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DESCRIBING FREE GROUPS, PART II: Π_4^0 HARDNESS AND NO Σ_2^0 BASIS

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ABSTRACT. We continue the study of free groups from a computability theoretic perspective. In particular, we show that for F_∞, the free group on a countable number of generators, the descriptions given in our preceding paper are the best possible.

This paper is a continuation of [2], and we refer to it for background, motivation, and definitions. In [2], we establish that F_∞ has a Π_4^0 definition in the language of groups, and hence that $I(F_\infty)$ is a Π_4^0 set. Consequently, the same is true about $I(FrGr)$, because a countable group is free iff it is isomorphic to F_∞ or F_n for some n, and the latter have $d-\Sigma_2^0$ definitions. Here we show that $I(F_\infty)$ and $I(FrGr)$ are m-complete Π_4^0. In [2], it is shown that every computable copy of F_∞ has a Π_2^0 basis. Here we prove that the result is sharp by constructing a computable copy of F_∞ with no Σ_2^0 basis.

In both constructions, a key issue is whether or not a particular tuple $\vec{a}$ can be included in a basis of the group we are building (if, indeed, that group is free and has a basis). In general, if a group is free with basis B (given to us "ahead of time"), then deciding this matter for the tuple $\vec{a}$ is computable relative to B, since there is an algorithm that we can perform on the formal words in B to determine if that tuple can be included in a basis. However, if we do not know the basis ahead of time, then there need not be any such algorithm. This suggests the need for the careful distinctions in the following definition.

Definition 1. i) A tuple of *elements* $\vec{a}$ in a free group F is *primitive* if that tuple can be included in a basis of F.

ii) A tuple of words $w_1(\vec{x}), w_2(\vec{x}), \ldots, w_n(\vec{x})$ over the tuple of variables $\vec{x}$ is *formally primitive* if this tuple can be included in a basis of the free group on the generators $\vec{x}$.

The following result from Bestvina and Feighn shows that a primitive element can be expressed as a formally non-primitive word over a tuple of elements.

Theorem 1 ([1]). *Let H be the free group on (a,b,c) and $w = a^2b^2c^3$. Then for any finite subset S of H, there is a homomorphism $\phi : H \to H$ such that $\phi(w) = a$ and ϕ is injective on S.*

Application of Theorem 1: In both of our constructions, at stage s we will have elements x_0, x_1, y_1, z_1, and we will have set $x_0 = x_1^2y_1^2z_1^3$, which is a *formally non-primitive word* on the elements x_1, y_1, z_1. Therefore, if x_1, y_1, z_1 satisfy no nontrivial relations among themselves, then we will guarantee that x_0 is a *non-primitive*

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element of the subgroup $\langle x_1, y_1, z_1\rangle$. (And hence, we will guarantee that x_0 is a non-primitive element of the group G we are building, since a primitive element is primitive in every subgroup that contains it.) That is, we currently guess that the element x_0 is not part of a basis for the group G we are constructing.

Now at some later stage t, we may want to change our mind and guess that x_0 is part of a basis for the group G. By this stage, we will have made a finite number of statements S about the elements x_1, y_1, z_1, saying that they *do not* satisfy certain relations. In particular, suppose we have made statements that imply $w_i(x_1, y_1, z_1) \neq 1$ for some words $w_1, \ldots, w_k$. We add two *new* generators b_0 and c_0. By the theorem, there is a homomorphism $\phi : \langle x_0, b_0, c_0\rangle \to \langle x_0, b_0, c_0\rangle$ such that $\phi(x_0^2 b_0^2 c_0^3) = x_0$ and $\phi(w_i(x_0, b_0, c_0)) \neq 1$ for $1 \leq i \leq k$. We make x_1, y_1, z_1 words on x_0, b_0, c_0 by setting $x_1 = \phi(x_0), y_1 = \phi(b_0), z_1 = \phi(c_0)$. This assignment respects everything we have already said, and, within the group we have constructed so far (generated by x_0, b_0, c_0 and some other elements that are obviously independent of these), x_0 is a primitive element. Thus, we can safely change our minds and guess that x_0 is part of a basis for the final group G.

