\alpha(L^{p,\lambda})$ in [11] to an entirely new way of constructing both the non-negative function h enjoying

$$|[b, I_\alpha]f| \lesssim I_\alpha h, \quad \|h\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\lambda}}, \quad |\nabla b| \in L^N$$

and its iterated form, a regularity (or trace, or integrability) theory of the commutators associated with Riesz potential integral operators I_α acting on Morrey spaces $L^{p,\lambda}$ covering Lebesgue spaces L^p, namely, to explore the precise values of $q > 0$ and the essential properties of both non-negative Borel measures μ on $\mathbb{R}^N$ and BMO symbols b such that

$$\sup_{\|f\|_{L^{p,\lambda}} \le 1} \int_{B(0,1)} |[b, I_\alpha]f|^q\, d\mu < \infty \quad \text{holds when} \quad \lambda > \alpha p \tag{1.1}$$

and for some constant $c > 0$,

$$\sup_{\|f\|_{L^{p,\lambda}} \le 1} \int_{B(0,1)} \exp\left(c|[b, I_\alpha]f|^q\right) d\mu < \infty \quad \text{holds when} \quad \lambda = \alpha p. \tag{1.2}$$

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1.2. **Conventions.** Before going into the next two sections detailing the above overview, let us agree to some conventions. The N-dimensional Euclidean space $\mathbb{R}^N$, the set of positive integers $\mathbb{Z}^+$, and the Euclidean open ball $B(x,r)$ with center x and radius r are our working base. On $\mathbb{R}^N$ the Morrey functions and John-Nirenberg functions – $f \in L^{p,\lambda}$ (where $1 \le p < \infty$ and $0 < \lambda \le N$) and $b \in \mathrm{BMO}$ (cf. [18]) mean respectively:

$$\|f\|_{L^{p,\lambda}} = \left[\sup_{x\in\mathbb{R}^N\ \&\ r>0} r^{\lambda-N}\int_{B(x,r)} |f(y)|^p\,dy\right]^{\frac{1}{p}} < \infty$$

and

$$\|b\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^N\ \&\ r>0} \fint_{B(x,r)} \left|b - \fint_{B(x,r)} b\right| < \infty,$$

where $\fint_{B(x,r)} b$ denotes the Lebesgue integral mean of b over $B(x,r)$.

Given $\alpha \in (0,N)$, denote the α-th order Riesz commutator with symbol b by

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - [I_\alpha(bf)](x) = \int_{\mathbb{R}^N} \frac{[b(x)-b(y)]f(y)}{|x-y|^{N-\alpha}}\,dy,$$

where

$$I_\alpha f(x) = \int_{\mathbb{R}^N} f(y)|x-y|^{\alpha-N}\,dy$$

represents the α-th order Riesz operator acting on a given function f in $\mathbb{R}^N$.

The notation $(\partial/\partial x)^s U$ represents the distributional derivative of U of order $\mathbf{s} = (s_1,\dots,s_N)$ with $s_j \in \mathbb{Z}^+$, i.e.,

$$\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} = \Big(\frac{\partial}{\partial x_1}\Big)^{s_1}\cdots\Big(\frac{\partial}{\partial x_N}\Big)^{s_N},$$

and given $m \in \mathbb{Z}^+$, $|D^m U|$ is used as the Euclidean norm of the m-th order gradient

$$D^m U = \left\{\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} U\right\}_{|\mathbf{s}|=m}, \quad \text{where} \quad |\mathbf{s}| = \sum_{j=1}^N s_j.$$

In the above and below, $\mathsf{X} \lesssim \mathsf{Y}$, $\mathsf{X} \gtrsim \mathsf{Y}$, and $\mathsf{X} \approx \mathsf{Y}$ are employed to denote that there exists a constant $c > 0$ such that $\mathsf{X} \le c\mathsf{Y}$, $\mathsf{X} \ge c\mathsf{Y}$, and $c^{-1}\mathsf{Y} \le \mathsf{X} \le c\mathsf{Y}$, separately. Moreover, $\int\cdots$ is used whenever an integral domain is clear and the integral is with respect to the Lebesgue measure.

2. Morrey commutators

By a Morrey commutator we mean $[b, I_\alpha]f$, where $f \in L^{p,\lambda}$. In this section, we revisit continuity of $[b, I_\alpha]$ with $b \in \mathrm{BMO}$ under the standard setting $\lambda > p\alpha$ via giving a new proof, and find out some properties of the commutators under the endpoint setting $\lambda = p\alpha$. To do so, we need the following Adams' extension of the classical Sobolev imbedding to Morrey spaces (cf. [3, Theorem 3.1] or [5, Theorem 4.1 & Remark 4.1]):

Theorem 2.1 (Adams)**.** *Let $\alpha \in (0,N)$ and $\lambda \in (0,N]$.*
(i) *If $1 < p < \frac{\lambda}{\alpha}$, then*

$$\|I_\alpha f\|_{L^{\frac{\lambda p}{\lambda-\alpha p},\lambda}} \lesssim \|f\|_{L^{p,\lambda}}.$$

(ii) *If* $1 < p = \frac{\lambda}{\alpha}$, *then*

$$\|I_\alpha f\|_{\mathrm{BMO}} \lesssim \|f\|_{L^{p,\lambda}}.$$

2.1. **All exponents** $p \in (1, \frac{\lambda}{\alpha})$**.** Following [10], which gives a new formulation of the predual to a Morrey space with $\lambda \in (0, N)$, we write

$$H^{p,\lambda} = \left\{ f \in L^p_{loc} : \ \|f\|_{H^{p,\lambda}} = \inf_w \Big(\int_{\mathbb{R}^N} |f(y)|^p w(y)^{1-p}\, dy \Big)^{\frac{1}{p}} < \infty \right\},$$

where L^p_{loc} is the class of all p-locally integrable functions on $\mathbb{R}^N$ and the infimum ranges over the class $A_1^{(N-\lambda)}$ of all non-negative weights w belonging to the weight class A_1 (whose definition and related A_p-weights may be read in e.g. [16], [25], and [26]) and satisfying

$$\int_{\mathbb{R}^N} w\, d\Lambda^{(\infty)}_{N-\lambda} = \int_0^\infty \Lambda^{(\infty)}_{N-\lambda}(\{x \in \mathbb{R}^N : w(x) > t\})\, dt \le 1.$$

In the above and below, for $0 < \lambda < N$ the symbol $\Lambda^{(\infty)}_{N-\lambda}$ is the Hausdorff capacity of order $N - \lambda$, as a set function on $\mathbb{R}^N$.

