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ON THE PREPERIODIC POINTS OF AN ENDOMORPHISM OF $\mathbb{P}^1 \times \mathbb{P}^1$ WHICH LIE ON A CURVE

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Abstract. Let X be a projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and φ be an endomorphism of degree ≥ 2 of $\mathbb{P}^1 \times \mathbb{P}^1$, given by two rational functions by $\varphi(z, w) = (f(z), g(w))$ (i.e., $\varphi = f \times g$), where all are defined over $\overline{\mathbb{Q}}$. In this paper, we prove a characterization of the existence of an infinite intersection of $X(\overline{\mathbb{Q}})$ with the set of φ-preperiodic points in $\mathbb{P}^1 \times \mathbb{P}^1$, by means of a binding relationship between the two sets of preperiodic points of the two rational functions f and g, in their respective $\mathbb{P}^1$-components. In turn, taking limits under the characterization of the Julia set of a rational function as the derived set of its preperiodic points, we obtain the same relationship between the respective Julia sets $\mathcal{J}(f)$ and $\mathcal{J}(g)$ as well. We then find various sufficient conditions on the pair (X, φ) and often on φ alone, for the finiteness of the set of φ-preperiodic points of $X(\overline{\mathbb{Q}})$. The finiteness criteria depend on the rational functions f and g, and often but not always, on the curve. We consider in turn various properties of the Julia sets of f and g, as well as their interactions, in order to develop such criteria. They include: topological properties, symmetry groups as well as potential theoretic properties of Julia sets.

1. Arakelov theory and the preperiodic points on a curve in $\mathbb{P}^1 \times \mathbb{P}^1$

1.1. Some important remarks.

1.2. Introduction. An endomorphism φ of a projective space, defined over a number field, determines an infinite set of algebraic preperiodic points. Let us denote by $\mathrm{Prep}_\varphi(X)$ the intersection that a projective curve X in this space may have with this set over $\overline{\mathbb{Q}}$. We may ask, is $\mathrm{Prep}_\varphi(X)$ Zariski dense? Under what conditions can this set be infinite? Is the curve X itself necessarily preperiodic under the morphism φ, which then would automatically provide for such an infinite intersection set?

These questions are closely related to various conjectures and problems which historically guided the evolution of algebraic dynamics. In particular, the Bogomolov conjecture [5] (settled by Ullmo and Zhang) and its generalization, as well as the original dynamical Manin-Mumford conjecture of S. Zhang, which has since been refined in [35] and expanded by him, into what is a list of conjectures, during a recent conference.[1]

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[1] Algebraic Dynamics Conference 2010, New York

Let us also mention the Ihara-Serre-Tate theorem, whose premise can be formulated in terms of an infinite set $\mathrm{Prep}_\varphi(X)$, for a special morphism φ as we will see further on.

For simplicity, we will confine ourselves to a special case of the initial problem. Namely, we will be looking at curves in $\mathbb{P}^1\times\mathbb{P}^1$ and endomorphisms of type $\varphi = f\times g$, where $\varphi(z,w) = (f(z), g(w))$. In so doing, we will also benefit from the large body of knowledge about preperiodic points and Julia sets of rational maps of $\mathbb{P}^1$ which we will make use of. Note that we do not restrict ourselves to polarized dynamical systems.

For recent, related developments to the current topic as well as for a better perspective, see [1], [6], [7], [8], [11], [17], [28], [35], [38].

1.3. **Preliminary remarks.** All curves, morphisms and rational functions considered throughout will be defined over $\overline{\mathbb{Q}}$, taken as a subfield of $\mathbb{C}$. For a given rational function f with $\deg(f) \geq 2$, defined over a number field K, we may look at the periodic points under iteration by f in $\mathbb{P}^1$ and, more generally, at the preperiodic points of f. The latter set is characterized as the set of points in $\mathbb{P}^1$ which have finite forward orbit under iteration by f, denoted here by $\mathrm{Prep}_f(\mathbb{P}^1)$. When $\deg(f) \geq 2$, those sets are infinite, and lie in $\mathbb{P}^1(\overline{\mathbb{Q}})$ since f is defined over a number field.

For a rational function f, $\deg(f) \geq 2$, we will denote by $\mathcal{J}(f)$, the Julia set of f, which may be defined in $\mathbb{P}^1(\mathbb{C})$ as the closure of the set of repelling periodic points of f, or alternatively, as the complement of the locus of normality (or of equicontinuity with respect to the chordal metric) of the family of iterates $\{f^n, n \geq 1\}$ of f, viewed as complex maps in $\mathbb{P}^1(\mathbb{C})$. However, for the purpose of the application we have in mind in this context, we will need still another characterization of the Julia set of a rational function f, to be proved in §1.6, as the derived set (or set of accumulation points denoted by S' for a given set S) of the set $\mathrm{Prep}_f(\mathbb{P}^1)$. For generalities about Julia sets and iteration of rational functions in general, see [2], [26], [29] as well as [32].

For a self-map $\varphi = f \times g$ of $\mathbb{P}^1 \times \mathbb{P}^1$, where f and g act componentwise, let us denote by $\mathrm{Prep}_\varphi(\mathbb{P}^1 \times \mathbb{P}^1)$, the set of φ-preperiodic points. For any given subset S of $\mathbb{P}^1 \times \mathbb{P}^1$, we will denote by $\mathrm{Prep}_\varphi(S)$, the set of φ-preperiodic points lying in $S(\overline{\mathbb{Q}})$. A point $P = (z,w) \in \mathbb{P}^1 \times \mathbb{P}^1$ has finite forward orbit under $\varphi = f \times g$ iff the forward orbits of z and w in $\mathbb{P}^1$ by f and g, respectively, are finite. We therefore have:

$$\mathrm{Prep}_\varphi(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Prep}_f(\mathbb{P}^1) \times \mathrm{Prep}_g(\mathbb{P}^1).$$

Remark 1.1. Let $\varphi = f \times g$ be an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$, given by rational maps f and g defined over some number field K, f not the identity map, with $\deg(f) = 1$ and $\deg(g) \geq 2$. Then $\mathrm{Prep}_f(\mathbb{P}^1)$ is finite, and it has one or two points since $f \in \mathrm{Aut}(\mathbb{P}^1)$. If with such an f the set of φ-preperiodic points $\mathrm{Prep}_\varphi(X)$ is to be infinite, then X has to be $\{a\} \times \mathbb{P}^1$, where a is one of the (at most two) fixpoints of f (for X is irreducible). We will require therefore that the projections from X to each $\mathbb{P}^1$ be surjective, in order to avoid such trivial cases. As we will do so, we may assume throughout, without loss of generality: $\deg_m(\varphi) = \min\{\deg(f), \deg(g)\} \geq 2$ instead of simply requiring $\deg(\varphi) \geq 2$.

Remark 1.2. Let X be an irreducible projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined over some number field K. It is then given by a single irreducible bihomogeneous polynomial $H \in K[z_0, z_1, w_0, w_1]$.

Remark 1.3. Note that we need not specify where the curve is defined in the above remark. It will be automatically defined over $\overline{\mathbb{Q}}$, as long as the map is defined over some number field and that we are assuming $\operatorname{Prep}_\varphi(X)$ is infinite. To see this, let $H(z, w)$ be an irreducible polynomial of two variables over some field, normalized to have has some coefficient $= 1$. If the curve is not defined over a number field, then the field K_0 generated over $\overline{\mathbb{Q}}$ by the coefficients of H is $\neq \overline{\mathbb{Q}}$. We can find a $\overline{\mathbb{Q}}$-automorphism τ of $\overline{K_0}$, the algebraic closure of K_0, such that $H^\tau \neq H$. But at the same time $H^\tau(Z, W) = H(Z, W)$, as those two normalized irreducible polynomials have in common infinitely many zeros, namely, the preperiodic points of f and g which lie in $\overline{\mathbb{Q}}$.

Remark 1.4. The above remark shows that, if the morphism is defined over $\overline{\mathbb{Q}}$, then: For any irreducible (normalized) polynomial $H(Z, W)$, the curve X defined in $\mathbb{P}^1 \times \mathbb{P}^1$ by the bihomogenized polynomial H^*, hosts finitely many φ-preperiodic points as soon as some coefficient of H is transcendental for example.

Remark 1.5. If one takes for $\varphi(x, y) = (x^n, y^m)$, for any $n, m \geq 2$, then we would have $\operatorname{Prep}_\varphi(\mathbb{P}^1 \times \mathbb{P}^1) = \boldsymbol{\mu}^* \times \boldsymbol{\mu}^*$ (where $\boldsymbol{\mu}$ is the set of all roots of unity and $\boldsymbol{\mu}^* = \boldsymbol{\mu} \cup \{0, \infty\}$). Using such a choice of φ, the Ihara-Serre-Tate theorem's premise can be interpreted as the curve having an infinite intersection with this set.

1.4. **Heights and their application to the preperiodic points on X.** Let X be an irreducible projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$, $\varphi = f \times g$ an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$, both defined over some number field K, with

$$\deg_m(\varphi) = \min\{\deg(f),\ \deg(g)\} \geq 2.$$

Let us denote by $d_1 \geq 2$ and $d_2 \geq 2$ the degrees of the rational functions f and g.

Let $\mathcal{L}_0 = \mathcal{O}(1)$ on $\mathbb{P}^1$, and let us fix the isomorphisms (as $f^* \mathcal{L}_0 \simeq \mathcal{O}(d_1)$, and similarly for g):

$$\varphi_1 \colon \mathcal{L}_0^{\otimes d_1} \to f^* \mathcal{L}_0 \quad \text{and} \quad \varphi_2 \colon \mathcal{L}_0^{\otimes d_2} \to g^* \mathcal{L}_0.$$

We now define the metrized line bundles:

$$\overline{\mathcal{L}}_1 = \big(\mathcal{O}(1),\, \|\cdot\|_1\big) \quad \text{and} \quad \overline{\mathcal{L}}_2 = \big(\mathcal{O}(1),\, \|\cdot\|_2\big),$$

where the two metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathcal{L}_0$ are defined to be the unique ones satisfying:

$$\|\cdot\|_1 = \Big(\varphi_1^* f^* \|\cdot\|_1\Big)^{\frac{1}{d_1}} \quad \text{and} \quad \|\cdot\|_2 = \Big(\varphi_2^* g^* \|\cdot\|_2\Big)^{\frac{1}{d_2}}.$$

Next, we justify the existence of the admissible metrized line bundles above.

The existence of such metrics on $\mathcal{L}_0$ is assured by a theorem of S. Zhang [37, Thm. 2.2, p. 290], which we quote next for convenience (it is hoped that the notation, which we have not changed in the quote, will not cause confusion).

Let K be an algebraically closed valuation field, X a projective variety over $\operatorname{Spec} K$, $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ a line bundle on X, with a continuous and bounded metric $\|\cdot\|$, $f \colon X \to X$ a surjective morphism, and $\varphi \colon \mathcal{L}^{\otimes d} \simeq f^* \mathcal{L}$ an isomorphism where $d > 1$ is an integer. We define $\|\cdot\|_n$ on $\mathcal{L}$ inductively as follows:

$$\|\cdot\|_1 = \|\cdot\|, \quad \|\cdot\|_n = \varphi^* f^* \|\cdot\|_{n-1}^{\frac{1}{d}}.$$

Theorem 1.6 (Zhang)**.**
(a) The metrics $\|\cdot\|_n$ on $\mathcal{L}$ converge uniformly to a metric $\|\cdot\|_0$ on $\mathcal{L}$. This means that the function $\log \frac{\|\cdot\|_n}{\|\cdot\|_1}$ converges uniformly on $X(K)$ to $\log \frac{\|\cdot\|_0}{\|\cdot\|_1}$.
(b) $\|\cdot\|_0$ is the unique (bounded and continuous) metric on $\mathcal{L}$ satisfying the equation

$$\|\cdot\|_0 = \left(\varphi^* f^* \|\cdot\|_0\right)^{\frac{1}{d}}.$$

(c) If φ changes to $\lambda\varphi$ with $\lambda \in K^*$, then $\|\cdot\|_0$ changes to $|\lambda|^{\frac{1}{d-1}} \|\cdot\|_0$.

Let us define the metrized line bundle

$$\overline{\mathcal{L}} = p_1^* \overline{\mathcal{L}}_1\big|_X \otimes p_2^* \overline{\mathcal{L}}_2\big|_X.$$

Let us also define the height relative to $\overline{\mathcal{L}}$, of a point

$$x \in X(\overline{\mathbb{Q}}), \quad h_{\overline{\mathcal{L}}}(x) = \frac{\deg\left(\overline{\mathcal{L}}\big|_{D_x}\right)}{\deg\left(D_x\right)_{\mathbb{Q}}},$$

where D_x is the Zariski closure of x in X, the degree of the field extension is the relative degree of x over the field of definition K and, similarly for $h_{\overline{\mathcal{L}}_i}(x)$ for points x of $\mathbb{P}^1(\overline{\mathbb{Q}})$ for the metrized line bundles $\overline{\mathcal{L}}_i$ defined above.

We now use the line bundles $\overline{\mathcal{L}}_i$ in question to characterize the preperiodic points of φ that lie on $X(\overline{\mathbb{Q}})$.

Proposition 1.7.
(1) $\forall z \in \mathbb{P}^1(\overline{\mathbb{Q}}), \quad h_{\overline{\mathcal{L}}_i}(z) \geq 0$ for $i = 1, 2$;
(2) $\forall x \in X(\overline{\mathbb{Q}}), \quad h_{\overline{\mathcal{L}}}(x) \geq 0$;
(3) $h_{\overline{\mathcal{L}}_1}(f(z)) = \deg(f) \cdot h_{\overline{\mathcal{L}}_1}(z)$ (similarly for g and $h_{\overline{\mathcal{L}}_2}$);
(4) $z \in \operatorname{Prep}_f\left(\mathbb{P}^1(\overline{\mathbb{Q}})\right) \iff h_{\overline{\mathcal{L}}_1}(z) = 0$ (similarly for g and $h_{\overline{\mathcal{L}}_2}$);
(5) A point $x \in X(\overline{\mathbb{Q}})$ is preperiodic for $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ $\left(x \in \operatorname{Prep}_\varphi(X)\right)$ if and only if $h_{\overline{\mathcal{L}}}(x) = 0$.