Note that in Theorem 1 we can find the homomorphism ϕ computably from the finite set S (expressed as words on a, b, c). Any homomorphism ϕ is completely determined by its assignment of a, b, c, and we can check computably whether ϕ is injective on the elements of S.

This result from Bestvina and Feighn is the main algebraic tool we apply in our constructions. In addition, to verify that our constructions do indeed produce the group or the sequence of groups with the required properties, we employ a number of other important notions and results from the algebraic theory of free groups.

Definition 2. Let G be a locally free group. A finitely generated subgroup K is a $*$-subgroup if it is a free factor of every finitely generated subgroup in which it is contained.

Theorem 2 (Takahasi [5])**.** *A locally free group is free iff every finite subset of elements belongs to some $*$-subgroup.*

Lemma 3 ([4])**.** *If G is a free factor of F and H is any subgroup of F, then $G \bigcap H$ is a free factor of H.*

Definition 3 ([3])**.** An element of a free group F is *almost primitive* if it is not primitive but is primitive in any proper subgroup of F containing it.

Theorem 4 ([3])**.** *Let F be a finitely generated free group with basis B. Let $B_1, \ldots, B_n$ be pairwise disjoint, non-empty subsets of B, and let F_j be the subgroup of F generated by B_j. Let $a_1, \ldots, a_n$ be elements so that a_j is an almost primitive element of F_j. Then $a = a_1 \cdots a_n$ is an almost primitive element of F.*

Corollary 5. *If H is a proper subgroup of $\langle a, b, c\rangle$ and $a^2b^2c^3 \in H$, then $a^2b^2c^3$ is primitive in H.*

Proof. Since the free group generated by a single element is isomorphic to the integers under addition, a^2 is almost primitive in $\langle a\rangle$, b^2 is almost primitive in $\langle b\rangle$, and c^3 is almost primitive in $\langle c\rangle$. Therefore, by the previous theorem, $a^2b^2c^3$ is almost primitive in $\langle a, b, c\rangle$, and hence it is primitive in H. $\square$

Note that by definition, almost primitive words are not primitive. The proof above then actually proves that $x^2y^2z^3$ is not a formally primitive word, which was stated earlier and is essential to our arguments. Here is the first of our two main results.

Theorem 6. *$I(F_\infty)$ and $I(FrGr)$ are m-complete Π^0_4.*

Proof. We reduce these problems to showing that $I(FrGr)$ is Σ^0_3-hard. Let B be any Π^0_4-complete set. So $B(n)$ is of the form $\forall m A(\langle n,m\rangle)$, where $A(\langle n,m\rangle)$ is a Σ^0_3 relation. If $I(FrGr)$ is Σ^0_3-hard, then there is a computable sequence of groups $(G_{\langle n,m\rangle})_{\langle n,m\rangle\in\omega}$ so that for all n,m, $\langle n,m\rangle \in A$ iff $G_{\langle n,m\rangle}$ is free. Furthermore, we assume—as will be the case in the construction below—that none of the groups $G_{\langle n,m\rangle}$ is the trivial group. Then, for each $n \in \omega$, let H_n be the free product of $G_{\langle n,m\rangle}$ for all $m \in \omega$. It is clear that $n \in B$ iff H_n is free iff $H_n \cong F_\infty$. The rest of this proof establishes that $I(FrGr)$ is Σ^0_3-hard.

Let $A = \{e : W_e$ is finite or cofinite $\} = \mathrm{Cof} \cup \mathrm{Fin}$. It is clear that A is Σ^0_3, but it is also complete at this level. We can see this because $\mathrm{Cof} \le_m A$ via $f(e) =$ the index for $W_e \oplus \omega$. We build a computable sequence of groups G_n such that G_n is free iff $n \in A$.

Fix some computable approximation $W_{n,s}$ to W_n such that at most one element enters at each stage. For convenience of the notation in our construction, we also assume $0 \notin W_n$ for all n. (That is, we really do not consider the standard c.e. sets $W_0, W_1, \ldots$, but instead consider the c.e. sets V_i, where $V_i = W_i - \{0\}$. Notice that V_i is finite or cofinite exactly if W_i is.)

We define the partial computable function $f(n,t)$.

Define $f(n,0) = 0$ for all n.

