

GENERIC STABILITY, FORKING, AND \mathfrak{p} -FORKING

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ABSTRACT. Abstract notions of “smallness” are among the most important tools that model theory offers for the analysis of arbitrary structures. The two most useful notions of this kind are forking (which is closely related to certain measure zero ideals) and thorn-forking (which generalizes the usual topological dimension). Under certain mild assumptions, forking is the finest notion of smallness, whereas thorn-forking is the coarsest.

In this paper we study forking and thorn-forking, restricting ourselves to the class of generically stable types. Our main conclusion is that in this context these two notions coincide. We explore some applications of this equivalence.

1. INTRODUCTION

1.1. **General overview.** It is well known that one of the most powerful tools that model theory offers for the analysis of structures is abstract notions of “bigness” and “dimension”.

Until recently, it was common to tie “bigness” and “dimension” together; that is, assume that in a given context the “correct” notion of bigness is the one that gives rise to a nice notion of dimension and independence. This approach recently changed when measure theory came into play in model theory, especially in the analysis of definable sets in \mathfrak{o} -minimal, and, more generally, “dependent” theories (theories without the independence property). It has become clear that in many situations one should look at more than one ideal on the algebra of definable sets, and the interplay between the different notions of bigness (or smallness) sheds much light on the structure.

The two most common notions of smallness in model theory are forking (which captures “algebraic” independence in stable and simple theories, and has recently been understood to correspond to certain measure 0 ideals), and thorn-forking, which in a sense captures the notion of “topological” or “analytic” dimension in many important cases. In stable theories these notions coincide. This is a nontrivial fact, which has interesting consequences. For example, in the theory of algebraically closed fields, this corresponds in a sense to the equality of algebraic and analytic dimensions in complex algebraic geometry. It is therefore a natural and interesting question to ask under which conditions forking and thorn-forking agree, and the consequences that can be concluded when the two notions coincide.

As a quick example of another possible implication of such an equivalence, Adler proved in [1] that any independence relation must lie between forking and \mathfrak{p} -forking.

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That is, forking is the finest notion of smallness, satisfying axioms of an independence relation [1], whereas thorn-forking is the coarsest. This implies that whenever forking and \mathfrak{p} -forking coincide, there is a *unique* independence relation.

It is known that in the presence of a definable order, forking is very different from thorn-forking. Therefore, if one wants to study situations when these notions coincide in an arbitrary theory, one needs to work (locally) with objects that are “far” from definable orders, that is, exemplify a certain kind of stable behavior.

In this paper we will concentrate on the class of generically stable types. This is in a sense the most general class of types which have certain basic properties related to stability. They have been studied a lot recently (e.g. [17], [20], [6], [16]), since generically stable types have been realized to play a crucial role in the analysis of many concrete theories such as algebraically closed, p -adic, and many other Henselian valued fields (e.g. [3], [5]).

We work with generically stable types in an arbitrary theory. In particular, our treatment includes the following classes in which we are most interested: generically stable types in dependent theories (theories without the independence property) and stable types in an arbitrary theory.

1.2. Forking and \mathfrak{p} -forking. We continue by describing and comparing the main concepts we are investigating, that is, forking and \mathfrak{p} -forking.

Let us fix a complete theory T and its monster (sufficiently saturated and homogeneous) model \mathfrak{C} . Throughout the paper, a, b, c, d will denote tuples (normally finite) of elements of \mathfrak{C} , A, B, C, D , will denote sets of parameters (subsets of \mathfrak{C}), M, N , will denote models of T (elementary submodels of \mathfrak{C}), and X, Y, Z will denote definable sets (with parameters) in \mathfrak{C} . All sets (of parameters) and models will have cardinality significantly smaller than the saturation of \mathfrak{C} . On the other hand, by compactness, all the definable sets are either finite or of the cardinality (or the saturation) of \mathfrak{C} .

Recall that if A is a set, we denote by $\text{Aut}(\mathfrak{C}/A)$ the group of automorphisms of \mathfrak{C} fixing A pointwise. This group acts on the algebra of definable sets in \mathfrak{C} . We will often say “ A -conjugate”, meaning “ $\text{Aut}(\mathfrak{C}/A)$ -conjugate”, and “ A -invariant”, meaning “ $\text{Aut}(\mathfrak{C}/A)$ -invariant”.

Forking (Definition 1.1) was defined by Shelah (e.g. [21]) to enable one to understand the more intricate parts of classification theory. A definable set X is said to *divide* over a set A if there are infinitely many A -conjugates $\langle X_i : i < \omega \rangle$ of X , which are almost disjoint (more precisely, there is some $k < \omega$ such that X_i are k -disjoint). Essentially, X *forks* over A if it belongs to the ideal on the algebra of definable sets “generated” by definable sets which divide over A . This ideal is sometimes referred to as the *forking ideal* over A .

Looking at the definition of forking, it is quite natural to relate this concept to a certain kind of measure, but it took quite a while until this connection was realized and fully explored. This line of thought was pioneered by Keisler in [9]. He introduced the notion of an A -invariant additive probability measure on the algebra of definable sets (we now refer to these as “Keisler measures”). It is not hard to see (and a very good exercise to understand the essence of nonforking) that the forking ideal is contained in the measure 0 ideal of any Keisler measure. Keisler also noted that whenever one worked in a dependent theory (Definition 1.5), there were enough invariant additive measures on definable sets to understand “nonforking” as something close to “commensurable”.

Recently, Keisler’s ideas have been successfully used in the study of o-minimal groups by Peterzil, Pillay and Hrushovski (e.g. [7], [15]) and in subsequent works on compact domination and invariant measures by Hrushovski, Pillay, and Simon (e.g. [6], [8]).

This is not, however, the only “mathematically concrete” meaning of the abstract notion of forking. Since it was defined, forking (or better nonforking) was understood as a notion which in some circumstances could generalize well-known independence relations such as linear independence in vector spaces and algebraic independence in the complex field. It was under this approach that during the nineties a lot of energy was put into understanding the theories in which nonforking had this independence-like behavior (simple theories). This is also why, when \mathfrak{p} -forking was introduced, it seemed to be a generalization of forking to contexts in which nonforking failed to provide this independence relation.

The notion of \mathfrak{p} -forking, Definition 1.2, was defined by Thomas Scanlon and the second author in the second author’s Ph.D. thesis. Its basic properties are investigated in [11]. The definition is similar to that of forking, but instead of dividing it is based on the notions of \mathfrak{p} -dividing and strong dividing, which require that the infinite family X_i of k -disjoint conjugates be uniformly definable, that is, of the form $X(a)$, with the parameter a varying in some definable set Y over A . This yields a much stronger notion of forking, hence a much coarser notion of smallness. If one takes $k = 2$ in the definition of \mathfrak{p} -dividing, one gets that $X(a)$ are fibers over a definable set Y , which gives us a picture of a notion of dimension (or independence) very close to the classic geometric notion one finds in complex or real analytic geometry.

A good example to consider in order to understand the difference between forking and \mathfrak{p} -forking is any o-minimal theory (or a reduct thereof). On the one hand, it is shown in [11] that in this case \mathfrak{p} -forking gives rise to the usual notion of independence in o-minimal theories. This is consistent with our intuition of \mathfrak{p} -forking corresponding to topological dimension. In particular, bounded intervals will rarely \mathfrak{p} -fork, but it is easy to see that they normally fork. Hence there are many more instances of forking than \mathfrak{p} -forking.

As we have pointed out before, nonforking has a different important meaning, and it is closer to commensurability. In this sense, in a compact topological group defined in a real closed field, a nonforking subset can always cover the group with finitely many translates, while a non \mathfrak{p} -forking subset would not necessarily do so (one can define an infinitesimal interval –or box– in any nonstandard expansion of the real field).

The above discussion shows that in the presence of definable order the \mathfrak{p} -forking ideal is generally much smaller than the forking ideal. However, if no order can be found, the situation is very different. Specifically, the second author showed in [11] that in a stable theory forking and \mathfrak{p} -forking coincide. This fact is not a natural consequence of the definitions. Recall that in stable theories forking often “captures” algebraic dimension, whereas \mathfrak{p} -forking corresponds to the topological one. In this sense, the equivalence between the two seems to be a generalization of the fact that, for example, in complex algebraic geometry the Krull dimension of an algebraic variety V (the maximum proper chain of irreducible subvarieties of V) is the same as the complex analytic dimension of V . Under this analogy, it is clear that whenever the two notions coincide, one should be able to apply the intuition

behind the geometric (analytic) definition into the formalism and powerful tools provided by the more combinatorial and stronger algebraic definition, which would hopefully give us a powerful combination.

It therefore makes sense to study the relation (and possible equivalence) of forking and \mathfrak{b} -forking in contexts involving types whose realizations are not easily in the domain of a partial order. One of the main goals of this paper is to prove that forking and \mathfrak{b} -forking coincide on a large class of types, which exemplify stable behavior “generically”. That is, definable order may be found on the set of their realizations, but it cannot be detected by looking only at generic extensions. Such types are called generically stable and were studied by the third author in [20]. The main purpose of this paper is precisely to study the relation between forking and \mathfrak{b} -forking in generically stable types.

The following are some formal definitions that we will need.

Definition 1.1. • A formula $\phi(x, a)$ is said to *divide* over A if there is an infinite A -indiscernible sequence $I = \langle a_i \rangle_{i \in \omega}$ with $a_0 = a$ and such that

$$\bigwedge_{i \in \omega} \phi(x, a_i)$$

is inconsistent.

In this case we say that the sequence I *witnesses* the dividing of $\phi(x, a)$.