Proof. Since

$$h_{\overline{\mathcal{L}}} = h_{\overline{\mathcal{L}}_1} \circ p_1 + h_{\overline{\mathcal{L}}_2} \circ p_2,$$

(2) follows from (1), and noting that a point x is preperiodic for φ iff its projections are preperiodic for f and g respectively, (5) follows from (4). Because of the choice of our metrized line bundles, (1),(3),(4) of the proposition can be proved essentially by invoking Northcott's theorem and functorial properties of the height. We will instead conclude by invoking a special case of a theorem of S. Zhang [37, Thm. 2.4, p. 292]. □

1.5. **Main theorem.** Now let X be a *nontrivial* (see Remark 1.2) projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$, $\varphi = f \times g$ an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\deg_m(\varphi) \geq 2$, both defined over some number field K.

We can now state the main theorem:

Theorem 1.8 (Main theorem)**.** *With the notation, the height $h_{\overline{\mathcal{L}}}$, and Julia set $\mathcal{J}$ defined above, we have:*

I. The following are equivalent:
(a) The set $\operatorname{Prep}_\varphi(X)$ is infinite.
(b) $\forall\, x = (z, w) \in X(\overline{\mathbb{Q}}), \quad z \in \operatorname{Prep}_f(\mathbb{P}^1) \iff w \in \operatorname{Prep}_g(\mathbb{P}^1)$.
(c) $\forall \epsilon > 0$, the set $\{x \in X(\overline{\mathbb{Q}}) \mid h_{\overline{\mathcal{L}}}(x) < \epsilon\}$ is infinite.

II. Furthermore, the above equivalent conditions imply the following:

(d) $\forall (z,w) \in X(\mathbb{C}), \quad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g)$.

Proof. We first prove Part I of the theorem.
(a) $\Rightarrow$ (c): For any $\epsilon > 0$ the set in (c) includes the set $\mathrm{Prep}_{\varphi}(X)$ of (a), since any member of $\mathrm{Prep}_{\varphi}(X)$ satisfies $h_{\overline{\mathcal{L}}}(x) = 0$ by Proposition 1.7.
(b) $\Rightarrow$ (a): As $\deg(f) \geq 2$, the set $\mathrm{Prep}_f(\mathbb{P}^1)$ is infinite, and since $p_1\big|_X$ is surjective we can find $w \in \mathbb{P}^1$ so that $x = (z,w) \in X$. By (b), since z is preperiodic for f we have that w must be preperiodic for g and, therefore, $x \in X$ is preperiodic for φ. This being the case for an infinite set of z's and therefore of x's, we get (a).
(c) $\Rightarrow$ (b): By (c) we have:

$$\varliminf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{\mathcal{L}}}(x) \leq 0$$

and by Proposition 1.7(2) we know that $h_{\overline{\mathcal{L}}} \geq 0$. Therefore:

$$\varliminf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{\mathcal{L}}}(x) = 0. \tag{1.1}$$

We will now apply a theorem of S. Zhang [37, Thm. 1.10, p. 289], which we quote next in full as it is the backbone of the proof, in order to obtain (b):

Theorem 1.9 (Zhang)**.** *If $\overline{\mathcal{L}}$ is an ample metrized line bundle, then*

$$e_1(\overline{\mathcal{L}}) \geq h_{\overline{\mathcal{L}}}(X) \geq \frac{e_1(\overline{\mathcal{L}}) + \cdots + e_d(\overline{\mathcal{L}})}{d},$$

where for $i = 1, 2, \ldots, d$, and

$$e_i(\overline{\mathcal{L}}) = \sup_{\operatorname{codim} Y = i} \inf_{x \in X \setminus Y} h_{\overline{\mathcal{L}}}(x),$$

X is any projective variety of $\dim d-1$ *over* $\operatorname{Spec} \mathbb{Q}$,

$$h_{\overline{\mathcal{L}}}(X) = \frac{c_1(\overline{\mathcal{L}})^d}{d\, c_1(\overline{\mathcal{L}}_{\mathbb{Q}})^{d-1}}$$

and, where Y runs through the set of reduced subvarieties of X.

We apply this to our case with $d = 2$. X is to be our curve X which we can consider as a projective variety over Spec $\mathbb{Q}$ (possibly disconnected then), and $\overline{\mathcal{L}}$ is to be our ample metrized line bundle

$$\overline{\mathcal{L}} = p_1^* \overline{\mathcal{L}}_1\big|_X \otimes p_2^* \overline{\mathcal{L}}_2\big|_X$$

as defined before. In our context, we can make the numbers $e_1(\overline{\mathcal{L}})$ and $e_2(\overline{\mathcal{L}})$ more explicit; we have:

$$e_1(\overline{\mathcal{L}}) = \varliminf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{\mathcal{L}}}(x) \text{ and } \quad e_2(\overline{\mathcal{L}}) = \inf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{\mathcal{L}}}(x).$$

We have already shown (given the expression of $e_1(\overline{\mathcal{L}})$ just written in equation (1.1)) that (c) implies $e_1(\overline{\mathcal{L}}) = 0$. It now follows by S. Zhang's theorem above that:

$$h_{\overline{\mathcal{L}}}(X) = \frac{c_1(\overline{\mathcal{L}})^2}{2\, c_1(\overline{\mathcal{L}}_{\mathbb{Q}})} = 0.$$

That is:

$$c_1(\overline{\mathcal{L}})^2 = \left(p_1^* \overline{\mathcal{L}}_1\big|_X \otimes p_2^* \overline{\mathcal{L}}_2\big|_X\right)^2 = 0. \tag{1.2}$$

We now prove the following three lemmas.

Lemma 1.10. *For the metrized line bundles* $p_1^*\,\overline{\mathcal{L}}_1\big|_X$, $p_2^*\,\overline{\mathcal{L}}_2\big|_X$ *and* $\overline{\mathcal{L}}$ *we have:*
(1) $e_j(p_i^*\,\overline{\mathcal{L}}_i\big|_X) \geq 0$, *for* $i, j = 1, 2$*;*
(2) $e_j(\overline{\mathcal{L}}) \geq 0$, *for* $j = 1, 2$.

Proof. The definition of the numbers $e_i(\cdot)$ involves taking inferior limits and infimums on the heights $h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X} = h_{\overline{\mathcal{L}}_i} \circ p_i$ and on $h_{\overline{\mathcal{L}}}$. The range of all those heights is positive in view of the positivity of the heights in Proposition 1.7(1) and (2). □

Lemma 1.11. $e_1(p_i^*\,\overline{\mathcal{L}}_i\big|_X) = 0$, *for* $i = 1, 2$.

Proof. The sets $\mathrm{Prep}_f(\mathbb{P}^1)$ and $\mathrm{Prep}_g(\mathbb{P}^1)$ are infinite, since both f and g have degree at least two. The projections $p_i\big|_X$ are surjective, so there are infinitely many points $x \in X(\overline{\mathbb{Q}})$ whose i-th projection is preperiodic for the corresponding rational function (f or g), that is, points x which are such that

$$h_{\overline{\mathcal{L}}_1} \circ p_i(x) = 0$$

by Proposition 1.7(4), and since

$$h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X}(x) = h_{\overline{\mathcal{L}}_i} \circ p_i(x) \quad \text{for } i = 1, 2,$$

we have

$$h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X}(x) = 0$$

for infinitely many points of $X(\overline{\mathbb{Q}})$ for $i = 1, 2$. Therefore $e_1(\overline{\mathcal{L}}_i\big|_X) \leq 0$ and since they are already positive by the previous lemma, they are equal to 0. □

Lemma 1.12. *For the ample metrized line bundles* $p_1^*\,\overline{\mathcal{L}}_1\big|_X$ *and* $p_2^*\,\overline{\mathcal{L}}_2\big|_X$ *we have:*
(1) $p_1^*\,\overline{\mathcal{L}}_1\big|_X^2 = 0$ *and* $p_2^*\,\overline{\mathcal{L}}_2\big|_X^2 = 0$*;*
(2) $p_1^*\,\overline{\mathcal{L}}_1\big|_X \cdot p_2^*\,\overline{\mathcal{L}}_2\big|_X = 0$.

Proof. We now apply the first inequality in Zhang's theorem to the ample metrized line bundles $p_1^*\,\overline{\mathcal{L}}_1\big|_X$ and $p_2^*\,\overline{\mathcal{L}}_2\big|_X$ to get, in view of the result of the previous lemma ($e_1(p_i^*\,\overline{\mathcal{L}}_i\big|_X) = 0$, for $i = 1, 2$), for each $i = 1, 2$:

$$h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X}(X) = 0.$$

Indeed, we get ≤ 0 at first, but the right hand side of the inequality in Zhang's theorem is ≥ 0 because the e_i's associated to these metrized line bundles are non-negative by Lemma 1.10(1) for both $j = 1, 2$, and therefore

$$h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X}(X) \geq 0.$$

Finally, we get that they are $= 0$. On the other hand,

$$h_{p_i^*\,\overline{\mathcal{L}}_i\big|_X}(X) = 0 \iff c_1(p_1^*\,\overline{\mathcal{L}}_1\big|_X)^2 = p_1^*\,\overline{\mathcal{L}}_1\big|_X^2 = 0.$$

This proves (1). Now (1) implies that the two symmetric terms in the intersection number in the identity (1.2) are zero and since the whole number is zero we have statement (2) of the lemma. □

We now continue with the proof of the theorem. Putting

$$\overline{\mathcal{L}}_\natural = p_1^* \overline{\mathcal{L}}_1\big|_X^{\otimes \deg(p_2)} \otimes p_2^* \overline{\mathcal{L}}_2\big|_X^{\otimes -\deg(p_1)},$$

we then have $\overline{\mathcal{L}}_\natural^2 = 0$. Now $\overline{\mathcal{L}}_\natural$ has degree 0 and $\overline{\mathcal{L}}_\natural^2 = 0$ so, by the Hodge index theorem (Faltings, Hriljac) we conclude that $\overline{\mathcal{L}}_\natural$ is numerically equivalent to 0. Therefore, we have for all $x \in X(\overline{\mathbb{Q}})$:

$$\deg(p_2) \cdot h_{p_1^* \overline{\mathcal{L}}_1}(x) = \deg(p_1) \cdot h_{p_2^* \overline{\mathcal{L}}_2}(x).$$

In particular,

$$h_{p_1^* \overline{\mathcal{L}}_1}(x) = 0 \iff h_{p_2^* \overline{\mathcal{L}}_2}(x) = 0.$$

Equivalently:

$$h_{\overline{\mathcal{L}}_1}(p_1(x)) = 0 \iff h_{\overline{\mathcal{L}}_2}(p_2(x)) = 0.$$

This means finally, using Proposition 1.7(4), which characterizes preperiodic points in the components in terms of the vanishing of the corresponding heights:

$$\forall\, x = (z, w) \in X(\overline{\mathbb{Q}}), \quad z \in \mathrm{Prep}_f(\mathbb{P}^1) \iff w \in \mathrm{Prep}_g(\mathbb{P}^1).$$

This was precisely what was to be proved, so we have (c) $\Rightarrow$ (b). □

1.6. **Julia sets as derived sets and proof of Part II of theorem.** We will now prove Part II of the theorem, assuming Theorem 1.13 considered further on in this section, and which states that the derived set (set of accumulation points) of $\mathrm{Prep}_f(\mathbb{P}^1)$ is the Julia set $\mathcal{J}(f)$. We need to prove the equivalence II(d), under the assumption of the equivalent properties (a), (b), (c) of Part I of Theorem 1.8.

Proof. Assuming therefore that the derived set of $\mathrm{Prep}_f(\mathbb{P}^1)$ is $\mathcal{J}(f)$ (and similarly for g, respectively), let us take $(z, w) \in X(\mathbb{C})$ with $z \in \mathcal{J}(f)$. Accordingly, z is then an accumulation point of $\mathrm{Prep}_f(\mathbb{P}^1)$. Let p_1 and p_2 denote the projections of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ onto the first and second $\mathbb{P}^1(\mathbb{C})$ components (containing $\mathcal{J}(f)$ and $\mathcal{J}(g)$ respectively). Let W be a (complex) open neighborhood of w in $\mathbb{P}^1$. Then, there is an open neighborhood U of (z, w) in $X(\mathbb{C})$, such that $p_2(U) \subset W$. The projections p_1, p_2 are open maps. Therefore, $V = p_1(U)$ is an open neighborhood of z in $\mathbb{P}^1$. Consequently, V must contain infinitely many points of $\mathrm{Prep}_f(\mathbb{P}^1)$, as z is an accumulation point of $\mathrm{Prep}_f(\mathbb{P}^1)$. Therefore, there are in U, infinitely many points (z', w') with $z' \in \mathrm{Prep}_f(\mathbb{P}^1)$. Since we are under the assumption that the curve contains infinitely many φ-preperiodic points, we have from the equivalence in Theorem 1.8(b), that each w' of this infinite set of points of U is preperiodic for g. That is, W contains, as p_2 is a finite map, infinitely many points of $\mathrm{Prep}_g(\mathbb{P}^1)$, namely the w''s. We conclude, since W was arbitrary, that w is likewise an accumulation point of $\mathrm{Prep}_g(\mathbb{P}^1)$, and therefore $w \in \mathcal{J}(g)$. By symmetry, we get $z \in \mathcal{J}(f)$ if we start with $w \in \mathcal{J}(g)$. This completes the proof of (d) and of Part II of the theorem. □

We will now show that the derived set of $\mathrm{Prep}_f(\mathbb{P}^1)$ for a rational function f is the Julia set $\mathcal{J}(f)$. This fact will complete the proof of the equivalence (d) of Theorem 1.8 II just established earlier. That the preperiodic points accumulate at any point in the Julia set is expected and is a well-known result. The additional fact needed is that this doesn't happen anywhere outside the Julia set. We now state the result:

Theorem 1.13. *The derived set of* $\mathrm{Prep}_f(\mathbb{P}^1)$ *satisfies:* $\mathrm{Prep}_f(\mathbb{P}^1)' = \mathcal{J}(f)$.