For $t > 0$, $f(n,t) = \max\{x : x \notin W_{n,s}, x < y, y \in W_{n,s} \setminus W_{n,s-1}\}$, where s is the least stage such that the cardinality of $W_{n,s}$ is t. In words, $f(n,t)$ is the greatest element not in $W_{n,s}$ less than whatever element entered W_n at stage s.

Note that if W_n is infinite, then $f(n,t)$ is total. Moreover, for W_n infinite, $n \in \mathrm{Cof}$ iff $\liminf_t f(n,t)$ is finite; in particular, if $n \in \mathrm{Cof}$, then $\liminf_t f(n,t) =$ the greatest element of $\overline{W}_n$. (Recall that we are assuming $0 \in \overline{W}_n$ for all n.)

The idea is to use $f(n,t)$ to "nest" below some element in our group so that if $\liminf_t f(n,t)$ is infinite, then we nest infinitely often and our group is not free, by the result of Takahasi on locally free groups. We use the result of Bestvina and Feighn as described above to show that we can collapse our nesting at any time in the construction. If we nest only finitely often or we ultimately collapse all but finitely many of our nestings, then the group will be free.

Construction of G_n.

Stage 0. We begin constructing G_n as the free group on a, b.

Stage $s+1$.

Case 1: $\neg\exists k \in W_{n,s+1} \setminus W_{n,s}$. (So we are guessing $n \in \mathrm{Fin}$.) Then we continue building G_n as intended at the end of stage s.

Case 2: $\exists k \in W_{n,s+1} \setminus W_{n,s}$. (So we are guessing $n \notin \mathrm{Fin}$.) Let t be the cardinality of $W_{n,s+1}$. We consider $f(n,t)$.

Subcase 1: $f(n,t-1) = f(n,t)$. (So we are guessing that $n \in \mathrm{Cof}$.) We make no changes, but continue our construction as intended at the end of stage s.

Subcase 2: $f(n,t) > f(n,t-1)$. (So we are guessing that $n \notin \mathrm{Cof}$.) Now we want to extend our nesting below a. For each i with $f(n,t-1) < i \leq f(n,t)$, we add new elements x_i, y_i, z_i and we set $x_i^2 y_i^2 z_i^3 = x_{i-1}$. (In the case that $i = 1$, we add x_1, y_1, z_1 so that $a = x_1^2 y_1^2 z_1^3$.) We now build G_n as the free group on $x_{f(n,t)}$, y_i, z_i for $i \leq f(n,t)$, b, and any b_k, c_k introduced during the construction because of subcase 3 (see below).

Subcase 3: $f(n,t) < f(n,t-1)$. (So we are guessing that $n \in \mathrm{Cof}$ and $f(n,t)$ is the greatest element in $\overline{W}_n$.) We want to collapse our nesting below a back down to $f(n,t)$ levels. We apply the Bestvina-Feighn result as outlined above to introduce new elements b_j, c_j and to put a relation on $x_{f(n,t-1)}$, $y_{f(n,t-1)}$, and $z_{f(n,t-1)}$ so that $x_{f(n,t-1)-1}$, b_j, c_j are independent and generate $x_{f(n,t-1)}$, $y_{f(n,t-1)}$, and $z_{f(n,t-1)}$. We next use the application again to introduce new elements b_{j+1}, c_{j+1} and to put some relation on $x_{f(n,t-1)-1}$, $y_{f(n,t-1)-1}$, and $z_{f(n,t-1)-1}$ so that $x_{f(n,t-1)-2}$, b_{j+1}, c_{j+1} are independent and generate $x_{f(n,t-1)-1}$, $y_{f(n,t-1)-1}$, and $z_{f(n,t-1)-1}$. We repeat this procedure until we have cut back to $x_{f(n,t)}$, $y_{f(n,t)}$, $z_{f(n,t)}$. We now build G_n as the free group on $x_{f(n,t)}$, y_i, z_i for $i \leq f(n,t)$, b and the generators b_k, c_k introduced during the construction.

End of Construction.

Remark about notation. If after reducing back to $f(n,t)$ we again go past it, we introduce completely new x_i, y_i, z_i, although there are not different names for these elements in the above construction.