- A formula *forks* over A if it implies a finite disjunction of formulas which fork over A .
- A type forks (divides) over A if it implies a formula which forks (divides) over A .

Definition 1.2. • A formula $\phi(x, a)$ is said to *strongly divide* over B if $a \notin \text{acl}(B)$ and there is a B -definable set X containing a such that for some k and for any a_1, \dots, a_k in X the formula

$$\bigwedge_{i=1}^k \phi(x, a_i)$$

is k -inconsistent.

- A formula $\phi(x, a)$ is said to *\mathfrak{b} -divide* over A if there is some $B \supset A$ such that $\phi(x, a)$ strongly divides over B .
- A formula *\mathfrak{b} -forks* over A if it implies a finite disjunction of formulas which fork over A .
- A type forks (\mathfrak{b} -forks) over A if it implies a formula which forks (\mathfrak{b} -forks) over A .

Notation. We write $a \downarrow_A B$ for $\text{tp}(a/B)$ does not fork over A , and $a \downarrow_A^{\mathfrak{b}} B$ for $\text{tp}(a/B)$ does not \mathfrak{b} -fork over A .

Definition 1.3. We define the U-rank and $U^{\mathfrak{b}}$ -rank of types to be the foundation rank of forking and \mathfrak{b} -forking, respectively. So, for example, $U(p(x)) \geq 0$ if and only if $p(x)$ is consistent and $U(p(x)) \geq \alpha + 1$ if and only if there is a forking extension $q(x)$ of $p(x)$ such that $U(q(x)) \geq \alpha$.

Definition 1.4. (i) We say that a type $p \in S(A)$ is *extensible* if it does not fork over A (equivalently, if it has a global nonforking extension). We will also say that b is extensible over A if $\text{tp}(b/A)$ is extensible.

- (ii) We say that a type $p \in \text{acl}(A)$ is *stationary* if it has a unique nonforking extension to any superset of A (equivalently, if it has a unique global nonforking extension). If $p(x)$ is stationary, then $p(x)|_B$ is defined to be the (unique) restriction to B of the global nonforking extension of $p(x)$.
- (iii) We call a sequence $I = \langle a_i : i < \lambda \rangle$ a (\mathfrak{b} -)Morley sequence over a set B based on a set A if
 - I is indiscernible over B .
 - For every $i < \lambda$, the type $\text{tp}(a_i/Ba_{<i})$ does not (\mathfrak{b} -)fork over A .
 If $A = B$, we omit it.

Note: By $a_{<i}$ we mean the sequence $\langle a_j : j < i \rangle$, which we often identify with the set $\{a_j : j < i\}$.

1.3. Local dependence. Although we are not assuming anything on the ambient theory in this paper, we will work with the following *local* versions of dependence.

Definition 1.5. (i) Let $I = \langle a_i : i \in O \rangle$ be an indiscernible sequence. Recall ([17]) that I is called *stable for a formula* $\varphi(x, y)$ if for every b either

$$\{i : \models \varphi(b, a_i)\}$$

or

$$\{i : \models \neg\varphi(b, a_i)\}$$

is finite. We will call I *stable* if it is stable for every formula $\varphi(x, y)$.

- (ii) Let $I = \langle a_i : i \in O \rangle$ be an indiscernible sequence with no last element. We say that I is *dependent for a formula* $\varphi(x, y)$ if for every b either

$$\{i : \models \varphi(b, a_i)\}$$

or

$$\{i : \models \neg\varphi(b, a_i)\}$$

is bounded. We will call I *dependent* if it is dependent for every formula $\varphi(x, y)$.

- (iii) Let $p(x) \in S(A)$. Then p is called *dependent* if every indiscernible sequence in p is dependent.
- (iv) A theory T is called *dependent* if every type p is dependent.

The following is clear from the definitions and compactness.

Observation 1.1. If $\langle b_i : i \in I \rangle$ is an infinite dependent indiscernible set, then it is stable. Moreover, for every $\varphi(x, y)$ there exists $k = k_\varphi < \omega$ such that for every $c \in \mathfrak{C}$ either $|\{i \in I : \varphi(b_i, c)\}| < k$ or $|\{i \in I : \neg\varphi(b_i, c)\}| < k$.

1.4. Generically stable types. Lascar and Poizat [10] identified a class of types such that, restricting oneself to realizations of those types, one gets stable behavior. Such types are called *stable*. Much more recently, the third author [20], based on Shelah [17], has introduced a much larger class of types, which behave in a very “stable-like” manner, particularly with respect to nonforking. Such types are called *generically stable*. The theory developed in [20] assumed that the ambient theory T was dependent. However (as was first noticed in [16]), this is not necessary once a strong enough definition is given.

First we remind the reader of the definitions of stable types and definability of types.

Definition 1.6.

- A definable or ∞ -definable set X will be called *stable* if for any formula $\phi(x, y)$ there are no infinite sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ with $a_i \in X$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$.
- A type $p(x) \in S(A)$ is *stable* if $X := \{c \in \mathfrak{C} \mid \mathfrak{C} \models p(c)\}$ is stable.

Definition 1.7. Let $p(x)$ be a type over the monster model $S(\mathfrak{C})$. We will say that $p(x)$ is *definable* if for any $\phi(x, y)$ there is a formula $d_p\phi(y)$ such that $\phi(x, a) \in p(x)$ if and only if $\models d_p\phi(a)$.

We call d_p the *definition schema* of p .

The type $p(x)$ is definable over A if for all ϕ the parameters used to define $d_p\phi$ are contained in A .

Finally, given a type $p(x) \in S(A)$ we will say that $p(x)$ is definable over $\text{acl}(A)$ if there is a global type extending $p(x)$ which is definable over $\text{acl}(A)$.

For the purpose of this paper, the following will be taken as the definition of a generically stable type. However, see Appendix A for the connections with related notions. Implications and connections between the different properties listed in the definition will be explored elsewhere.

Definition 1.8. A type $p \in S(A)$ is called *generically stable* if

- *Definability:* p is definable over $\text{acl}(A)$ by a definition schema d_p .
- *Stationarity:* p is stationary over $\text{acl}(A)$, that is, any extension of p to $\text{acl}(A)$ is stationary.
- *Symmetry:* For any $a \models p$ and b which is extensible over A , we have

$$a \underset{A}{\downarrow} b \iff b \underset{A}{\downarrow} a.$$

- *Generic dependence:* Every (equivalently, some) Morley sequence in p is dependent.

The class of generically stable types as defined here includes the following classes.

- Stable types and, more generally, stably dominated types [3] in an arbitrary theory.
- Generically stable types in dependent theories [20].
- Generically stable types in an arbitrary theory, as defined by Pillay and Tanović in [16]. As a matter of fact, Definition 1.8 is essentially equivalent to the one of Pillay and Tanović (see the appendix).

Note that generic stability is not necessarily closed under extensions. In fact, it is easy to see that p is stable if and only if every extension of it is generically stable.

1.5. Outline. The paper is divided as follows. Section 2 contains a characterization of forking for generically stable types and some consequences concerning generically stable weight. In Section 3 we prove the equivalence of forking and \mathfrak{b} -forking for generically stable types. Finally, Section 4 contains some applications.

Results in Section 2 also appear in some preprints of the second and the third authors, which were distributed, but are not intended for publication. Results in Section 3 appear in the first author's Master's thesis which was written under the supervision of the second author.

2. GENERIC STABILITY AND FORKING

The main result of this section is Theorem 2.4, which gives a nontrivial characterization of forking for formulas, given generic stability of the type of the parameters. This result is quite interesting by itself, and will be crucial in Section 3. In the last subsection we apply the characterization above to certain questions concerning weight.

2.1. Basic properties. We begin by recalling some basic facts about forking and \mathfrak{p} -forking. Some of these are well known (see for example [11]), and others follow immediately from the definitions.

Fact 2.1. The following hold.

$a \downarrow_A^{\mathfrak{p}} b$ implies $a \downarrow_A b$ for any a, b, A .

Extension: If $A \subset B$ and $\text{tp}(a/B)$ does not fork (\mathfrak{p} -fork) over A , then for any $C \supset B$ there is some $a' \models \text{tp}(a/B)$ such that $\text{tp}(a'/C)$ does not fork (\mathfrak{p} -fork) over A .

Monotonicity: If $\text{tp}(a/Abc)$ does not fork (\mathfrak{p} -fork) over A , then $\text{tp}(a/Ab)$ does not fork (\mathfrak{p} -fork) over A and $\text{tp}(a/Abc)$ does not fork (\mathfrak{p} -fork) over Ab .

Left Monotonicity: If $\text{tp}(ac/Ab)$ does not fork (\mathfrak{p} -fork) over A , then $\text{tp}(a/Ab)$ does not fork (\mathfrak{p} -fork) over A .

Reflexivity: $\text{tp}(a/Ab)$ does not fork (\mathfrak{p} -fork) over A if and only if $\text{tp}(a/\text{acl}(Ab))$ does not fork (\mathfrak{p} -fork) over A .

Left transitivity: If $\text{tp}(a/Abc)$ does not fork (\mathfrak{p} -fork) over Ab and $\text{tp}(b/Ac)$ does not fork over A , then $\text{tp}(ac/Ab)$ does not fork (\mathfrak{p} -fork) over A .

Let us list some basic properties of generically stable types. The proof of the following fact is common knowledge by now, and we will not say much about it.

Theorem 2.2 (Morley sequences in generically stable types).