It is known that if $\alpha \in (0, N)$, $p \in (1, \infty)$, $0 < \alpha p < \lambda$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\lambda}$, then

$$[b, I_\alpha] : \ L^{p,\lambda} \to L^{q,\lambda} \Longleftrightarrow b \in \mathrm{BMO}\,.$$

See also [19], [13], [14], [22] and the references therein.

Our contribution to this direction is to present an alternative proof to the above known result.

Theorem 2.2. *Let*

$$0 < \lambda, \alpha < N, \ 1 < p < \frac{\lambda}{\alpha}, \ p' = \frac{p}{p-1}, \ \bar{p} = \frac{\lambda p}{\lambda - \alpha p}, \ \& \ \bar{p}' = \frac{\bar{p}}{\bar{p}-1}.$$

Then the following three statements are equivalent:
(i) $b \in \mathrm{BMO}$*;*
(ii) $[b, I_\alpha] : \ H^{p,\lambda} \to H^{\bar{p},\lambda}$ *is bounded;*
(iii) $[b, I_\alpha] : \ L^{\bar{p}',\lambda} \to L^{p',\lambda}$ *is bounded.*

Proof. Because (ii)⇒(iii)⇒(i) follows from the dual pairing

$$\int ([b, I_\alpha] f) g = -\int ([b, I_\alpha] g) f \quad \forall\ f \in H^{p,\lambda} \quad \& \quad g \in L^{\bar{p}',\lambda}$$

and the known implication in [14]-[13], it remains to prove (i)⇒(ii). Suppose $b \in$ BMO. We rely on the following basic estimate, which appeared in [14, Lemma 2], [13, Lemma 2] and [22, Lemma 1]:

$$\big([b, I_\alpha] f\big)^{\#} \lesssim \|b\|_{\mathrm{BMO}} \Big([M(I_\alpha f)^r]^{\frac{1}{r}} + [M_{\alpha t} f^t]^{\frac{1}{t}} \Big) \tag{2.1}$$

for

$$1 < r, t < p < \frac{N}{\alpha} \ \& \ f \ge 0.$$

Here and henceforth,

$$f^{\#}(x) = \sup_{x\in B} \fint_B \left| f - \fint_B f \right|$$

is the sharp function of f at x, where the supremum is taken over all Euclidean balls B containing x, and

$$M_\beta f(x) = \sup_{r>0} r^\beta \fint_{B(x,r)} |f|, \quad 0 \le \beta \le N$$

is the fractional maximal function of f; in particular, M_0 is the standard Hardy-Littlewood maximal operator M.

Instead of solving the original problem, we work on the predual space $H^{p,\lambda}$. To see this, without loss of generality, we may assume $0 \le f \in H^{p,\lambda}$ and $0 \le g \in L^{q',\lambda}$, where q' is the conjugate of $q = \bar{p}$. Due to the following duality inequality (see [25, p. 146] but also the proof of [10, Theorem 4.5]),

$$\left| \int \big([b, I_\alpha]f\big) g \right| \lesssim \int \big([b, I_\alpha]f\big)^{\#} Mg, \tag{2.2}$$

from (2.1) we need to first handle the following product:

$$P_1 := \left(\inf_{w\in A_1^{(N-\lambda)}} \int \big[M(I_\alpha f)^r\big]^{\frac{q}{r}} w^{1-q} \right)^{\frac{1}{q}} \|Mg\|_{L^{q',\lambda}}.$$

But $w \in A_1$, so $w^{1-q} \in A_q$. However, via the reverse Hölder inequality, we have $w^{1-q} \in A_{q/r}$ for some $r > 1$. So, via A_p weight theory, we use [10, Theorem 4.7 (ii)] to derive

$$\begin{aligned} P_1 &\lesssim \left(\inf_{w\in A_1^{(N-\lambda)}} \int (I_\alpha f)^q w^{1-q} \right)^{\frac{1}{q}} \|g\|_{L^{q',\lambda}} \\ &\lesssim \|f\|_{H^{p,\lambda}} \|g\|_{L^{q',\lambda}}. \end{aligned}$$

Next, we look at

$$P_2 := \left(\inf_{w\in A_1^{(N-\lambda)}} \int \big[M_{\alpha t}(f^t)\big]^{\frac{q}{t}} w^{1-q} \right)^{\frac{1}{q}} \|Mg\|_{L^{q',\lambda}}.$$

For this, we need the following.

Lemma 2.3. *If $w \in A_1$, then there exists a $\delta > 0$ such that $w^t \in A_1$ whenever $0 < t < 1 + \delta$. In particular, this implication holds for all t close to 1. Moreover, if c_1 denotes the "A_1 constant for w", i.e.,*

$$\fint_Q w \le c_1 \inf_Q w \quad \forall \text{ coordinate cubes } Q \subseteq \mathbb{R}^N,$$

then, with c_t the "A_1 constant for w^t", we have

$$c_t \le \begin{cases} c_1^t, & 0 < t < 1, \\ (c_1 c_{RH})^t, & 1 < t < 1 + \delta, \end{cases}$$

where c_{RH} stands for the reverse Hölder constant.

In fact, when $0 < t < 1$, we have that for any coordinate cube Q,

$$\fint_Q w^t \leq \Big(\fint_Q w\Big)^t \leq c_1^t \inf_Q w^t.$$

While there is a $\delta > 0$ such that the reverse Hölder inequality holds with constant $c_{RN} > 0$ (cf. [26, p. 230]), consequently, under $1 < t < 1 + \delta$,

$$\Big(\fint_Q w^t\Big)^{\frac{1}{t}} \leq c_{RH} \fint_Q w \leq c_{RH} c_1 \inf_Q w.$$

Now, we use this lemma as follows: Write

$$w^{1-q} = \Theta^{1-\frac{q}{t}}, \quad \text{where} \quad \Theta = w^{\frac{\frac{q}{t}-1}{\frac{q}{t}-1}} \in A_1$$

since the map $w \to w^t$ is bijective on A_1. Thus, we have

$$P_2 \lesssim \|M_{\alpha t}(f^t)\|^{\frac{1}{t}}_{H^{\frac{q}{t},\lambda}} \|g\|_{L^{q',\lambda}} \lesssim \|f^t\|^{\frac{1}{t}}_{H^{\frac{p}{t},\lambda}} \|g\|_{L^{q',\lambda}}$$

thanks to

$$M_{\alpha t}(f^t) \lesssim I_{\alpha t}(f^t)$$

and [10, Theorem 4.7]. But again using Lemma 2.3 we see that

$$\begin{aligned} P_3 : \quad &= \quad \left(\inf_{w \in A_1^{(N-\lambda)}} \int (f^t)^{\frac{p}{t}} w^{1-\frac{p}{t}} \right)^{\frac{1}{p}} \|g\|_{L^{q',\lambda}} \\ &\lesssim \quad \left(\inf_{\Theta \in A_1^{(N-\lambda)}} \int f^p \Theta^{1-p} \right)^{\frac{1}{p}} \|g\|_{L^{q',\lambda}} \\ &\lesssim \quad \|f\|_{H^{p,\lambda}} \|g\|_{L^{q',\lambda}}, \end{aligned}$$

thereby reaching

$$P_2 \lesssim P_3 \lesssim \|f\|_{H^{p,\lambda}} \|g\|_{L^{q',\lambda}}.$$

The previous estimates for P_1 and P_2, along with duality, yield the desired result. □

2.2. **The borderline $p = \frac{\lambda}{\alpha} > 1$.** The foregoing argument does not apply to the endpoint case $p = \frac{\lambda}{\alpha}$. But we can settle this via a direct application of (2.1).

