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MINKOWSKI VALUATIONS ON L^p-SPACES

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Abstract. A complete classification is obtained of continuous $\mathrm{GL}(n)$-equivariant Minkowski valuations on $L^p(\mathbb{R}^n, |x|dx)$. As a consequence, a characterization of the moment body for functions is obtained.

1. Introduction

Denote by $\mathcal{K}^n$ the collection of convex bodies in $\mathbb{R}^n$, that is, of nonempty compact convex subsets of $\mathbb{R}^n$. Let $\mathcal{W} \subseteq \mathcal{K}^n$ and let $\mathcal{G}$ be an abelian semigroup. A function $\mathrm{Z} : \mathcal{W} \to \mathcal{G}$ is called a valuation if

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}\, K + \mathrm{Z}\, L,$$

for all $K, L \in \mathcal{W}$ with $K \cup L, K \cap L \in \mathcal{W}$. Starting with Dehn's solution of Hilbert's Third Problem in 1901, the theory of real-valued valuations on convex bodies plays an important role in geometry. Perhaps the best known result is Hadwiger's classification of continuous rigid motion invariant valuations on $\mathcal{K}^n$ (see [16, 21] and [1, 2, 3, 4, 5, 6, 12, 18, 19, 20, 28, 37, 38] for further results on real-valued valuations). More recently, Minkowski valuations, that is, valuations $\mathrm{Z} : \mathcal{W} \to \mathcal{K}^n$, where addition in $\mathcal{K}^n$ is Minkowski addition, have attracted a lot of interest (see [15, 17, 23, 24, 25, 26, 43, 45, 46]). Ludwig [25] proved that two important operators on convex bodies can be characterized as Minkowski valuations with certain invariance properties with respect to the special linear group: projection bodies and moment bodies. Whereas projection bodies appear in the classification of valuations on Sobolev spaces [27], this is not the case for moment bodies. In this paper, moment bodies for functions are introduced on suitable function spaces and are completely characterized by their valuation and invariance properties.

For $1 \le p < \infty$, denote by $L^p(\mathbb{R}^n, |x|dx)$ the space of measurable functions $f : \mathbb{R}^n \to [-\infty, \infty]$ such that their L^p-norm $\|f\|_{L^p(\mathbb{R}^n,|x|dx)}$ is finite, where

$$\|f\|_{L^p(\mathbb{R}^n,|x|dx)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p |x|\, dx \Big)^{1/p}.$$

Here $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. We will call a function $\Phi : L^p(\mathbb{R}^n, |x|dx) \to \mathcal{K}^n$ a Minkowski valuation if

$$\Phi(f \vee g) + \Phi(f \wedge g) = \Phi(f) + \Phi(g),$$

for all $f, g \in L^p(\mathbb{R}^n, |x|dx)$ and $\Phi(0) = \{o\}$. Here $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$. The analogous definition of a real-valued valuation on a general L^p-space

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has been studied (see [47]). It turns out that on $L^p(\mathbb{R}^n, |x|dx)$ there are interesting Minkowski valuations, whereas on $L^p(\mathbb{R}^n, |x|^q dx)$, where $q > 1$, it is more natural to consider so-called L^q-Minkowski valuations, that is, valuations with respect to L^q-Minkowski addition. The study of L^q-Minkowski valuations is part of the important L^q-Brunn-Minkowski theory (see, for example, [7, 8, 14, 25, 31, 32, 33, 34, 35, 36]), but we will restrict our attention to the classical case $q = 1$.

For $f \in L^1(\mathbb{R}^n, |x|dx)$, we define the moment vector of f, $\overline{m}(f)$, as

$$\overline{m}(f) = \int_{\mathbb{R}^n} f(x)x\,dx.$$

For a convex body $K \in \mathcal{K}^n$, we denote by $h(K, \cdot)$ its support function (see Section 2). For $f \in L^1(\mathbb{R}^n, |x|dx)$, we define the moment body of f, $\overline{\mathrm{M}}(f)$, as the convex body with support function

$$h(\overline{\mathrm{M}}(f), u) = \int_{\mathbb{R}^n} |f(x)||x \cdot u|\,dx$$

for $u \in \mathbb{R}^n$. These definitions are generalizations of the notion of moment vector and moment body of a convex body. For example, if $K \in \mathcal{K}^n$, then

$$\overline{m}(\chi_K) = \int_K x\,dx,$$

which is equal to $m(K)$, the usual moment vector of K. Here χ_K stands for the characteristic function of K (see Section 2). Also, if $K \in \mathcal{K}^n$, then

$$h(\overline{\mathrm{M}}(\chi_K), u) = \int_K |x \cdot u|\,dx$$

for $u \in \mathbb{R}^n$. So $\overline{\mathrm{M}}(\chi_K)$ is equal to $\mathrm{M}\,K$, the usual moment body of K. It should be noted that the centroid body ΓK of a convex body $K \in \mathcal{K}^n$ with positive volume $V(K)$ defined as $\Gamma K = V(K)^{-1}\,\mathrm{M}\,K$ is a classical notion in convex geometry (see [9, 12, 40, 42]) that has attracted much attention in recent years (see, for example, [7, 9, 10, 11, 13, 14, 15, 25, 29, 30, 33, 36, 39, 48]).

Let $\mathrm{GL}(n)$ be the general linear group (set of all invertible linear transformations) on $\mathbb{R}^n$. If $A \in \mathrm{GL}(n)$ and $f \in L^p(\mathbb{R}^n, |x|dx)$, then Af is defined as $Af = f \circ A^{-1}$, and clearly $Af \in L^p(\mathbb{R}^n, |x|dx)$. A Minkowski valuation $\Phi : L^p(\mathbb{R}^n, |x|dx) \to \mathcal{K}^n$ is called $\mathrm{GL}(n)$-equivariant if

$$\Phi(Af) = |\det A| A\Phi(f),$$

for every $f \in L^p(\mathbb{R}^n, |x|dx)$ and every $A \in \mathrm{GL}(n)$. The main result in this paper is the following theorem.

Theorem 1.1. *A function $\Phi : L^p(\mathbb{R}^n, |x|dx) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous $\mathrm{GL}(n)$-equivariant Minkowski valuation if and only if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with the property that there exist real numbers $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma|\alpha|^p$ and $0 \leq H(\alpha) \leq \delta|\alpha|^p$ for all $\alpha \in \mathbb{R}$, and*

$$\Phi(f) = \overline{m}(G \circ f) + \overline{\mathrm{M}}(H \circ f)$$

for every $f \in L^p(\mathbb{R}^n, |x|dx)$.

A direct application of Theorem 1.1 is the following theorem.

Theorem 1.2. *A function $\Phi : L^p(\mathbb{R}^n, |x|dx) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous $\mathrm{GL}(n)$-equivariant homogeneous Minkowski valuation if and only if there exist constants $a, b \in \mathbb{R}$ with $b \geq 0$ such that*

$$\Phi(f) = a\,\overline{m}(|f|^p) + b\,\overline{\mathrm{M}}(|f|^p)$$

for all $f \in L^p(\mathbb{R}^n, |x|dx)$.

Recall that $\Phi : L^p(\mathbb{R}^n, |x|dx) \to \mathcal{K}^n$ is homogeneous if there exists $\beta \in \mathbb{R}$ such that $\Phi(\alpha f) = |\alpha|^\beta \Phi(f)$, for all $\alpha \neq 0$ and for all $f \in L^p(\mathbb{R}^n, |x|dx)$. Notice that in Theorem 1.2, if $p = 1$, then we obtain a characterization of the moment body of functions in $L^1(\mathbb{R}^n, |x|dx)$ since $\overline{\mathrm{M}}(|f|) = \overline{\mathrm{M}}(f)$ for all $f \in L^1(\mathbb{R}^n, |x|dx)$.

The critical ingredient in the proof of Theorem 1.1 is Ludwig's result on the moment vector and the moment body of convex bodies. To state the precise result here, we need some definitions. Denote by $\mathrm{SL}(n)$ the special linear group (set of all invertible linear transformations with determinant 1) on $\mathbb{R}^n$. Let $\mathcal{P}_0^n$ be the set of all convex polytopes in $\mathbb{R}^n$ that contain the origin. For $A \in \mathrm{SL}(n)$ and $P \in \mathcal{P}_0^n$, AP is defined as $AP = \{Ax : x \in P\}$. Clearly $AP \in \mathcal{P}_0^n$. A Minkowski valuation $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{K}^n$ is called $\mathrm{SL}(n)$-equivariant if $\mathrm{Z}(AP) = A\,\mathrm{Z}\,P$ for every $P \in \mathcal{P}_0^n$ and every $A \in \mathrm{SL}(n)$, and is called homogeneous of degree $\beta \in \mathbb{R}$ if $Z(\alpha P) = \alpha^\beta\,\mathrm{Z}\,P$ for every $\alpha > 0$.

Theorem 1.3 (Ludwig [25])**.** *If $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{K}^n$, where $n \geq 2$, is an $\mathrm{SL}(n)$-equivariant Minkowski valuation that is homogeneous of degree $n+1$, then there exist $a, b \in \mathbb{R}$ with $b \geq 0$ such that*

$$\mathrm{Z}\,P = a\,m(P) + b\,\mathrm{M}\,P$$

for every $P \in \mathcal{P}_0^n$.

2. Preliminaries

Let $(X, \mathfrak{F}, \mu)$ be a measure space, that is, X is an arbitrary set, $\mathfrak{F}$ is a σ-algebra on X, and $\mu : \mathfrak{F} \to [0, \infty]$ is a measure. The L^p-space $L^p(\mu)$, where $1 \leq p < \infty$, is the collection of μ-measurable functions $f : X \to [-\infty, \infty]$ that satisfy

$$\int_X |f|^p \, d\mu < \infty.$$

For $f \in L^p(\mu)$, the L^p-norm of f denoted by $\|f\|_p$ is defined as

$$\|f\|_p = \Big(\int_X |f|^p \, d\mu \Big)^{1/p}.$$

The functional $\|\cdot\|_p : L^p(\mu) \to \mathbb{R}$ is a semi-norm. In $L^p(\mu)$, functions that are equal almost everywhere with respect to μ (a.e. $[\mu]$) are identified, so then $\|\cdot\|_p : L^p(\mu) \to \mathbb{R}$ becomes a norm and $L^p(\mu)$ becomes a normed linear space. A function $f \in L^p(\mu)$ is said to be essentially supported if there exists an $E \in \mathfrak{F}$ with $\mu(E) < \infty$ such that

$$\int_{X \setminus E} |f| \, d\mu = 0.$$

In such a case we say that f is essentially supported on E.