- (i) If $p \in S(A)$ is generically stable, then every Morley sequence in p is a stable indiscernible set.
- (ii) Let $p \in S(A)$ be generically stable, let $\theta(x, a)$ be the ϕ -definition of $p(x)$ and let $\langle b_i \rangle_{i \in \omega}$ be a Morley sequence of p . Then $\theta(\mathfrak{C}, a)$ is definable over b_1, \dots, b_n for some finite n .
- (iii) Let $p \in S(A)$ be generically stable and stationary. Then p is definable over A .

Proof. The first item is a well-known consequence of symmetry and stationarity, combined with Observation 1.1. The second item follows exactly like Lemmas 4.2 and 4.3 in [20] or Proposition 1 in [16]. The last item follows from the fact that the definition schema in this case is over A . \square

The following is an easy consequence of stationarity and definability:

Proposition 2.3 (Nonforking on generically stable types). *Let $\text{tp}(a/A)$ be a generically stable type. Then the following hold.*

- **Transitivity:** Let $A \subset B \subset C$. Then $a \downarrow_A C$ if and only if $a \downarrow_A B$ and $a \downarrow_B C$.
- **Bounded forking:** For any $C \supset A$, there are boundedly many $q \in S(C)$ extending p such that q does not fork over A .
- **Definability:** If $p \in S(B)$ and $A \subset B$, then p is definable over $\text{acl}(A)$ if and only if p does not fork over A .

2.2. Characterization of forking for generically stable parameters.

Theorem 2.4. *Suppose that $p(x)$ is an extensible type over A such that $p(x) \cup \{\phi(x, b)\}$ forks over A . Assume furthermore that $\text{tp}(b/A)$ is generically stable. Then given any ω -Morley $\langle b_i \rangle$ sequence in $\text{tp}(b/A)$ the type*

$$p(x) \cup \{\phi(x, b_i)\}_{i \in \omega}$$

is inconsistent.

Proof. Suppose otherwise, so that $\phi(x, b)$ forks over A , $p(x) := \text{tp}(b/A)$ is generically stable and there is a Morley sequence $\bar{b} := \langle b_i \rangle_{i \in \omega}$ such that

$$p(x) \cup \{\phi(x, b_i)\}_{i \in \omega}$$

is consistent, realized by some element d .

We will construct an infinite sequence $\bar{c} := \langle c_i \rangle_{i \in \omega}$ inductively as follows:

Given any n , suppose we have constructed c_0, \dots, c_n and let c_{n+1} realize a non-forking extension of $\text{tp}(b/A)$ to $A\bar{b}c_0 \dots c_n$. By monotonicity we have $c_{n+1} \downarrow_A \bar{b}c_0 \dots c_n$, and by symmetry (note that $d \models p$, which is extensible over A) and monotonicity $d \downarrow_A c_{n+1}$.

It follows that $I := \bar{b} \hat{\ } \bar{c}$ is a Morley sequence in $\text{tp}(b/A)$, hence a stable indiscernible set. Since $d \downarrow_A c_i$, but $\phi(x, c_i)$ forks over A , we clearly have $\neg\phi(x, c_i) \in \text{tp}(d/Ac_i)$ for all i .

But then both sets $\{e \in I \mid \phi(d, e)\}$ and $\{e \in I \mid \neg\phi(d, e)\}$ have infinitely many elements, contradicting stability of I . \square

This essentially yields equivalence between dividing and forking when generically stable parameters are involved. The only additional requirement on $\phi(x, b)$ in the corollary below is that it has a realization whose type over A is extensible, which is quite a mild assumption.

Corollary 2.5. *Suppose that $p(x)$ is an extensible type over A such that $p(x) \cup \{\phi(x, b)\}$ forks over A , and $\text{tp}(b/A)$ is generically stable. Then $\phi(x, b)$ divides over A .*

2.3. Generic stability, weight, and dp-minimality. The results in this subsection are not relevant for the rest of the paper, and they address questions which are much closer to [12] (and the last section of [20]). However, they are related to the general theme of the paper (generic stability and forking), and since they show another application of the results above (particularly Theorem 2.4), we decided to include them.

The main question we are addressing here is understanding the generically stable weight of an arbitrary type in an arbitrary structure. In a sense, this notion aims to capture the “generically stable part” of an arbitrary type. This study continues [20], where similar questions were addressed under the assumption that T is dependent.

Definition 2.1. (i) Given a type $p \in S(A)$, we define the *generically stable pre-weight* of p , $\text{gstpw}(p)$, to be the supremum of all ordinals α such that there exists a set $\{b_i : i < \alpha\}$, which is an A -independent set of realizations of generically stable types over A , and $a \not\downarrow_A b_i$ holds for all $i < \alpha$.

Given $\{b_i : i < \alpha\}$ as above, we say that they *witness* that p has a generically stable pre-weight at least α .

- (ii) We define the *generically stable weight* of p , $\text{gstw}(p)$, to be the supremum of $\text{gstpw}(q)$ for all nonforking extensions q of p .
- (iii) We say that p has *rudimentarily finite generically stable pre-weight* if there is no witness of $\text{gstpw}(p) \geq \omega$.

We say that p has *rudimentarily finite generically stable weight* if every nonforking extension of it has rudimentarily finite generically stable pre-weight.

Note that a priori it could be the case that p has infinite but rudimentarily finite generically stable pre-weight.

The main purpose of this subsection is to prove that in a strong theory every type has rudimentarily finite generically stable weight. This generalizes the main result of Section 8 of [20] in two ways: first, the theory was assumed to be dependent (and the proof actually made use of that). Second, the definition of generically stable weight there was not quite satisfactory because it used dividing instead of forking. In addition, the proof that we offer here (using Theorem 2.4) is significantly simpler than the one in [20].

Recall that in [19] Shelah defined the notions of strong dependence and several ranks associated with it. Based on Shelah's ideas, randomness patterns, dp-rank and dp-minimality were defined in [12], and strong theories were defined in [1].

Definition 2.2. Let $p(x) \in S(A)$, $k \leq \omega$.

- (i) We say that the sequence $\langle a, (I_\alpha)_{\alpha < k}, (\phi_\alpha)_{\alpha < k} \rangle$ is a *randomness pattern of depth k for $p(x)$* if $a \models p(x)$, $I_\alpha = \langle b_i^\alpha : i < \omega \rangle$ are mutually A -indiscernible sequences (by which we mean that I_α is indiscernible over $AI_{\neq \alpha}$ for all α), and

$$a \models \bigwedge_{\alpha < k} \left[\phi_\alpha(x, b_0^\alpha) \wedge \bigwedge_{i \neq 0} \phi_\alpha(x, b_i^\alpha) \right].$$

A type $p(x)$ is *dp-minimal* if there are no randomness patterns of depth 2 for $p(x)$.

A type $p(x)$ is *strongly dependent* if there are no randomness patterns of depth ω for $p(x)$.

- (ii) We say that the sequence $\langle a, (I_\alpha)_{\alpha < k}, (\phi_\alpha)_{\alpha < k} \rangle$ is a *dividing pattern of depth k for $p(x)$* if $a \models p(x)$, $I_\alpha = \langle b_i^\alpha : i < \omega \rangle$ are mutually A -indiscernible sequences, $\varphi_\alpha(a, b_0^\alpha)$ holds for all α , and I_α witnesses that $\phi_\alpha(x, b_0^\alpha)$ divides over A .

A type $p(x)$ is *strong* if there are no dividing patterns of depth ω for $p(x)$.

- (iii) A theory is called dp-minimal/strongly dependent/strong if every type is dp-minimal/strongly dependent/strong, respectively.

Randomness patterns provide another way of characterizing dependent types.

Fact 2.6. Let $p \in S(A)$. Then p is dependent if and only if there is no randomness pattern for p of depth ω with $\phi_\alpha = \phi$ for all α .

The following fact relates randomness and dividing patterns. It is Lemma 2.10 in [12].

Lemma 2.7. *Let $p(x)$ be a type over a set A , let $I = \langle b_i \rangle_{i \in \mathbb{O}}$ be a sequence indiscernible over A , and let $\varphi(x, y)$ be a formula such that $p(x) \cup \varphi(x, b_i)$ is consistent for some (all) i and $\{\varphi(x, b_i)\}_{i \in \mathbb{O}}$ is k -inconsistent for some $k \in \mathbb{N}$. Then $p(x) \cup \{\varphi(x, b_l)\} \cup \{\neg\varphi(x, b_i)\}_{i \neq l}$ is consistent for all l .*

By the lemma, we have the following:

Corollary 2.8. *If a type p is strongly dependent, then it is strong.*

As a matter of fact (see e.g. [12]), if p is dependent, then it is strong if and only if it is strongly dependent.

Theorem 2.9. *Let A be a set, a be a tuple, and let $\{b_\alpha\}_{\alpha < \lambda}$ be an independent set of tuples satisfying generically stable types over A . Assume furthermore that $a \not\downarrow_A b_\alpha$ for all α .*

Then there is a dividing pattern $\langle a, (I_\alpha)_{\alpha < \lambda}, (\varphi_\alpha)_{\alpha < \lambda} \rangle$ for $\text{tp}(a/A)$ of depth λ such that I_α starts with b_α . Hence there is such a randomness pattern in p as well.

Proof. We construct $I_\alpha = \langle b_i^\alpha : i < \omega \rangle$ by induction on α such that I_α is a Morley sequence starting over $AI_{<\alpha}$ starting with b_α . It is not hard to check that (just as in stable theories) generic stability ensures that the standard construction works and, moreover, the obtained sequences satisfy the following: I_α is a Morley sequence over $AI_{\neq\alpha}$ based on A . In particular, I_α is an indiscernible sequence over $AI_{\neq\alpha}$ (see e.g. Lemma 1.3 in [12] for a more elaborate argument in the stable case; the same proof works here).