Theorem 2.4. *Let $1 < p = \frac{\lambda}{\alpha}$ and $0 < \lambda, \alpha < N$. If $0 < \|b\|_{\mathrm{BMO}} < \infty$, then there exists a constant $c > 0$ such that*

$$\sup_{\|f\|_{L^{p,\lambda}} \leq 1} \int_{B(0,1)} \exp\Big[\frac{c|[b, I_\alpha]f(x)|}{\|b\|_{\mathrm{BMO}}}\Big]\, dx < \infty.$$

Proof. Without loss of generality, we may assume $f \geq 0$. We first prove that for each $\gamma > 0$ there is a constant $c > 0$ such that

$$\int_{B(0,1)} \exp[\gamma g(x)]\, dx \lesssim \int_{B(0,2)} \exp[c\gamma g^{\#}(x)]\, dx \tag{2.3}$$

holds for $g \geq 0$. In fact, following the arguments for [10, Remark 4.6], [16, Theorem 3.8] and [23, Lemmas 3.5-3.6 & Theorem 3.7], we can use the Calderón-Zygmund decomposition for a cube $2Q$ (cf. [17, p. 127]) with $Q \subseteq B(0,1) \subseteq 2Q \subseteq B(0,2)$ to

obtain two constants $c_1, c_2 > 0$ independent of both $0 \le g \in L^{q_0}(B(0,2))$ for some $q_0 \in [1,\infty)$ and $q \ge q_0$ such that

$$\int_{B(0,1)} [g(x)]^q \, dx \le c_1 c_2^q \int_{B(0,2)} [g^{\#}(x)]^q \, dx.$$

This, along with

$$\exp t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \quad \forall\, t > 0,$$

yields (2.3).

Next, (2.1) suggests to us to handle two terms. The first one is

$$\int_{B(0,2)} \big[M(I_\alpha f)^r\big]^{\frac{k}{r}} \, dx \le c(k) \int_{B(0,2)} \big[I_\alpha f(x)\big]^k \, dx \quad \text{for integer} \quad k > r,$$

where (cf. [20, p. 12])

$$c(k) = \frac{(k/r) 2^{k/r} 5^N}{(k/r) - 1}.$$

Now since $0 \le f \in L^{p,\lambda}$ with $\lambda = \alpha p$, $I_\alpha f \in \mathrm{BMO}$ follows from Theorem 2.1 (ii) with $\alpha = \frac{\lambda}{p}$. By the well-known John-Nirenberg theorem (cf. [18]), there is a constant $c_3 > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_3^k}{k!} \int_{B(0,2)} \big[I_\alpha f(x)\big]^k \, dx < \infty.$$

This, in turn, induces a constant $c_{3,*} > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_{3,*}^k}{k!} \int_{B(0,2)} \big[M(I_\alpha f)^r(x)\big]^{k/r} \, dx < \infty. \tag{2.4}$$

The second one is

$$\int_{B(0,2)} \big[M_{\alpha t} f^t(x)\big]^{\frac{k}{t}} \, dx \lesssim \int_{B(0,2)} \big[I_{\alpha t} f^t(x)\big]^{k/t} \, dx \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

This follows from $M_{\alpha t} f^t \lesssim I_{\alpha t} f^t$. Note also that

$$f^t \in L^{p/t,\lambda} \quad \& \quad (\alpha t)(p/t) = \lambda.$$

So $I_{\alpha t} f^t \in \mathrm{BMO}$ follows by Theorem 2.1(ii). Consequently, there is a constant $c_4 > 0$ such that

$$\sum_{k=0}^{\infty} \frac{c_4^k}{k!} \int_{B(0,2)} \big[M_{\alpha t} f^t(x)\big]^{k/t} \, dx < \infty. \tag{2.5}$$

A combination of (2.1) and (2.3) yields that for $\gamma > 0$ and $\|f\|_{L^{p,\lambda}} \le 1$,

$$\begin{aligned}
&\int_{B(0,1)} \exp\Big[\frac{\gamma|[b,I_\alpha]f(x)|}{\|b\|_{\mathrm{BMO}}}\Big]\,dx \\
&\lesssim \int_{B(0,2)} \exp\Big[\frac{c\gamma\big([b,I_\alpha]f\big)^{\#}(x)}{\|b\|_{\mathrm{BMO}}}\Big]\,dx \\
&\lesssim \int_{B(0,2)} \exp\Big[cc'\gamma\Big(\big[M(I_\alpha f)^r(x)\big]^{\frac{1}{r}} + \big[M_{\alpha t}f^t(x)\big]^{\frac{1}{t}}\Big)\Big]\,dx \\
&\lesssim \left(\int_{B(0,2)} \exp\Big[2cc'\gamma\big[M(I_\alpha f)^r(x)\big]^{\frac{1}{r}}\Big]\,dx\right)^{\frac{1}{2}} \\
&\quad\times \left(\int_{B(0,2)} \exp\Big[2cc'\gamma\big[M_{\alpha t}f^t(x)\big]^{\frac{1}{t}}\Big]\,dx\right)^{\frac{1}{2}},
\end{aligned}$$

where $c' > 0$ is the constant in front of $\|b\|_{\mathrm{BMO}}$ of (2.1).