If $f : X \to [-\infty, \infty]$ is μ-measurable, then the essential supremum of f, ess sup f, is defined by

$$\operatorname{ess\,sup} f = \inf\{\gamma : \gamma \in \mathbb{R},\ \mu(f^{-1}((\gamma, \infty])) = 0\},$$

where $\inf \emptyset = \infty$, and the L^∞-norm of f, $\|f\|_\infty$, is defined by

$$\|f\|_\infty = \text{ess sup}\,|f|.$$

It should be noted that $|f(x)| \leq \|f\|_\infty$, a.e. $[\mu]$. If $f : X \to [-\infty, \infty]$ is μ-measurable and satisfies $\|f\|_\infty < \infty$, then f is called essentially bounded. The collection of essentially bounded functions is denoted by $L^\infty(\mu)$. In $L^\infty(\mu)$, functions that are equal almost everywhere with respect to μ are identified, so then $\|\cdot\|_\infty : L^\infty(\mu) \to \mathbb{R}$ becomes a norm and $L^\infty(\mu)$ becomes a normed linear space.

In this paper, we consider two measure spaces, namely, $(\mathbb{R}^n, \mathfrak{M}, \lambda)$ and $(\mathbb{R}^n, \mathfrak{M}, \mu_n)$, where $\mathfrak{M}$ is the collection of Lebesgue measurable sets in $\mathbb{R}^n$, λ is Lebesgue measure and μ_n is the measure defined as

$$\mu_n(E) = \int_E |x|\, dx$$

for $E \in \mathfrak{M}$. Note that $L^p(\mu_n) = L^p(\mathbb{R}^n, |x|dx)$. We will write $L^p(\mathbb{R}^n)$ instead of $L^p(\lambda)$. Also, if $f \in L^p(\mathbb{R}^n)$, it is customary to write $\int_{\mathbb{R}^n} f(x)\, dx$ in place of $\int_{\mathbb{R}^n} f\, d\lambda$. Similarly, if $f \in L^p(\mu_n)$, we will write $\int_{\mathbb{R}^n} f(x)|x|\, dx$ in place of $\int_{\mathbb{R}^n} f\, d\mu_n$. For simplicity, we will always write measurable functions instead of λ-measurable functions or μ_n-measurable functions. It should be noted that if $E \in \mathfrak{M}$, then $\mu_n(E) = 0$ if and only if $\lambda(E) = 0$, so we will simply write a.e. for almost everywhere with respect to μ_n as well as almost everywhere with respect to λ. Hence, $L^\infty(\mu_n) = L^\infty(\mathbb{R}^n)$ and the two spaces have the same norms, so we will denote both of them by $\|\cdot\|_\infty$. For $f \in L^p(\mu_n)$, we will use the notation $\|f\|_{L^p(\mu_n)}$ (or $\|f\|_{L^p(\mathbb{R}^n, |x|dx)}$) to denote the L^p-norm of f in the space $L^p(\mu_n)$ instead of $\|f\|_p$, that is,

$$\|f\|_{L^p(\mu_n)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p |x|\, dx \Big)^{1/p}.$$

We will reserve the notation $\|f\|_p$ for the L^p-norm of f in the space $L^p(\mathbb{R}^n)$, that is,

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p\, dx \Big)^{1/p}.$$

The characteristic function (or indicator function) of $E \in \mathfrak{F}$ is denoted by χ_E and is defined as

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in X \backslash E. \end{cases}$$

A function $s \in L^p(\mu)$ is called simple if there exist pairwise disjoint sets $E_1,\ldots,E_k \in \mathfrak{F}$ with finite measures and $\alpha_1,\ldots,\alpha_k \in \mathbb{R}$ such that

$$s = \sum_{i=1}^k \alpha_i \chi_{E_i}.$$

We will need some theorems from analysis. The first one is the following theorem (see page 75 in Rudin's book [41]).

Theorem 2.1. *The function $\Psi : L^p(\mu) \to L^1(\mu)$ defined as $\Psi(f) = |f|^p$, for $f \in L^p(\mu)$, is continuous.*

The second theorem is a standard result in analysis.

Theorem 2.2. *If $f : X \to [0, \infty]$ is μ-measurable, then there exists a sequence of simple functions $\{s_i\}$ on X such that*

(i) $0 \le s_1(x) \le s_2(x) \le ... \le f(x)$, *for all $x \in X$, and*
(ii) $s_i(x) \to f(x)$, *for all $x \in X$.*

The third is labeled as a proposition and is a small modification of Lebesgue's Dominated Convergence Theorem. A proof of it can be found in [47].

Proposition 2.3. *Let $\{f_i\}$ be a sequence of μ-measurable functions on X. Suppose that $f(x) = \lim_{i\to\infty} f_i(x)$ exists a.e. $[\mu]$. If there exists a sequence of functions $\{g_i\}$ in $L^1(\mu)$ such that $g(x) = \lim_{i\to\infty} g_i(x)$ exists a.e. $[\mu]$ with $g \in L^1(\mu)$ and*

$$\lim_{i\to\infty} \int_X g_i \, d\mu = \int_X g \, d\mu,$$

and if

$$|f_i(x)| \le g_i(x), \text{ a.e. } [\mu],$$

then $f \in L^1(\mu)$ and

$$\lim_{i\to\infty} \int_X |f_i - f| \, d\mu = 0.$$

Thus,

$$\lim_{i\to\infty} \int_X f_i \, d\mu = \int_X f \, d\mu.$$

The fourth theorem is another standard result in analysis.

Theorem 2.4. *If $\{f_i\}$ is a convergent sequence in $L^p(\mu)$ with limit f, then $\{f_i\}$ has a subsequence, $\{f_{i_j}\}$, such that $f_{i_j}(x) \to f(x)$ a.e. $[\mu]$.*

A closed rectangle R in $\mathbb{R}^n$ is a set of the form

$$R = \{(x_1, ..., x_n) : x_i \in \mathbb{R}, a_i \le x_i \le b_i\},$$

where $a_i, b_i \in \mathbb{R}$ and $a_i \le b_i$. If $|b_i - a_i| = |b_j - a_j|$ for all $i, j = 1, ..., n$, then R is called a closed cube. For $K, L \subseteq \mathbb{R}^n$, $K\Delta L = K\backslash L \cup L\backslash K$ is called the symmetric difference of the sets K and L. The following theorem is a standard result in real analysis.

Theorem 2.5. *If $\varepsilon > 0$ and $E \in \mathfrak{M}$ with $\lambda(E) < \infty$, then there exist closed cubes $Q_1, Q_2, ..., Q_k \subseteq \mathbb{R}^n$ such that $\lambda\Big(E\Delta \bigcup_{i=1}^{k} Q_i\Big) < \varepsilon$.*

We also need some tools from convex geometry. A function $g : \mathbb{R}^n \to \mathbb{R}$ is called sublinear if it satisfies both:

(i) $g(\alpha u) = \alpha g(u)$, for all $u \in \mathbb{R}^n$ and all $\alpha \ge 0$, and
(ii) $g(u + v) \le g(u) + g(v)$, for all $u, v \in \mathbb{R}^n$.

The support function of a convex body $K \in \mathcal{K}^n$, $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined as

$$h(K, u) = \max\{x \cdot u : x \in K\}$$

for $u \in \mathbb{R}^n$. It should be clear that the support function of a convex body is sublinear. Denote by S^{n-1} the unit sphere on $\mathbb{R}^n$. Note that (i) shows that $h(K, \cdot)$ is uniquely determined by $h(K, u)$ for all $u \in S^{n-1}$. It is well known in convex geometry that for every sublinear function $g : \mathbb{R}^n \to \mathbb{R}$ there exists a unique convex

body $K \in \mathcal{K}^n$ such that $h(K,u) = g(u)$ for all $u \in \mathbb{R}^n$ (see page 48 in Gruber's book [12]).

If $K, L \in \mathcal{K}^n$, then the Hausdorff distance, $\delta(K,L)$, of K and L is defined as

$$\delta(K,L) = \sup\{|h(K,u) - h(L,u)| : u \in S^{n-1}\}.$$

The space $\mathcal{K}^n$ is a metric space when equipped with the Hausdorff metric. Another well-known result in convex geometry is that if $\{K_i\}$ is a sequence in $\mathcal{K}^n$ and $K \in \mathcal{K}^n$, then $K_i \to K$ in $\mathcal{K}^n$ if and only if for every $u \in S^{n-1}$, $h(K_i,u) \to h(K,u)$ (see page 54 of Schneider's book [42]).

The Minkowski sum of two convex bodies $K, L \in \mathcal{K}^n$, $K + L$, is defined as

$$K + L = \{x + y : x \in K,\, y \in L\}.$$

It is easy to check that the Minkowski sum of two convex bodies is also a convex body with support function:

$$h(K + L, \cdot) = h(K, \cdot) + h(L, \cdot).$$

If $\alpha \geq 0$ and $K \in \mathcal{K}^n$, then αK is defined as $\alpha K = \{\alpha x : x \in K\}$. Trivially, αK is a convex body with support function $h(\alpha K, \cdot) = \alpha h(K, \cdot)$. Lastly, if $y \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$, then $y + K$ is defined as $y + K = \{y + x : x \in K\}$. Trivially, for any $\alpha \in \mathbb{R}$, $\alpha y + K$ is a convex body with support function $h(\alpha y + K, u) = \alpha y \cdot u + h(K,u)$, for $u \in \mathbb{R}^n$.

3. Proof of Theorem 1.1

The following theorem is a restatement of one part of Theorem 1.1.

Theorem 3.1. *If $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ are continuous functions with the property that there exist real numbers $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma|\alpha|^p$ and $0 \leq H(\alpha) \leq \delta|\alpha|^p$ for all $\alpha \in \mathbb{R}$, then the function $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ defined for $f \in L^p(\mu_n)$ by*

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x) x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for $u \in \mathbb{R}^n$ is a continuous $\mathrm{GL}(n)$*-equivariant Minkowski valuation.*

Proof. First, we show that $\Phi(f) \in \mathcal{K}^n$. Note that $|h(\Phi(f), u)| < \infty$ for every $f \in L^p(\mu_n)$ and every $u \in \mathbb{R}^n$ since

$$\begin{aligned}
|h(\Phi(f), u)| &= \Big| \int_{\mathbb{R}^n} [(G \circ f)(x) x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx \Big| \\
&\leq \int_{\mathbb{R}^n} (|(G \circ f)(x)| + |(H \circ f)(x)|)|x||u|\, dx \\
&\leq \int_{\mathbb{R}^n} (\gamma|f(x)|^p + \delta|f(x)|^p)|x||u|\, dx \\
&= (\gamma + \delta)|u| \int_{\mathbb{R}^n} |f(x)|^p |x|\, dx < \infty.
\end{aligned}$$

Also note that $h(\Phi(f),\cdot)$ is sublinear since for $u\in\mathbb{R}^n$ and $\beta\geq 0$,

$$\begin{aligned}h(\Phi(f),\beta u)&=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot\beta u+(H\circ f)(x)|x\cdot\beta u|]\,dx\\&=\beta\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx\\&=\beta h(\Phi(f),u),\end{aligned}$$

and for $u,v\in\mathbb{R}^n$,

$$\begin{aligned}h(\Phi(f),u+v)&=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot(u+v)+(H\circ f)(x)|x\cdot(u+v)|]\,dx\\&\leq\int_{\mathbb{R}^n}[(G\circ f)(x)(x\cdot u+x\cdot v)+(H\circ f)(x)(|x\cdot u|+|x\cdot v|)]\,dx\\&=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx\\&\quad+\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot v+(H\circ f)(x)|x\cdot v|]\,dx\\&=h(\Phi(f),u)+h(\Phi(f),v).\end{aligned}$$

It follows that $h(\Phi(f),\cdot)=h(K,\cdot)$ for a unique convex body K and $\Phi(f)=K\in\mathcal{K}^n$.