Proof. For any periodic point z of the Julia set of f, we have

$$\mathcal{J}(f) \subset \mathcal{O}^-(z)'.$$

A proof of this can be found in [29, Thm. 4, p. 33]. Therefore, we have that

$$\mathcal{J}(f) \subset \mathrm{Prep}_f(\mathbb{P}^1)'.$$

We now only need to exclude possible accumulation points of $\mathrm{Prep}_f(\mathbb{P}^1)$ in $\mathcal{F}$ (the Fatou set of f). Let z_k be a convergent sequence of distinct elements of $\mathrm{Prep}_f(\mathbb{P}^1)$ converging to $w \in \mathcal{F}$. Each z_k has an iterate which is a periodic point of f. Only finitely many of the z_k may have their periodic iterate in $\mathcal{J}(f)$, since z_k converges to a point in $\mathcal{F}$. However, $\mathcal{F}$ may only contain finitely many periodic points of f. Therefore, infinitely many of the z_k's must have the same periodic iterate a in $\mathcal{F}$, and the subsequence of distinct elements of $\mathrm{Prep}_f(\mathbb{P}^1)$ which they form must again converge to w. We may also deduce that this common periodic iterate a is not an exceptional point, since the backward orbit of a by the choice of a, contains infinitely many elements, and exceptional points have finite backward orbits. In other words, we have the following:
If $w \in \mathcal{F}$ is an accumulation point of $\mathrm{Prep}_f(\mathbb{P}^1)$, then it is also an accumulation point of the backward orbit of a nonexceptional periodic point a of f, with $a \in \mathcal{F}$. To complete the proof of the theorem, we need to show this is not possible. In fact, quite the opposite is true. □

Lemma 1.14. *Let $a \in \mathcal{F}$ be a nonexceptional periodic point of f. Then, the derived set of $\mathcal{O}^-(a)$ satisfies:*

$$\mathcal{O}^-(a)' = \mathcal{J}(f).$$

Proof. See [3, p. 581]. For a detailed proof, one may consult [27]. □

Another approach to prove this result might be via the equidistribution theory of the preperiodic points on X with respect to a measure which under the two projections maps to the canonical measures attached to f and g. For this approach see [6] and the references therein.

2. Finiteness criteria for $\mathrm{Prep}_\varphi(X)$ based on the topology of Julia sets

2.1. **Introduction.** As the subsequent results of this paper will show, the assumption of an infinite set of preperiodic points for a pair (X, φ) forces the two Julia sets in each $\mathbb{P}^1$ component, corresponding to the rational maps f and g of $\varphi = f \times g$, to have many similarities. We will develop in this section several criteria, based on topological properties of the Julia sets, such that any asymmetry in these properties between the two components will cause $\mathrm{Prep}_\varphi(X)$ to be finite instead.

2.2. Pairing up Julia sets.

Definition 2.1. A rational map will be called *exceptional* if it is conformally conjugate by some $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ to a rational map with Julia set in:

$$\{\mathbb{P}^1,\ \partial\mathbb{D}\ (\textit{unit circle}),\ \mathbb{I} = [-1, 1]\}.$$

In that case, its Julia set is the image by σ^{-1} of the Julia set in the list. Analogously, we will call a Julia set $\mathcal{J}(f)$ *exceptional*, if its image by some $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ is a set $T \in \{\mathbb{P}^1, \partial\mathbb{D}, \mathbb{I}\}$ and then, we will write $\mathcal{J}(f) \sim T$. This will be the case iff it is $\mathbb{P}^1$, a circle in $\mathbb{P}^1$ or a segment of a circle in $\mathbb{P}^1$.

Theorem 2.2. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ defined over $\overline{\mathbb{Q}}$. If $\mathrm{Prep}_\varphi(X)$ is infinite, then we have:*

$$\forall T \in \{\mathbb{P}^1, \partial\mathbb{D}, \mathbb{I}\} \quad \mathcal{J}(f) \sim T \iff \mathcal{J}(g) \sim T$$

and in particular: f is exceptional iff g is exceptional.

Proof. Let p_1 and p_2 be the left and right projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto its $\mathbb{P}^1$ components. From the assumption $\mathrm{Prep}_\varphi(X)$ is infinite, we have by Theorem 1.8 II(d) the equivalence:

$$\forall (z, w) \in X(\mathbb{C}), \quad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g). \tag{2.1}$$

It is clear from this that:

$$\mathcal{J}(f) \sim \mathbb{P}^1 \iff \mathcal{J}(f) = \mathbb{P}^1 \iff \mathcal{J}(g) = \mathbb{P}^1 \iff \mathcal{J}(g) \sim \mathbb{P}^1.$$

We may now assume $\mathcal{J}(f) \sim \partial\mathbb{D}$ or $\mathbb{I}$. By conjugating if necessary, we may assume first, that both Julia sets $\mathcal{J}(f)$, $\mathcal{J}(g)$ are compacts of $\mathbb{C} \subset \mathbb{P}^1$ and we may also assume $\mathcal{J}(f) = \partial\mathbb{D}$ or $\mathbb{I} = [-1, 1]$. We need to show that $\mathcal{J}(g)$ is a circle or a segment of a circle, respectively. Let $(z, w) \in X$ with $z \in \mathcal{J}(f)$, $w \in \mathcal{J}(g)$ and both away from the branch locus of p_1 and p_2 respectively and in addition in the case $\mathcal{J}(f) = \mathbb{I}$, z away from the endpoints of $\mathbb{I}$ as well. Then, there is some small open set, say U_0 in X which is mapped conformally onto open neighborhoods of z and w by $p_{1,0} = p_1\big|_{U_0}$ and $p_{2,0} = p_2\big|_{U_0}$ respectively. Then, for sufficiently small r,

$$\varphi = p_{2,0} \circ (p_{1,0}^{-1})\big|_{D(z,r)}$$

maps $D(z, r)$ conformally onto the open set $\varphi\big(D(z, r)\big)$ such that, by choosing r smaller still, the connected analytic curve $\mathcal{J}(g) \cap \varphi\big(D(z, r)\big)$ is the image of $D(z, r) \cap \mathcal{J}(f)$.

By the well-known expanding property of a rational function on its Julia set (see [29, Ch. 2, §5, Thm. 2]), there is some integer $n_0 \geq 1$ such that

$$g^{n_0}\Big(\mathcal{J}(g) \cap \varphi\big(D(z, r)\big)\Big) = \mathcal{J}(g).$$

So we conclude that $\mathcal{J}(g)$ is the image by the rational map g^{n_0} (holomorphic on $\mathcal{J}(g)$) of the connected analytic curve $\mathcal{J}(g) \cap \varphi(D(z, r))$. So $\mathcal{J}(g)$ is a connected analytic curve, and it is closed (setwise) since it is a Julia set. Now, the Fatou set $\mathcal{F}(g) = \mathbb{P}^1 \setminus \mathcal{J}(g)$ has finitely many components as it has regular boundary. As such the number of such can be one, or two. If it is one, then $\mathbb{P}^1 \setminus \mathcal{J}(g)$ is connected. In that case $\mathcal{J}(g)$ is contractible, so it is an analytic Jordan arc (closed). If it is two, then it is an analytic Jordan curve (smooth boundary, therefore accessible from each component). The map g sends components onto components (see [29, Ch. 2, §6, Thm. 1]). In the second case (Jordan curve), g may permute the exterior and interior components of $\mathcal{J}(g)$, and we may then replace g by its iterate g^2 and so fix each of them (and keep the same Julia set). Therefore in either case, $\mathcal{J}(g)$ is the Julia set of a rational function (g or g^2) which bounds a component fixed by the function. The Julia set is then either a circle in $\mathbb{P}^1$, or a closed segment of a circle in $\mathbb{P}^1$ by [29, Ch. 5, §3, Thm. 2]. We have shown that, if $\mathcal{J}(f)$ is a circle or segment of a circle, then so is $\mathcal{J}(g)$. To prove the theorem we also need to show that we can't have a circle on one side and a segment of a circle on the other. By symmetry, it is enough to show that if $\mathcal{J}(f) = \partial\mathbb{D}$, then $\mathcal{J}(g)$ cannot be a segment of a circle in $\mathbb{P}^1$. By conjugating g, and since a segment of a circle in $\mathbb{P}^1$ can be

sent onto $\mathbb{I}$ by $\mathrm{Aut}(\mathbb{P}^1)$, we may just prove that $\mathcal{J}(g) \neq \mathbb{I}$ if $\mathcal{J}(f) = \partial\mathbb{D}$. We treat this case in the:

Lemma 2.3. *With the assumption* $\left|\mathrm{Prep}_{\varphi}(X)\right| = \infty$*, we cannot have simultaneously* $\mathcal{J}(f) = \partial\mathbb{D}$ *and* $\mathcal{J}(g) = \mathbb{I}$.

Proof. Let $H(z,w) = a_0(z)\,w^m + a_1(z)\,w^{m-1} + \cdots + a_m(z) = 0$ be the affine equation of X and let us consider the map

$$\psi(z) = \prod_{p_1(P)=z} p_2(P)^{e_{p_2}(P)},$$

where $e_{p_2}(P)$ is the ramification index at P. It is a well-defined holomorphic self-map of $\mathbb{P}^1$ and coincides with the rational function $R(z) = \frac{a_m(z)}{a_0(z)}$ up to sign. By the equivalence in equation (2.1) above, each factor $p_2(P)$ of the product $\psi(z)$ belongs to the interval $\mathbb{I} = [-1,1]$ when $z \in \partial\mathbb{D}$. Since $\mathbb{I} = [-1,1]$ is stable by multiplication, ψ sends the entire unit circle into $\mathbb{I}$. As the unit circle is connected and compact, the image must be a closed interval in $\mathbb{I}$. Next we will show that a rational map cannot do that. □

Lemma 2.4. *If a rational function* h *maps the unit circle* $\partial\mathbb{D}$ *into a circle* C *of* $\mathbb{P}^1$*, then its image is the full circle* C.

Proof. Let $\sigma \in PSL(2,\mathbb{C})$ which maps the circle C onto $\partial\mathbb{D}$. Then $h_0 = \sigma \circ h$ maps the unit circle into the unit circle and is still rational. As the image must contain an open segment of the circle, there is some integer p such that the rational function $h_1 = h_0^p$ sends the unit circle onto the unit circle. As such, h_1 must be a quotient of two (finite) Blaschke products. However, it is easy to see that if some power of a rational function is a quotient of two Blaschke products, then it is itself such a quotient. We can write the quotient h_1 of Blaschke products as:

$$h_1(z) = u_0\, z^{n_1} \prod_{i=1}^{i=n_2} \Big(\frac{z-a_i}{1-\bar{a}_i z}\Big)^{m_i}\ ,\ a_i \in \mathbb{D},\ n_1 \in \mathbb{Z},\ m_i \in \mathbb{Z}\setminus\{0\},\ |u_0| = 1.$$

Since h_0 has the same roots and poles as h_0, in order to show that h_0 is a quotient of Blaschke products it is enough to show that the algebraic multiplicity in h_0 of a_i and $1/\bar{a}_i$ adds up to 0. But this is obvious as this sum is p times the sum they make in h_0. The constant in h_0 must be of modulus one since its power is, and z appears with integral power in h_0 since it is a rational function.

Now, since h_0 is such a quotient, it maps the unit circle onto the unit circle, and since σ is a homeomorphism of $\mathbb{P}^1$, that means h had to map $\partial\mathbb{D}$ onto C as well. □

This then concludes the proof of the previous lemma, as well as that of the theorem. □

We now use this theorem to give concrete examples. Let a_i, $i = 1,\ldots,n$ be algebraic numbers (0 and repetition possible), $|a_i| < 1$ and $u \in \overline{\mathbb{Q}}$, $|u| = 1$. Let $b_1, b_2, \ldots, b_6$ be six algebraic numbers no four of which lie on a circle or a line and let $g_1, g_2, \ldots, g_6$ be arbitrary rational functions with no zeros or poles at the b_i's. Also define $p_i(w) = \prod_{j\neq i}(z-b_j)$, $i = 1,\ldots,6$. Put:

$$f_0(z) = u \prod_{i=1}^{i=n} \frac{z-a_i}{1-\bar{a}_i z} \quad \text{and} \quad g(w) = \sum_{i=1}^{i=6} \frac{b_i\, p_i(w)}{p_i(b_i)\, g_i(b_i)} g_i(w).$$

We then have the:

Proposition 2.5. *For any choice of parameters a_i, u, b_i and rational functions g_i as above, and $\sigma \in PSL(2,\mathbb{C})$, let $f = f_0^\sigma$ (conjugate), with f_0 and g as above and put $\varphi = f \times g$. Then, for any curve X in $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Prep}_\varphi(X)$ is finite.*

Proof. Since f_0 is a Blaschke product (by construction) its Julia set is the unit circle. Therefore the Julia set of f is some circle in $\mathbb{P}^1$. If $\mathrm{Prep}_\varphi(X)$ were to be infinite, then by the theorem (Thm. 2.2), it would force the Julia set of g to be a circle in $\mathbb{P}^1$. We conclude with a lemma:

Lemma 2.6. *The rational function g as defined above cannot have a circle for a Julia set for any choice of the parameters a_i, u, b_i, g_i.*

Proof. Suppose $\mathcal{J}(g)$ is a circle. Then the two complementary simply connected components of $\mathcal{F}(g) = \mathbb{P}^1 \setminus \mathcal{J}(g)$ cannot host three distinct periodic points of g, as then two of them would have to be in the same component. However, we constructed g so that the six points $b_1, b_2, \ldots, b_6$ are fixed points of

$$g(w) = \sum_{i=1}^{i=6} \frac{b_i\, p_i(w)}{p_i(b_i)\, g_i(b_i)} g_i(w).$$

Indeed, the p_i's vanish for every b_j, $j \neq i$. Therefore,

$$g(b_i) = \frac{b_i\, p_i(b_i)}{p_i(b_i)\, g_i(b_i)} g_i(b_i) = b_i.$$

As no three fixed points of g may lie in the Fatou set of g as we have seen, four out of the six b_i's have to lie on the circle $\mathcal{J}(g)$, but that would contradict the initial assumption on the points b_i that this not be so. Therefore, $\mathcal{J}(g)$ cannot be a circle of $\mathbb{P}^1$. □

By the lemma we would reach a contradiction if $\mathrm{Prep}_\varphi(X)$ were infinite, so $\mathrm{Prep}_\varphi(X)$ must be finite. □

Generally, as a consequence of the theorem we have the criterion:

Let us say that a rational function is of type T_i if its Julia set is a set of the following kind T_i, where:
$T_1 = \mathbb{P}^1$; $T_2 =$ a circle in $\mathbb{P}^1$; $T_3 =$ a segment of a circle in $\mathbb{P}^1$; $T_4 =$ any other subset of $\mathbb{P}^1$. With this we have:

Corollary 2.7. *For any curve X in $\mathbb{P}^1 \times \mathbb{P}^1$, and any $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, defined over $\overline{\mathbb{Q}}$, with f and g rational functions of different types we have: $\mathrm{Prep}_\varphi(X)$ is finite.*

Proof. A direct consequence of the theorem. □

2.3. **Connectedness of the Julia sets as a criterion.** We will show in this section that if $\mathrm{Prep}_\varphi(X)$ is to be infinite, then the Julia sets of f and g are simultaneously connected or disconnected.