The last estimate, together with (2.4) and (2.5), derives the desired inequality, via choosing a small $\gamma > 0$. □

Remark 2.5. Let $p \ge \frac{\lambda}{\alpha} > 1$, $0 < \lambda \le N$, $f \in L^{p,\lambda}$ be supported in $B(0,1)$ with $\|f\|_{L^{p,\lambda}} \le 1$, and $b \in L^\infty$ with

$$\|b\|_{os} = \sup_{(x,r)\in\mathbb{R}^N\times(0,\infty)} \sup_{y\in B(x,r)} |b(x) - b(y)| > 0.$$

(i) If $p = \frac{\lambda}{\alpha}$, then

$$\int_{B(0,1)} \exp\left[c\frac{|[b,I_\alpha]f(x)|}{\|b\|_{os}}\right]\,dx < \infty \tag{2.6}$$

is valid for any non-negative constant c less than

$$c_{\alpha,\lambda} = \frac{\lambda}{N-\alpha}\Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{1}{p}-1},$$

where ω_{N-1} is the surface area of $B(0,1)$.
(ii) If $p > \frac{\lambda}{\alpha}$, then

$$\sup_{x\in B(0,1)} \frac{|[b,I_\alpha]f(x)|}{\|b\|_{os}} \le \left(\frac{Np-\lambda}{\alpha p-\lambda}\right) 2^{\alpha-\frac{\lambda}{p}}.$$

Moreover,

$$[b,I_\alpha] : L^{p,\lambda} \to C^{\alpha-\frac{\lambda}{p}}$$

under $\|b\|_{os} < \infty$ and $\lambda/p < \alpha \le N/p$. This corresponds nicely to the Morrey type embedding below (cf. [10, Remark 3.4]),

$$I_\alpha : L^{p,\lambda} \to C^{\alpha-\frac{\lambda}{p}} \quad \text{under} \quad \frac{\lambda}{p} < \alpha \le \frac{N}{p},$$

extending the famous Morrey's lemma (cf. [20, p. 30, 1.53 Theorem]):

$$|\nabla u| \in L^{p,\lambda} \ \&\ \lambda < p \Longrightarrow u \in C^{1-\frac{\lambda}{p}}.$$

3. Regularity properties

Concerning the continuity of an operator, I_α and $[b, I_\alpha]$ (under $b \in \mathrm{BMO}$) behave very similarly. Thus, it is reasonable to expect that the regularity estimates for the Morrey commutators are similar to those for the Morrey potentials under some further assumptions on b since BMO functions are only defined with respect to the Lebesgue N-measure. Below is a summary of the regularity estimates for the Morrey potentials established in [11] (see also [1] regarding an early result of the case $\lambda = N$).

Theorem 3.1 (Adams-Xiao)**.** *Let μ be a non-negative Borel measure on $\mathbb{R}^N$ such that*

$$\sup_{(r,x)\in(0,\infty)\times\mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \textit{for} \quad N \geq d > \lambda - \alpha\, p \geq 0.$$

(i) *If $1 < p < \frac{\lambda}{\alpha}$ and $0 < \lambda < N$, then*

$$\sup_{\|f\|_{L^{p,\lambda}}\leq 1} \int_{B(0,1)} |I_\alpha f|^q \, d\mu < \infty \quad \textit{for} \quad q < \frac{dp}{\lambda - \alpha p}$$

and

$$\sup_{\|f\|_{L^{p,\lambda}}\leq 1} \int_{B(0,1)} \frac{|I_\alpha f|^{\tilde{p}}}{[\ln(1+|I_\alpha f|)]^\gamma} \, d\mu < \infty \quad \textit{for} \quad \tilde{p} = \frac{dp}{\lambda - \alpha p} \ \& \ \gamma > 2.$$

(ii) *If $1 < p = \frac{\lambda}{\alpha}$ and $0 < \lambda \leq N$, then there exists a constant $c > 0$ such that*

$$\sup_{\|f\|_{L^{p,\lambda}}\leq 1} \int_{B(0,1)} \exp\left(c|I_\alpha f|^q\right) d\mu < \infty$$

holds for

$$\textit{either} \quad (\lambda, q) \in (0, N) \times (0, 1] \quad \textit{or} \quad (\lambda, q) = \Big(N, \frac{N}{N-1}\Big).$$

3.1. **All exponents $p \in (1, \frac{\lambda}{\alpha})$.** Based on Theorem 3.1 (i), we discover the following result.

Theorem 3.2. *Let $|\nabla b| \in L^N$, $\alpha \in (0, N)$, $p > 1$, and μ be a non-negative Borel measure on $\mathbb{R}^N$ such that*

$$\sup_{(r,x)\in(0,\infty)\times\mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \textit{holds for} \quad N \geq d > \lambda - \alpha\, p > 0.$$

Then

$$\sup_{\|f\|_{L^{p,\lambda}}\leq 1} \int_{B(0,1)} |[b, I_\alpha] f|^q \, d\mu < \infty$$

holds for

$$\textit{either} \quad (\lambda, q) \in (0, N) \times \Big(0, \frac{dp}{\lambda - \alpha p}\Big) \quad \textit{or} \quad (\lambda, q) = \Big(N, \frac{dp}{\lambda - \alpha p}\Big).$$

Proof. First of all, let us control the case

$$(\lambda, q) = \Big(N, \frac{dp}{\lambda - \alpha p}\Big).$$

Under this, we consider two subcases as follows.

Subcase 1: $\alpha = m \in \mathbb{Z}^+$. Set

$$U = [b, I_\alpha]f \quad \& \quad K_\alpha(x,y) = |x-y|^{\alpha - N}.$$

If $|\mathbf{s}| = m$, then according to [3, p. 769], we have

$$|U(x)| \lesssim I_m \left(\sum_{|\mathbf{s}|=m} \left| \left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} U(x) \right| \right) \tag{3.1}$$

holds almost everywhere with respect to the N-Lebesgue measure. Note that if $m = 1$, then

$$\left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} [b, I_\alpha](x) = \left[b, \left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} K_\alpha\right] f(x) + \left(\left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} b(x) \right) (I_\alpha f)(x).$$

Thus, if $f \in L^{\bar{q},\lambda}$ (in which case $\bar{q} > 1$ will be selected later on, but $\lambda = N$ is always assumed), then Hölder's inequality and Theorem 2.1 yield

$$\begin{aligned} \left\| \left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} [b, I_\alpha] f \right\|_{L^{\bar{q},\lambda}} &\leq \left\| \left[b, \left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} K_\alpha\right] f \right\|_{L^{\bar{q},\lambda}} + \left\| \left(\left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} b(x) \right) (I_\alpha f) \right\|_{L^{\bar{q},\lambda}} \\ &\lesssim \|b\|_{\mathrm{BMO}} \|f\|_{L^{\bar{q},\lambda}} + \left\| |D^1 b| \right\|_{L^{\lambda,\lambda}} \|f\|_{L^{\bar{q},\lambda}} \\ &\lesssim \left\| |\nabla b| \right\|_{L^N} \|f\|_{L^{\bar{q},\lambda}}. \end{aligned}$$

This last estimate and (3.1) give

$$|[b, I_\alpha]f| \lesssim I_\alpha h, \tag{3.2}$$

where

$$h = \sum_{|\mathbf{s}|=1} \left| \left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} [b, I_\alpha] f \right|$$

and

$$\|h\|_{L^{\bar{q},\lambda}} \lesssim \left\| |\nabla b| \right\|_{L^N} \|f\|_{L^{\bar{q},\lambda}}.$$