Now we prove that Φ is a Minkowski valuation on $L^p(\mu_n)$. Note that $G(0)=0$ and $H(0)=0$, which imply that $h(\Phi(0),u)=0$ for every $u\in\mathbb{R}^n$. Thus, $\Phi(0)=\{o\}$. Next, let $f,g\in L^p(\mu_n)$ and $u\in\mathbb{R}^n$. Clearly, we have for all $x\in\mathbb{R}^n$,

$$(G\circ(f\vee g))(x)x\cdot u+(G\circ(f\wedge g))(x)x\cdot u=(G\circ f)(x)x\cdot u+(G\circ g)(x)x\cdot u$$

and

$$(H\circ(f\vee g))(x)|x\cdot u|+(H\circ(f\wedge g))(x)|x\cdot u|=(H\circ f)(x)|x\cdot u|+(H\circ g)(x)|x\cdot u|.$$

Thus,

$$\begin{aligned}&h(\Phi(f\vee g),u)+h(\Phi(f\wedge g),u)\\&\quad=\int_{\mathbb{R}^n}[(G\circ(f\vee g))(x)x\cdot u+(H\circ(f\vee g))(x)|x\cdot u|]\,dx\\&\qquad+\int_{\mathbb{R}^n}[(G\circ(f\wedge g))(x)x\cdot u+(H\circ(f\wedge g))(x)|x\cdot u|]\,dx\\&\quad=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx\\&\qquad+\int_{\mathbb{R}^n}[(G\circ g)(x)x\cdot u+(H\circ g)(x)|x\cdot u|]\,dx\\&\quad=h(\Phi(f),u)+h(\Phi(g),u).\end{aligned}$$

Hence, $h(\Phi(f\vee g)+\Phi(f\wedge g),u)=h(\Phi(f)+\Phi(g),u)$. So, $\Phi(f\vee g)+\Phi(f\wedge g)=\Phi(f)+\Phi(g)$. Therefore, Φ is a Minkowski valuation on $L^p(\mu_n)$.

Now we show that Φ is $\mathrm{GL}(n)$-equivariant. Let $f \in L^p(\mu_n)$, $A \in \mathrm{GL}(n)$ and $u \in \mathbb{R}^n$. Observe that

$$\begin{aligned}
h(\Phi(Af), u) &= \int_{\mathbb{R}^n} [(G \circ (Af))(x)x \cdot u + (H \circ (Af))(x)|x \cdot u|]\, dx \\
&= \int_{\mathbb{R}^n} [(G \circ f)(A^{-1}x)x \cdot u + (H \circ f)(A^{-1}x)|x \cdot u|]\, dx \\
&= \int_{\mathbb{R}^n} [(G \circ f)(y)Ay \cdot u + (H \circ f)(y)|Ay \cdot u|]|\det A|\, dy \\
&= |\det A| \int_{\mathbb{R}^n} [(G \circ f)(y)y \cdot A^T u + (H \circ f)(y)|y \cdot A^T u|]\, dy \\
&= |\det A| h(\Phi(f), A^T u) \\
&= |\det A| h(A\Phi(f), u) \\
&= h(|\det A| A\Phi(f), u).
\end{aligned}$$

Therefore, $\Phi(Af) = |\det A| A\Phi(f)$, so Φ is $\mathrm{GL}(n)$-equivariant.

Lastly, we show that $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ is continuous. Let $f \in L^p(\mu_n)$ and let $\{f_i\}$ be any sequence in $L^p(\mu_n)$ that converges to f. We wish to show that $\{\Phi(f_i)\}$ converges to $\Phi(f)$, which is equivalent to showing that for every $u \in S^{n-1}$, $h(\Phi(f_i), u) \to h(\Phi(f), u)$. We will show this by showing that for each $u \in S^{n-1}$, every subsequence, $\{h(\Phi(f_{i_j}), u)\}$, of $\{h(\Phi(f_i), u)\}$ has a subsequence, $\{h(\Phi(f_{i_{j_k}}), u)\}$ that converges to $h(\Phi(f), u)$. Let $\{f_{i_j}\}$ be a subsequence of $\{f_i\}$. Then we still have that $\{f_{i_j}\}$ converges to f in $L^p(\mu_n)$. Hence, by Theorem 2.4, there is a subsequence, $\{f_{i_{j_k}}\}$, of $\{f_{i_j}\}$ that satisfies $f_{i_{j_k}}(x) \to f(x)$ a.e. Let $u \in S^{n-1}$. Since G and H are continuous, we also have that

$$(G \circ f_{i_{j_k}})(x)x \cdot u + (H \circ f_{i_{j_k}})(x)|x \cdot u| \to (G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|, \text{ a.e.}$$

Since $|G(\alpha)| \le \gamma|\alpha|^p$ and $0 \le H(\alpha) \le \delta|\alpha|^p$ for all $\alpha \in \mathbb{R}$ we get

$$|(G \circ f_{i_{j_k}})(x)x \cdot u + (H \circ f_{i_{j_k}})(x)|x \cdot u|| \le (\gamma + \delta)|f_{i_{j_k}}(x)|^p|x||u|, \text{ a.e.}$$

Also, since $f_{i_{j_k}} \to f$ in $L^p(\mu_n)$, applying Theorem 2.1 gives

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} (\gamma + \delta)|f_{i_{j_k}}(x)|^p|x||u|\, dx = \int_{\mathbb{R}^n} (\gamma + \delta)|f(x)|^p|x||u|\, dx,$$

and that the limit above is finite. Therefore, Proposition 2.3 gives

$$\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_{i_{j_k}})(x)x \cdot u + (H \circ f_{i_{j_k}})(x)|x \cdot u|]\, dx \\
= \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx,
\end{aligned}$$

and that the limit above is finite. Hence, $h(\Phi(f_{i_{j_k}}), u) \to h(\Phi(f), u)$. Therefore, $\Phi : L^p(\mu) \to \mathbb{R}$ is continuous. □

To prove the other part of Theorem 1.1 (as restated in Theorem 3.16 below), we require the following lemmas.

Lemma 3.2. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \ge 2$, is a $\mathrm{GL}(n)$-equivariant Minkowski valuation, then for each $\alpha \in \mathbb{R}$, $\mathrm{Z}_\alpha : \mathcal{P}_0^n \to \mathcal{K}^n$ defined as $\mathrm{Z}_\alpha P = \Phi(\alpha\chi_P)$ for $P \in \mathcal{P}_0^n$ is an $\mathrm{SL}(n)$-equivariant Minkowski valuation that is homogeneous of degree $n + 1$.*

Proof. First, we will show that Z_α is a Minkowski valuation on $\mathcal{P}_0^n$. Let $P, Q \in \mathcal{P}_0^n$ with $P \cup Q \in \mathcal{P}_0^n$. Observe that if $\alpha \geq 0$, then

$$\alpha\chi_{P\cup Q} = \alpha\chi_P \vee \alpha\chi_Q \text{ and } \alpha\chi_{P\cap Q} = \alpha\chi_P \wedge \alpha\chi_Q.$$

Also, if $\alpha < 0$, then

$$\alpha\chi_{P\cap Q} = \alpha\chi_P \vee \alpha\chi_Q \text{ and } \alpha\chi_{P\cup Q} = \alpha\chi_P \wedge \alpha\chi_Q.$$

Thus, the valuation property of Φ gives

$$\begin{aligned}\mathrm{Z}_\alpha(P\cup Q) + \mathrm{Z}_\alpha(P\cap Q) &= \Phi(\alpha\chi_{P\cup Q}) + \Phi(\alpha\chi_{P\cap Q})\\ &= \Phi(\alpha\chi_P \vee \alpha\chi_Q) + \Phi(\alpha\chi_P \wedge \alpha\chi_Q)\\ &= \Phi(\alpha\chi_P) + \Phi(\alpha\chi_Q)\\ &= \mathrm{Z}_\alpha P + \mathrm{Z}_\alpha Q.\end{aligned}$$

Therefore, Z_α is a Minkowski valuation on $\mathcal{P}_0^n$.

Now, we show that Z_α is $\mathrm{SL}(n)$-equivariant. Since Φ is $\mathrm{GL}(n)$-equivariant, we get that for every $P \in \mathcal{P}_0^n$ and every $A \in \mathrm{SL}(n)$,

$$\begin{aligned}\mathrm{Z}_\alpha(AP) &= \Phi((\alpha\chi_P)\circ A^{-1})\\ &= \Phi(A(\alpha\chi_P))\\ &= |\det A| A\Phi(\alpha\chi_P)\\ &= A\,\mathrm{Z}_\alpha P.\end{aligned}$$

Hence, Z_α is $\mathrm{SL}(n)$-equivariant.

Lastly, we show that Z_α is homogeneous of degree $n+1$. Since Φ is $\mathrm{GL}(n)$-equivariant, we get that for every $P \in \mathcal{P}_0^n$ and every $\beta > 0$,

$$\begin{aligned}\mathrm{Z}_\alpha(\beta P) &= \Phi((\alpha\chi_P)\circ(\beta I)^{-1})\\ &= \Phi((\beta I)(\alpha\chi_P))\\ &= |\det(\beta I)|(\beta I)\Phi(\alpha\chi_P)\\ &= \beta^{n+1}\,\mathrm{Z}_\alpha P.\end{aligned}$$

Note that I is the identity map on $\mathbb{R}^n$. Therefore, Z_α is homogeneous of degree $n+1$. □

Lemma 3.3. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous $\mathrm{GL}(n)$-equivariant Minkowski valuation, then there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that for every $P \in \mathcal{P}_0^n$ and every $\alpha \in \mathbb{R}$,*

$$h(\Phi(\alpha\chi_P), u) = G(\alpha)\int_P x\cdot u\,dx + H(\alpha)\int_P |x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$.