Proposition 2.8. *Let $\mathrm{Prep}_\varphi(X)$ be infinite, and $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ defined over $\overline{\mathbb{Q}}$. Assume $\mathrm{Prep}_\varphi(X)$ is infinite. Then, $\mathcal{J}(f)$ is connected iff $\mathcal{J}(g)$ is connected.*

Proof. By symmetry, it is clearly enough to show one implication. Suppose $\mathcal{J}(f)$ is connected. Since $\mathrm{Prep}_{\varphi}(X)$ is infinite, we have the equivalence of Theorem 1.8 II(d), which we write as follows:

$$p_1^{-1}\big(\mathcal{J}(f)\big) = p_2^{-1}\big(\mathcal{J}(g)\big).$$

As p_1 is finite, the number of connected components of $p_1^{-1}\big(\mathcal{J}(f)\big)$ is finite too. However, the equality above implies that the image of these by p_2 is all of $\mathcal{J}(g)$. Since each connected piece of $p_1^{-1}\big(\mathcal{J}(f)\big)$ goes by p_2 onto a connected piece of $\mathcal{J}(g)$, after mapping them by p_2 their number may only decrease and, in any case, it is finite. Therefore, $\mathcal{J}(g)$ has finitely many connected components. Therefore $\mathcal{J}(g)$ is connected, since it has only finitely many components (see [29, Ch. 2, §5, Thm. 3]). □

This provides a simple finiteness criterion:

Corollary 2.9. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, defined over $\overline{\mathbb{Q}}$. If only one of $\mathcal{J}(f)$, $\mathcal{J}(g)$ is connected, then $\mathrm{Prep}_{\varphi}(X)$ is finite.*

Proof. A direct consequence of the proposition. □

2.4. **Finiteness criteria for $\mathrm{Prep}_{\varphi}(X)$ based on a "saturation" property.**

Proposition 2.10. *Let the curve X in $\mathbb{P}^1 \times \mathbb{P}^1$ be determined by an irreducible polynomial in $\mathbb{C}[Z, W]$ given by*

$$H(Z, W) = P_1(Z)\, Q_2(W) - P_2(Z)\, Q_1(W),$$

where $R = \frac{P_1}{P_2}$ and $S = \frac{Q_1}{Q_2}$ are fractions in reduced form. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ defined over $\overline{\mathbb{Q}}$. Assume $\mathrm{Prep}_{\varphi}(X)$ is infinite. Then, we have the following:

(1) $R\big(\mathrm{Prep}_f(\mathbb{P}^1)\big) = S\big(\mathrm{Prep}_g(\mathbb{P}^1)\big)$.
(2) $R\big(\mathcal{J}(f)\big) = S\big(\mathcal{J}(g)\big)$.
(3) $R^{-1}\Big(R\big(\mathrm{Prep}_f(\mathbb{P}^1)\big)\Big) = \mathrm{Prep}_f(\mathbb{P}^1)$ *and* $S^{-1}\Big(S\big(\mathrm{Prep}_g(\mathbb{P}^1)\big)\Big) = \mathrm{Prep}_g(\mathbb{P}^1)$.
(4) $R^{-1}\Big(R\big(\mathcal{J}(f)\big)\Big) = \mathcal{J}(f)$ *and* $S^{-1}\Big(S\big(\mathcal{J}(g)\big)\Big) = \mathcal{J}(g)$.

Proof. This is a consequence of the two equivalences we get from the assumption that $\mathrm{Prep}_{\varphi}(X)$ is infinite from Theorem 1.8, namely:

$$\forall (z, w) \in X(\mathbb{C}), \qquad z \in \mathrm{Prep}_f(\mathbb{P}^1) \iff w \in \mathrm{Prep}_g(\mathbb{P}^1), \tag{2.2}$$

$$\forall (z, w) \in X(\mathbb{C}), \qquad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g). \tag{2.3}$$

To prove (1) and (2), let (T_1, T_2) be either $(\mathrm{Prep}_f(\mathbb{P}^1), \mathrm{Prep}_g(\mathbb{P}^1))$ or $(\mathcal{J}(f), \mathcal{J}(g))$. Let $z_0 \in T_1$ and $x_0 = R(z_0)$. We then solve $R(z_0) = S(w)$ for w in $\mathbb{P}^1$. Let w_0 be a solution. Then, $(z_0, w_0) \in X$ and $z_0 \in T_1$ so, by the equivalence (whichever is appropriate), we must have $w_0 \in T_2$ and thus $x_0 = S(w_0) \in S(T_2)$. To prove (3) or (4) it is enough to do it either for R or for S, by symmetry. Let $z_1 \in R^{-1}(R(T_1))$. Then, there is some $z_0 \in T_1$ with $R(z_1) = R(z_0)$. By solving $R(z_0) = S(w)$ for w in $\mathbb{P}^1$ we can find some w_0 with $(z_0, w_0) \in X$ and, by the equivalence, $w_0 \in T_2$. Since $R(z_1) = R(z_0) = S(w_0)$, the point $(z_1, w_0) \in X$ as well. We apply the equivalence once more to conclude that $z_1 \in T_1$. This shows that $R^{-1}(R(T_1)) \subset T_1$; the other inclusion is trivial. □

Corollary 2.11. *Let the curve X in $\mathbb{P}^1 \times \mathbb{P}^1$ be determined by an irreducible polynomial in $\mathbb{C}[Z, W]$ given by*

$$H(Z, W) = P_1(Z)\, Q_2(W) - P_2(Z)\, Q_1(W),$$

where $R = \frac{P_1}{P_2}$ and $S = \frac{Q_1}{Q_2}$ are fractions in reduced form. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ defined over $\overline{\mathbb{Q}}$. Suppose:

$$\Big(R(\mathcal{J}(f)) \setminus S(\mathcal{J}(g))\Big) \cup \Big(S(\mathcal{J}(g)) \setminus R(\mathcal{J}(f))\Big) \neq \emptyset.$$

Then, $\mathrm{Prep}_{\varphi}(X)$ *is finite.*

Proof. This is a direct consequence of the theorem, as the symmetric difference of the sets can only be nonempty if we do not have the equality (2) of the theorem. □

Corollary 2.12. *With the same data (curve and morphism) as in the corollary above, suppose given two points a, b in $\mathbb{C}$ such that $a \in \mathcal{J}(f)$ and $b \notin \mathcal{J}(f)$ with $R(a) = R(b) = 0$. Then,* $\mathrm{Prep}_{\varphi}(X)$ *is finite.*

Proof.

$$b \in R^{-1}\big(R(a)\big) \subset R^{-1}\big(R(\mathcal{J}(f))\big)$$

as $a \in \mathcal{J}(f)$, but $b \notin \mathcal{J}(f)$ contradicts assertion (4) of the theorem. □

2.5. **Generalization of the Ihara-Serre-Tate theorem for special classes of curves.** The Ihara-Serre-Tate theorem assumes the curve has an infinite intersection with the set $\boldsymbol{\mu}^* \times \boldsymbol{\mu}^*$ (where $\boldsymbol{\mu}$ is the set of all roots of unity and $\boldsymbol{\mu}^* = \boldsymbol{\mu} \cup \{0, \infty\}$). In our context this amounts to assuming that it has an infinite intersection with $\mathrm{Prep}_f(\mathbb{P}^1) \times \mathrm{Prep}_g(\mathbb{P}^1)$, for some $\varphi = f \times g$ which is such that $\mathrm{Prep}_f(\mathbb{P}^1) = \mathrm{Prep}_g(\mathbb{P}^1) = \boldsymbol{\mu}^*$. Obvious candidates for φ are the maps $\varphi_{n,m}(z, w) = (z^n, w^m)$ with $n, m \geq 2$. Note in particular that for a map f with $\mathrm{Prep}_f(\mathbb{P}^1) = \boldsymbol{\mu}^*$ we have $\mathcal{J}(f) = \partial\mathbb{D}$.

We can ask whether the following extension of the Ihara-Serre-Tate theorem is true: If f and g are two rational maps of degree at least two, with Julia set equal to the unit circle $\partial\mathbb{D}$, and if the curve has an infinite intersection with $\mathrm{Prep}_f(\mathbb{P}^1) \times \mathrm{Prep}_g(\mathbb{P}^1)$ can we conclude: first, to the rationality of the curve and/or to its being very special and second, to a specific type of equation for the curve?
The assumption on the preperiodic points:

$$\mathrm{Prep}_f(\mathbb{P}^1) = \mathrm{Prep}_g(\mathbb{P}^1) = \boldsymbol{\mu}^*$$

is only slightly changed from that of the I-S-T theorem to become:

$$\mathcal{J}(f) = \partial\mathbb{D} \quad \text{and} \quad \mathcal{J}(g) = \partial\mathbb{D}$$

so that the hypothesis on the preperiodic points is that they accumulate on the unit circle (no roots of unity are required).

2.5.1. *Curves determined by the equality of two rational functions* $\mathbf{R}(\mathbf{z}) = \mathbf{S}(\mathbf{w})$. We shall now prove a similar theorem for the class of curves, given by an irreducible polynomial of $\mathbb{C}[z, w]$ of the form: $H(z, w) = P_1(z)\, Q_2(w) - P_2(z)\, Q_1(w)$, where $R = \frac{P_1}{P_2}$ and $S = \frac{Q_1}{Q_2}$ are rational functions in reduced form. We assume that we are given a pair of rational functions f and g defined over some number field, and whose Julia sets are the unit circle.

Theorem 2.13. *Let the polynomial*

$$H(z,w) = P_1(z)\,Q_2(w) - P_2(z)\,Q_1(w)$$

be irreducible in $\mathbb{C}[z,w]$*, where either* $\deg(P_1) \neq \deg(P_2)$ *or* $\deg(Q_1) \neq \deg(Q_2)$*. Let* f *and* g *be rational functions with coefficients in* $\overline{\mathbb{Q}}$*, both of which have* $\partial\mathbb{D}$ *as their Julia set. If there exist infinitely many pairs* (z,w) *for which the polynomial* H *vanishes, where* z *and* w *are subject to being preperiodic points of* f *and* g *respectively, then* $R = \frac{P_1}{P_2}$ *and* $S = \frac{Q_1}{Q_2}$ *are quotients of (finite) Blaschke products.*

Proof. Putting $\varphi = f \times g$, we then have $\deg_m(\varphi) \geq 2$, since we assume both f and g have the $\partial\mathbb{D}$ as their Julia set. We now consider the projective curve X determined by H^* in $\mathbb{P}^1 \times \mathbb{P}^1$. The assumption on the curve is that it has an infinite intersection with $\mathrm{Prep}_f(\mathbb{P}^1) \times \mathrm{Prep}_g(\mathbb{P}^1) = \mathrm{Prep}_\varphi(\mathbb{P}^1 \times \mathbb{P}^1)$. That is, $\mathrm{Prep}_\varphi(X)$ is infinite. We now are in a position to apply Theorem 1.8 II(d) to the pair (X,φ). This gives us the equivalence:

$$\forall (z,w) \in X(\mathbb{C}), \quad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g). \tag{2.4}$$

We may assume by renaming the P_i's and Q_i's that $\deg(P_1) > \deg(P_2)$. We may also assume by rescaling H by a constant factor, if necessary, that P_1 is monic. If we write

$$H(z,w) = c_0(w)\,z^n + c_1(w)\,z^{n-1} + \cdots + c_n(w),$$

by comparing it to

$$H(z,w) = P_1(z)\,Q_2(w) - P_2(z)\,Q_1(w),$$

we can identify in particular the rational functions $\frac{c_n(w)}{c_0(w)}$ and $\frac{Q_1(w)}{Q_2(w)} = S(w)$, in view of the assumption $\deg(P_1) > \deg(P_2)$ and since P_1 is monic. The map

$$\psi(w) = \prod_{p_2(P)=w} p_1(P)^{e_{p_1}(P)}$$

is a well-defined holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$ ("norm"), where $e_{p_1}(P)$ is the ramification index at P. It coincides up to sign with the rational function $\frac{c_n(w)}{c_0(w)}$ and therefore (up to sign) with $S(w) = \frac{Q_1(w)}{Q_2(w)}$. By the equivalence (2.4) above between Julia sets, both of which are the unit circle, $\psi(w)$ is now a rational function sending the unit circle onto itself (each root z of $H(z,w) = 0$ once w is fixed $w \in \partial\mathbb{D}$ is also in $\partial\mathbb{D}$). A rational map sending the unit circle to itself is a quotient of (finite) Blaschke products (we can see that, by dividing ψ by a factor of the form $\frac{z-a_i}{1-\overline{a_i}z}$, for each zero or pole a_i in $\mathbb{D} \setminus \{0\}$ and by some z^n, the resulting function, say η, as well as its inverse $\frac{1}{\eta}$, are holomorphic in $\mathbb{D}$ and still send $\partial\mathbb{D}$ to itself, we can finish with the maximun modulus principle). From Proposition 2.10(4) with $\mathcal{J}(f) = \mathcal{J}(g) = \partial\mathbb{D}$ we have $R(\partial\mathbb{D}) = S(\partial\mathbb{D})$ and, because S is now a quotient of Blaschke products, $S(\partial\mathbb{D}) = \partial\mathbb{D}$. Therefore, $R(\partial\mathbb{D}) = \partial\mathbb{D}$ and R is such a quotient as well. □

Corollary 2.14. *Suppose* H *is an ireducible polynomial in* $\mathbb{C}[z,w]$ *with*

$$H(z,w) = F(z) - G(w), \text{ where } F \text{ and } G \text{ are two polynomials.}$$

Let f *and* g *be rational functions with coefficients in* $\overline{\mathbb{Q}}$ *which both have* $\partial\mathbb{D}$ *as their Julia set. If there exist infinitely many pairs* (z,w) *for which the polynomial* H *vanishes, where* z *and* w *are subject to being preperiodic points of* f *and* g *respectively,*

then:

$$H(Z,W) = a\,Z^m + b\,W^n, \qquad |\frac{b}{a}| = 1.$$

Proof. From the theorem, the premise of which is satisfied since the denominators have degree 0, we know that $F(z)$ and $G(w)$ are quotients of Blaschke products. The only way such a quotient can be a polynomial is if it is of the form $u_0 z^n$ with $|u_0| = 1$. We deduce from this that F and G are z^n and w^m (where n and m are the degrees of F and G) up to a constant factor of modulus one. We then get the conclusion of the theorem. □

2.5.2. *Curves satisfying some topological criterion.* We now prove another finiteness result, where we assume something about the branch locus of the branched covers of $\mathbb{P}^1$ given by the projections: $p_i \colon X \to \mathbb{P}^1$.