Next, we take $\theta > 0$ and write the I_θ potential of the commutator as

$$\begin{aligned} I_\theta\big(bI_1 f - I_1(bf)\big) &= I_\theta(bI_1 f) - bI_{\theta+1} f + bI_{\theta+1} f - I_{\theta+1}(bf) \\ &= [b, I_{\theta+1}]f - [b, I_\theta](I_1 f), \end{aligned}$$

or

$$[b, I_{\theta+1}]f = [b, I_\theta](I_1 f) + I_\theta\big([b, I_1]f\big). \tag{3.3}$$

Below, we see that identity (3.3) is the key. In fact, from the previous case $m = 1$, we have upon setting $\theta = 1$ in (3.3),

$$|[b, I_2]f| \lesssim I_1 g + I_2 h, \tag{3.4}$$

where

$$0 \leq g \in L^{\tilde{p},\lambda}, \ \tilde{p} = \frac{\lambda p}{\lambda - p}, \ \& \ 0 \leq h \in L^{p,\lambda};$$

i.e., we apply (3.2) first with $\bar{q} = \tilde{p}$ and then with $\bar{q} = p$. So, the trace of the commutator can be determined by (3.4). Notice that the remark (ii) of [4, Theorem 4] ensures that

$$\int_{B(0,1)} |I_2 h|^q \, d\mu < \infty \quad \text{for} \quad q = \frac{dp}{\lambda - 2p}$$

and

$$\int_{B(0,1)} |I_1 g|^r \, d\mu < \infty \quad \text{for} \quad r = \frac{d\tilde{p}}{\lambda - \tilde{p}} = \frac{dp}{\lambda - 2p}.$$

As a result, we obtain that under $|\nabla b| \in L^N$,

$$\int_{B(0,1)} |[b, I_m]f|^{\frac{dp}{N-mp}} \, d\mu < \infty \quad \text{for} \quad m = 1, 2.$$

Now, assume that the assertion is valid for $\alpha = m$. When $\alpha = m+1$, we apply (3.3) with $\theta = m$ to derive

$$|[b, I_{m+1}]f| \lesssim |[b, I_m](I_1 f)| + I_m\big([b, I_1]f\big) \lesssim \sum_{j=0}^{m} I_{m+1-j} f_j, \tag{3.5}$$

where

$$0 \le f_j \in L^{\frac{\lambda p}{\lambda - jp}, \lambda} \quad \text{for} \quad j = 0, 1, ..., m$$

and

$$\int_{B(0,1)} |I_{m+1-j} f_j|^q \, d\mu < \infty \quad \text{for} \quad q = \frac{dp}{\lambda - (m+1)p}.$$

Clearly,

$$\int_{B(0,1)} |[b, I_{m+1} f]|^q \, d\mu < \infty \quad \text{for} \quad q = \frac{dp}{\lambda - (m+1)p}.$$

So, the general case follows by induction.

Subcase 2: $m - 1 < \alpha < m \in \mathbb{Z}^+$. If $m = 1$, i.e., $0 < \alpha < 1$, then according to [4, (3.2)] we have

$$|[b, I_\alpha]f| \lesssim I_\alpha \left(\sum_{|\mathbf{s}|=1} \left| \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} I_{1-\alpha}\big([b, I_\alpha]f\big) \right| \right). \tag{3.6}$$

Hence, it is enough to verify that under $|\nabla b| \in L^N$ and $|\mathbf{s}| = 1$ one has

$$f \in L^{p,\lambda} \Longrightarrow h = \left| \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} I_{1-\alpha}\big([b, I_\alpha]f\big) \right| \in L^{p,\lambda}. \tag{3.7}$$

Let us form the operator

$$T_\alpha(b, f)(x) = \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} \Big(I_{1-\alpha}[b, I_\alpha]f \Big),$$

where f may be assumed to be smooth whenever needed. After extending $T_\alpha(b, f)$ to $T_\xi(b, f)$, where $\xi = \zeta + i\eta$ and $0 < \zeta < 1$, we see that $T_\xi(b, \cdot)$ is an analytic family of operators. It remains to check the boundedness of the extended operator on L^p in the cases $\zeta = 0, 1$.

For $\zeta = 0$, we have

$$\begin{aligned} T_{i\eta}(b, f) &= \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} \Big(I_{1-i\eta}\big(b I_{i\eta} f - I_{i\eta}(bf)\big) \Big) \\ &= \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} I_{1-i\eta}[b, I_{i\eta}]f. \end{aligned}$$

Note that $\left(\frac{\partial}{\partial x}\right)^{\mathbf{s}} I_{1-i\eta}$ exists as a bounded linear operator on L^p. Thus

$$\|T_{i\eta}(b, f)\|_{L^{p,\lambda}} \lesssim \|[b, I_{i\eta}]f\|_{L^{p,\lambda}} \lesssim \|b\|_{\mathrm{BMO}} \|f\|_{L^{p,\lambda}} \lesssim \big\||\nabla b|\big\|_{L^N} \|f\|_{L^{p,\lambda}}. \tag{3.8}$$

For $\zeta = 1$, we get

$$\begin{aligned} T_{1+i\eta}(b,f) &= \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}}\Big(I_{-i\eta}\big(bI_{1+i\eta}f - I_{1+i\eta}(bf)\big)\Big) \\ &= I_{-i\eta}\left(\Big(\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}}b\Big)I_{1+i\eta}f\right) + I_{-i\eta}\big([b,\tilde{K}]f\big), \end{aligned}$$

where

$$\tilde{K}(x,\cdot) = \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}}|x-\cdot|^{1+i\eta-N}$$

is a Calderón-Zygmund integral kernel. As a result, we have

$$\|[b,\tilde{K}]\|_{L^{p,\lambda}} \lesssim \|b\|_{\mathrm{BMO}}\|f\|_{L^{p,\lambda}} \lesssim \|\nabla b\|_{L^N}\|f\|_{L^{p,\lambda}}$$

and

$$\left\|(\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}}b)I_{1+i\eta}f\right\|_{L^{p,\lambda}} \lesssim \big\||\nabla b|\big\|_{L^N}\|f\|_{L^{p,\lambda}}.$$

Also note that $I_{-i\eta}$ generates a Calderón-Zygmund integral kernel. So

$$\|T_{1+i\eta}(b,f)\|_{L^{p,\lambda}} \lesssim \big\||\nabla b|\big\|_{L^N}\|f\|_{L^{p,\lambda}}. \tag{3.9}$$

Using (3.8) and (3.9) as well as Stein's interpolation theorem (cf. [12, p. 209, Theorem 3.3]), we obtain that under $|\nabla b| \in L^N$,

$$\|T_\xi(b,f)\|_{L^{p,\lambda}} \lesssim \big\||\nabla b|\big\|_{L^N}\|f\|_{L^{p,\lambda}}, \quad 0 < \Re\xi = \zeta < 1. \tag{3.10}$$

Thus, T_ξ is a bilinear analytic family of operators. Consequently, (3.10) is true for $\xi = \alpha \in (0,1)$. Therefore, (3.6), (3.7) and [8, p. 193, Theorem 7.2.2] yield the required result under $0 < \alpha < 1$.