Verification. First, note that if W_n is finite, then after some stage we will always be in Case 1. Then we never again add new generators, so G_n is a finitely generated free group. From now on, assume W_n is infinite. Next we need to establish the following two easy claims about the construction and the behavior of $f(n,t)$.

Claim 1. If $1 \leq m \notin W_n$, then there is a stage s by which our assignments of x_i, y_i, and z_i for $1 \leq i \leq m$ settle down and are never subsequently collapsed.

Proof. First, the construction guarantees that if $1 \leq f(n,t)$ and $|W_{n,s}| = t$, then for all i with $1 \leq i \leq f(n,t)$, the elements x_i, y_i, z_i are defined by stage s, and those assignments remain valid unless $f(n,v) < f(n,t)$ for some $v > t$. (This is because in Subcase 2 of Case 2, we define x_i, y_i, z_i for all i with $f(n,t-1) < i \leq f(n,t)$; and in Subcase 3, we collapse only down to $f(n,t)$ levels.)

Let s be the first stage such that $W_n \restriction m = W_{n,s} \restriction m$, and let $t = |W_{n,s}|$. Then from stage $s+1$ onward, we consider $f(n,v)$ only for $v > t$. But by the definition of f, $f(n,v) \geq m$ for $v > t$. □

Claim 2. For each n, G_n is a locally free group.

Proof. The group G_n is generated (not necessarily freely) by the collection X of the following elements: a, b; the x_i, y_i, z_i that are never cancelled; and the elements b_k, c_k introduced in the stages where we entered Subcase 3 of Case 2. Of course, any finitely generated subgroup H of G_n is a subgroup of a group generated by a finite subset $S \subset X$. Let i^* be the largest index on an x, y or z element appearing in S. Then H is a subgroup of the subgroup H' generated by x_{i^*}, y_i, z_i for $i \leq i^*$, b, and all of the b_k, c_k appearing in S. By construction, H' is generated *freely* by these elements, so H is a subgroup of a free group, so H is free. □

We are now ready to verify the main proposition.

Proposition 7. *W_n is cofinite iff G_n is free.*

Proof. ($\Rightarrow$) Suppose m is the greatest number such that $m \notin W_n$. By the above claim, let s be a stage by which x_i, y_i, and z_i for $i \leq m$ have settled down. $\liminf_t f(n,t) = m$, so any x_i, y_i, z_i introduced for $i > m$ will eventually be collapsed. Then a generating set for G_n is x_m, y_i, z_i for $i \leq m$, b, and all the b_k, c_k introduced during the construction. (Remark: There will be infinitely many of these b_k, c_k unless after some stage $W_{n,s}$ enumerates all the elements in increasing order, in which case we will always be in Case 2, Subcase 1.) All of these elements are independent by the construction, so G_n is the free group on this generating set.

($\Leftarrow$) Suppose W_n is not cofinite. Then $\liminf_t f(n,t) = \infty$ and there are infinitely many elements that never enter W_n. By the above claim, we will eventually settle on some x_i, y_i, z_i for each i, so there is an infinite sequence $\{x_i, y_i, z_i\}_{i \in \omega}$ with $x_1^2 y_1^2 z_1^3 = a$ and $x_i^2 y_i^2 z_i^3 = x_{i-1}$ for $i \geq 2$. All of these elements along with b and all the b_k, c_k introduced form a generating set (but, as we shall see, not a basis) for G_n. By Takahasi's result (Theorem 2), the following claim proves that the locally free group G_n is not free and finishes the proof of the proposition, and thus the theorem.

Claim 3. a is not part of any $*$-subgroup.

Proof. Suppose that H is a $*$-subgroup of G_n and $a \in H$. Let l be the least number such that $L = \langle x_{l+1}, y_{l+1}, z_{l+1} \rangle$ is not a subgroup of H. There must be an l like this, since H is finitely generated. Then $x_l \in H$ (we take $a = x_0$ if $l = 1$) and $H \bigcap L$ is a proper subgroup of L. Then by Corollary 5, x_l is primitive in $H \bigcap L$. Let F be any finitely generated subgroup of G_n which contains H and L. Since H is a $*$-subgroup, it is a free factor of F. Then by Lemma 3, $H \bigcap L$ is a free factor of L. This implies that x_l is primitive in L, contradicting the fact that x_l is a formally non-primitive word on the basis elements $x_{l+1}, y_{l+1}, z_{l+1}$ of L. □

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Now we give our second main result.