Since I_α is a Morley sequence starting with b_α and $a \not\downarrow_A b_\alpha$, by Theorem 2.4 the sequence I_α witnesses that $\text{tp}(a/Ab_\alpha)$ divides over A . So we have obtained a dividing pattern as required. By Lemma 2.7, this also yields a randomness pattern. \square

The main result of this subsection now follows immediately:

Corollary 2.10. *Let p be a strong generically stable type. Then p has a rudimentarily finite generically stable pre-weight. In particular, in a strong theory, every type has a rudimentarily finite generically stable weight.*

One can ask the following

Question 2.1. Let $p \in S(A)$ be a strong type. Does it necessarily have rudimentarily finite generically stable weight?

Note that if not, there would be $B \supseteq A$, $a \models p$, $a \not\downarrow_A B$, and $\{b_i : i < \omega\}$ generically stable and independent over B such that $a \not\downarrow_B b_i$ for all i . The main problem is that it is not clear that b_i are generically stable over A . One question is therefore whether one can assume without loss of generality that $b_i \not\downarrow_A B$.

The following is another result related to Theorem 2.9. Note that it does not follow from Theorem 2.9 directly because we only assume that one of the elements is generically stable. Here we need to work in a dependent theory.

Theorem 2.11. *Let a, b, c be elements and A be a subset of a model M of a dependent theory T . If $\text{tp}(b/A)$ is generically stable, $b \not\downarrow_A c$, $\text{tp}(c/A)$ is extensible (does not fork over A), $\text{tp}(a/Ab)$ forks over A , and $\text{tp}(a/Ac)$ divides over A , then $\text{tp}(a/A)$ is not dp -minimal.*

Proof. Let a, b, c, A be as in the statement of the theorem.

Let I be a Morley sequence in $\text{tp}(b/A)$ over Ac based on A . By Theorem 2.4, I witnesses that $\text{tp}(a/Ab)$ divides over A . Note also that $\text{tp}(I/A)$ is generically stable.

Now $I \perp_A c, c \perp_A I$ (by symmetry and generic stability), so we can apply Corollary 4.14 in [12] in order to obtain the desired conclusion. \square

3. FORKING AND \mathfrak{p} -FORKING FOR GENERICALLY STABLE TYPES

In this section we will analyze cases in which forking is the same as \mathfrak{p} -forking when generically stable types are involved. The following concepts will be key in order to achieve \mathfrak{p} -forking.

Let $p(x, b) := \text{tp}(a/Ab)$ be a stationary generically stable type, and let $\phi(x, y)$ be any formula (over A). Since the concept of generic stability is invariant under automorphisms, we know that for any $b' \models \text{tp}(b/A)$ the type $p(x, b')$ is a stationary generically stable type. It follows that if $\theta(y, b)$ is the $\phi(x, y)$ -definition of $\text{tp}(a/Ab)$ (note that by stationarity, θ can be taken over Ab), then $\theta(y, b')$ will be the $\phi(x, y)$ -definition of $\text{tp}(a/Ab')$. It is natural, therefore, to consider the following equivalence relation:

$$b_1 R_\phi b_2 \iff \forall y (\theta(y, b_1) \iff \theta(y, b_2)).$$

In other words, $b_1 R_\phi b_2$ if and only if the ϕ -global types given by ϕ -definitions of $p(x, b_1)$ and $p(x, b_2)$ coincide. That is, if and only if $p_\phi(x, b_1)|\mathfrak{C} = p_\phi(x, b_2)|\mathfrak{C}$.

Lemma 3.1. *Let $p(x, b)$ be a stationary generically stable type. Given realizations b_1, b_2 of $\text{tp}(b/A)$ we will define $b_1 E b_2$ if and only if $p(x, b_1) \cup p(x, b_2)$ does not fork over both Ab_1 and Ab_2 . Then the following hold.*

- (i) E defines an equivalence relation on the realizations of $\text{tp}(b/A)$.
- (ii) For b_1, b_2 realizations of $\text{tp}(b/A)$ we have that $b_1 E b_2$ if and only if $b_1 R_\phi b_2$ for all ϕ over A .
- (iii) $p(x, b)$ forks over A if and only if $[b]_E$ has unboundedly many A -conjugates, equivalently, if and only if $[b]_{R_\phi}$ has infinitely many A -conjugates for some ϕ .

Proof. The first item follows easily from stationarity (even without assuming generic stability), but in this case it follows trivially from (ii) and (iii).

For (ii) note that $\pi(x) = p(x, b_1) \cup p(x, b_2)$ does not fork over both Ab_1 and Ab_2 if and only if $\pi(x)$ is included in both $p(x, b_1)|\mathfrak{C}$ and $p(x, b_2)|\mathfrak{C}$, which are the unique global nonforking extensions of $p(x, b_1), p(x, b_2)$, respectively.

So assume $b_1 R_\phi b_2$ for all ϕ ; then (by the definitions) $p(x, b_1)|\mathfrak{C} = p(x, b_2)|\mathfrak{C} := p^*$. Since $p(x, b_i) \subseteq p(x, b_i)|\mathfrak{C} = p^*$, clearly $b_1 E b_2$ holds.

On the other hand, assume $b_1 E b_2$. In particular, this means that $\pi(x)$ is a non-forking extension of both $p(x, b_i)$, stationary over $Ab_1 b_2$. But then by stationarity both $p(x, b_1)|\mathfrak{C}$ and $p(x, b_2)|\mathfrak{C}$ must be equal to $\pi(x)$, thus completing the proof of (ii).

The equivalence of the statements in (iii) follow immediately by (ii) and compactness. So it is enough to prove that $p(x, b)$ forks over A if and only if $[b]_{R_\phi}$ has infinitely many A -conjugates for some ϕ .

By Proposition 2.3, $p(x, b)$ forks over A if and only if for some $\phi(x, y)$, the ϕ -definition $\theta(y, b)$ of $p(x, b)$ is not over $\text{acl}(A)$. It follows by compactness that this

happens if and only if the set $\theta(\mathfrak{C}, b)$ has infinitely many A -conjugates, which by definition is equivalent to $[b]_{R_\phi}$ having infinitely many A -conjugates. \square

Remark 3.2. In general, in order to get stationarity one must go to the algebraic closure of the parameter set. It is therefore quite useful to notice that in Lemma 3.1 the tuple b can be an enumeration of the algebraic closure of a set, in which case R_ϕ would be an equivalence relation on the finite subtuple of b needed for the ϕ -definition of $p(x, b)$, so it can be interpreted as an equivalence relation on finite tuples, thus allowing for the compactness arguments above to work.

We will now continue with the following (somewhat surprising) consequence of Theorem 2.4. One can view it as a kind of “weak transitivity of \mathfrak{b} -forking for generically stable types”: given a stationary generically stable type p over A , a non- \mathfrak{b} -forking extension of it p_1 , and a nonforking extension p_2 of p_1 to a generically stable parameter, we have that p_2 does not \mathfrak{b} -fork over A .

As a matter of fact, the type p needs only to be a stationary type with an invariant global extension (the full generic stability is not used). Also, the relation $\downarrow^{\mathfrak{b}}$ can be replaced with any relation \downarrow^* which is A -invariant and satisfies extension and monotonicity.

Lemma 3.3. *Let $p(x, b) := \text{tp}(a/Ab)$ be a stationary generically stable type such that $a \downarrow_A^{\mathfrak{b}} b$. Then for any c such that $\text{tp}(c/Ab)$ is generically stable, the nonforking extension $p(x, b)|_{Abc}$ does not \mathfrak{b} -fork over A .*

Proof. Notice that the class of $q(x, b, c) \in S(Abc)$ which do not \mathfrak{b} -fork over A is invariant under $\text{Aut}(\mathfrak{C}/A)$. We can therefore find an enumeration

$$\langle q_\alpha : \alpha < 2^{|A|+|L|} \rangle$$

of all types $q(x, y, z) \in S(A)$ such that for any $b'c' \models \text{tp}(bc/A)$ the type $q_\alpha(x, b', c')$ does not \mathfrak{b} -fork over A . Denote $\lambda = |A| + |L|$. Let $\langle c_\sigma : \sigma < (2^\lambda)^+ \rangle$ be a Morley sequence in $\text{tp}(c/Ab)$. Finally, let $d \models \text{tp}(a/Ab)$ be such that $d \downarrow_A^{\mathfrak{b}} b(c_\sigma)$. If we show that $\text{tp}(d/Abc) = p(x, b)|_{Abc}$, the proof would be complete.

Since by monotonicity $d \downarrow_A^{\mathfrak{b}} bc_\sigma$ for all σ , we know that for any $\sigma \in (2^\lambda)^+$ there is some $\alpha \in 2^\lambda$ such that $\text{tp}(d, b, c_\sigma) = q_\alpha(x, y, z)$. By the cardinalities of the sequences, we can find some α and an ω -sequence $\sigma_1, \sigma_2, \dots$ such that

$$\text{tp}(d, b, c_{\sigma_i}) = q_\alpha(x, y, z)$$

for $i \in \omega$.

But $\langle c_{\sigma_i} \rangle$ is a Morley sequence in $\text{tp}(c/Ab)$, so by Theorem 2.4 we know that $q_\alpha(x, b, c)$ does not fork over Ab (since $q(x, b)$ is generically stable, hence extensible over Ab). This type extends $\text{tp}(d/Ab) = \text{tp}(a/Ab) = p(x, b)$, which by stationarity implies that $q_\alpha(x, b, c) = p|_{Abc}$, as required. \square

We will now prove the main result of this section.