Theorem 2.15. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ be defined over $\overline{\mathbb{Q}}$ and such that $\mathcal{J}(f) = \mathcal{J}(g) = \partial\mathbb{D}$.*
Let

$$\mathcal{S} = p_1^{-1}(\mathbb{P}^1 \setminus \partial\mathbb{D}) \cup p_2^{-1}(\mathbb{P}^1 \setminus \partial\mathbb{D}).$$

Suppose at least one connected component of $\mathcal{S}$ is simply connected. Then we have: If $\mathrm{Prep}_\varphi(X)$ is infinite, then X is rational.

Proof. Let U_0 denote a simply connected component of $\mathcal{S}$. Then let $\mathrm{Prep}_\varphi(X)$ be infinite. Then we have $\forall (z,w) \in X(\mathbb{C})$, $z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g)$ by Theorem 1.8 II(d). This is the same thing as saying (by taking complements in $\mathbb{P}^1$ and lifting):

$$p_1^{-1}(\mathbb{P}^1 \setminus \partial\mathbb{D}) = p_2^{-1}(\mathbb{P}^1 \setminus \partial\mathbb{D}).$$

That is, the components above $\mathbb{D}$ or $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$ are the same when we take inverse images by either p_1 or p_2. Therefore, each component of the set $\mathcal{S}$ is mapped by p_1 and p_2 to either $\mathbb{D}$ or $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$. In particular the simply connected component U_0 maps to either $\mathbb{D}$ or $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$ by both p_1 and p_2. By replacing if necessary p_i by $\tilde{p_i} = \sigma_0 \circ p_i$, where $\sigma_0(z) = \frac{1}{z}$, we may assume it is $\mathbb{D}$ in each case. Let us keep track of it by saying $p_i = \tilde{p_i}^{\epsilon_i}$ with $\epsilon_i = \pm 1$ (according as to which is the case). Since U_0 is simply connected, we can find a conformal map $\theta \colon U_0 \to \mathbb{D}$. We note that, by the choice of U_0, each restriction $p_i\big|_{U_0} \colon U_0 \to p_i(U_0)$ is a finite covering, say n_i to 1. Then, so is each $\tilde{p_i}$ from U_0 onto $\mathbb{D}$ since σ_0 is conformal. We now consider the compositions $\tilde{p_i} \circ \theta^{-1}$. They define exactly n_i to 1 holomorphic self-maps of $\mathbb{D}$ for some positive integers n_i. By a theorem of Fatou, which we quote next, they must be (finite) Blaschke products.

Theorem 2.16 (Fatou)**.** *A holomorphic m to 1 map of the unit disc onto itself is rational (in particular it must be a finite Blaschke product).*

Proof. The proof can be found in [31, Satz 1]. □

Let us denote by $B_i(z)$ the two Blaschke products. Let $H(z,w) = 0$ be the (affine) equation of X. Since U_0 is an open set in X we have by definition:

$$\forall\, Q \in U_0, \quad H\big(p_1(Q), p_2(Q)\big) = 0.$$

In terms of $\tilde{p_i}$ this reads:

$$\forall\, Q \in U_0, \quad H\big(\tilde{p_1}(Q)^{\epsilon_1}, \tilde{p_2}(Q)^{\epsilon_2}\big) = 0.$$

But $\theta^{-1}\colon \mathbb{D} \to U_0$ is a homeomorphism, so we may replace Q by $\theta^{-1}(z)$ with $z \in \mathbb{D}$. The equation now reads:

$$\forall z \in \mathbb{D}, \quad H\Big(\tilde{p_1}(\theta^{-1}(z))^{\epsilon_1}, \tilde{p_2}(\theta^{-1}(z))^{\epsilon_2}\Big) = 0.$$

Since we established above that $\tilde{p_i} \circ \theta^{-1}(z) = B_i(z)$ (rational functions), we get finally:

$$\forall z \in \mathbb{D}, \quad H\big(B_1(z)^{\epsilon_1}, B_2(z)^{\epsilon_2}\big) = 0.$$

We conclude therefore that the curve X is rational. □

It would also follow from the theorem that H must be of some special type. We will not expand on this here.

Corollary 2.17. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ be defined over $\overline{\mathbb{Q}}$ and such that $\mathcal{J}(f) = \mathcal{J}(g) = \partial\mathbb{D}$. Assume that the branch locus of either projection p_1 or p_2, is entirely in $\mathbb{P}^1 \setminus \mathbb{D}$ or, entirely in $\overline{\mathbb{D}}$. Then we have: If $|\mathrm{Prep}_{\varphi}(X)| = \infty$, then X is a rational curve.*

Proof. Let us say that $T \in \{\mathbb{D},\ \mathbb{P}^1 \setminus \overline{\mathbb{D}}\}$ is such that p_i (i can be either one) is unramified above T. Then, any component of $p_i^{-1}(T)$ along with the restriction of p_i to it, will give rise to an unramified finite cover of the simply connected open set T. As such, the component in question will itself be simply connected. We conclude now by the theorem. □

3. Finiteness criteria for $\mathrm{Prep}_{\varphi}(X)$ based on the symmetry groups of Julia sets

3.1. **Introduction.** G. Julia showed in [20] that if f and g are two rational functions that commute, then they have the same Julia set. One can then ask the question as to how many rational functions of a given degree share the same Julia set. It turns out that this question is intimately connected with the symmetries that the Julia set may possess, and in most cases the answer is: finitely many. We will discuss some results of G. M. Levin on symmetries of Julia sets and use them to derive some criteria for the finiteness of $\mathrm{Prep}_{\varphi}(X)$.

Symmetries consist on the one hand of *local symmetries*, coming for example from other rational functions sharing the same Julia set as a given one. On the other hand, there are symmetries consisting of subgroups of $PSL(2,\mathbb{C})$ leaving the Julia set invariant by their action. We will consider both cases and use them to prove finiteness results for preperiodic points on curves.

3.2. Symmetries of Julia sets of rational functions.

3.2.1. *Linear-fractional symmetries of Julia sets.*

Definition 3.1. Let h be a rational function, $\deg(h) \geq 2$, and let $\mathcal{J}(h)$ be its Julia set. Let us define the linear-fractional symmetry group of the function h to be

$$\Sigma_h = \{\sigma \in PSL(2,\mathbb{C}) \mid \forall z \in \mathbb{P}^1,\ h(\sigma(z)) = h(z)\}.$$

For any subset T of $\mathbb{P}^1$, let us define the stabilizer group

$$\Sigma(T) = \{\sigma \in PSL(2,\mathbb{C}) \mid \sigma(T) = T\}.$$

In particular, the group of linear-fractional symmetries of the Julia set of h will be denoted: $\Sigma\big(\mathcal{J}(h)\big)$.

Remark 3.2. Let h be a rational function, $\deg(h) \geq 2$ and $\sigma \in PSL(2, \mathbb{C})$, such that: $h \circ \sigma = h$. Then the Julia set of h is invariant by σ, that is,

$$\Sigma_h \subset \Sigma\big(\mathcal{J}(h)\big)$$

by the complete invariance characterization of the Julia set (see [2, Thm. 4.2.2]).

3.2.2. *Local symmetries.* The notion of local symmetry, which we recall next, will be mentioned only to the extent needed for the intelligence of Levin's results, as well as their applicability to our problem. For more details, see [22]. Let h be a rational function of $\deg(h) \geq 2$, and let μ_h be the unique h-invariant measure of maximal entropy on $\mathcal{J}(h)$ (for the existence of such a measure see: [24] or, [14, Thm., p. 46] and [25, Thm., p. 27]).

Definition 3.3. Let $\varphi \neq Id$ be holomorphic in a ball $B(a, r)$, with center in the Julia set $\mathcal{J}(h)$, and radius $r > 0$. We say that φ is a *local symmetry* on $\mathcal{J}(h)$ iff the following invariance property holds:

$$x \in B(a, r) \cap \mathcal{J}(h) \iff \varphi(x) \in \varphi\big(B(a, r)\big) \cap \mathcal{J}(h). \tag{3.1}$$

For exceptional cases of $\mathcal{J}(h)$: there exists $\alpha > 0$ such that

$$\mu_h\big(\varphi(T)\big) = \alpha \cdot \mu_h(T),$$

for every Borel set $T \subset B(a, r)$ for which the restriction $\varphi\big|_T$ is injective.

Definition 3.4. A set of functions $\mathcal{H}$ holomorphic in a ball $B(a, r)$, $a \in \mathcal{J}(h)$, is said to be a *nontrivial family of symmetries* on $\mathcal{J}(h)$ iff:
(1) every function φ of $\mathcal{H}$ is a local symmetry;
(2) $\mathcal{H}$ is a normal family on $B(a, r)$ and for every $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{H}$ with $\varphi_n \to \varphi$, φ is not a constant.

3.2.3. *Orbifolds.* Orbifolds were introduced by Thurston [33] and treated extensively in [9]. We will recall the definition and properties of orbifolds of a particular kind, namely those naturally associated to a *critically finite* rational function h (in other words, h such that the post-critical set $\mathcal{P}(h) = \overline{\bigcup_{n>0} h^n(\mathcal{C}(h))}$ is finite).

Let h be a critically finite rational map, with finite post-critical set $\mathcal{P}(h)$. The *orbifold* $\mathcal{O}_h$ of h will be the Riemann sphere $\mathbb{P}^1$ together with the set of singular points $\mathcal{P}(h)$, and with a function $n\colon \mathbb{P}^1 \to \mathbb{N} \cup \{\infty\}$ such that $n(z) = 1$ for $z \notin \mathcal{P}(h)$ and ≥ 2 otherwise. The set of numbers $\{n(z) \mid z \in \mathcal{P}(h)\}$ is further determined, so that

$$n(z) = \operatorname{lcm}\{n(t)\, e_h(t) \mid t \in h^{-1}(z)\},$$

where $e_h(t)$ is the ramification index of h at t (and lcm is the lowest common multiple). This insures that $h\colon \mathcal{O}_h \to \mathcal{O}_h$ is a covering of orbifolds. That is,

$$h\colon \mathbb{P}^1 \setminus \{z \mid n(z) = \infty\} \to \mathbb{P}^1 \setminus \{z \mid n(z) = \infty\}$$

is a holomorphic branched cover with $n(h(z)) = e_h(z)\, n(z)$. One can also define the Euler characteristic of such an orbifold,

$$\chi(\mathcal{O}_h) = 2 - \sum_{z\in\mathcal{P}(h)} \Big(1 - \frac{1}{n(z)}\Big).$$

The Riemann-Hurwitz formula holds: $\chi(\mathcal{O}_h) = \deg(h)\, \chi(\mathcal{O}_h)$, and since h is of degree at least two, $\chi(\mathcal{O}_h) = 0$. Thus, we have the name parabolic (or Euclidean)

orbifold. It is clear that $\chi(\mathcal{O}_h) = 0$ puts a constraint on such possible functions:

$$n\colon \mathbb{P}^1 \to \mathbb{N} \cup \{\infty\}.$$

There are indeed only six types of such orbifolds. For a list which we need not reproduce, and a justification of the following remark, we refer to [9, 12, 22].

Remark 3.5. This classification of such orbifolds reveals that the Julia sets of rational maps which are critically finite with parabolic orbifold can only be (up to conformal equivalence): $\mathbb{P}^1$, $\partial\mathbb{D}$, or $\mathbb{I} = [-1, 1]$.

3.2.4. *Results of G. M. Levin.* G. M. Levin [22], pursuing some work of A. E. Eremenko [12], has shown that outside the exceptional cases, the group of linear-fractional symmetries of the Julia set of a rational function is finite. He has also shown the finiteness for any fixed degree, in most cases, of the number of rational functions of that degree sharing their Julia set. His results apply to the cases of *exceptional* Julia sets, provided we now require that the maps share their unique measure of maximal entropy (this is stronger since this measure has the Julia set as its support). We now devote the next section to describing some of these results, and derive some useful consequences.

Let f be a rational function. Let us define the set $R_d(f)$ as follows:

$$R_d(f) = \begin{cases} \{h \in \mathbb{C}(x) \mid \deg(h) = d \text{ and } \mathcal{J}(h) = \mathcal{J}(f)\} \\ \quad \text{if f } \textit{is not} \text{ critically finite with parabolic orbifold;} \\ \\ \{h \in \mathbb{C}(x) \mid \deg(h) = d \text{ and } \mu_h = \mu_f\} \\ \quad \text{if f } \textit{is} \text{ critically finite with parabolic orbifold.} \end{cases}$$

We have the following theorem of Levin:

Theorem 3.6 (Levin)**.** *The function f is critically finite with parabolic orbifold iff there exists an infinite nontrivial family of symmetries on $\mathcal{J}(f)$.*

Proof. See [22, Thm. 1]. □

Note that in the theorem above, the notion of an infinite nontrivial family of symmetries requires the limit of any convergent sequence of the family to be non-constant by definition. The point of this theorem is that each new member h of $R_d(f)$ adds another local symmetry because of the invariance $h(\mathcal{J}(f)) = \mathcal{J}(f)$. When infinitely many of them exist, we fall into the case of an f which is *critically finite with parabolic orbifold* and, therefore, an exceptional rational function. We now examine another result of Levin, which elucidates, in particular, the relationship of two rational functions with a common preperiodic point.