If $m > 1$, then an application of (3.3) and the above treatment of $m = 1$ gives the assertion. As a matter of fact, since $1 \le m-1 < \alpha < m$, there is a $\theta \in (0,1)$ such that $\alpha = m-1+\theta$. When $m = 2$, from (3.3) and the handling of $m = 1$ there are two non-negative functions

$$f_1 \in L^{\frac{\lambda p}{\lambda - p},\lambda} \quad \& \quad f_0 \in L^{p,\lambda}$$

such that

$$|[b,I_\alpha]f| \lesssim I_\theta f_1 + I_{1+\theta}f_0 \approx I_\theta f_1 + I_\alpha f_0$$

with

$$\int_{B(0,1)} |I_\theta f_1|^{\frac{dp}{\lambda-\alpha p}}\,d\mu < \infty, \ \int_{B(0,1)} |I_\alpha f_0|^{\frac{dp}{\lambda-\alpha p}}\,d\mu < \infty, \ \ \& \ \ \lambda = N.$$

Of course, this last argument verifies the result in the case $m = 2$. A similar argument works for other values of m through induction.

Next, let us turn to the case

$$(\lambda,q) \in (0,N) \times \Big(0, \frac{dp}{\lambda-\alpha p}\Big).$$

We settle this according to two subcases.

Subcase 1: $\alpha = m \in \mathbb{Z}^+$. This follows from Theorem 3.1(i) and the argument for the previously discussed *Subcase 1* with

$$q < \frac{dp}{\lambda-2p} \ \ \& \ \ r < \frac{d\tilde{p}}{\lambda-\tilde{p}}$$

as well as

$$q < \frac{dp}{\lambda-(m+1)p}$$

in place of

$$q = \frac{dp}{\lambda - 2p} \quad \& \quad r = \frac{d\tilde{p}}{\lambda - \tilde{p}}$$

as well as

$$q = \frac{dp}{\lambda - (m+1)p}.$$

Subcase 2: $m - 1 < \alpha < m \in \mathbb{Z}^+$. If $m = 1$, i.e., $0 < \alpha < 1$, then since it has just been proved that $|\nabla b| \in L^N$ ensures that

$$T_\alpha(b, \cdot) : L^p \to L^p \quad \text{for} \quad 1 < p < \frac{N}{\alpha},$$

we now use the Garcia-Cuerva and Rubio de Francia extrapolation theorem (cf. [16, pp. 411-412]) to get boundedness of $T_\alpha(b, \cdot)$ on the weighted Lebesgue space $L^p(w)$ with w being an A_p weight:

$$T_\alpha(b, \cdot) : L^p(w) \to L^p(w) \quad \text{for} \quad 1 < p < \frac{N}{\alpha}.$$

Consequently, $T^*_\alpha(b, \cdot)$ (the dual of $T_\alpha(b, \cdot)$) and $p' = \frac{p}{p-1}$ satisfy

$$T^*_\alpha(b, \cdot) : L^{p'}(w) \to L^{p'}(w).$$

Furthermore, choosing a weight $w \in A_p$ so that $w^{-p} \in A_1^{(N-\lambda)}$, we get

$$T^*_\alpha(b, \cdot) : H^{p',\lambda} \to H^{p',\lambda}.$$

Taking the dual of $T^*_\alpha(b, \cdot)$, we achieve

$$T_\alpha(b, \cdot) : L^{p,\lambda} \to L^{p,\lambda}, \quad \text{where} \quad 1 < p < \frac{\lambda}{\alpha} < \frac{N}{\alpha}.$$

Thus, this last boundedness plus (3.6) yields a non-negative function h such that

$$|[b, I_\alpha]f| \lesssim I_\alpha h \quad \& \quad \|h\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\lambda}},$$

thereby implying the desired result via Theorem 3.1(i).

If $m > 1$, then there is a $\beta \in (0, 1)$ and m non-negative functions $\{f_j\}_{j=0}^{m-1}$ (by (3.3)) such that $\alpha = m - 1 + \beta$ and (see also (3.5))

$$|[b, I_\alpha]f| \lesssim \sum_{j=0}^{m-1} I_{m-1-j+\beta} f_j$$

with

$$f_j \in L^{\frac{\lambda p}{\lambda - jp}, \lambda} \quad \text{for} \quad j = 0, 1, ..., m-1$$

and

$$\int_{B(0,1)} |I_{m-1-j+\beta} f_j|^q \, d\mu < \infty \quad \forall\, q < \frac{dp}{\lambda - \alpha p}.$$

As a result, we get by induction (but also by the above-treated case $m = 1$, i.e., $0 < \alpha < 1$)

$$\int_{B(0,1)} |[b, I_\alpha f]|^q \, d\mu < \infty \quad \forall\, q < \frac{dp}{\lambda - \alpha p},$$

as desired. □

Remark 3.3. Two comments are in order:
(i) Under $|\nabla b| \in L^N$, $\lambda \in (0, N]$ and $\alpha \in (0, N)$, we can prove a bit more; in fact, since there is a capacitary strong-type inequality (CSI) for $C_\alpha(\cdot; L^p)$ (but not for $C_\alpha(\cdot; L^{p,\lambda})$, $0 < \lambda < N$; see also [9]), one can get the following new trace inequality, actually a CSI:

$$\int |[b, I_\alpha]f|^p \, dC_\alpha(\cdot; L^p) = \int_0^\infty C_\alpha\big(\{x : |[b, I_\alpha]f(x)| > t\}; L^p\big) \, dt^p \lesssim \|f\|_{L^p}^p.$$

(ii) It is conjectured that Theorem 3.2 is true for $b \in I_\alpha(L^{\frac{\lambda}{\alpha},\lambda}) \subseteq \mathrm{BMO}$ with $0 < \alpha < \lambda \le N$ (cf. [4]&[6]).

3.2. **The borderline** $p = \frac{\lambda}{\alpha} > 1$**.** Regarding the regularity estimates for the Morrey commutators at the endpoint case $\alpha p = \lambda$, we first get the following result.