Theorem 3.4. *Let $p(x, b) := \text{tp}(a/Ab)$ be a generically stable type. Then if $p(x, b)$ forks over A , it \mathfrak{b} -forks over A .*

Before we start we will need an easy claim.

Claim 3.5. We may assume that $p(x, b)$ is stationary.

Proof. Let $p(x, b)$ be any generically stable type, let \bar{b} be an enumeration of $\text{acl}(Ab)$ and let $p(x, \bar{b})$ be $\text{tp}(a/Ab)$. It follows from Fact 2.1 and Proposition 2.3 that

- $p(x, \bar{b})$ is stationary,
- $p(x, \bar{b})$ does not fork over A if and only if $p(x, b)$ does not fork over A , and
- $p(x, \bar{b})$ does not \mathfrak{p} -fork over A if and only if $p(x, b)$ does not \mathfrak{p} -fork over A .

This immediately yields the claim. \square

Proof of Theorem 3.4. By the claim above, we may assume that $p(x, b)$ is stationary. This means we can define the relations R_ϕ as in the discussion preceding Lemma 3.1, and by the same lemma, $p(x, b)$ forks over A if and only if for some ϕ the class $[b]_\phi$ has infinitely many A -conjugates. So let us assume this is the case for some $\phi = \phi(x, y)$.

Now assume towards contradiction that $p(x, b)$ does not \mathfrak{p} -fork over A , that is, $a \downarrow_A^{\mathfrak{p}} b$. Let $\langle a_i : i < \omega \rangle$ be a Morley sequence in $\text{tp}(a/Ab)$. By Lemma 3.3 we know that $\text{tp}(a_{n+1}/Aba_1 \dots a_n)$ does not \mathfrak{p} -fork over A , that is, $\langle a_i \rangle$ is a \mathfrak{p} -Morley sequence over Ab based on A . An easy induction using monotonicity and left transitivity in Fact 2.1 shows that

$$a_{n+1}a_n \dots a_1 \downarrow_A^{\mathfrak{p}} b.$$

However, by Theorem 2.2 we know that for some k the $\phi(x, y)$ -definition $\theta(y, b)$ of $p(x, b)$ is definable over a_1, \dots, a_n by some formula $\eta(y, a_1, \dots, a_n)$. It follows that the formula

$$\forall y (\eta(y, z_1, \dots, z_k) \Leftrightarrow y \in [b]_\phi)$$

is in $\text{tp}(a_1, \dots, a_k/Ab)$ and that it clearly strongly divides over A , contradicting $a_k \dots a_1 \downarrow_A^{\mathfrak{p}} b$. \square

Note that in the proof above, the relation $\downarrow^{\mathfrak{p}}$ can be replaced with any relation \downarrow^* which is automorphism-invariant, and satisfies extension, monotonicity and left transitivity. (It is an easy and very insightful exercise to show, using the definition of \mathfrak{p} -forking, that any such \downarrow^* would imply non \mathfrak{p} -forking.)

Corollary 3.6 now follows immediately. This is a generalization of Theorem 2.20 in [4], where the result is proved assuming rosiness of the theory.

Corollary 3.6. *Let $A \subset B \subset C$ and let $\text{tp}(a/B)$ be stable. Then*

- $\text{tp}(a/B)$ forks over A if and only if it \mathfrak{p} -forks over A .
- $\text{tp}(a/C)$ forks over B if and only if it \mathfrak{p} -forks over B .

Proof. By definition (or Fact 2.1) any type which \mathfrak{p} -forks must fork. So in either item we only need to prove that if the type forks, then it \mathfrak{p} -forks.

The first item follows immediately from Theorem 3.4. The second item is also immediate, once we realize that, since $\text{tp}(a/C)$ is an extension of a stable type, it must be (generally) stable, too. \square

Remark 3.7. It is not true that any forking extension of a generically stable type is a \mathfrak{p} -forking extension. Even assuming dependence for the whole structure, an easy example can be found in a nonstandard model of $(\mathbb{Q}, +, 0, <_{(-1,1)})$ where \mathbb{Q} is the set of rational numbers, $+$ is the usual addition and $<_{(-1,1)}$ is the less than restricted to the interval $(-1, 1)$ (so that $a <_{(-1,1)} b$ if and only if $-1 < a < b < 1$).

The type $p(x)$ at infinity (defined as $x - b$ is $<_{(-1,1)}$ -incomparable to 0 for all $b \in \mathbb{Q}$) is generically stable. However, if $a \models p$, then $p(x) \cup 0 <_{(-1,1)} (x - a)$ does not \mathfrak{p} -fork over \mathbb{Q} , so we can find a complete non \mathfrak{p} -forking extension of $a \models p$ and then $p(x) \cup 0 <_{(-1,1)} (x - a)$ to $\mathbb{Q}a$. This type will not be generically stable, it will not \mathfrak{p} -fork over \mathbb{Q} , but it will fork over \mathbb{Q} .

4. SOME APPLICATIONS OF THE EQUIVALENCE BETWEEN FORKING AND \mathfrak{p} -FORKING

As we mentioned in the introduction, non \mathfrak{p} -forking codes a geometric independence notion, which is not captured by $-$ or, better said, it is different from the one captured by $-$ nonforking outside stable (or simple) structures. The combined analysis of both independence notions seems to be something with quite a lot of potential, which has yet to be studied. Also, as has been the case with forking outside simple theories, it is quite likely that the study of non \mathfrak{p} -forking could be quite fruitful even without any assumptions on the ambient theory.

In this section we start with some results which we hope may provide a starting point towards such an analysis.

4.1. A small observation on stable types. The motivation for this example comes from the theory of algebraically closed valued fields (ACVFs) and in particular from the study of stable types and stably dominated types. In [3] and in more recent work it has become quite clear that one of the main tools in this study is stable domination. Even in [3] it is clear that one of the more difficult issues that one needs to deal with when studying the theory of stably dominated types (even in the context of ACVFs) is the fact that it is quite hard to reduce the base of stably dominated types. The following fact (called “descent” in [3]) is one of the most technically difficult theorems in the first part of [3] and requires quite technical lemmas. It seems that a simpler proof should be available with a good understanding of how the space of stable types behaves, even in the context of ACVF.

Theorem 4.1 (Theorem 4.9 in [3]). *Let $\text{tp}(a/B)$ be a stably dominated type and let $A \subset B$ be such that $a \perp_A B$ and such that $\text{tp}(B/A)$ has an A -invariant extension. Then $\text{tp}(a/A)$ is also stably dominated.*

Even though the following is not a proof of descent, it does provide some relevant and useful information in the context of ACVFs for which we don’t know any other proofs.

Definition 4.1. Let T be any theory. We will say T has *bounded finite U-rank for stable types* if given any tuple \bar{x} (in possibly different sorts of T) there is some finite number N such that the U-rank of any stable types with variables \bar{x} is less than N .

In the proof of the theorem below we will make use of the fact that certain restrictions of a stable type are stable. It is still open whether in an arbitrary theory a nonforking restriction of a stable type is stable (for dependent theories this was shown in Lemma 2.26 in [4]). It turns out that for *co-nonforking* restrictions this is true in general and is quite easy:

Lemma 4.2. *Let $\text{tp}(b/A)$ be unstable and $\text{tp}(b/Ac)$ be stable. Then $c \not\perp_A b$. In fact, $\text{tp}(c/Ab)$ divides over A .*

Proof. Suppose otherwise. Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence witnessing the order property (possibly with external parameters) on $\text{tp}(b/A)$, and without loss of generality $b_0 = b$. Since I is an indiscernible sequence in $\text{tp}(b/A)$ and $\text{tp}(c/Ab)$ does not divide over A , clearly without loss of generality I is indiscernible over Ac , hence an indiscernible sequence in $\text{tp}(a/Ac)$ witnessing the order property, which is a contradiction. \square

Theorem 4.3. *Let T be a dependent theory with bounded finite $U^{\mathfrak{p}}$ -rank for stable types and let a, b, c be tuples such that $b \perp_A c$ and both $\text{tp}(a/Ab)$ and $\text{tp}(a/Ac)$ are stable. Then $\text{tp}(a/A)$ is stable.*

Proof. Fix a and b such that $\text{tp}(a/Ab)$ is stable. Then the theorem states that if there is any c such that $b \perp_A c$ and $\text{tp}(a/Ac)$ is stable, then $\text{tp}(a/A)$ was already stable. Choose such c so that $U(\text{tp}(a/Ac))$ has the largest possible U -rank (which exists by the definition of bounded finite U -rank for stable types).

We are going to show that under the assumptions of the previous paragraph, it is necessarily the case that $a \perp_A c$. This will immediately imply that $\text{tp}(a/A)$ is stable (in a dependent theory, a nonforking restriction of a stable type is stable; see for example Corollary 2.27 in [4]). So assume towards contradiction that $a \not\perp_A c$. By Corollary 3.6 we know that $a \not\perp_A^{\mathfrak{p}} c$.

Claim 4.4. There is d such that $\text{tp}(a/Ad)$ is stable, $U(\text{tp}(a/Ac)) = U(\text{tp}(a/Ad))$, $b \perp_A d$ and $\text{tp}(a/Ad)$ \mathfrak{p} -divides over A .