Theorem 3.7 (Levin)**.** *Suppose $g \in R(f) = \bigcup_{d\geq 2} R_d(f)$ and one of the following conditions is satisfied:*
1) There is a point a that is preperiodic for f and periodic and repulsive for g.
2) The limit set P'_f of the iterates of the critical points of f is finite and contains no neutral irrational periodic points of f.
Then, either f and g are critically finite and have a common parabolic orbifold, or for some positive integers l, k we have

$$f^l \circ g^k = f^{2l}.$$

Proof. See [22, Thm. 3] as well as [23]. □

3.3. **Consequences of the finiteness of symmetries.**

3.3.1. *A basic finiteness theorem.* We recall that a rational map f is called *exceptional* if its Julia set is mapped by $\mathrm{Aut}(\mathbb{P}^1)$ to a set in

$$\{\mathbb{P}^1,\ \partial\mathbb{D}\ (\textit{unit circle}),\ \mathbb{I} = [-1, 1]\}.$$

Proposition 3.8. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, N an integer and f a nonexceptional rational map defined over $\overline{\mathbb{Q}}$. Then, $\mathrm{Prep}_{\varphi}(X)$ is finite, for all but finitely many of the rational maps $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ and $\deg(\varphi) \leq N$, as g varies in $\overline{\mathbb{Q}}(X)$, subject only to the degree constraints.*

Proof. Let us denote by A, the set of all rational functions g defined over $\overline{\mathbb{Q}}$, of degree bounded by N, such that $\mathrm{Prep}_{\varphi}(X)$ is infinite with $\varphi = f \times g$, where f is nonexceptional. Let us also put $A_d = \{g \in A,\ \deg(g) = d\}$. To prove the proposition, in view of the fact that

$$A = \bigcup_{d \leq N} A_d,$$

a finite union, it is enough to show that A_d is finite for any degree $d \geq 2$. We first need the following lemma:

Lemma 3.9. *All rational functions $g \in A$ share the same invariant measure of maximal entropy.*

Proof. Given a rational function $g \in A$ by Theorem 1.8, we get the equivalences:

$$\forall P = (z, w) \in X(\mathbb{C}), \qquad z \in \mathrm{Prep}_f(\mathbb{P}^1) \iff w \in \mathrm{Prep}_g(\mathbb{P}^1), \tag{3.2}$$

$$\forall P = (z, w) \in X(\mathbb{C}), \qquad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g). \tag{3.3}$$

This means that as sets we have:

$$p_1^{-1}\big(\mathrm{Prep}_f(\mathbb{P}^1)\big) = p_2^{-1}\big(\mathrm{Prep}_g(\mathbb{P}^1)\big) \quad \text{and} \quad p_1^{-1}\big(\mathcal{J}(f)\big) = p_2^{-1}\big(\mathcal{J}(g)\big).$$

We deduce that as sets, we have in both instances (as p_2 is surjective):

$$p_2\big(p_1^{-1}(\mathrm{Prep}_f(\mathbb{P}^1)\big) = \mathrm{Prep}_g(\mathbb{P}^1) \quad \text{and} \quad p_2\big(p_1^{-1}(\mathcal{J}(f))\big) = \mathcal{J}(g).$$

That is to say, that the set of preperiodic points $\mathrm{Prep}_g(\mathbb{P}^1)$ of g (and respectively, the Julia set $\mathcal{J}(g)$ of g) is entirely determined by $\mathrm{Prep}_f(\mathbb{P}^1)$ and X alone (respectively $\mathcal{J}(f)$ and X alone) and this holds for any such g in A. Let us denote those common sets by $\mathrm{Prep}_A(\mathbb{P}^1)$ and $\mathcal{J}_A$.

Since $\mathcal{J}(f)$ (or f) is not exceptional by assumption, it follows from Theorem 2.2 that $\mathcal{J}_A$ (or g, for any $g \in A$) isn't either. All g in A have a common set of preperiodic points. From this, we will deduce that they also share their invariant measure of maximal entropy. Let us take any two $g_1,\ g_2 \in A$ and a point $z \in \mathrm{Prep}_A(\mathbb{P}^1)$, periodic and repulsive for g_1. Then z is preperiodic for g_2 since they share $\mathrm{Prep}_A(\mathbb{P}^1)$ and, by Theorem 3.7 (see [22, Thm. 3] and [23]), and because $\mathcal{J}_A$ is not exceptional (see Remark 3.5 above), we deduce that

$$g_1^l \circ g_2^k = g_1^{2l},$$

where l and k are two positive integers ($l = k$ if $\deg(g_1) = \deg(g_2)$, such as when both $g_1, g_2 \in A_d$).

This is enough to force any two $g_1,\ g_2 \in A$ to have the same invariant measure of maximal entropy $\mu_{g_1} = \mu_{g_2}$. More generally, since the choice of $g_1,\ g_2 \in A$ was

arbitrary, this insures that all $g \in A$ have the same common maximal invariant measure μ. $\square$

Let us now assume that for some $d \leq N$, the set A_d is infinite. We can then find a sequence g_n in A_d converging to some rational function $\tilde{g}$ everywhere outside of a finite set $\mathcal{C}$. This can be seen from the fact that rational functions of degree d have a fixed number of coefficients which vary in a fixed power of $\mathbb{P}^1$. There is therefore a convergent subsequence in A_d, which converges locally uniformly to a rational function $\tilde{g}$ outside of a finite set. In fact, as the following lemma will show by a normality argument applied to the family A_d, $\tilde{g}$ must be a constant.

We now prove another lemma to achieve the proof of the proposition.

Lemma 3.10. *With $(g_n)_{n\geq 1}$ an infinite sequence of A_d as above, converging to $\tilde{g}$ outside of a finite set of points $\mathcal{C}$, we have the following:*
(1) The limit function $\tilde{g} \equiv c$, a constant, with $c \in \mathcal{C} \subset \mathcal{J}_A$.
(2) Let $g_0 \in A_d$. There exists a positive integer k_0, such that for all $k \geq k_0$, $g_0^k(c)$ is not a critical point of g_0.

Proof. All the arguments for the proof of the lemma and the remainder of the proof of the proposition are basically contained in [23, §1]. We reproduce them here since they have been adapted to our needs somewhat.

We have $\mathcal{C} \subset \mathcal{J}_A$ because the family of functions of $\{g_n, n \geq 1\}$ must be normal outside of their common Julia set $\mathcal{J}_A$, and therefore convergent there. This means that $\mathcal{C}$ must lie in $\mathcal{J}_A$.

If $\tilde{g}$ were not a constant c, given the fact that all $g \in A_d$ are nonexceptional (see Remark 3.5), the convergent sequence of rational functions $(g_n)_{n\geq 1}$ forms an infinite nontrivial family of symmetries, contrary to [23, Theorem 1]. Therefore for some c, we have $\tilde{g} \equiv c$. The Julia set $\mathcal{J}_A$, being closed and invariant by all the $g \in A_d$, must have $c \in \mathcal{J}_A$. We also have $c \in \mathcal{C}$, because otherwise for large fixed n, the family of iterates of g_n would be normal at c and put c in the complement of $\mathcal{J}_A$ since the latter is common to all the g_n. This proves (1).

If (2) were false, then there would be a periodic orbit in $\mathcal{J}_A$ containing a critical point, which is impossible. $\square$

We now finish the proof of the proposition. Let $U \subset \mathcal{J}_A \setminus \mathcal{C}$ be any nonempty, small compact neighborhood of a point in $\mathcal{J}_A \setminus \mathcal{C}$. Set $U_n = g_n(U)$. We have $U_n \to \{c\}$. On the other hand, $\mu(U_n) \geq \mu(U) > 0$ for all n, as μ is common to all g_n. Pick an integer $k > k_0$ such that $d^{k-k_0}\mu(U) \geq 1$. Then, for this k, in view of Part (2) of the lemma just proved, and the fact that $U_n \to \{c\}$, there is an n_0 such that for all $n > n_0$, $g_0^{k-k_0}$ is injective on $g_0^{k_0}(U_n)$. This in turn insures that

$$\mu(g_0^k(U_n)) = \mu(g_0^{k-k_0}(g_0^{k_0}(U_n))) = d^{k-k_0}\mu(g_0^{k_0}(U_n)) \geq d^{k-k_0}\mu(U_n).$$

Therefore, we have

$$\forall n > n_0, \quad \mu(g_0^k(U_n)) \geq d^{k-k_0}\mu(U_n) \geq d^{k-k_0}\mu(U) \geq 1.$$

This is a contradiction, since $\mu(g_0^k(U_n)) \to 0$ as U_n shrinks to $\{c\}$, and therefore, A_d must be finite, and so must be A. $\square$

3.3.2. *The groups $\Sigma(\mathcal{J}(f))$.* Which groups do occur as $\Sigma(\mathcal{J}(f))$? If $\mathcal{J}(f) = \mathbb{P}^1$ or a circle, the question is trivial. We know they are finite in all cases of interest to us. A finite subgroup of $PSL(2, \mathbb{C})$ contains only elliptic elements (besides the

identity). It is then conjugate in $PSL(2,\mathbb{C})$ to a subgroup of $PSU(2,\mathbb{C})$ and as such, it must be conjugate to one of the following:
(1) A cyclic group.
(2) A dihedral group.
(3) The symmetry group of a regular tetrahedron, octahedron or icosahedron inscribed in the sphere S^2.

All these types occur. For example, rational functions whose Julia set has the symmetries of the icosahedron can be found in a paper of P. Doyle and C. McMullen [10, Prop. 5.3, p. 166].

3.4. **Some consequences derived from the group $\Sigma(\mathcal{J}(f))$.** We now make use of the transfer to the Julia sets of the component rational functions of φ, of linear-fractional symmetries the curve might have in each $\mathbb{P}^1$, which occur under the assumption of an infinite set $\mathrm{Prep}_{\varphi}(X)$.

Let us define a special kind of symmetry group of the curve. We consider the curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, and such that the set $\mathrm{Prep}_{\varphi}(X)$ is infinite. Given a pair of elements σ, $\tau \in PSL(2,\mathbb{C})$ we may define an automorphism $\omega = \sigma \times \tau$ of $\mathbb{P}^1 \times \mathbb{P}^1$, where σ and τ act on each component independently. In particular, suppose some nontrivial automorphism $\omega = \sigma \times Id$ of $\mathbb{P}^1 \times \mathbb{P}^1$ leaves the curve X invariant in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us denote by $\mathcal{A}_L(X)$ the subgroup of $PSL(2,\mathbb{C})$ such that $\mathcal{A}_L(X) \times 1$ is the following subgroup of $\mathrm{Aut}(X)$:

$$\mathcal{A}_L(X) = \{\sigma \in PSL(2,\mathbb{C}) \mid \omega = \sigma \times Id \in \mathrm{Aut}(X)\}.$$

The action of an element σ of $\mathcal{A}_L(X)$ on a point $P = (z,w) \in \mathbb{P}^1 \times \mathbb{P}^1$ may be written as: $P^{\sigma} = (\sigma(z), w)$.

We similarly define $\mathcal{A}_R(X)$ on the right.

Theorem 3.11. *Let (X,φ), $\deg_m(\varphi) \geq 2$, be given in $\mathbb{P}^1 \times \mathbb{P}^1$. Assume f or g is not exceptional. If $\mathcal{A}_L(X) \not\subset \Sigma(\mathcal{J}(f))$, then $\mathrm{Prep}_{\varphi}(X)$ is finite.*

Proof. Assume to the contrary that $\mathcal{A}_L(X) \not\subset \Sigma(\mathcal{J}(f))$ and that $\mathrm{Prep}_{\varphi}(X)$ is infinite. Since the latter set is infinite we first get by Theorem 1.8(b) the equivalence:

$$\forall P = (z,w) \in X(\mathbb{C}), \quad z \in \mathrm{Prep}_f(\mathbb{P}^1) \iff w \in \mathrm{Prep}_g(\mathbb{P}^1).$$

We now claim that the set $\mathrm{Prep}_f(\mathbb{P}^1)$ is invariant by the group $\mathcal{A}_L(X)$. That is:

$$\mathcal{A}_L(X) \subset \Sigma(\mathrm{Prep}_f(\mathbb{P}^1)).$$

To prove the claim, let $z \in \mathrm{Prep}_f(\mathbb{P}^1)$ and $\sigma_0 \in \mathcal{A}_L(X)$. We can find some $w \in \mathbb{P}^1$ such that $P = (z,w) \in X$ (p_1 is surjective). Now X is invariant by the action of $\mathcal{A}_L(X)$, so we have $P^{\sigma_0} = (\sigma_0(z), w) \in X$. But $z \in \mathrm{Prep}_f(\mathbb{P}^1)$, so by the equivalence just recalled above we must have, as $P = (z,w)$ is on the curve, that $w \in \mathrm{Prep}_g(\mathbb{P}^1)$. Since for the same w, $P^{\sigma_0} = (\sigma_0(z), w) \in X$ and $w \in \mathrm{Prep}_{\varphi}(\mathbb{P}^1)$, using the equivalence again we have $\sigma_0(z) \in \mathrm{Prep}_f(\mathbb{P}^1)$. So $\mathrm{Prep}_f(\mathbb{P}^1)$ is invariant by σ_0.

At this point, to be able to say anything about the Julia set of f we need the lemma:

Lemma 3.12. *Let $\sigma \in PSL(2,\mathbb{C})$ leave $\mathrm{Prep}_f(\mathbb{P}^1)$ invariant. Then the Julia set $\mathcal{J}(f)$ of f is also invariant by σ, that is,*

$$\Sigma(\mathrm{Prep}_f(\mathbb{P}^1)) \subset \Sigma(\mathcal{J}(f)).$$

Proof. By Theorem 1.13, the derived set of $\mathrm{Prep}_f(\mathbb{P}^1)$ is the Julia set $\mathcal{J}(f)$ of f. Since σ is a homeomorphism of $\mathbb{P}^1$ it maps derived sets to derived sets, that is, $\mathcal{J}(f)$ into $\mathcal{J}(f)$. The Julia set $\mathcal{J}(f)$ must therefore be invariant by σ. □

We now conclude the proof of the theorem. By Lemma 3.12 above, the Julia set $\mathcal{J}(f)$ must be invariant by σ_0, since $\mathrm{Prep}_f(\mathbb{P}^1)$ is. This contradicts the choice of σ_0 we started with, which satisfied $\sigma_0 \in \mathcal{A}_L(X) \setminus \Sigma(\mathcal{J}(f))$. □

Corollary 3.13. *Let X be given by the equality of two rational functions $R(z) = S(w)$, $\varphi = f \times g$ and $\deg_m(\varphi) \geq 2$. Assume f or g is not exceptional.*
If either

$$\Sigma_R \not\subset \Sigma\big(\mathcal{J}(f)\big) \text{ or } \Sigma_S \not\subset \Sigma\big(\mathcal{J}(g)\big)$$

(where the sets Σ_S, Σ_R are as defined in Definition 3.1), then $\mathrm{Prep}_\varphi(X)$ is finite.

Proof. It is enough to note that $\Sigma_R \subset \mathcal{A}_L(X)$ and $\Sigma_S \subset \mathcal{A}_R(X)$ (where $\mathcal{A}_R(X)$ is the symmetric construct on the right) by construction of X. We conclude by the theorem. □

Example 3.14. We wish to find a family of pairs (X, φ), satisfying the assumptions of the corollary and thus produce examples of curves with equation of type $R(z) = S(w)$ with finite set $\mathrm{Prep}_\varphi(X)$.

We will have done so if, given a nonexceptional rational function f which we choose (we may assume $\mathcal{J}(f)$ a compact of $\mathbb{C}$ by conjugating it if necessary), we can find a way to produce at will, elements of $PSL(2, \mathbb{C})$ of finite order not leaving the Julia set $\mathcal{J}(f)$ (which is not exceptional) invariant.