Theorem 8. *There is a computable copy F of F_∞ with no Σ_2^0 basis.*

Proof. First, note that any Σ_2^0 basis is actually a Δ_2^0 basis. Indeed, as we enumerate the basis B, we also enumerate the reduced words on B of length ≥ 2; every nontrivial element of our copy of F_∞ lies in exactly one of these two sets, so we can decide whether an element is in the basis or not. Therefore, it suffices to show that there is a computable copy of F_∞ with no Δ_2^0 basis.

Recall that the set of all primitive elements is computable in any basis for a free group, so we build our copy F so that no $\phi_e(x,s)$ is a computable approximation witnessing that the set of all primitive elements is Δ_2^0. For each e, we build a group G_e to diagonalize against ϕ_e, and we build F as the free product of all G_e. No matter how the diagonalization proceeds, in this construction (unlike the last one), the group G_e is free, so F is a copy of F_∞. Assume, without loss of generality, that for all e, x, s, if $\phi_e(x,s)$ is defined, it has value 0 or 1.

We begin our construction by putting a_e into G_e as a potential primitive element. At stage s, for each $e \leq s$ let t_s be the largest t (less than s) such that $\phi_{e,s}(a_e, t)$ converges. If $\phi_{e,s}(a_e, t_s) = \phi_{e,s}(a_e, t_{s-1})$, we continue on our intended course of construction. If $\phi_{e,s}(a_e, t_s) \neq \phi_{e,s}(a_e, t_{s-1})$, we act as follows.

Case 1. If $\phi_{e,s}(a_e, t_{s-1}) = 0$ (or is undefined) and $\phi_{e,s}(a_e, t_s) = 1$, then we add new elements $x_{e,k}, y_{e,k}, z_{e,k}$ and set $a_e = x_{e,k}^2 y_{e,k}^2 z_{e,k}^3$ where k is the number of times we have been in Case 1 for this e. Until we see a subsequent change in the approximation ϕ_e, we build G_e as a free group with these three elements as basis elements along with $b_{e,i}, c_{e,i}$ for $i < k$.

Case 2. If $\phi_{e,s}(a_e, t_{s-1}) = 1$ (or is undefined) and $\phi_{e,s}(a_e, t_s) = 0$, we use the application of Theorem 1 to add new elements $b_{e,k}, c_{e,k}$ and set $x_{e,k}, y_{e,k}, z_{e,k}$ equal to words on $a_e, b_{e,k}$, and $c_{e,k}$ as determined by the homomorphism of that theorem. Here k is the greatest number such that $x_{e,k}, y_{e,k}, z_{e,k}$ are defined. Until we see a subsequent change in the approximation ϕ_e, we build G_e as a free group with basis $\{a_e, b_{e,i}, c_{e,i}\}_{i \leq k}$. This ends the construction.

Verification. First, we argue that $F \cong F_\infty$. Since F is the free product of the groups G_e, we must show that each G_e is free. If $\phi_e(a_e, s)$ has a limit in s, then after some stage s, we are never in Case 1 or Case 2 again, and we build G_e as a finitely generated free group. Suppose now that $\phi_e(a_e, s)$ does not have a limit because it alternates between 0 and 1 infinitely often. Then a generating set for G_e is $\{a_e, b_{e,i}, c_{e,i}\}_{i \in \omega}$. These elements are all independent, however, so it is actually a basis for G_e.

Claim 4. The set of primitive elements is not Δ_2^0.

Proof. Suppose it is, and let ϕ_e be the computable approximation that witnesses so. Then there is a value t such that $\forall s \geq t\ \phi_e(a_e, s) = \phi_e(a_e, t)$. If $\phi_e(a_e, t) = 0$, then at some point in the construction we were in Case 2 and never in Case 1 again, so a_e is primitive, a contradiction. Then $\phi_e(a_e, t) = 1$, which means that at some point in the construction we were in Case 1 and never again in Case 2. Then a_e is non-primitive because it is a formally non-primitive word on some independent $x_{e,k}, y_{e,k}$, and $z_{e,k}$. This contradicts the value of $\phi_e(a_e, t)$. □

□

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