Proof. By definition of \mathfrak{p} -forking, $\text{tp}(a/Ac)$ implies a disjunction $\bigvee_{i \leq n} \phi_i(x, c_i)$, each of which \mathfrak{p} -divide over A ; let $\bar{c} := c_0, \dots, c_n$. We can choose (by extension, and a standard argument where we compose with an automorphism) $\bar{c}' \equiv_{Ac} \bar{c}$ so that $b \perp_A c\bar{c}'$ and then choose $\bar{c}'' \equiv_{Acb} \bar{c}'$ so that $a \perp_{Acb} \bar{c}''$. By left transitivity we have $ab \perp_{Ac} \bar{c}''$, so $a \perp_{Ac} \bar{c}''$ by left monotonicity of nonforking; see Fact 2.1. Since

$$\text{tp}(a/Ac) \vdash \bigvee_{i \leq n} \phi_i(x, c_i'')$$

for $\bar{c}'' := c_0'', \dots, c_n''$, we know that $a \models \phi_i(x, c_i'')$ for some i ; we may assume without loss of generality that $i = 0$. So $U(\text{tp}(a/Ac)) = U(\text{tp}(a/Acc_0''))$, $\text{tp}(a/Acc_0'')$ is a stable type which \mathfrak{p} -divides over A and $b \perp_A cc_0''$ ($b \perp_A c\bar{c}''$). Let $d := cc_0''$. \square

So we may replace c with d and assume that $\text{tp}(a/Ac)$ \mathfrak{p} -divides over A . Let e be such that some formula in $\text{tp}(a/Ac)$ strongly divides over e .

Claim 4.5. We may assume without loss of generality that $a \perp_{Ac} e$ and $b \perp_A ce$.

Proof. The proof is actually quite close to the proof of Claim 4.4, but we include it for completeness. It follows from the definition that the fact that a formula $\phi(x, c)$ strongly divides over Ae depends only on $\text{tp}(c/Ae)$. First, choose $e' \equiv_{Ac} e$ such that $b \perp_A e'c$ and let $e'' \equiv_{Abc} e'$ be such that $a \perp_{Abc} e''$. So in particular $\text{tp}(a/Ac)$ has a formula which strongly divides over Ae'' and $b \perp_A e''c$. It follows by left transitivity that $ab \perp_{Ac} e''$, which implies that $a \perp_{Ac} e''$. So e'' has all the properties we need. \square

Now the proof is quite easy. First, notice that it is clear that $\text{tp}(a/Aec)$ forks (in fact it has a formula which strongly divides) over Ae , so in particular

$$U(a/Ae) > U(a/Ace) = U(a/Ac).$$

Now, since by strong dividing $c \in \text{acl}(Aae)$ and $\text{tp}(a/Abe)$ is stable, we know that $\text{tp}(c/Aeb)$ is stable. But $b \downarrow_{Ae} c$, so by Lemma 4.2, $\text{tp}(c/Ae)$ is stable. Finally, notice that $\text{tp}(a/Ace)$ is stable and $\text{tp}(c/Ae)$ is stable, so $\text{tp}(ac/Ae)$ is stable, which in particular implies that $\text{tp}(a/Ae)$ is stable. This contradicts the maximality of $U(\text{tp}(a/Ac))$. \square

Remark 4.6. The assumption of T dependent in Theorem 4.3 may be replaced by T rosy or by assuming that A is a model of T :

In the proof of Theorem 4.3 we only use the assumption of T dependent to show that whenever $\text{tp}(a/A)$ is unstable, then $a \downarrow_A c$ implies $\text{tp}(a/Ac)$ is unstable. By Lemma 2.26 in [4] this happens whenever T is dependent or rosy. It is also well known that this follows when A is a model of T without any further assumptions on the theory.

4.2. Coordinatization, strong dividing, and \mathfrak{b} -modularity. One of the things that may seem strange in the definition of \mathfrak{b} -forking is the nature of the extra parameter that comes in the definition of \mathfrak{b} -dividing. It turns out that this seems to be related to coordinatization (or lack of coordinatization) in theories with finite $U^{\mathfrak{b}}$ -rank.

This subsection is mainly about \mathfrak{b} -coordinatization and \mathfrak{b} -1-basedness and might therefore seem to be independent of the rest of the paper. However, the reader should be aware that questions about (forking) coordinatization and 1-basedness are central to stability and simplicity theory, and the fact that one can give necessary conditions for both \mathfrak{b} -coordinatization and \mathfrak{b} -1-basedness of particular types does provide a powerful tool for their forking analogues whenever forking and \mathfrak{b} -forking agree (see for example Corollary 4.11).

We will start with the following definitions needed to state the result.

Definition 4.2. We will say that $\text{tp}(a/A)$ is *coordinatizable by \mathfrak{b} -minimal types* if there are a_0, a_1, \dots, a_N such that $a_n \in \text{acl}(Aa)$ for all n , $a_N = a$ and that $\text{tp}(a_{n+1}/Aa_0 \dots a_n)$ has $U^{\mathfrak{b}}$ -rank one.

We will say that $\text{tp}(a/A)$ is *coordinatizable by minimal types* if there are a_0, a_1, \dots, a_N such that $a_n \in \text{acl}(Aa)$ for all n , $a_N = a$ and that $\text{tp}(a_{n+1}/Aa_0 \dots a_n)$ has U -rank one.

Whenever a_0, a_1, \dots, a_N witness that $\text{tp}(a/A)$ is coordinatizable by (\mathfrak{b} -)minimal types, we will say that a_0, a_1, \dots, a_N is a (\mathfrak{b} -)coordinatizing sequence of $\text{tp}(a/A)$.

We will also need the following.

Definition 4.3. We will say that $p(x) \in S(A)$ has *acl-witnesses for \mathfrak{b} -forking* if given any realization a of p , if $p(x)$ has a nonalgebraic non \mathfrak{b} -forking extension, then there exists $a_0 \in \text{acl}(Aa)$ such that $\text{tp}(a/Aa_0)$ is nonalgebraic and $a \not\downarrow_A^{\mathfrak{b}} a_0$.

Theorem 4.7. *Let $p(x) := \text{tp}(a/A)$ be a type of finite $U^{\mathfrak{b}}$ -rank in an arbitrary theory T such that given any $a' \equiv_A a$, any $B \supset A$, $B \subseteq \text{acl}(Aa')$, and any $c \in \text{acl}(a'B)$, the type $\text{tp}(c/B)$ has acl-witnesses for \mathfrak{b} -forking. Then $p(x)$ is coordinatizable by \mathfrak{b} -minimal types.*

The following lemma is the key to the proof of Theorem 4.7. We decided to separate it because it does not assume finite $U^{\mathfrak{b}}$ -rank and sheds some light on the strength of having acl-witnesses for \mathfrak{b} -forking.

Lemma 4.8. *Let $p(x)$ be a nonalgebraic type of ordinal valued $U^{\mathfrak{p}}$ -rank, $a \models p$ such that every extension of $p(x)$ to $B \subseteq \text{acl}(Aa)$ has acl-witnesses for \mathfrak{p} -forking. Then there is some $a_0 \in \text{acl}(aA)$ such that $U^{\mathfrak{p}}(\text{tp}(a/Aa_0)) = 1$.*

Proof. Let $a'_0 \in \text{acl}(aA)$ be such that $0 < U^{\mathfrak{p}}(a/Aa'_0)$ is as small as possible (a'_0 could be \emptyset), and assume towards a contradiction that $U^{\mathfrak{p}}(\text{tp}(a/Aa'_0)) = k > 1$. By hypothesis $\text{tp}(a/Aa'_0)$ has acl-witnesses for \mathfrak{p} -forking, so there is some $a_0 \in \text{acl}(Aa'_0) = \text{acl}(Aa)$ such that

$$1 \leq U^{\mathfrak{p}}(a/Aa'_0a_0) < U^{\mathfrak{p}}(a/Aa'_0) = k,$$

contradicting minimality of k . \square

Proof of Theorem 4.7. Let $p(x) := \text{tp}(a/A)$ be a type as in the statement of the theorem. If $U^{\mathfrak{p}}(p(x)) = 1$ there is nothing to prove. We will prove the theorem by induction on $n := U^{\mathfrak{p}}(p(x))$.

If $n \geq 2$, then by Lemma 4.8 there is some $a_{n-1} \in \text{acl}(Aa)$ such that

$$U^{\mathfrak{p}}(\text{tp}(a/Aa_{n-1})) = 1.$$

By Lascar inequalities for $U^{\mathfrak{p}}$ -rank (see [11])

$$n = U^{\mathfrak{p}}(a/A) = U^{\mathfrak{p}}(aa_{n-1}/A) = U^{\mathfrak{p}}(a/Aa_{n-1}) + U^{\mathfrak{p}}(a_{n-1}/A) = 1 + U^{\mathfrak{p}}(a_{n-1}/A).$$

It follows that $U^{\mathfrak{p}}(a_{n-1}/A) = n - 1 < n$, and by induction hypothesis (it is clear by the ‘‘hereditary’’ character of the statement of the theorem that the conditions hold for $\text{tp}(a_{n-1}/A)$) there are a_0, \dots, a_{n-2} in $\text{acl}(Aa_{n-1})$ such that $\text{tp}(a_{i+1}/Aa_0 \dots a_i)$ have $U^{\mathfrak{p}}$ -rank one. By transitivity of algebraic closure, a_0, \dots, a_{n-1}, a is a coordinatizing sequence for $p(x)$ (note that $\text{tp}(a/Aa_0 \dots a_{n-1})$ is not algebraic because $a_0 \dots a_{n-2} \in \text{acl}(Aa_{n-1})$), as required. \square

The analogue of Theorem 4.7 for supersimple theories has exactly the same proof, and it is very likely that this analogue is known to people who worked in coordinatization theorems as a step in the proof coordinatization for 1-based theories (see Corollary 4.12 and notice that all the proofs work if we replace \mathfrak{p} -forking with forking). However, in the case of \mathfrak{p} -forking, this intermediate step does provide sufficient conditions for coordinatization of types, and those conditions may be satisfied by some non \mathfrak{p} -1-based theories.