First, let us show that given such an element σ we can find examples. We may just take for $R(z)$,

$$R(z) = \prod_{k=1}^{k=\mathrm{ord}(\sigma)} \sigma^k(z)$$

(or any other rational symmetric function in the $\sigma^k(z)$'s) and for S any rational function such that the polynomial we get by clearing denominators in $R(z) = S(w)$ is irreducible. Then, clearly $\sigma \in \Sigma_R$, and we can conclude to the finiteness of $\mathrm{Prep}_\varphi(X)$ by the corollary.

To find appropriate σ's, we can choose elliptic elements with fixed points at b, c confined in an area of the plane, and the argument of the multiplier of σ smaller than a certain bound, for example so that an iterate of it moves $\mathcal{J}(f)$.

3.5. **Further consequences derived from the group $\Sigma(\mathcal{J}(f))$.** We now develop a result which, out of a curve X with infinite set $\mathrm{Prep}_\varphi(X)$, produces as many *finite* cases of $\mathrm{Prep}_\varphi(Y)$ as we want (note the morphism φ is the same), by simply producing curves Y which map to X in a particular way.

Proposition 3.15. *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve, $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, φ defined over $\overline{\mathbb{Q}}$. Assume $\mathrm{Prep}_\varphi(X)$ infinite, and either $\mathcal{J}(f)$ or $\mathcal{J}(g) \neq \mathbb{P}^1$ and let Y be another curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Assume further that there exists a morphism $\psi = h \times Id\colon Y \to X$, determined by a single rational map h given by $\psi(z, w) = (h(z), w)$ such that:*

$$\Sigma_h \not\subset \Sigma(\mathcal{J}(f)).$$

Then, $\mathrm{Prep}_\varphi(Y)$ is finite.

Proof. Suppose to the contrary that $\mathrm{Prep}_\varphi(Y)$ is infinite. Since we already have $\mathrm{Prep}_\varphi(X)$ infinite, we have by Theorem 1.8 II(d):

$$\forall\,(z,w) \in X(\mathbb{C}), \qquad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g), \tag{3.4}$$

$$\forall\,(z,w) \in Y(\mathbb{C}), \qquad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g). \tag{3.5}$$

We need a lemma:

Lemma 3.16. *Under these conditions the Julia set $\mathcal{J}(f)$ of f is completely invariant under h. That is,*

$$h^{-1}(\mathcal{J}(f)) = \mathcal{J}(f) = h(\mathcal{J}(f)).$$

Proof. Let $z \in \mathcal{J}(f)$, and let $w \in \mathbb{P}^1$ with $(z,w) \in X$. Then, by the equivalence for X we have $w \in \mathcal{J}(g)$. Since ψ sends Y onto X, there is some $z' \in \mathbb{P}^1$ such that $\psi(z',w) = (h(z'),w) = (z,w)$. Now, by the equivalence for Y, we have $z' \in \mathcal{J}(f)$, since $w \in \mathcal{J}(g)$ (same w). So each time $h(z') \in \mathcal{J}(f)$, we have $z' \in \mathcal{J}(f)$; in other words:

$$h^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f).$$

To show $h\big(\mathcal{J}(f)\big) = \mathcal{J}(f)$, we start from $z' \in \mathcal{J}(f)$, find a w with $(z',w) \in Y$, and get $w \in \mathcal{J}(g)$ by the equivalence for Y. Finally $\psi(z',w) = (h(z'),w)$ and $w \in \mathcal{J}(g)$. Another application of the equivalence for X gives us $h(z') \in \mathcal{J}(f)$. Since z' was arbitrary, we get the inclusion

$$h(\mathcal{J}(f)) \subset \mathcal{J}(f).$$

We now have $h^{-1}\big(\mathcal{J}(f)\big) \subset \mathcal{J}(f)$ and $h\big(\mathcal{J}(f)\big) \subset \mathcal{J}(f)$. From the latter inclusion we easily get:

$$\mathcal{J}(f) \subset h^{-1}\big(h(\mathcal{J}(f))\big) \subset h^{-1}\big(\mathcal{J}(f)\big),$$

that is, $h^{-1}\big(\mathcal{J}(f)\big) \supset \mathcal{J}(f)$, which, coupled with the inclusion in the other direction, gives us an equality.

By the surjectivity of h we have $h\Big(h^{-1}\big(\mathcal{J}(f)\big)\Big) = \mathcal{J}(f)$. From what we already know, the left hand side is $h\big(\mathcal{J}(f)\big)$, so we get the other equality. □

Let $z \in \mathcal{J}(f)$ and $\sigma \in \Sigma_h$. Then, $h(z) \in \mathcal{J}(f)$ by the lemma. Since $\sigma \in \Sigma_h$ we have:

$$h(\sigma(z)) = h(z) \in \mathcal{J}(f).$$

Therefore, $\sigma(z) \in h^{-1}\big(\mathcal{J}(f)\big) = \mathcal{J}(f)$, again by the lemma. Therefore,

$$\Sigma_h \subset \Sigma\big(\mathcal{J}(f)\big).$$

This contradicts the assumption of the proposition. □

Example 3.17. Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $H(z,w) = 0$ be its affine equation. Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$ with say $\mathrm{Prep}_f(\mathbb{P}^1) \neq \mathbb{P}^1$. Let σ be an element of finite order of $PSL(2,\mathbb{C})$ such that $\sigma \notin \Sigma(\mathcal{J}(f))$. We can find plenty of them as $\mathrm{Prep}_f(\mathbb{P}^1) \neq \mathbb{P}^1$, and let

$$h(z) = \prod_{k=1}^{k=\mathrm{ord}(\sigma)} \sigma^k(z)$$

(or any other rational function symmetric in the iterates of σ). Then we have $\Sigma_h \not\subset \Sigma(\mathcal{J}(f))$. We may then apply the proposition. If $\mathrm{Prep}_\varphi(X)$ is infinite,

then for any component Y_0 of the curve with affine equation $H(h(z), w) = 0$ (clear denominators) we can conclude that: $\mathrm{Prep}_{\varphi}(Y_0)$ is finite.

4. Finiteness criteria for $\mathrm{Prep}_{\varphi}(X)$ from potential theory

4.1. **Introduction.** In this part we will look exclusively at curves with affine equation

$$F(z) - G(w) = 0,$$

where F and G are polynomials over $\overline{\mathbb{Q}}$. These curves turn out to have the property that the functions F and G transform the Julia sets of the component rational maps of $\varphi = f \times g$ in a manner which is amenable to potential theory. The main tool will be the classical theorem of Fekete, which describes how logarithmic capacity changes when the underlying set is mapped by a polynomial. Using Fekete's theorem, and the equivalence we have derived in Theorem 1.8 II(d) for a pair (X, φ) for which $\mathrm{Prep}_{\varphi}(X)$ is assumed infinite, we will show that we can get a numerical relationship between the logarithmic capacities of the two Julia sets involved: $\mathcal{J}(f)$, $\mathcal{J}(g)$. This in turn will provide some criteria for the finiteness of the set $\mathrm{Prep}_{\varphi}(X)$, for a pair (X, φ) individually, as well as for families of such pairs depending on parameters. We start with definitions and generalities, the section following that will establish a needed positivity result for capacities of Julia sets and, finally, the last section will derive a criterion from Fekete's theorem.

4.2. **Definition and generalities on capacity and Hausdorff dimension.**

4.2.1. *Logarithmic capacity and transfinite diameter for a compact set.*

Definition 4.1. Let E be a compact set in the complex plane. We define its diameter of order n as:

$$d_n = \max_{z_i \in E} \prod_{1 \le i < j \le n} |z_i - z_j|^{\frac{2}{n(n-1)}}.$$

d_n is a decreasing sequence and has a limit

$$d_\infty = \lim_{n \to \infty} d_n$$

called the transfinite diameter of E.

The second-order diameter is the usual diameter, so the transfinite diameter is finite for compact sets. It is zero for finite sets and also invariant by translation, and furthermore it scales by the modulus in a dilation; see [19, Thm. 16.2.2].

We now define and recall some generalities about the capacity of compact sets. We will extend the definition and properties to arbitrary sets afterwards. Capacity will only mean here as is customary, logarithmic capacity.

Let $\mathcal{M}(E)$ be the set of Borel probability measures on the compact set $E \subset \mathbb{C}$. So, for $\mu \in \mathcal{M}(E)$ we have $\mu(E) = 1$.

Definition 4.2. The *energy integral* of the normalized measure μ on E is defined as:

$$\mathcal{I}_E[\mu] = \int_E \int_E \log \frac{1}{|z_1 - z_2|} \, d\mu(z_1) d\mu(z_2).$$

The above integral is defined as the limit as $N \to +\infty$ of

$$\mathcal{I}_E[\mu]_N = \int_E \int_E \min\left(\log \frac{1}{|z_1 - z_2|}, N \right) \, d\mu(z_1) d\mu(z_2).$$

Definition 4.3. The *logarithmic capacity* $\gamma(E)$ of E is defined as:

$$\gamma(E) = \begin{cases} 0 & \text{if } \mathcal{I}_E[\mu] = +\infty \text{ for all } \mu \in \mathcal{M}(E); \\ \\ e^{-V} & \text{if } V(E) = \inf_{\mu \in \mathcal{M}(E)} \mathcal{I}_E[\mu] < +\infty. \end{cases}$$

Now let E be again a compact set in the complex plane.

Theorem 4.4. *If $V(E) < +\infty$, then $V(E) > -\infty$, and there is a unique element $\nu \in \mathcal{M}(E)$ such that:*

$$\mathcal{I}_E[\nu] = V(E).$$

Proof. The proof can be found in [19, Thm. 16.4.3]. □

This unique measure is called the *equilibrium distribution* of E.

We now recall the relationship between the transfinite diameter and the logarithmic capacity.

Theorem 4.5. *For a compact set $E \subset \mathbb{C}$ we have $\gamma(E) = d_\infty(E)$.*

Proof. The proof can be found in [19, Thm. 16.4.4]. □

Remark 4.6. It follows from the above theorem that for compact sets, the capacity is bounded by the actual diameter of the set and, therefore, is finite.

4.2.2. *Logarithmic capacity for arbitrary Borel sets.* We now define capacitability for arbitrary sets in $\mathbb{C}$.

Definition 4.7. The *inner capacity* of an arbitrary set $E \in \mathbb{C}$ is defined as

$$\gamma_i(E) = \sup\{\gamma(K) \mid K \text{ compact}, K \subset E\},$$

and the *outer capacity* of the set E, which we denote by γ_e, is defined as

$$\gamma_e(E) = \inf\{\gamma(U) \mid U \text{ open}, U \supset E\}.$$

We say that the set E is *capacitable* if $\gamma_i(E) = \gamma_e(E)$, in which case we call this common value, denoted by $\gamma(E)$, the (logarithmic) capacity of E.

We already know that compact sets are capacitable, but we have more generally:

Theorem 4.8. *Every Borel set in $\mathbb{C}$ is capacitable.*

Proof. The proof can be found in the more general setting of $\mathbb{R}^n$ in [21, Ch. 2, §2, Thm. 2.8]. □

Corollary 4.9. *For any Borel set E in $\mathbb{C}$ we have:*

$$\gamma(E) = \lim_{n \to +\infty} \gamma(E \cap K_n)$$

for any sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$, whose interiors exhaust $\mathbb{C}$.

Proof. Borel sets are capacitable. Also note that γ is monotone increasing, so lim and sup are the same here, and that any compact subset of E is contained in one of the $E \cap K_n$ for some n. Therefore:

$$\gamma(E) = \sup_{K \subset E,\, K \text{ compact}} \gamma(K) \;\leq\; \sup_n \gamma(E \cap K_n) \;\leq\; \gamma(E).$$

□

Remark 4.10. Let us also recall that the logarithmic capacity γ is countably subadditive on Borel sets, and that finite and countable sets have zero capacity.

4.2.3. *Hausdorff measure and Hausdorff dimension.* We will give the definition of Hausdorff measure using squares with sides parallel to the coordinate axes. One can use open sets, closed sets, balls, squares or convex sets, etc. to define it; the various measures thus defined agree [13, p. 7]. Generalities about Hausdorff measure and dimension can be found in [34, Ch. 3, §4] or [13, §1.2]. Let $h(t)$ be any continuous increasing function with $h(0) = 0$. A cover of the set $E \in \mathbb{C}$, by squares $s_i, i \in I$, where I is at most countable, the sides being parallel to the coordinate axes and of length less than $\rho > 0$, will be called a ρ-cover of E.

Definition 4.11. The Hausdorff measure Λ_h associated to h is defined as:

$$\Lambda_h(E) = \lim_{\rho \to 0} \Lambda_h^\rho(E), \tag{4.1}$$

$$\Lambda_h^\rho(E) = \inf \sum_{i \in I} h(s_i), \tag{4.2}$$

where the infimum is taken over all such covers, and where E is a Borel set of $\mathbb{C}$. In the particular case of $h(t) = t^\alpha$ for some $\alpha > 0$, the measure Λ_h will be denoted Λ_α and $\Lambda_\alpha(E)$ is called the α-dimensional measure of E.

Definition 4.12. For any E we have $0 \leq \Lambda_\alpha(E) \leq \infty$, and there is a unique value, denoted $\dim(E)$, of α such that we have the following:

$$\Lambda_\alpha(E) = \begin{cases} +\infty & \text{if } 0 \leq \alpha < \dim(E); \\ 0 & \text{if } \dim(E) < \alpha < \infty. \end{cases}$$

This value, $\dim(E)$, is called the *Hausdorff dimension* of E.

4.3. **Capacity and Hausdorff dimension of Julia sets.** This part will establish a result needed in the proof of Theorem 4.18 in the next section, namely, that the Julia sets we encounter always have positive logarithmic capacity. We introduce a notion of affine Julia set which coincides with the Julia set if compact.

Notation. Let f be a rational function of degree at least two. Let $\mathcal{J}(f)$ be its Julia set. We then denote the set obtained by dropping the point at infinity (if it happens to be in it) by:

$$\mathcal{J}_0(f) = \mathcal{J}(f) \setminus \{\infty\} = \mathcal{J}(f) \cap \mathbb{C}.$$

Note that $\mathcal{J}_0(f)$ is a closed subset of $\mathbb{C}$ as $\mathcal{J}(f)$ is closed in $\mathbb{P}^1$ and $\mathcal{J}_0(f)$ coincides with the Julia set (as defined in $\mathbb{P}^1$) if it is compact. Let us call this set $\mathcal{J}_0(f)$, the *affine Julia set* of f.

Proposition 4.13. *The logarithmic capacity $\gamma(\mathcal{J}_0(f))$, of the affine Julia set $\mathcal{J}_0(f)$ for a rational function of degree at least two, is positive.*

Before we prove this we need a lemma.