Theorem 3.4. *Let* $|\nabla b| \in L^N$, $\alpha \in \mathbb{Z}^+ \cap (0, N)$, $p = \frac{\lambda}{\alpha} = \frac{N}{\alpha} > 1$, *and* μ *be a non-negative Borel measure on* $\mathbb{R}^N$ *such that*

$$\sup_{(r,x)\in(0,\infty)\times\mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \textit{holds for} \quad N \ge d > 0.$$

Then there is a constant $c > 0$ *such that*

$$\sup_{\|f\|_{L^{p,\lambda}} \le 1} \int_{B(0,1)} \exp\big(c|[b, I_\alpha]f|^q\big) \, d\mu < \infty \quad \forall\, q \in \Big(0, \frac{N}{N-1}\Big).$$

Proof. First of all, we show that if

$$E_t = \{x \in \mathbb{R}^N : \ |[b, I_\alpha]f(x)| > t\} \cap B(0,1) \quad \forall\, t > 0,$$

then

$$C_1(E_t; L^N) \lesssim t^{-N} \quad \forall \text{ large} \quad t > 0.$$

To get this, we use the special case $\lambda = N$ of (3.5):

$$|[b, I_m]f| \lesssim \sum_{j=0}^{m-1} I_{m-j} f_j, \quad \text{where} \quad 0 \le f_j \in L^{\frac{pN}{N-jp}}$$

and [8, Theorem 5.5.1 (d)] to obtain a constant $\kappa > 0$ such that if

$$E_{j,t} = \Big\{x \in \mathbb{R}^N : \ I_{m-j} f_j(x) > \frac{t}{\kappa}\Big\} \cap B(0,1), \quad j = 0, 1, ..., m-1,$$

then

$$C_1(E_t; L^N) \lesssim \sum_{j=0}^{m-1} \Big(C_{m-j}\big(E_{j,t}; L^{\frac{N}{m-j}}\big)\Big)^{\frac{(N-1)(m-j)}{N-m+j}}$$

and hence, it follows from [2] (cf. [20, 1.86 Lemma]) and CSI in [4] (cf. Remark 3.3 (i)) that

$$\begin{aligned} \int_{t_0}^\infty C_1(E_t; L^N) \, dt^q &\lesssim \sum_{j=0}^{m-1} \Big(\int_{t_0}^\infty C_{m-j}(E_{j,t}; L^{\frac{N}{m-j}}) \, dt^{r_j}\Big)^{\frac{(N-1)(m-j)}{N-m+j}} \\ &\lesssim \sum_{j=0}^{m-1} \|f\|_{L^{r_j}}^q \qquad (3.11) \\ &< \infty \end{aligned}$$

holds for $t_0 > 1$ being big enough, where

$$r_j = \frac{\big(\frac{N}{m-j} - 1\big)q}{N-1} \quad \& \quad q < N.$$

Next, using [8, Theorem 5.1.12 & Corollary 5.1.14] it follows that for the given μ,

$$C_1(E_t; L^N) \geq \left(\ln \frac{1}{\mu(E_t)}\right)^{1-N}, \tag{3.12}$$

where $\mu(E_t)$ is of course assumed to be less than 1. As a result of (3.11) and (3.12), it follows that

$$\int_{t_0}^{2t_0} \left(\ln \frac{1}{\mu(E_t)}\right)^{1-N} dt^q \lesssim 1 \tag{3.13}$$

for $t_0 > 1$ being big enough. Note that the constant on the right-hand side of (3.13) is independent of f, t_0, and $\|f\|_{L^{p,\lambda}} \leq 1$. Thus (3.13) yields

$$\left(\ln \frac{1}{\mu(E_{t_0})}\right)^{1-N} t_0^q \lesssim 1,$$

whence

$$\mu(E_t) \lesssim \exp(-c_1 t^{\frac{q}{N-1}})$$

holds for some constant $c_1 > 0$ and $t \geq t_0$. Consequently, there is a constant $c \in (0, c_1)$ such that

$$\begin{aligned}
\int_{B(0,1)} \exp\big(c|[b, I_\alpha]f|^r\big)\, d\mu &= \int_0^\infty \mu(E_t)\, d\exp(ct^r) \\
&= \Big(\int_0^{t_0} + \int_{t_0}^\infty\Big)\mu(E_t)\, d\exp(ct^r) \\
&\lesssim \mu\big(B(0,1)\big)\big(\exp(ct_0^r) - 1\big) \\
&\quad + c\int_{t_0}^\infty \exp(-c_1 t^{\frac{q}{N-1}} + ct^r)\, dt^r \\
&< \infty
\end{aligned}$$

holds for all $r < \frac{q}{N-1} < \frac{N}{N-1}$. □

Remark 3.5. Theorem 3.4 is optimal in the sense that if f is a nice function, say $f \in C_0^\infty(\mathbb{R}^N)$, then $[b, I_\alpha]f$ can blow up on a zero set of $C_1(\cdot, L^N)$, i.e.,

$$C_1\big(\{x \in \mathbb{R}^N : \ |[b, I_\alpha]f(x)| = \infty\}; L^N\big) = 0.$$

The proof uses (3.2) and [8, Theorem 5.5.1 (d)], yielding that if

$$E_j = \{x \in \mathbb{R}^N : \ I_{m-j} f_j(x) = \infty\} \quad \& \quad E = \{x \in \mathbb{R}^N : \ |[b, I_\alpha]f(x)| = \infty\},$$

then $E \subseteq \bigcup_{j=0}^m E_j$ and hence

$$C_1(E; L^N) \lesssim \sum_{j=0}^{m-1} C_1(E_j; L^N) \lesssim \sum_{j=0}^{m-1} C_{m-j}\big(E_j; L^{\frac{N}{m-j}}\big)^{\frac{(N-1)(m-j)}{N-m+j}} = 0.$$

When a hypothesis on b stronger than $|\nabla b| \in L^N$ is placed, the following becomes a natural thing.

Theorem 3.6. *Let $b \in I_\alpha(L^{\frac{N}{\alpha}})$, $\alpha \in \mathbb{Z}^+ \cap (0,N)$, $p > 1$, and μ be a non-negative Borel measure on $\mathbb{R}^N$ such that*

$$\sup_{(r,x)\in(0,\infty)\times\mathbb{R}^N} \frac{\mu(B(x,r))}{r^d} < \infty \quad \textit{holds for} \quad N \geq d > \lambda - \alpha\, p = 0.$$

Then there is a constant $c > 0$ such that

$$\sup_{\|f\|_{L^{p,\lambda}}\leq 1} \int_{B(0,1)} \exp\left(c|[b, I_\alpha]f|^q\right) d\mu < \infty$$

holds for

$$\textit{either} \quad (\lambda, q) \in (0,N) \times (0,1] \quad \textit{or} \quad (\lambda, q) = \Big(N, \frac{p}{p-1}\Big).$$

Proof. The assertion follows immediately from Theorem 3.1(ii) and the following.