Definition 4.4. Let $p(x) \in S(A)$ be any type in a theory T such that $p(x)$ has ordinal valued $U^{\mathfrak{p}}$ -rank. We will say that $p(x)$ has *definable strong dividing* whenever $\text{tp}(a/B)$ is an extension of $p(x)$ over $B \supset A$ which \mathfrak{p} -divides over A , and there is $e' \in \text{acl}(Aa)$ such that $\text{tp}(a/B)$ contains a formula which strongly divides over Ae' .

Because strong dividing is something which can be ultimately witnessed by formulas, definable strong dividing is related in some way to definable choice in the sense that one wants to find, given a family of definable sets, a function which, given the defining parameter, produces a realization of the corresponding element of the family. Of course, in order to have strong dividing, some nonalgebraicity must be preserved, which is the reason why definable choice is not enough to prove definable strong dividing. This may seem like a small issue, but, as we will see later, it turns out that definable strong dividing implies something quite close to 1-basedness, which means that this subtlety is quite significant.

The following lemma will be key for both Lemma 4.10 and Theorem 4.13.

Lemma 4.9. *Let $q(x) = \text{tp}(a/A)$ be a type with ordinal valued U^b -rank and definable strong dividing. Let $B \supset A$ be a set such that $\text{tp}(a/B)$ \mathfrak{b} -divides over A . Then there is some $a' \in \text{acl}(Aa) \cap \text{dcl}(B)$ such that $\text{tp}(a'/Aa)$ \mathfrak{b} -divides over A .*

Proof. By definable strong dividing, there is some $e' \in \text{acl}(Aa)$ such that $\text{tp}(a/B)$ strongly divides over Ae' . By definition, this implies that $a \models \phi(x, a')$, which strongly divides over Ae' for some subtuple $a' \in \text{dcl}(B)$.

By hypothesis $e' \in \text{acl}(Aa)$, and by strong dividing $a' \in \text{acl}(Ae'a) \setminus \text{acl}(Ae')$. This in particular implies that $a' \in \text{acl}(Aa)$, as required. \square

Lemma 4.10. *Let $p(x)$ be a type in a theory T with ordinal valued U^b -rank and definable strong dividing. Then $p(x)$ has acl -witnesses for \mathfrak{b} -forking.*

Proof. If $U^b(p(x)) = 1$ there is nothing to prove. Otherwise, by definition of U^b -rank, there is some $B \supset A$ such that $U^b(a/B) \geq 1$ and $\text{tp}(a/B)$ \mathfrak{b} -divides over A . By Lemma 4.9 there is $a_0 \in \text{acl}(Aa) \cap \text{dcl}(B)$ such that $a \not\downarrow_A a_0$. Since $\text{tp}(a/Aa_0)$ is implied by $\text{tp}(a/B)$, we know that in particular $\text{tp}(a/Aa_0)$ is nonalgebraic, and

$$U^b(\text{tp}(a/A)) > U^b(\text{tp}(a/Aa_0)) \geq U^b(\text{tp}(a/B)) \geq 1,$$

which completes the proof of the lemma. \square

Corollary 4.11. *Let $p(x)$ be a type with bounded finite U^b -rank and such that every extension of $p(x)$ has definable strong dividing. Then $p(x)$ is coordinatizable by \mathfrak{b} -minimal types. If $p(x)$ is stable, then it is coordinatizable by minimal types.*

Corollary 4.11 is strongly related to the coordinatization of 1-based types in supersimple theories. In fact, with the right definitions, a type with bounded U^b -rank and definable strong dividing is quite close to being \mathfrak{b} -1-based.

Definition 4.5. Let $p(x) \in S(A)$ be any type. We define $p(x)$ to be \mathfrak{b} -1-based if given any realization a of $p(x)$ and any $B \supset A$ there is $c \in \text{acl}(Aa) \cap \text{acl}(B)$ such that $a \downarrow_{Ac}^{\mathfrak{b}} B$.

Let $p(x) \in S(A)$ be any type. Following the notation of Makkai's \mathfrak{a} -models, we define $p(x)$ to be \mathfrak{a} - \mathfrak{b} -1-based if given any realization a of $p(x)$ and any ω -saturated $M \supset A$ there is $c \in \text{acl}(Aa) \cap M$ such that $a \downarrow_{Ac}^{\mathfrak{b}} M$.

Corollary 4.12 (of Theorem 4.7). *Let T be a theory of finite U^b -rank such that every type $p(x)$ is \mathfrak{a} - \mathfrak{b} -one-based. Then any type $p(x)$ is coordinatizable.*

Proof. By Theorem 4.7 it is enough to show that any type $p(x) \in S(A)$ with a nonalgebraic \mathfrak{b} -forking extension $q(x)$ has acl -witness for \mathfrak{b} -forking. Suppose $q(x) \in S(B)$ is a nonalgebraic type with $B \supset A$ and let $a \models q$; so in particular $a \not\downarrow_A^{\mathfrak{b}} B$. Let M be a (slightly saturated) model such that $a \downarrow_B^{\mathfrak{b}} M$, so that by transitivity $a \not\downarrow_A^{\mathfrak{b}} M$. By the definition of \mathfrak{a} - \mathfrak{b} -1-based, there is some $c \in \text{acl}(aA) \cap M$ such that $a \downarrow_{Ac}^{\mathfrak{b}} M$. By transitivity again $a \not\downarrow_A^{\mathfrak{b}} c$, so c witnesses \mathfrak{b} -forking, as required. \square

Theorem 4.13. *Let $p(x)$ be a type of ordinal-valued U^b -rank such that any extension has definable strong dividing. Then $p(x)$ is \mathfrak{a} - \mathfrak{b} -1-based.*

Proof. We will assume otherwise and show that there is an infinite forking chain (contradicting the well foundedness of the U^b -rank of $p(x)$). Let $p(x) = \text{tp}(a/A)$

and let $M \supset A$ be a set, and assume that for any $c \in \text{acl}(Aa) \cap M$ we have $a \not\downarrow_{Ac}^{\mathfrak{b}} M$. We will inductively define elements $a_i \in \text{acl}(Aa) \cap M$ such that $\text{tp}(a/Aa_0 \dots a_{i+1})$ \mathfrak{b} -divides over $Aa_0 \dots a_i$. Suppose we have constructed the sequence up to some $n \in \mathbb{N}$.

Notice that by hypothesis $a \not\downarrow_{Aa_0 \dots a_n}^{\mathfrak{b}} M$ so that (by ω -saturation of M) $\text{tp}(a/M)$ \mathfrak{b} -divides over $Aa_0 \dots a_n$. By Lemma 4.9, there is some $a_{n+1} \in \text{acl}(Aa_0 \dots a_n) \cap M$ such that $\text{tp}(a/Aa_0 \dots a_n a_{n+1})$ \mathfrak{b} -divides over $Aa_0 \dots a_n$. Since $\text{acl}(Aa) \supset \text{acl}(Aa_0 \dots a_n)$, a_{n+1} has all the properties we need and

$$\langle \text{tp}(a/Aa_0 \dots a_i) \rangle_{i \in \mathbb{N}}$$

is a forking sequence which contradicts that $p(x)$ has ordinal-valued $U^{\mathfrak{b}}$ -rank. \square

Remark 4.14. Note that we do not actually need every extension of p to have definable strong dividing. It is enough if, given $a \models p$, the assumption holds for any extension of p to any $B \subseteq \text{acl}(Aa)$.

Theorem 4.13 is a nice result, and may be quite useful to prove (\mathfrak{b}) -1-basedness of particular types of ordinal valued $U^{\mathfrak{b}}$ -rank. However, it seems that strong dividing allows us to approximate coordinatization even when one does not have 1-basedness. Notice that in Lemma 4.10, we did not actually use the fact that *every* \mathfrak{b} -dividing extension of a type $\text{tp}(a/A)$ could be witnessed by strong dividing over Ae' for $e' \in \text{acl}(Aa)$. We only needed that for *some* \mathfrak{b} -dividing extension this could be achieved. All of the proofs follow by replacing the hypothesis of definable strong dividing for the following weaker version of weak definable strong dividing. Definable strong dividing seemed to us to be the more natural notion, which is the reason we proved the results using this hypothesis.

Definition 4.6. Let $p(x) \in S(A)$ be any type in a theory T such that $p(x)$ has ordinal valued $U^{\mathfrak{b}}$ -rank. We will say that $p(x)$ has *weak definable strong dividing* if either $U^{\mathfrak{b}}(p(x)) = 1$ or there is some $B \supset A$ and a nonalgebraic forking extension $q(x) \in S(B)$ of $p(x)$ such that for some $e' \in \text{acl}(Aa)$ $q(x)$ implies a formula which strongly divides over Ae' .

Remark 4.15. It is quite easy to see that definable strong dividing implies weak definable strong dividing, and this implication sheds light on the reason why we must single out the \mathfrak{b} -minimal case in Definition 4.6.