Proof. Apply Lemma 3.2 and then Theorem 1.3 to get that for each $\alpha \in \mathbb{R}$, there exist $a_\alpha, b_\alpha \in \mathbb{R}$ with $b_\alpha \geq 0$ such that for every $P \in \mathcal{P}_0^n$,

$$h(\Phi(\alpha\chi_P), u) = a_\alpha\int_P x\cdot u\,dx + b_\alpha\int_P |x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$. Define $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ by $G(\alpha) = a_\alpha$ and $H(\alpha) = b_\alpha$. Let us show that G and H are continuous. For $P, Q \in \mathcal{P}_0^n$ and $u \in S^{n-1}$ define the matrix

$$M(P,Q,u) = \begin{pmatrix} \int_P x \cdot u \, dx & \int_P |x \cdot u| \, dx \\ \int_Q x \cdot u \, dx & \int_Q |x \cdot u| \, dx \end{pmatrix}.$$

Clearly, there are $P_0, Q_0 \in \mathcal{P}_0^n$ and $u_0 \in S^{n-1}$ such that $\det M(P_0, Q_0, u_0) \neq 0$. Hence, from the two equations

$$h(\Phi(\alpha\chi_{P_0}), u_0) = G(\alpha) \int_{P_0} x \cdot u_0 \, dx + H(\alpha) \int_{P_0} |x \cdot u_0| \, dx,$$
$$h(\Phi(\alpha\chi_{Q_0}), u_0) = G(\alpha) \int_{Q_0} x \cdot u_0 \, dx + H(\alpha) \int_{Q_0} |x \cdot u_0| \, dx,$$

we get

$$G(\alpha) = \Big(\frac{\int_{Q_0} |x \cdot u_0| \, dx}{\det M(P_0,Q_0,u_0)}\Big) h(\Phi(\alpha\chi_{P_0}),u_0) - \Big(\frac{\int_{P_0} |x \cdot u_0| \, dx}{\det M(P_0,Q_0,u_0)}\Big) h(\Phi(\alpha\chi_{Q_0}),u_0)$$

$$H(\alpha) = -\Big(\frac{\int_{Q_0} x \cdot u_0 \, dx}{\det M(P_0,Q_0,u_0)}\Big) h(\Phi(\alpha\chi_{P_0}), u_0) + \Big(\frac{\int_{P_0} x \cdot u_0 \, dx}{\det M(P_0,Q_0,u_0)}\Big) h(\Phi(\alpha\chi_{Q_0}), u_0).$$

Note that the above gives $G(0) = H(0) = 0$ since $\Phi(0) = \{o\}$. Now let $\alpha \in \mathbb{R}$ and $\{\alpha_i\}$ be a sequence in $\mathbb{R}$ such that $\alpha_i \to \alpha$. Observe that for $P \in \mathcal{P}_0^n$,

$$\begin{aligned} \|\alpha_i\chi_P - \alpha\chi_P\|_{L^p(\mu_n)} &= |\alpha_i - \alpha| \|\chi_P\|_{L^p(\mu_n)} \\ &= |\alpha_i - \alpha| \Big(\int_{\mathbb{R}^n} |\chi_P(x)|^p |x| \, dx \Big)^{1/p}. \end{aligned}$$

Hence, $\|\alpha_i\chi_P - \alpha\chi_P\|_{L^p(\mu_n)} \to 0$ since $\alpha_i \to \alpha$. Thus, $\{\alpha_i\chi_{P_0}\}$ converges to $\alpha\chi_{P_0}$ in $L^p(\mu_n)$, and $\{\alpha_i\chi_{Q_0}\}$ converges to $\alpha\chi_{Q_0}$ in $L^p(\mu_n)$. Hence, the continuity of Φ gives $h(\Phi(\alpha_i\chi_{P_0}), u_0) \to h(\Phi(\alpha\chi_{P_0}), u_0)$ and $h(\Phi(\alpha_i\chi_{Q_0}), u_0) \to h(\Phi(\alpha\chi_{Q_0}), u_0)$. Thus $G(\alpha_i) \to G(\alpha)$ and $H(\alpha_i) \to H(\alpha)$. Therefore, both G and H are continuous. □

Lemma 3.4. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ is a Minkowski valuation, then for each $k \in \mathbb{N}$,*

$$\Phi\Big(\sum_{i=1}^k \alpha_i \chi_{E_i}\Big) = \sum_{i=1}^k \Phi(\alpha_i \chi_{E_i}),$$

for all pairwise disjoint sets $E_1,\ldots,E_k \in \mathfrak{M}$ with finite measures and for all $\alpha_1,\ldots,\alpha_k \in \mathbb{R}$ such that either all α_i's are nonnegative or all α_i's are nonpositive.

Proof. First assume that all α_i's are nonnegative. We use induction on k. The case $k = 1$ is trivially true. Assume that

$$\Phi\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) = \sum_{i=1}^{k-1} \Phi(\alpha_i \chi_{E_i}),$$

for every pairwise disjoint sets $E_1,...,E_{k-1} \in \mathfrak{M}$ with finite measures and for all nonnegative $\alpha_1,...,\alpha_{k-1} \in \mathbb{R}$. Let $E_1,...,E_k \in \mathfrak{M}$ be pairwise disjoint sets with finite measures and let $\alpha_1,...,\alpha_k \in \mathbb{R}$ be nonnegative. It follows that

$$\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) \vee \alpha_k \chi_{E_k} = \sum_{i=1}^{k} \alpha_i \chi_{E_i} \quad \text{and} \quad \Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) \wedge \alpha_k \chi_{E_k} = 0.$$

Thus, the valuation property of Φ and the induction assumption give

$$\begin{aligned}
\Phi\Big(\sum_{i=1}^{k} \alpha_i \chi_{E_i}\Big) &= \Phi\Big(\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) \vee \alpha_k \chi_{E_k}\Big) + \Phi(0) \\
&= \Phi\Big(\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) \vee \alpha_k \chi_{E_k}\Big) + \Phi\Big(\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) \wedge \alpha_k \chi_{E_k}\Big) \\
&= \Phi\Big(\sum_{i=1}^{k-1} \alpha_i \chi_{E_i}\Big) + \Phi(\alpha_k \chi_{E_k}) \\
&= \sum_{i=1}^{k-1} \Phi(\alpha_i \chi_{E_i}) + \Phi(\alpha_k \chi_{E_k}) \\
&= \sum_{i=1}^{k} \Phi(\alpha_i \chi_{E_i}).
\end{aligned}$$

The proof when all the α_i's are nonpositive is the same as the nonnegative case when "$\vee$" is swapped with "$\wedge$" and "nonnegative" is swapped with "nonpositive." $\square$

Let $\mathcal{P}^n$ be the set of all convex polytopes in $\mathbb{R}^n$. For $P \in \mathcal{P}^n$ we will denote by P^o the interior of P.

Lemma 3.5. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that for every $Q \in \mathcal{P}_0^n$ and every $\alpha \in \mathbb{R}$,*

$$h(\Phi(\alpha\chi_Q), u) = G(\alpha) \int_Q x \cdot u \, dx + H(\alpha) \int_Q |x \cdot u| \, dx$$

for $u \in \mathbb{R}^n$, then for every $P \in \mathcal{P}^n$ and every $\alpha \in \mathbb{R}$,

$$h(\Phi(\alpha\chi_P), u) = G(\alpha) \int_P x \cdot u \, dx + H(\alpha) \int_P |x \cdot u| \, dx$$

for $u \in \mathbb{R}^n$.

Proof. The proof where $P \in \mathcal{P}_0^n$ or $P \in \mathcal{P}^n$ with $\lambda(P) = 0$ is trivial. So let $P \in \mathcal{P}^n \backslash \mathcal{P}_0^n$ with $\lambda(P) > 0$. The following technique is suggested by Ludwig [22]. Let Q be the convex hull of P and $\{o\}$. There exist $k \in \mathbb{N}$ and convex polyhedral cones C_i with apex at o, for $i = 1, ..., k$, such that the C_i's have pairwise disjoint interiors and such that $P = \bigcup_{i=1}^{k} P_i$, where $P_i = P \cap C_i$ and $Q = \bigcup_{i=1}^{k} Q_i$, where $Q_i = Q \cap C_i$. Note that $Q_i \in \mathcal{P}_0^n$. Now for $i = 1, ..., k$, let $R_i \in \mathcal{P}_0^n$ be such that

$P_i \cup R_i = Q_i$ and $P_i^o \cap R_i^o = \emptyset$. Let $\alpha \in \mathbb{R}$. Note that

$$\alpha\chi_Q = \alpha\chi_{\bigcup_{i=1}^k Q_i} = \alpha\chi_{\bigcup_{i=1}^k Q_i^o} = \sum_{i=1}^k \alpha\chi_{Q_i^o},$$

and similarly

$$\alpha\chi_P = \sum_{i=1}^k \alpha\chi_{P_i^o}.$$

Also, $\alpha\chi_{Q_i^o} = \alpha\chi_{P_i^o} + \alpha\chi_{R_i^o}$. Thus, by Lemma 3.4,

$$\Phi(\alpha\chi_Q) = \sum_{i=1}^k \Phi(\alpha\chi_{Q_i^o}) = \sum_{i=1}^k \Phi(\alpha\chi_{P_i^o}) + \sum_{i=1}^k \Phi(\alpha\chi_{R_i^o}) = \Phi(\alpha\chi_P) + \sum_{i=1}^k \Phi(\alpha\chi_{R_i}).$$

Thus for $u \in \mathbb{R}^n$,

$$\begin{aligned} h(\Phi(\alpha\chi_Q), u) &= h(\Phi(\alpha\chi_P) + \sum_{i=1}^k \Phi(\alpha\chi_{R_i}), u) \\ &= h(\Phi(\alpha\chi_P), u) + \sum_{i=1}^k h(\Phi(\alpha\chi_{R_i}), u). \end{aligned}$$

Now since $Q, R_i \in \mathcal{P}_0^n$, we get that

$$\begin{aligned} h(\Phi(\alpha\chi_P), u) &= h(\Phi(\alpha\chi_Q), u) - \sum_{i=1}^k h(\Phi(\alpha\chi_{R_i}), u) \\ &= G(\alpha)\int_Q x\cdot u\,dx + H(\alpha)\int_Q |x\cdot u|\,dx \\ &\quad - \sum_{i=1}^k \Big[G(\alpha)\int_{R_i} x\cdot u\,dx + H(\alpha)\int_{R_i} |x\cdot u|\,dx\Big] \\ &= G(\alpha)\int_{Q\setminus(\bigcup_{i=1}^k R_i)} x\cdot u\,dx + H(\alpha)\int_{Q\setminus(\bigcup_{i=1}^k R_i)} |x\cdot u|\,dx \\ &= G(\alpha)\int_P x\cdot u\,dx + H(\alpha)\int_P |x\cdot u|\,dx. \end{aligned}$$

□

For $r, R \in \mathbb{R}$ with $0 \le r < R$, let $A[r, R)$ denote the annulus

$$A[r, R) = \{x : x \in \mathbb{R}^n,\ r \le |x| < R\}.$$

Lemma 3.6. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \ge 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \ge 0$ and $G(0) = H(0) = 0$ such that for every $P \in \mathcal{P}^n$ and every $\alpha \in \mathbb{R}$,*

$$h(\Phi(\alpha\chi_P), u) = G(\alpha)\int_P x\cdot u\,dx + H(\alpha)\int_P |x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$, then for all $R > 0$ and all $E \in \mathfrak{M}$ with $E \subseteq A[0, R)$ and every $\alpha \in \mathbb{R}$,

$$h(\Phi(\alpha\chi_E), u) = G(\alpha)\int_E x\cdot u\,dx + H(\alpha)\int_E |x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$.