Lemma 3.7. *Let $b \in I_\alpha(L^{\frac{N}{\alpha}})$. Then for any $f \in L^{p,\lambda}$ there is a non-negative function $h \in L^{p,\lambda}$ such that*

$$|[b, I_\alpha]f| \lesssim I_\alpha h \quad \textit{with} \quad \|h\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\lambda}}$$

provided $\alpha = m \in \mathbb{Z}^+ \cap (0,N)$, $\lambda \in (0,N]$, and $1 < p \leq \frac{\lambda}{\alpha}$.

So, it remains to verify this lemma. Suppose

$$U = [b, I_\alpha]f, \ K_\alpha(x,y) = |x-y|^{\alpha-N}, \ \& \ f \in L^{p,\lambda}.$$

By the Leibniz rule there are constants $c_{|\mathbf{t}|,m}$ depending only on $|\mathbf{t}|$ and m such that (cf. [7])

$$\begin{aligned}\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} U(x) &= \Big[b, \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} K_\alpha\Big] f(x) \\ &\quad + \sum_{|\mathbf{t}|=1}^{|\mathbf{t}|=N-1} c_{|\mathbf{t}|,m} \int \Big(\frac{\partial}{\partial x}\Big)^{t} b(x) \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}-\mathbf{t}} K_\alpha(x,y) f(y)\, dy \\ &\quad + \Big(\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} b(x)\Big)(I_\alpha f)(x) \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

Note that

$$\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} K_\alpha(x, \cdot)$$

is a Calderón-Zygmund integral kernel, and $b \in I_\alpha(L^{\frac{N}{\alpha}}) \subseteq \mathrm{BMO}$ thanks to Theorem 2.1(ii). So $f \in L^{p,\lambda}$ yields

(3.14)

$$\|T_1\|_{L^{p,\lambda}} = \Big\|\Big[b, \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} K_\alpha\Big] f\Big\|_{L^{p,\lambda}} \lesssim \|b\|_{\mathrm{BMO}} \|f\|_{L^{p,\lambda}} \lesssim \big\||D^m b|\big\|_{L^{N/m}} \|f\|_{L^{p,\lambda}}$$

by the well-known $L^{p,\lambda}$-boundedness of the commutator $[b, K]$; see e.g.[15, Theorem 2.3] or [21, Theorem 3.3].

Next, the conclusion that $f \in L^{p,\lambda}$ and $b \in I_\alpha L^{\frac{N}{\alpha}}$ imply

$$T_3 = \Big(\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} b\Big)(I_\alpha f) \in L^{p,\lambda} \tag{3.15}$$

follows from the fact that $|D^m b| \in L^{N/m}$ and the following two situations:

Situation 1: If $N \geq \lambda > mp$, then an application of Hölder's inequality and Theorem 2.1(i) yields

$$\int_{B(z,r)} \Big|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} b(x)(I_\alpha f)\Big|^p \lesssim \left(\int_{B(z,r)} \Big|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} b(x)\Big|^{\lambda/m}\right)^{\frac{pm}{\lambda}} \left(\int_{B(z,r)} |I_m f|^{\frac{p\lambda}{\lambda-mp}}\right)^{\frac{\lambda-pm}{\lambda}}$$
$$\lesssim r^{N-\lambda} \big\||D^m b|\big\|^p_{L^{N/m}} \|f\|^p_{L^{p,\lambda}}.$$

Situation 2: If $N \geq \lambda = mp$, then the Hölder inequality and the Sobolev inequality for higher order derivatives on balls (cf. [24, Theorem 3.1]) are applied to derive that for $N > \lambda = mp$,

$$\begin{aligned}\int_{B(z,r)} \Big|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} b(x)(I_\alpha f)\Big|^p &\lesssim \big\||D^m b|\big\|^p_{L^{N/m}} \left(\int_{B(z,r)} |I_m f|^{\frac{pN}{N-pm}}\right)^{\frac{N-pm}{N}} \\ &\lesssim \big\||D^m b|\big\|^p_{L^{N/m}} \int_{B(z,r)} |f|^p \\ &\lesssim r^{N-\lambda} \big\||D^m b|\big\|^p_{L^{N/m}} \|f\|^p_{L^{p,\lambda}}.\end{aligned}$$

The case $N = \lambda = mp$ can be treated by using $\|I_m f\|_\infty$.

Finally, let us control the middle terms T_2. By Hölder's inequality, Theorem 2.1(i), and the Sobolev embedding for higher order derivatives over balls once again, we get

$$\begin{aligned}&\int_{B(z,r)} \left|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{t}} b(x) \left(\int \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}-\mathbf{t}} K_\alpha(x,y) f(y)\, dy\right)\right|^p \\ &\lesssim \int_{B(z,r)} \Big|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{t}} b(x)\Big|^p \big|I_{|\mathbf{t}|} f\big|^p \\ &\lesssim \int_{B(z,r)} |D^{|\mathbf{t}|} b|^p \big|I_{|\mathbf{t}|} f\big|^p \\ &\lesssim \left(\int_{B(z,r)} |D^{|\mathbf{t}|} b|^{\frac{\lambda}{|\mathbf{t}|}}\right)^{\frac{p|\mathbf{t}|}{\lambda}} \left(\int_{B(z,r)} \big|I_{|\mathbf{t}|} f\big|^{\frac{p\lambda}{\lambda-p|\mathbf{t}|}}\right)^{\frac{\lambda-p|\mathbf{t}|}{\lambda}} \\ &\lesssim r^{N-\lambda} \big\||D^m b|\big\|^p_{L^{N/m}} \|f\|^p_{L^{p,\lambda}},\end{aligned}$$

whence

$$\left\|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{t}} b(x) \left(\int \Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}-\mathbf{t}} K_\alpha(x,y) f(y)\, dy\right)\right\|_{L^{p,\lambda}} \lesssim \big\||D^m b|\big\|_{L^{N/m}} \|f\|_{L^{p,\lambda}}. \tag{3.16}$$

Putting (3.14)-(3.15)-(3.16) together, we get

$$h = \sum_{|\mathbf{s}|=m} \Big|\Big(\frac{\partial}{\partial x}\Big)^{\mathbf{s}} [b, I_m] f\Big| \in L^{p,\lambda}, \tag{3.17}$$

whence completing the argument for the lemma via (3.1). □

Remark 3.8. It is conjectured that Theorem 3.6 is valid for non-integral $\alpha \in (0, N)$. On the other hand, it follows readily from

$$|[b, I_\alpha] f| \leq \|b\|_{os} I_\alpha |f| \quad \& \quad \big\||f|\big\|_{L^{p,\lambda}} = \big\|f\big\|_{L^{p,\lambda}}$$

that Theorems 3.4 and 3.2 are trivially true for $b \in L^\infty$.

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