Whenever $U^{\mathfrak{b}}(a/A) \geq 1$ we can, by definition, find some b such that $\text{tp}(a/Ab)$ \mathfrak{b} -divides over A and $\text{tp}(a/Ab)$ is nonalgebraic. The existence of b' and e' given by Definition 4.4 witness the conclusion of Definition 4.6. However, when $\text{tp}(a/A)$ has $U^{\mathfrak{b}}$ -rank one, we cannot find any witness of \mathfrak{b} -dividing (much less strong dividing) where $\text{tp}(a/Ab')$ is nonalgebraic. On the other hand, if we give Definition 4.6 without any assumptions on the algebraicity of $\text{tp}(a/Ab')$, then one can trivially imply (by letting $a = b'$) that *every* type has weak definable strong dividing.

If one looks at the proof of Theorem 4.7 and Lemma 4.10, it follows that in a stable theory any example of lack of coordinatization provides an example where the parameter that one needs to go from \mathfrak{b} -dividing to strong dividing plays an essential role. It is hard to define this precisely, other than saying that in any theory with an instance where coordinatization fails, one can come up with an instance of \mathfrak{b} -dividing where the strong dividing parameter cannot be chosen within the definable

closure of the parameters of the original type. Probably the easiest way to show this is the following known examples.

The simplest example where the strong dividing parameter is needed is the affine plane (say in the language $\mathcal{L} := \{l(x, y, z)\}$) where $l(a, b, c)$ if and only if a, b, c are colinear. Then the formula $l(x, a, b)$ \mathfrak{p} -forks over \emptyset , but in order to get an instance of strong dividing witnessing this, one needs to fix some $e \models l(x, a, b)$:

$$l(x, a, b) \models (l(x, a, b) \wedge x \neq e) \vee (x = e);$$

the first formula strongly divides over e and the second one over \emptyset .

On the other hand, take $\mathcal{L} := \{E(x, y), l(x, y, z)\}$, where E is an equivalence relation with infinitely many classes, each of which is an affine plane defined by l (so $l(a, b, c)$ implies that $aEbEc$ and a, b, c are colinear). This is a standard example of a superstable theory of finite rank, the generic type of which is not coordinatizable.

One of the consequences of Corollary 4.11 is that there is no way to witness that $l(x, a, b)$ \mathfrak{p} -forks over \emptyset by getting strong dividing over \emptyset because this would prove that the generic type in the definable family of affine planes is coordinatizable.

The above discussion is not particular to these examples. In fact, we believe any example of a superstable (superrosy) theory with finite U-rank (U^P -rank) where one can witness lack of weak definable strong dividing can give rise in the same way to a superstable (superrosy) theory of finite rank where coordinatization fails.

APPENDIX A. GENERIC STABILITY

The purpose of the appendix is to show that our treatment covers all the known examples of “stable-like” types.

Remark A.1. A stable type is generically stable.

Proof. Stationarity and definability are in [10]. Generic dependence is clear (as a matter of fact, a stable type is *dependent* and extensible, hence in particular generically dependent). Symmetry is probably well known as well and can be found in e.g. [14], Theorem 3.7. \square

In [20], generically stable types are studied, assuming that the theory is dependent. A type p in a dependent theory is called generically stable if there is a Morley sequence in p which is an indiscernible set. In this case, generic dependence is automatic, definability and stationarity is proven in Section 4 of [20], and symmetry is the “Strong Symmetry Lemma”, Lemma 8.5 in [20].

Finally, generically stable types in an arbitrary theory have recently been studied by Pillay and Tanović. In [16], a global type p^* is said to be generically stable over a set A if it is A -invariant, and every sequence in p^* over A is a stable indiscernible set.

Definability and stationarity of such types are shown in Proposition 1 in [16]. Dependence of Morley sequences follows from the definition.

Symmetry does not appear in the current version of [16], but it is known. The following theorem (and proof) was communicated to us by Anand Pillay. We thank him for kindly letting us include it here. Let us call a type $p \in S(A)$ PT-generically stable if it has a global extension, which is generically stable according to [16].

Theorem A.2 (Pillay). *Let $\text{tp}(a/A)$ be a (stationary) PT-generically stable type and let b extensible over A such that $a \perp_A b$. Then $b \perp_A a$.*

Proof. It is easy to see that the Theorem follows from the following claim.

Claim A.3. Let $\phi(x, y)$ be a formula over A and let $q(y) \in S(A)$ be an extensible type. Then for some (any) realization b of q , $\phi(x, b)$ is in p if and only if for every global nonforking extension $q'(y)$ of q , we have $\phi(a', y) \in q'$.

Proof. Suppose $\phi(x, b) \in p$ (for some/any realization b of q). Let $\psi(y)$ be the ϕ definition of p , so that in particular $\psi(y) \in q$.

Suppose for a contradiction that $\neg\phi(a, y) \in q'$ for some global nonforking extension q' of q . Let a_1, a_2, \dots be a Morley sequence in p over A with $a = a_1$. Since q' does not divide over A , we know that $\{\neg\phi(a_i, y) \wedge \psi(y) : i < \omega\}$ is consistent, and so is realized by some d . But $\langle a_i \rangle_{i \in \omega}$ is a Morley sequence of a PT-generically stable type, hence a stable indiscernible set. This implies that $\neg\phi(x, d)$ must be in the global nonforking extension of $p(x)$. But we already assumed $\models \psi(d)$, a contradiction.

The other direction goes by taking negations. □

□

Note that Claim A.3 actually proves the following:

Corollary A.4. *Let $\text{tp}(a/A)$ be (stationary) PT-generically stable, $\text{tp}(b/A)$ extendible, and assume that $\varphi(a, b)$ holds. Then $\varphi(x, b)$ forks over A if and only if $\varphi(a, y)$ forks over A .*

Hence just as in Lemma 8.5 in [20], we can conclude “strong symmetry” in the general context.

Lemma A.5 (Strong Symmetry Lemma). *Let $\text{tp}(a/A)$ be PT-generically stable and $\text{tp}(b/A)$ be extensible. Then $a \downarrow_A b$ if and only if $b \downarrow_A a$. Moreover, if $a \downarrow_A b$ and $\text{tp}(a/A)$ is stationary, then $\text{tp}(b/A)$ has a unique nonforking extension to Aa , which equals $\text{tp}(b/Aa)$.*

Finally, note that any extension of a generically stable type $p \in S(A)$ (as defined in this paper) to $\text{acl}(A)$ is PT-generically stable. Hence our definition is essentially equivalent to the one in [16].

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REFERENCES

- [1] Hans Adler, *A geometric introduction to forking and thorn-forking*, J. Math. Log. **9** (2009), no. 1, 1–20. MR2665779
- [2] Darío García, *Tipos genéricamente estables, bifurcación y thorn bifurcación*, M. A. thesis, Universidad de los Andes, Bogotá, 2009.
- [3] Deirdre Haskell, Ehud Hrushovski, and Donald Macpherson, *Stable domination and independence in algebraically closed valued fields*, Lecture Notes in Logic, vol. 30, Association for Symbolic Logic, Chicago, IL, 2008. MR2369946 (2010c:03002)
- [4] Assaf Hasson and Alf Onshuus, *Stable types in rosy theories*, J. Symbolic Logic **75** (2010), no. 4, 1211–1230. MR2767965

- [5] Ehud Hrushovski and Ben Martin, *Zeta functions from definable equivalence relations*, preprint, 2008.
- [6] Ehud Hrushovski and Anand Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc. **13** (2011), no. 4, 1005–1061. MR2800483 (2012e:03069)
- [7] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay, *Groups, measures, and the NIP*, J. Amer. Math. Soc. **21** (2008), no. 2, 563–596. MR2373360 (2008k:03078)
- [8] Ehud Hrushovski, Anand Pillay, and Pierre Simon, *Generically stable and smooth measures in NIP theories*, submitted.
- [9] H. Jerome Keisler, *Measures and forking*, Ann. Pure Appl. Logic **34** (1987), no. 2, 119–169. MR890599 (88i:03052)
- [10] Daniel Lascar and Bruno Poizat, *An introduction to forking*, J. Symbolic Logic **44** (1979), no. 3, 330–350. MR540665 (80k:03030)
- [11] Alf Onshuus, *Properties and consequences of thorn-independence*, J. Symbolic Logic **71** (2006), no. 1, 1–21. MR2210053 (2007f:03047)
- [12] Alf Onshuus and Alexander Usvyatsov, *On dp-minimality, strong stability and weight*, J. Symbolic Logic **76** (2011), no. 3, 737–758. MR2849244
- [13] ———, *b -orthogonality and domination in unstable theories*, Fund. Math. **214** (2011), no. 3, 241–268. MR2845725
- [14] ———, *Stable domination and weight*, Ann. Pure Appl. Logic **162** (2011), no. 7, 544–560. MR2781094
- [15] Ya'acov Peterzil, *Returning to semi-bounded sets*, J. Symbolic Logic **74** (2009), no. 2, 597–617. MR2518813
- [16] Anand Pillay and Predrag Tanović, *Generic stability, regularity, and quasi-minimality*, in preparation.
- [17] Saharon Shelah, *Classification theory for elementary classes with the dependence property—a modest beginning*, Sci. Math Japan **59** (2004), no. 2, 265–316. Special issue on set theory and algebraic model theory. MR2062198 (2005m:03063)
- [18] ———, *Dependent first order theories, continued*, Israel J. Math. **173** (2009), 1–60. MR2570659
- [19] ———, *Strongly dependent theories*, Submitted (Sh863).
- [20] Alexander Usvyatsov, *Generically stable types in dependent theories*, Journal of Symbolic Logic **74** (2009), no. 1, 216–250. MR2499428 (2010b:03037)
- [21] S. Shelah, *Classification theory and the number of nonisomorphic models*, Second edition, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR1083551 (91k:03085)

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