Proof. Let $R > 0$ and let $E \in \mathfrak{M}$ with $E \subseteq A[0, R)$. Since $\lambda(E) < \infty$, by Theorem 2.5, there exists a sequence $\{K_i\}$ of finite unions of closed cubes in $\mathcal{R}^n$ such that $\lambda(K_i \Delta E) < 1/i$. Since $E \subseteq A[0, R)$ and $\lambda(K_i \Delta E) < 1/i$, it is clear that there exists a $C > 0$ such that $E, K_i \subseteq A[0, C)$. For $\alpha \in \mathbb{R}$, observe that

$$\begin{aligned}\|\alpha\chi_{K_i} - \alpha\chi_E\|_{L^p(\mu_n)} &= |\alpha|\Big(\int_{\mathbb{R}^n} |\chi_{K_i}(x) - \chi_E(x)|^p |x|\, dx\Big)^{1/p} \\ &\le |\alpha|\Big(\int_{\mathbb{R}^n} |\chi_{K_i}(x) - \chi_E(x)|^p C\, dx\Big)^{1/p} \\ &= |\alpha| C^{1/p} \lambda(K_i \Delta E)^{1/p},\end{aligned}$$

so $\|\alpha\chi_{K_i} - \alpha\chi_E\|_{L^p(\mu_n)} \to 0$. Thus, $\{\alpha\chi_{K_i}\}$ converges to $\alpha\chi_E$ in $L^p(\mu_n)$. But for each K_i there exist $P_{i_1}, ..., P_{i_k} \in \mathcal{P}^n$ with pairwise disjoint interiors such that $K_i = \bigcup_{j=1}^k P_{i_j}$. Since

$$\alpha\chi_{K_i} = \sum_{j=1}^k \alpha\chi_{P_{i_j}^o},$$

Lemma 3.4 gives us

$$\Phi(\alpha\chi_{K_i}) = \sum_{j=1}^k \Phi(\alpha\chi_{P_{i_j}^o}) = \sum_{j=1}^k \Phi(\alpha\chi_{P_{i_j}}).$$

Thus, for $u \in S^{n-1}$,

$$\begin{aligned}h(\Phi(\alpha\chi_{K_i}), u) &= \sum_{j=1}^k h(\Phi(\alpha\chi_{P_{i_j}}), u) \\ &= \sum_{j=1}^k \Big[G(\alpha)\int_{P_{i_j}} x \cdot u\, dx + H(\alpha)\int_{P_{i_j}} |x \cdot u|\, dx\Big] \\ &= G(\alpha)\int_{\bigcup_{j=1}^k P_{i_j}} x \cdot u\, dx + H(\alpha)\int_{\bigcup_{j=1}^k P_{i_j}} |x \cdot u|\, dx \\ &= G(\alpha)\int_{K_i} x \cdot u\, dx + H(\alpha)\int_{K_i} |x \cdot u|\, dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\Big|h(\Phi(\alpha\chi_{K_i}), u) - \Big(G(\alpha)\int_E x \cdot u\, dx + H(\alpha)\int_E |x \cdot u|\, dx\Big)\Big| \\ &\le |G(\alpha)|\Big|\int_{K_i} x \cdot u\, dx - \int_E x \cdot u\, dx\Big| + |H(\alpha)|\Big|\int_{K_i} |x \cdot u|\, dx - \int_E |x \cdot u|\, dx\Big| \\ &\le (|G(\alpha)| + |H(\alpha)|)\int_{\mathbb{R}^n} |\chi_{K_i}(x) - \chi_E(x)||x|\, dx \\ &\le (|G(\alpha)| + |H(\alpha)|)\int_{\mathbb{R}^n} |\chi_{K_i}(x) - \chi_E(x)| C\, dx \\ &\le (|G(\alpha)| + |H(\alpha)|) C\lambda(K_i \Delta E).\end{aligned}$$

Now the continuity of Φ gives

$$\begin{aligned} h(\Phi(\alpha\chi_E), u) &= \lim_{i\to\infty} h(\Phi(\alpha\chi_{K_i}), u) \\ &= G(\alpha)\int_E x\cdot u\,dx + H(\alpha)\int_E |x\cdot u|\,dx. \end{aligned}$$

Therefore, for all $u \in \mathbb{R}^n$,

$$h(\Phi(\alpha\chi_E), u) = G(\alpha)\int_E x\cdot u\,dx + H(\alpha)\int_E |x\cdot u|\,dx.$$

□

Lemma 3.7. *Let $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ be a Minkowski valuation and let $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ be continuous functions with $H \geq 0$ and $G(0) = H(0) = 0$. If $W \subseteq L^p(\mu_n)$ satisfies*

(i) *$f \vee 0, f \wedge 0 \in W$ whenever $f \in W$, and*
(ii) *for every nonnegative and every nonpositive $f \in W$ and for all $u \in \mathbb{R}^n$,*

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G\circ f)(x)x\cdot u + (H\circ f)(x)|x\cdot u|]\,dx,$$

then for all $f \in W$,

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G\circ f)(x)x\cdot u + (H\circ f)(x)|x\cdot u|]\,dx$$

for all $u \in \mathbb{R}^n$.

Proof. For $f \in W$, $f \vee 0$ is a nonnegative function in W while $f \wedge 0$ is a nonpositive function in W. Hence, the valuation property of Φ together with $G(0) = H(0) = 0$ gives that for all $u \in \mathbb{R}^n$,

$$\begin{aligned} h(\Phi(f), u) &= h(\Phi(f) + \Phi(0), u) \\ &= h(\Phi(f\vee 0) + \Phi(f\wedge 0), u) \\ &= h(\Phi(f\vee 0), u) + h(\Phi(f\wedge 0), u) \\ &= \int_{\mathbb{R}^n} [(G\circ(f\vee 0))(x) + (G\circ(f\wedge 0))(x)]x\cdot u\,dx \\ &\quad + \int_{\mathbb{R}^n} [(H\circ(f\vee 0))(x) + (H\circ(f\wedge 0))(x)]|x\cdot u|\,dx \\ &= \int_{\mathbb{R}^n} [(G\circ f)(x)x\cdot u + (H\circ f)(x)|x\cdot u|]\,dx. \end{aligned}$$

□

Let $L^p_a(\mu_n)$ be the set of $f \in L^p(\mu_n)$ such that there exist $r, R \in \mathbb{R}$ with $0 < r < R$ and $\chi_{\mathbb{R}^n\setminus A[r,R)} f = 0$, and let $L^p_a(\mathbb{R}^n)$ be the set of $f \in L^p(\mathbb{R}^n)$ such that there exist $r, R \in \mathbb{R}$ with $0 < r < R$ and $\chi_{\mathbb{R}^n\setminus A[r,R)} f = 0$.

Lemma 3.8. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that for all $R > 0$ and all $E \in \mathfrak{M}$ with $E \subseteq A[0, R)$ and every $\alpha \in \mathbb{R}$,*

$$h(\Phi(\alpha\chi_E), u) = G(\alpha)\int_E x\cdot u\,dx + H(\alpha)\int_E |x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$, then for every simple function $s \in L_a^p(\mu_n)$,

$$h(\Phi(s), u) = \int_{\mathbb{R}^n} [(G \circ s)(x) x \cdot u + (H \circ s)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$.

Proof. If $s \in L_a^p(\mu_n)$ is a nonnegative or a nonpositive simple function, then there exist $r, R \in \mathbb{R}$ with $0 < r < R$ and there exist pairwise disjoint sets $E_1,...,E_k \in \mathfrak{M}$ with $E_i \subseteq A[r, R)$ and $\alpha_1,...,\alpha_k \in \mathbb{R}$ such that $s = \sum_{i=1}^k \alpha_i \chi_{E_i}$. Hence, by Lemma 3.4,

$$\begin{aligned}
h(\Phi(s), u) &= h(\sum_{i=1}^k \Phi(\alpha_i \chi_{E_i}), u) \\
&= \sum_{i=1}^k h(\Phi(\alpha_i \chi_{E_i}), u) \\
&= \sum_{i=1}^k \Big[G(\alpha_i) \int_{E_i} x \cdot u\, dx + H(\alpha_i) \int_{E_i} |x \cdot u|\, dx \Big] \\
&= \int_{\mathbb{R}^n} \Big[\Big(\sum_{i=1}^k G(\alpha_i) \chi_{E_i}(x) \Big) x \cdot u\, dx + \Big(\sum_{i=1}^k H(\alpha_i) \chi_{E_i}(x) \Big) |x \cdot u|\, dx \Big] \\
&= \int_{\mathbb{R}^n} [(G \circ s)(x) x \cdot u\, dx + (H \circ s)(x)|x \cdot u|\, dx].
\end{aligned}$$

Note that the last equality comes from the fact that $G(0) = H(0) = 0$,

$$G \circ s = \sum_{i=1}^k G(\alpha_i) \chi_{E_i} \quad \text{and} \quad H \circ s = \sum_{i=1}^k H(\alpha_i) \chi_{E_i}.$$

Finally, apply Lemma 3.7 with W equal to the collection of simple functions in $L_a^p(\mu_n)$ to get that for every simple function $s \in L_a^p(\mu_n)$,

$$h(\Phi(s), u) = \int_{\mathbb{R}^n} [(G \circ s)(x) x \cdot u + (H \circ s)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$. □

Proposition 3.9. *The spaces $L_a^p(\mu_n)$ and $L_a^p(\mathbb{R}^n)$ are equal.*

Proof. Let f be a measurable function such that there exist $r, R \in \mathbb{R}$ with $0 < r < R$ and $\chi_{\mathbb{R}^n \setminus A[r,R)}(x) f(x) = 0$ a.e. Observe that

$$\begin{aligned}
\|f\|_{L^p(\mu_n)} &\le \Big(\int_{A[r,R)} |f(x)|^p R\, dx \Big)^{1/p} \\
&= R^{1/p} \|f\|_p \\
&\le R^{1/p} \Big(\int_{A[r,R)} |f(x)|^p \frac{|x|}{r}\, dx \Big)^{1/p} \\
&= \Big(\frac{R}{r} \Big)^{1/p} \|f\|_{L^p(\mu_n)}.
\end{aligned}$$

Thus,

$$\|f\|_{L^p(\mu_n)} \leq R^{1/p}\|f\|_p \leq \left(\frac{R}{r}\right)^{1/p}\|f\|_{L^p(\mu_n)}.$$

This completes the proof. □

Lemma 3.10. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that*

$$h(\Phi(s), u) = \int_{\mathbb{R}^n} [(G \circ s)(x)x \cdot u + (H \circ s)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for every simple function $s \in L^p_a(\mu_n)$, then

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$.

Proof. Let $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$ be nonnegative. Then there exist $r, R \in \mathbb{R}$ with $0 < r < R$ such that $\chi_{\mathbb{R}^n \setminus A[r,R)}(x)f(x) = 0$ a.e. By Proposition 3.9, $f \in L^p_a(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Now by Theorem 2.2, there exists a sequence of simple functions $\{s_i\}$ on $\mathbb{R}^n$ such that $0 \leq s_1(x) \leq s_2(x) \leq ... \leq f(x)$, a.e. and $s_i(x) \to f(x)$, a.e. Note that we also have $|s_i(x)| \leq \|f\|_\infty < \infty$, a.e. Let $u \in S^{n-1}$. Since G and H are continuous, we also have that

$$(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u| \to (G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|, \text{ a.e.}$$

Also, note that all s_i are essentially supported on $A[r, R)$ and since $G(0) = H(0) = 0$, we also have that $G \circ s_i$ and $H \circ s_i$ are essentially supported on $A[r, R)$. Thus,

$$\begin{aligned}|(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|| &\leq |(G \circ s_i)(x) + (H \circ s_i)(x)||x| \\ &= |((G + H) \circ s_i)(x)||x| \\ &\leq \sup\{|(G + H)(\alpha)| : 0 \leq \alpha \leq \|f\|_\infty\}|x|\chi_{A[r,R]}(x),\end{aligned}$$

almost everywhere and

$$\begin{aligned}&\int_{\mathbb{R}^n} \sup\{|(G + H)(\alpha)| : 0 \leq \alpha \leq \|f\|_\infty\}|x|\chi_{A[r,R]}(x)\, dx \\ &\quad = \sup\{|(G + H)(\alpha)| : 0 \leq \alpha \leq \|f\|_\infty\}\mu_n(A[r, R)) < \infty.\end{aligned}$$

Therefore, Lebesgue's Dominated Convergence Theorem yields

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} [(G\circ s_i)(x)x\cdot u+(H\circ s_i)(x)|x\cdot u|]\, dx = \int_{\mathbb{R}^n} [(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\, dx,$$

and that the limit above is finite. Next, since $|s_i(x) - f(x)|^p \leq |f(x)|^p$ a.e. and $|f|^p \in L^1(\mu_n)$, Lebesgue's Dominated Convergence Theorem gives

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} |s_i - f|^p |x|\, dx = 0.$$

Hence, $\|s_i - f\|_{L^p(\mu_n)} \to 0$, so $\{s_i\}$ converges to f in $L^p(\mu_n)$. Now the continuity of Φ gives

$$\begin{aligned} h(\Phi(f), u) &= \lim_{i\to\infty} h(\Phi(s_i), u) \\ &= \lim_{i\to\infty} \int_{\mathbb{R}^n} [(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|]\, dx \\ &= \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx. \end{aligned}$$

Hence,

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$. If $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$ is nonpositive, then $-f$ is nonnegative. There exist $r, R \in \mathbb{R}$ with $0 < r < R$ such that $\chi_{\mathbb{R}^n \setminus A[r,R)}(x)f(x) = 0$ a.e. Again by Proposition 3.9, $f \in L^p_a(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Apply Theorem 2.2 to $-f$, to get a sequence of simple functions $\{s_i\}$ on $\mathbb{R}^n$ such that $0 \geq s_1(x) \geq s_2(x) \geq ... \geq f(x)$, a.e. and $s_i(x) \to f(x)$, a.e. Hence,

$$(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u| \to (G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|, \text{ a.e.}$$

Note that we still have $|s_i(x)| \leq \|f\|_\infty < \infty$, a.e. Thus,

$$\begin{aligned} |(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|| &\leq |(G \circ s_i)(x) + (H \circ s_i)(x)||x| \\ &= |((G + H) \circ s_i)(x)||x| \\ &\leq \sup\{|(G + H)(\alpha)| : -\|f\|_\infty \leq \alpha \leq 0\}|x|\chi_{A[r,R]}(x), \end{aligned}$$

almost everywhere and

$$\begin{aligned} &\int_{\mathbb{R}^n} \sup\{|(G + H)(\alpha)| : -\|f\|_\infty \leq \alpha \leq 0\}|x|\chi_{A[r,R]}(x)\, dx \\ &\quad = \sup\{|(G + H)(\alpha)| : -\|f\|_\infty \leq \alpha \leq 0\}\mu_n(A[r, R)) < \infty. \end{aligned}$$

Therefore, Lebesgue's Dominated Convergence Theorem yields

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} [(G\circ s_i)(x)x\cdot u+(H\circ s_i)(x)|x\cdot u|]\, dx = \int_{\mathbb{R}^n} [(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\, dx,$$

and that the limit above is finite. Similarly to the nonnegative case, we get

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$. Finally, for the general case, apply Lemma 3.7 with $W = L^p_a(\mu_n) \cap L^\infty(\mu_n)$ to get

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for all $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$. □

Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and denote by B_n the unit ball in $\mathbb{R}^n$.

Lemma 3.11. *For any sequence of positive real numbers $\{\beta_j\}$ there exists a sequence of balls $\{E_j : E_j = r_jB_n + c_je_1\}$ in $\mathbb{R}^n$ satisfying*

(i) $c_j > r_j$ *for* $j \geq 1$,
(ii) $0 < r_j < 1$ *for* $j \geq 1$,
(iii) $c_j > c_{j-1} + 2$ *for* $j \geq 2$, *and*
(iv) $(c_j + r_j)\lambda(E_j) = \beta_j$ *for* $j \geq 1$.

Furthermore, such a sequence $\{E_j\}$ is pairwise disjoint and satisfies

$$\lim_{j\to\infty} \frac{c_j - r_j}{c_j + r_j} = 1.$$

Proof. First note that $(c_j + r_j)\lambda(E_j) = \beta_j$ if and only if $(c_j + r_j)r_j^n\kappa_n = \beta_j$ if and only if $c_j = \dfrac{\beta_j}{r_j^n\kappa_n} - r_j$. Clearly, there exist $r_1 > 0$ satisfying $0 < r_1 < 1$ and $\dfrac{\beta_1}{r_1^n\kappa_n} - r_1 > r_1$. So let $c_1 = \dfrac{\beta_1}{r_1^n\kappa_n} - r_1$ and we have $c_1 > r_1$. Next, for $k \in \mathbb{N}$ assume that there exist $E_j = r_jB_n + c_je_1$ for $j = 1, ..., k-1$ such that (i)-(iv) hold. Observe that there exists $r_k > 0$ satisfying $0 < r_k < 1$ and

$$\frac{\beta_k}{r_k^n\kappa_n} - r_k > c_{k-1} + 2 > 1 > r_k.$$

So let $c_k = \dfrac{\beta_k}{r_k^n\kappa_n} - r_k$ and we have $c_k > c_{k-1} + 2 > r_k$. Thus, $E_1, ..., E_k$ satisfy (i)-(iv). It follows by induction that there exists a sequence of balls $\{E_j : E_j = r_jB_n + c_je_1\}$ in $\mathbb{R}^n$ satisfying (i)-(iv).

The fact that the sequence $\{E_j\}$ is pairwise disjoint follows from

$$c_{j-1} + r_{j-1} < c_{j-1} + 1 < c_j - 1 < c_j - r_j$$

for $j \geq 2$. Lastly,

$$\lim_{j\to\infty} \frac{c_j - r_j}{c_j + r_j} = 1$$

comes from the facts that $0 < r_j < 1$ and $c_j \to \infty$. □

Lemma 3.12. *Let $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ be continuous functions with $H \geq 0$ and $G(0) = H(0) = 0$. If for all $u \in S^{n-1}$ and all $f \in L^p(\mu_n)$ and every sequence $\{f_k\}$ in $L^p_a(\mu_n) \cap L^\infty(\mu_n)$ that converges to f in $L^p(\mu_n)$,*

$$\left|\limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx\right| < \infty,$$

then there exist $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma|\alpha|^p$ and $|H(\alpha)| \leq \delta|\alpha|^p$ for all $\alpha \in \mathbb{R}$.

Proof. First, suppose for all $\gamma \geq 0$ that there exists $\alpha \in \mathbb{R}$ such that

$$|G(\alpha)| > \gamma|\alpha|^p.$$

Since $G(0) = 0$, the above holds with $\alpha \neq 0$. Therefore, there exists a sequence $\{\alpha_i\}$ in $\mathbb{R}\backslash\{0\}$ such that

$$|G(\alpha_i)| > 2^i|\alpha_i|^p$$

for all $i \in \mathbb{N}$. There even exists a subsequence $\{\alpha_{i_j}\}$ of $\{\alpha_i\}$ such that either

$$G(\alpha_{i_j}) > 2^{i_j}|\alpha_{i_j}|^p$$

for all $j \in \mathbb{N}$, or

$$-G(\alpha_{i_j}) > 2^{i_j}|\alpha_{i_j}|^p$$

for all $j \in \mathbb{N}$. Define a sequence of balls $\{E_j : E_j = r_j B_n + c_j e_1\}$ in $\mathbb{R}^n$ as in Lemma 3.11 with $\beta_j = \dfrac{1}{2^{i_j}|\alpha_{i_j}|^p}$ for all $j \in \mathbb{N}$. Then we have that $\{E_j\}$ satisfies

(i) $c_j > r_j$ for $j \geq 1$,
(ii) $0 < r_j < 1$ for $j \geq 1$,
(iii) $c_j > c_{j-1} + 2$ for $j \geq 2$, and
(iv) $(c_j + r_j)\lambda(E_j) = \dfrac{1}{2^{i_j}|\alpha_{i_j}|^p}$ for $j \geq 1$.

Also, $\{E_j\}$ is pairwise disjoint and satisfies

$$\lim_{j\to\infty} \frac{c_j - r_j}{c_j + r_j} = 1.$$

Now for each $k \in \mathbb{N}$ define $f_k : \mathbb{R}^n \to [-\infty, \infty]$ by

$$f_k = \sum_{j=1}^{k} \alpha_{i_j} \chi_{E_j}.$$

Then each $f_k \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$. Also, define $f : X \to [-\infty, \infty]$ by

$$f = \sum_{j=1}^{\infty} \alpha_{i_j} \chi_{E_j}.$$

Observe that $0 \leq |f_1(x)|^p \leq |f_2(x)|^p \leq \ldots \leq \infty$, for all $x \in \mathbb{R}^n$ and $|f_k(x)|^p \to |f(x)|^p$, for all $x \in \mathbb{R}^n$, so Lebesgue's Monotone Convergence Theorem implies that $|f|^p$ is measurable and

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^p |x|\, dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k(x)|^p |x|\, dx \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} |\alpha_{i_j}|^p \int_{E_j} |x|\, dx \\
&\leq \sum_{j=1}^{\infty} |\alpha_{i_j}|^p (c_j + r_j)\lambda(E_j) \\
&= \sum_{j=1}^{\infty} \frac{1}{2^{i_j}} \\
&\leq 1.
\end{aligned}$$

Thus $f \in L^p(\mu_n)$.

Observe that $|f_k - f| \leq |f_k| + |f| \leq 2|f|$. Thus, $|f_k(x) - f(x)|^p \leq 2^p |f(x)|^p$ for all $x \in \mathbb{R}^n$. But $2^p|f|^p$ is integrable; thus, Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k(x) - f(x)|^p |x|\, dx = 0.$$

Hence, $\|f_k - f\|_{L^p(\mu_n)} \to 0$, so $\{f_k\}$ converges to f in $L^p(\mu_n)$. If $G(\alpha_{i_j}) > 2^{i_j}|\alpha_{i_j}|^p$ for all $j \in \mathbb{N}$, then let $u = e_1$ to get

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx \\
&= \limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot e_1 + (H \circ f_k)(x)|x \cdot e_1|]\, dx \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[G(\alpha_{i_j}) \int_{E_j} x \cdot e_1\, dx + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&\geq \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[2^{i_j}|\alpha_{i_j}|^p (c_j - r_j)\lambda(E_j) + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[2^{i_j}|\alpha_{i_j}|^p (c_j - r_j)\frac{1}{(c_j + r_j)2^{i_j}|\alpha_{i_j}|^p} + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[\frac{c_j - r_j}{c_j + r_j} + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&= \infty.
\end{aligned}$$

Note that the last equality comes from the facts that $H \geq 0$ and $\lim_{j\to\infty} \frac{c_j - r_j}{c_j + r_j} = 1$. If $-G(\alpha_{i_j}) > 2^{i_j}|\alpha_{i_j}|^p$ for all $j \in \mathbb{N}$, then let $u = -e_1$ to get

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx \\
&= \limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot (-e_1) + (H \circ f_k)(x)|x \cdot (-e_1)|]\, dx \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[-G(\alpha_{i_j}) \int_{E_j} x \cdot e_1\, dx + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&\geq \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[2^{i_j}|\alpha_{i_j}|^p (c_j - r_j)\lambda(E_j) + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[\frac{c_j - r_j}{c_j + r_j} + H(\alpha_{i_j}) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&= \infty.
\end{aligned}$$

In either case we have

$$\Big|\limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx\Big| = \infty$$

for some $u \in S^{n-1}$.

Next, assume that there exist $\gamma \geq 0$ such that $|G(\alpha)| \leq \gamma|\alpha|^p$ for all $\alpha \in \mathbb{R}$ and for all $\delta \geq 0$ there exists $\alpha \in \mathbb{R}$ such that

$$|H(\alpha)| > \delta|\alpha|^p.$$

Since $H(0) = 0$, the above holds with $\alpha \neq 0$. Therefore, there exists a sequence $\{\alpha_j\}$ in $\mathbb{R}\backslash\{0\}$ such that

$$|H(\alpha_j)| > 2^j|\alpha_j|^p$$

for all $j \in \mathbb{N}$. But $q \geq 0$, so we have

$$H(\alpha_j) > 2^j|\alpha_j|^p$$

for all $j \in \mathbb{N}$. Again, define a sequence of balls $\{E_j : E_j = r_jB_n + c_je_1\}$ in $\mathbb{R}^n$ as in Lemma 3.11 with $\beta_j = \dfrac{1}{2^j|\alpha_j|^p}$ for all $j \in \mathbb{N}$. Then we have that $\{E_j\}$ satisfies

(i) $c_j > r_j$ for $j \geq 1$,
(ii) $0 < r_j < 1$ for $j \geq 1$,
(iii) $c_j > c_{j-1} + 2$ for $j \geq 2$, and
(iv) $(c_j + r_j)\lambda(E_j) = \dfrac{1}{2^j|\alpha_j|^p}$ for $j \geq 1$.

Also, $\{E_j\}$ is pairwise disjoint and satisfies $\lim\limits_{j\to\infty} \dfrac{c_j - r_j}{c_j + r_j} = 1$. Now for each $k \in \mathbb{N}$, define $f_k : \mathbb{R}^n \to [-\infty, \infty]$ by

$$f_k = \sum_{j=1}^{k} \alpha_j \chi_{E_j}.$$

Then each $f_k \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$. Also, define $f : X \to [-\infty, \infty]$ by

$$f = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}.$$

Then $f \in L^p(\mu_n)$ and $\{f_k\}$ converges to f in $L^p(\mu_n)$. Now let $u = e_1$ to get

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx \\
&= \limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot e_1 + (H \circ f_k)(x)|x \cdot e_1|]\, dx \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[G(\alpha_j) \int_{E_j} x \cdot e_1\, dx + H(\alpha_j) \int_{E_j} |x \cdot e_1|\, dx\Big] \\
&\geq \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[G(\alpha_j) \int_{E_j} x \cdot e_1\, dx + 2^j|\alpha_j|^p(c_j - r_j)\lambda(E_j)\Big] \\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} \Big[G(\alpha_j) \int_{E_j} x \cdot e_1\, dx + \frac{c_j - r_j}{c_j + r_j}\Big] \\
&= \infty.
\end{aligned}$$

Note that the last equality comes from the facts that $\lim_{j\to\infty} \frac{c_j - r_j}{c_j + r_j} = 1$ and

$$\begin{aligned}\lim_{j\to\infty} \left| G(\alpha_j) \int_{E_j} x \cdot e_1 \, dx \right| &\leq \lim_{j\to\infty} \gamma |\alpha_j|^p \int_{E_j} |x \cdot e_1| \, dx \\ &\leq \lim_{j\to\infty} \gamma |\alpha_j|^p (c_j + r_j) \lambda(E_j) \\ &= \lim_{j\to\infty} \gamma |\alpha_j|^p \frac{1}{2^j |\alpha_j|^p} \\ &= 0.\end{aligned}$$

□

Lemma 3.13. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that*

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x) x \cdot u + (H \circ f)(x) |x \cdot u|] \, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$, then there exist $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma |\alpha|^p$ and $|H(\alpha)| \leq \delta |\alpha|^p$ for all $\alpha \in \mathbb{R}$.

Proof. Let $u \in S^{n-1}$, $f \in L^p(\mu_n)$ and $\{f_k\}$ be a sequence in $L^p_a(\mu_n) \cap L^\infty(\mu_n)$ that converges to f in $L^p(\mu_n)$. The continuity of Φ gives

$$h(\Phi(f), u) = \lim_{k\to\infty} h(\Phi(f_k), u) = \lim_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x) x \cdot u + (H \circ f_k)(x) |x \cdot u|] \, dx.$$

Hence,

$$\begin{aligned}&\left| \limsup_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x) x \cdot u + (H \circ f_k)(x) |x \cdot u|] \, dx \right| \\ &= \left| \lim_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x) x \cdot u + (H \circ f_k)(x) |x \cdot u|] \, dx \right| \\ &= |h(\Phi(f), u)| < \infty.\end{aligned}$$

Therefore, by Lemma 3.12, there exist $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma |\alpha|^p$ and $|H(\alpha)| \leq \delta |\alpha|^p$ for all $\alpha \in \mathbb{R}$. □

Lemma 3.14. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that*

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x) x \cdot u + (H \circ f)(x) |x \cdot u|] \, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$, then

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x) x \cdot u + (H \circ f)(x) |x \cdot u|] \, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p_a(\mu_n)$.

Proof. By Lemma 3.13 there exist $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma |\alpha|^p$ and $|H(\alpha)| \leq \delta |\alpha|^p$ for all $\alpha \in \mathbb{R}$. Let $f \in L^p_a(\mu_n)$ be nonnegative. By Proposition 3.9, $f \in L^p_a(\mathbb{R}^n)$. Now by Theorem 2.2, there exists a sequence of simple functions $\{s_i\}$ on

$\mathbb{R}^n$ such that $0 \leq s_1(x) \leq s_2(x) \leq ... \leq f(x)$, a.e. and $s_i(x) \to f(x)$, a.e. Let $u \in S^{n-1}$. Now, since G and H are continuous, we have that

$$(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u| \to (G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|, \text{ a.e.}$$

We also have

$$\begin{aligned} |(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|| &\leq |(G \circ s_i)(x) + (H \circ s_i)(x)||x| \\ &\leq (|(G \circ s_i)(x)| + |(H \circ s_i)(x)|)|x| \\ &\leq (\gamma|s_i(x)|^p + \delta|s_i(x)|^p)|x| \\ &\leq (\gamma + \delta)|s_i(x)|^p|x| \\ &\leq (\gamma + \delta)|f(x)|^p|x| \end{aligned}$$

almost everywhere. But $(\gamma + \delta)|f|^p| \cdot | \in L^1(\mathbb{R}^n)$, so Lebesgue's Dominated Convergence Theorem yields

$$\lim_{i \to \infty} \int_{\mathbb{R}^n} [(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|]\, dx = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx,$$

and that the limit above is finite. Next, since $|s_i(x) - f(x)|^p \leq |f(x)|^p$ a.e. and $|f|^p \in L^1(\mu_n)$, Lebesgue's Dominated Convergence Theorem gives

$$\lim_{i \to \infty} \int_{\mathbb{R}^n} |s_i(x) - f(x)|^p|x|\, dx = 0.$$

Hence, $\|s_i - f\|_{L^p(\mu_n)} \to 0$, so $\{s_i\}$ converges to f in $L^p(\mu_n)$. Now the continuity of Φ and the fact that $s_i \in L^p_a(\mu_n) \cap L^\infty(\mu_n)$ give

$$\begin{aligned} h(\Phi(f), u) &= \lim_{i \to \infty} h(\Phi(s_i), u) \\ &= \lim_{i \to \infty} \int_{\mathbb{R}^n} [(G \circ s_i)(x)x \cdot u + (H \circ s_i)(x)|x \cdot u|]\, dx \\ &= \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx. \end{aligned}$$

Hence,

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$. If $f \in L^p_a(\mu_n)$ is nonpositive, then $-f$ is nonnegative. Again by Proposition 3.9, $f \in L^p_a(\mathbb{R}^n)$. Apply Theorem 2.2 to $-f$ to get a sequence of simple functions $\{s_i\}$ on $\mathbb{R}^n$ such that $0 \geq s_1(x) \geq s_2(x) \geq ... \geq f(x)$, a.e. and $s_i(x) \to f(x)$, a.e. Similar to the nonnegative case, we get

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$. Finally, for the general case, apply Lemma 3.7 with $W = L^p_a(\mu_n)$ to get

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for all $f \in L^p_a(\mu_n)$. $\square$

Lemma 3.15. *If $\Phi : L^p(\mu_n) \to \mathcal{K}^n$, where $n \geq 2$, is a continuous Minkowski valuation and if there exist continuous functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ with $H \geq 0$ and $G(0) = H(0) = 0$ such that*

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p_a(\mu_n)$, then

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx$$

for all $u \in \mathbb{R}^n$, for every $f \in L^p(\mu_n)$.

Proof. By Lemma 3.13 there exist $\gamma, \delta \geq 0$ such that $|G(\alpha)| \leq \gamma|\alpha|^p$ and $|H(\alpha)| \leq \delta|\alpha|^p$ for all $\alpha \in \mathbb{R}$. Let $f \in L^p(\mu_n)$ be nonnegative. Now for each $k \in \mathbb{N}$ define $f_k : \mathbb{R}^n \to [-\infty, \infty]$ by

$$f_k = \sum_{j=0}^{k} \chi_{E_j} f,$$

where $E_j = A[1/2^{j+1}, 1/2^j) \cup A[j+1, j+2)$ for $j \geq 0$. Then $0 \leq f_1(x) \leq f_2(x) \leq ... \leq f(x)$, a.e. and $f_k(x) \to f(x)$, a.e. Let $u \in S^{n-1}$. Now, since G and H are continuous, we have that

$$(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u| \to (G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|, \text{ a.e.}$$

We also have

$$\begin{aligned}
|(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|| &\leq |(G \circ f_k)(x) + (H \circ f_k)(x)||x| \\
&\leq (|(G \circ f_k)(x)| + |(H \circ f_k)(x)|)|x| \\
&\leq (\gamma|f_k(x)|^p + \delta|f_k(x)|^p)|x| \\
&\leq (\gamma + \delta)|f_k(x)|^p|x| \\
&\leq (\gamma + \delta)|f(x)|^p|x|
\end{aligned}$$

almost everywhere. But $(\gamma + \delta)|f|^p| \cdot | \in L^1(\mathbb{R}^n)$, so Lebesgue's Dominated Convergence Theorem yields

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx = \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx,$$

and that the limit above is finite. Next, since $|f_k(x) - f(x)|^p \leq |f(x)|^p$ a.e. and $|f|^p \in L^1(\mu_n)$, Lebesgue's Dominated Convergence Theorem gives

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k(x) - f(x)|^p|x|\, dx = 0.$$

Hence, $\|f_k - f\|_{L^p(\mu_n)} \to 0$, so $\{f_k\}$ converges to f in $L^p(\mu_n)$. Now the continuity of Φ and the fact that $f_k \in L^p_a(\mu_n)$ give

$$\begin{aligned}
h(\Phi(f), u) &= \lim_{k\to\infty} h(\Phi(f_k), u) \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} [(G \circ f_k)(x)x \cdot u + (H \circ f_k)(x)|x \cdot u|]\, dx \\
&= \int_{\mathbb{R}^n} [(G \circ f)(x)x \cdot u + (H \circ f)(x)|x \cdot u|]\, dx.
\end{aligned}$$

Hence,

$$h(\Phi(f),u)=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx$$

for all $u\in\mathbb{R}^n$. If $f\in L^p(\mu_n)$ is nonpositive, then again for each $k\in\mathbb{N}$ define $f_k:\mathbb{R}^n\to[-\infty,\infty]$ by

$$f_k=\sum_{j=0}^{k}\chi_{E_j}f,$$

where $E_j=A[1/2^{j+1},1/2^j)\cup A[j+1,j+2)$ for $j\geq 0$. Then $0\geq f_1(x)\geq f_2(x)\geq ...\geq f(x)$, a.e. and $f_k(x)\to f(x)$, a.e. Similar to the nonnegative case, we get

$$h(\Phi(f),u)=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx$$

for all $u\in\mathbb{R}^n$. Finally, for the general case, apply Lemma 3.7 with $W=L^p(\mu_n)$ to get

$$h(\Phi(f),u)=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx$$

for all $u\in\mathbb{R}^n$, for all $f\in L^p(\mu_n)$. □

Theorem 3.16. *If $\Phi:L^p(\mu_n)\to\mathcal{K}^n$, where $n\geq 2$, is a continuous* GL(n)*-equivariant Minkowski valuation, then there exist continuous functions $G:\mathbb{R}\to\mathbb{R}$ and $H:\mathbb{R}\to\mathbb{R}$ with the property that there exist real numbers $\gamma,\delta\geq 0$ such that $|G(\alpha)|\leq\gamma|\alpha|^p$ and $0\leq H(\alpha)\leq\delta|\alpha|^p$ for all $\alpha\in\mathbb{R}$, and*

$$h(\Phi(f),u)=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx$$

for $u\in\mathbb{R}^n$, for all $f\in L^p(\mu_n)$.

Proof. Apply Lemmas 3.3, 3.5, 3.6, 3.8, 3.10, 3.12, 3.13, 3.14, and 3.15. □

Note that Theorem 3.1 combined with Theorem 3.16 gives Theorem 1.1.

4. Proof of Theorem 1.2

Suppose that $\Phi:L^p(\mu_n)\to\mathcal{K}^n$ is a continuous GL(n)-equivariant homogeneous Minkowski valuation. Then there exists $\beta\in\mathbb{R}$ such that

$$\Phi(\alpha f)=|\alpha|^\beta\Phi(f)$$

for all $\alpha\neq 0$ and all $f\in L^p(\mu_n)$. In other words,

$$h(\Phi(\alpha f),u)=|\alpha|^\beta h(\Phi(f),u)$$

for all $\alpha\neq 0$, all $u\in\mathbb{R}^n$ and all $f\in L^p(\mu_n)$. By Theorem 3.16, there exist continuous functions $G:\mathbb{R}\to\mathbb{R}$ and $H:\mathbb{R}\to\mathbb{R}$ with the property that there exist real numbers $\gamma,\delta\geq 0$ such that $|G(\alpha)|\leq\gamma|\alpha|^p$ and $0\leq H(\alpha)\leq\delta|\alpha|^p$ for all $\alpha\in\mathbb{R}$, and

$$h(\Phi(f),u)=\int_{\mathbb{R}^n}[(G\circ f)(x)x\cdot u+(H\circ f)(x)|x\cdot u|]\,dx$$

for $u\in\mathbb{R}^n$, for all $f\in L^p(\mu_n)$. So for every $\alpha\in\mathbb{R}$ and every $P\in\mathcal{P}_0^n$,

$$h(\Phi(\alpha\chi_P),u)=G(\alpha)\int_P x\cdot u\,dx+H(\alpha)\int_P|x\cdot u|\,dx$$

for $u \in \mathbb{R}^n$. For $P, Q \in \mathcal{P}_0^n$ and $u \in S^{n-1}$ define the matrix

$$M(P,Q,u) = \begin{pmatrix} \int_P x \cdot u \, dx & \int_P |x \cdot u| \, dx \\ \int_Q x \cdot u \, dx & \int_Q |x \cdot u| \, dx \end{pmatrix}.$$

Clearly, there are $P_0, Q_0 \in \mathcal{P}_0^n$ and $u_0 \in S^{n-1}$ such that $\det M(P_0, Q_0, u_0) \neq 0$. Hence, from the two equations

$$h(\Phi(\alpha\chi_{P_0}), u_0) = G(\alpha) \int_{P_0} x \cdot u_0 \, dx + H(\alpha) \int_{P_0} |x \cdot u_0| \, dx,$$
$$h(\Phi(\alpha\chi_{Q_0}), u_0) = G(\alpha) \int_{Q_0} x \cdot u_0 \, dx + H(\alpha) \int_{Q_0} |x \cdot u_0| \, dx,$$

we get

$$G(\alpha) = \Big(\frac{\int_{Q_0} |x \cdot u_0| \, dx}{\det M(P_0, Q_0, u_0)}\Big) h(\Phi(\alpha\chi_{P_0}), u_0) - \Big(\frac{\int_{P_0} |x \cdot u_0| \, dx}{\det M(P_0, Q_0, u_0)}\Big) h(\Phi(\alpha\chi_{Q_0}), u_0),$$

$$H(\alpha) = -\Big(\frac{\int_{Q_0} x \cdot u_0 \, dx}{\det M(P_0, Q_0, u_0)}\Big) h(\Phi(\alpha\chi_{P_0}), u_0) + \Big(\frac{\int_{P_0} x \cdot u_0 \, dx}{\det M(P_0, Q_0, u_0)}\Big) h(\Phi(\alpha\chi_{Q_0}), u_0).$$

But we have

$$h(\Phi(\alpha\chi_P), u) = |\alpha|^\beta h(\Phi(\chi_P), u)$$

for all $\alpha \neq 0$, all $u \in \mathbb{R}^n$ and all $P \in \mathcal{P}_0^n$. Thus, we have some constants $a, b \in \mathbb{R}$ such that for all $\alpha \neq 0$,

$$\begin{cases} G(\alpha) = a|\alpha|^\beta, \\ H(\alpha) = b|\alpha|^\beta. \end{cases}$$

But for some $\gamma \geq 0$, $|a||\alpha|^\beta \leq \gamma|\alpha|^p$ for all $\alpha \neq 0$, which occurs if and only if $\beta = p$ or $a = 0$. If $a = 0$, then $G = 0$. If $\beta = p$, then

$$G(\alpha) = a|\alpha|^p$$

for all $\alpha \in \mathbb{R}$. Also for some $\delta \geq 0$, $|b||\alpha|^\beta \leq \delta|\alpha|^p$ for all $\alpha \neq 0$, which occurs if and only if $\beta = p$ or $b = 0$. If $b = 0$, then $H = 0$. If $\beta = p$, then

$$H(\alpha) = b|\alpha|^p$$

for all $\alpha \in \mathbb{R}$, which also makes $b \geq 0$ since $q \geq 0$. In all scenarios, we have constants $a, b \in \mathbb{R}$ with $b \geq 0$ such that

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} (a|f(x)|^p x \cdot u + b|f(x)|^p |x \cdot u|) \, dx$$

for $u \in \mathbb{R}^n$, for all $f \in L^p(\mu_n)$.

On the other hand, let $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ be a function with the property that there exist constants $a, b \in \mathbb{R}$ with $b \geq 0$ such that

$$h(\Phi(f), u) = \int_{\mathbb{R}^n} (a|f(x)|^p x \cdot u + b|f(x)|^p |x \cdot u|) \, dx$$

for $u \in \mathbb{R}^n$, for all $f \in L^p(\mu_n)$. Then by Theorem 3.1, $\Phi : L^p(\mu_n) \to \mathcal{K}^n$ is a continuous $\mathrm{GL}(n)$-equivariant Minkowski valuation. Also, trivially Φ is homogeneous.

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