Lemma 4.14. *Let E be a closed set in $\mathbb{C}$ such that $E_1 = \sigma(E)$ is compact, where $\sigma \in PSL(2, \mathbb{C})$ is a Möbius map. If the logarithmic capacity $\gamma(E_1)$ of E_1 is positive, then so is the capacity $\gamma(E)$ of E.*

Proof. First, both E and E_1 are capacitable, as they are Borel sets. Let

$$\sigma(z) = \frac{az+b}{cz+d}, \text{ with } a,b,c,d \in \mathbb{C} \quad \text{and} \quad ad-bc=1.$$

We may assume $c \neq 0$, since otherwise E itself must be compact, and then we get the positivity from

$$\gamma(E_1) = |\frac{a}{d}|\,\gamma(E)$$

since capacity scales by the modulus in a dilation and is invariant by translation. Since

$$\gamma(E) = \sup_K \gamma(E \cap K),$$

where K runs through the compact sets of $\mathbb{C}$ and because γ is a monotone increasing function on compact sets, in order to conclude $\gamma(E) > 0$, it is enough to prove that, there is some compact set K of $\mathbb{C}$ such that $\gamma(E \cap K) > 0$. Consider the family of compact sets $\sigma(E \cap K)$, where K runs through the compact sets of $\mathbb{C}$. This family of compact subsets of E_1 exhausts E_1, and since compact sets are capacitable we have:

$$\gamma(E_1) = \sup_{K \text{ compact of } \mathbb{C}} \gamma\big(\sigma(E \cap K)\big).$$

Now since E_1 has positive capacity there must be a compact set K of $\mathbb{C}$ such that

$$\gamma\big(\sigma(E \cap K)\big) > 0.$$

From Theorem 4.4 we deduce, since $\sigma(E \cap K)$ is a compact set with positive capacity, that there is a probability measure $\nu \in \mathcal{M}\big(\sigma(E \cap K)\big)$, such that:

$$\gamma\big(\sigma(E \cap K)\big) = \mathcal{I}_{\sigma(E\cap K)}[\nu] = \int_{\sigma(E\cap K)} \int_{\sigma(E\cap K)} \log \frac{1}{|z_1 - z_2|}\, d\nu(z_1) d\nu(z_2).$$

Letting $\mu = \nu \circ \sigma$ we get a Borel probability measure $\mu \in \mathcal{M}(E \cap K)$ on the compact set $E \cap K$, and by a change of variable $z_i = \sigma(u_i)$ we get:

$$\gamma\big(\sigma(E \cap K)\big) = \int_{\sigma(E\cap K)} \int_{\sigma(E\cap K)} \log \frac{1}{|z_1 - z_2|}\, d\nu(z_1) d\nu(z_2) \tag{4.3}$$

$$= \int_{E\cap K} \int_{E\cap K} \log \frac{1}{|\sigma(u_1) - \sigma(u_2)|}\, d\mu(u_1) d\mu(u_2) \tag{4.4}$$

$$= \int_{E\cap K} \int_{E\cap K} \log \frac{1}{|u_1 - u_2|}\, d\mu(u_1) d\mu(u_2) - \log|ad-bc| \tag{4.5}$$

$$+ \int_{E\cap K} \int_{E\cap K} \log|(cu_1+d)(cu_2+d)|\, d\mu(u_1) d\mu(u_2).$$

Since E_1 is compact, σ does not have its pole $-\frac{d}{c}$ in E, and as E is closed, the pole lies at a positive distance A from E so that

$$|u_i - \frac{-d}{c}| \geq A > 0 \text{ for i = 1, 2 and } u_i \in E$$

and so for $u_i \in E \cap K$ for any K as well. We then have:

$$I_0 = \int_{E\cap K} \int_{E\cap K} \log|(cu_1+d)(cu_2+d)|\, d\mu(u_1) d\mu(u_2) \geq 2\log(|c|A) > -\infty$$

and since we have:

$$\mathcal{I}_{E\cap K}[\mu] = \mathcal{I}_{\sigma(E\cap K)}[\nu] - I_0$$

we now get from the definition of capacity

$$\gamma(E) \geq \gamma(E \cap K) \geq e^{-\mathcal{I}_{E\cap K}[\mu]} \geq \gamma(\sigma(E \cap K)) \times e^{I_0} \geq \gamma(\sigma(E \cap K)) \times (|c|A)^2 > 0. \tag{4.6}$$

□

We now proceed with the proof of Proposition 4.13.

Proof. In order to reduce the argument to the compact case we will first conjugate our rational function to have a compact Julia set. The proposition is clear if $\mathcal{J}(f) = \mathbb{P}^1$ since then $\mathcal{J}_0(f) = \mathbb{C}$ and $\mathbb{C}$ has positive (infinite) logarithmic capacity.

If $\mathcal{J}(f) \neq \mathbb{P}^1$, then we can pick some $\alpha \in \mathbb{C}\backslash\mathcal{J}(f) \neq \emptyset$ and by using a Möbius map $\sigma(z) = \frac{1}{z-\alpha}$, send $\mathcal{J}(f)$ to a compact subset $\sigma(\mathcal{J}(f))$, which is equal to $\mathcal{J}(\sigma f \sigma^{-1})$ (see [2, Thm. 3.1.4] for this and generally for elementary properties of Julia sets), where $\sigma f \sigma^{-1}$ is also a rational function of degree at least two. By Garber's theorem [16, Thm. 1, p. 499], which guarantees a positive Hausdorff dimension for the Julia set of a rational function if it lies in the closed unit ball, and a theorem of Beardon [4, Appendix, pp. 734-735], which shows that Hausdorff dimension is invariant by a Möbius map, we get that this compact set $\sigma(\mathcal{J}(f))$ has positive Hausdorff dimension. By Frostman's theorem, which says in particular that a compact set in the plane has positive logarithmic capacity if it has positive Hausdorff dimension ([15, Thm. 1, p. 86] or in [34, Thm. III. 19]), we conclude that $\sigma(\mathcal{J}(f))$ has positive capacity.

Let $E = \mathcal{J}_0(f)$ be the affine Julia set of f. If $E = \mathcal{J}_0(f) = \mathcal{J}(f)$, that is, $\mathcal{J}(f)$ is already compact, then, with the previous discussion (without the use of a Möbius map) we already know that E has positive capacity. If not, then $E = \mathcal{J}(f) \setminus \{\infty\}$ and therefore, if we let $E_1 = \sigma(E)$, we get: $E_1 = \sigma\big(\mathcal{J}(f)\big) \setminus \{\sigma(\infty)\}$. Since the capacity of a Borel set is unchanged by adding or removing from it, a Borel set of zero capacity, this set has the same capacity as $\sigma(\mathcal{J}(f))$ which, by the previous case, has positive capacity.

We can now apply to the pair of sets $E_1 = \sigma(E)$, the previous Lemma 4.14, from which we conclude that E, and therefore $\mathcal{J}_0(f)$, has positive capacity, as asserted. □

4.4. **Fekete's theorem and its consequences for the curve.** We now mention a theorem of Fekete, which will be our main tool in deriving information about the finiteness of preperiodic points on curves of a special type, namely whose affine equation is $F(z) - G(w) = 0$, for polynomials F and G.

Theorem 4.15 (Fekete)**.** *Let K be a compact set of $\mathbb{C}$ and let*

$$q(z) = \sum_{j=0}^{j=d} a_j z^j \in \mathbb{C}[z] \text{ , where } a_d \neq 0 \text{ and } d \geq 1.$$

Then

$$\gamma\big(q^{-1}(K)\big) = \Big(\frac{\gamma(K)}{|a_d|}\Big)^{\frac{1}{d}}.$$

Proof. The proof can be found in [30, Ch. 5, §2, Thm. 5.2.5]. □

We will now use this theorem to derive a result which will provide us the basis for a criterion of finiteness for the special type of curves X mentioned above.

Theorem 4.16. *Let U, W be closed sets in $\mathbb{C}$, $F = a_d z^d + \cdots$, $G = b_h w^h + \cdots$ be two complex polynomials of degrees d and h respectively. Let X_0 be the affine curve in $\mathbb{C} \times \mathbb{C}$ defined by the equation $F(z) - G(w) = 0$. Let us suppose further that for any point $(z, w) \in X_0$, we have:*

$$z \in U \iff w \in W.$$

Then, we have in terms of the capacities, the following equation:

$$|a_d|\, \gamma(U)^d = |b_h|\, \gamma(W)^h.$$

We first prove a lemma.

Lemma 4.17. *Under the assumptions of the theorem, we have the following:*
(i) $F(U) = G(W)$.
(ii) $F^{-1}\big(F(U)\big) = U$ and $G^{-1}\big(G(W)\big) = W$.

Proof. To prove (i), let $z_0 \in U$. We then solve $F(z_0) = G(w)$ for w in $\mathbb{C}$. Let w_0 be a solution. Then $(z_0, w_0) \in X_0$ and $z_0 \in U$, so, by the assumption of the lemma we must have $w_0 \in W$. In other words $F(z_0) = G(w_0) \in G(W)$, so that we have $F(U) \subset G(W)$ and symmetrically we have $G(W) \subset F(U)$ and therefore $F(U) = G(W)$.

To prove (ii), we shall do it only for U, as it is completely symmetric. We only need to prove that $F^{-1}\big(F(U)\big) \subset U$. Let $z_0 \in F^{-1}\big(F(U)\big)$. This means that there is some $z_1 \in U$ such that $F(z_0) = F(z_1)$. We solve $F(z_1) = G(w)$ for w and let w_1 be a solution. Then $(z_1, w_1) \in X_0$ by construction and since $z_1 \in U$ we must have $w_1 \in W$. But we also have $F(z_0) = G(w_1)$ with $w_1 \in W$ so that $z_0 \in U$. We have therefore $F^{-1}\big(F(U)\big) \subset U$. □

We now return to the proof of the theorem.

Proof. Let us denote by T the common set obtained from the equality (i) of Lemma 4.17: $F(U) = G(W)$. We now claim that T is closed. Indeed, since the map $F\colon \mathbb{C} \to \mathbb{C}$ defined by the polynomial F is proper, F^{-1} takes compact sets to compact sets. Let $t_n = F(z_n)$ be a sequence in T, $z_n \in U$, converging to some $t_0 \in \mathbb{C}$. Then we have:

$$z_n \in F^{-1}(\overline{B(t_0, 1)}) \quad \text{for} \quad n \geq n_0.$$

As this set is a compact subset of $\mathbb{C}$, some subsequence z_{n_k} converges to some z_0. Since U is closed and $z_{n_k} \in U$, we have $z_0 \in U$ and by continuity $t_0 \in T$. Therefore, T is closed.

Now let K be a compact subset of $\mathbb{C}$ and let us denote by $T_K = T \cap K$ the compact intersection. As F is proper, the family $F^{-1}(T_K)$ is a family of compact sets exhausting $F^{-1}(T) = F^{-1}(F(U)) = U$ by (ii) of Lemma 4.17. Therefore:

$$\gamma(U) = \sup_K \gamma\big(F^{-1}(T_K)\big);$$

similarly we have:

$$\gamma(W) = \sup_K \gamma\big(G^{-1}(T_K)\big).$$

We now use Fekete's theorem (Theorem 4.15) to compute the capacities of the compact sets above:

$$\gamma(F^{-1}(T_K)) = \Big(\frac{\gamma(T_K)}{|a_d|}\Big)^{\frac{1}{d}}$$

and similarly

$$\gamma(G^{-1}(T_K)) = \Big(\frac{\gamma(T_K)}{|b_h|}\Big)^{\frac{1}{h}}.$$

By eliminating between the above two identities, the term $\gamma(T_K)$, which is finite (possibly 0) by Remark 4.6 as T_K is compact for each K, we obtain the following new identity:

$$|a_d|\,\gamma(F^{-1}(T_K))^d = |b_h|\,\gamma(G^{-1}(T_K))^h.$$

Therefore

$$\sup_K \Big(|a_d|\,\gamma(F^{-1}(T_K))^d\Big) = \sup_K \Big(|b_h|\,\gamma(G^{-1}(T_K))^h\Big)$$

as K runs through the compact subsets of $\mathbb{C}$ and since γ is monotone increasing and the compact sets $F^{-1}(T_K)$ and $G^{-1}(T_K)$ exhaust U and W respectively we obtain:

$$|a_d|\gamma(U)^d = |b_h|\gamma(W)^h.$$

□

We now make the above Theorem 4.16 more explicit.

Theorem 4.18. *Let X be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, φ defined over $\overline{\mathbb{Q}}$. Let $\mathcal{J}_0(f) = \mathcal{J}(f) \cap \mathbb{C}$ be the affine Julia set of f and similarly $\mathcal{J}_0(g)$ for g. Let us assume the following:*
(1) $\mathrm{Prep}_\varphi(X)$ is infinite.
(2) The equation of the affine curve X_0 in $\mathbb{C}\times\mathbb{C}$ of X is of the form $F(z)-G(w) = 0$, where $F = a_d z^d + \cdots$, $G = b_h w^h + \cdots$ are two polynomials of degrees d and h respectively over $\overline{\mathbb{Q}}$.
Then we have:

$$|a_d|\,\gamma(\mathcal{J}_0(f))^d = |b_h|\,\gamma(\mathcal{J}_0(g))^h,$$

where both sides are nonzero. In addition, both sides are necessarily finite, if one of the capacities $\gamma(\mathcal{J}_0(f))$, $\gamma(\mathcal{J}_0(g))$ is finite. This is the case in particular, when one of the Julia sets $\mathcal{J}(f)$, $\mathcal{J}(g)$ does not contain ∞.

We now derive a criterion as an immediate consequence of the theorem as:

Corollary 4.19. *With the premises of the theorem, $\mathrm{Prep}_\varphi(X)$ will be finite provided:*

$$|a_d|\,\gamma(\mathcal{J}_0(f))^d \neq |b_h|\,\gamma(\mathcal{J}_0(g))^h.$$

We now continue with the proof of the theorem.

Proof. Applying Theorem 1.8 II(d) to this curve which satisfies $|\mathrm{Prep}_\varphi(X)| = \infty$, we get the equivalence:

$$\forall\,(z,w) \in X(\mathbb{C}), \quad z \in \mathcal{J}(f) \iff w \in \mathcal{J}(g).$$

Since the curve X has affine equation $H(z,w) = F(z) - G(w) = 0$, its bihomogeneous equation must be given by:

$$H^*(z_0,z_1,w_0,w_1) = w_1^h F^*(z_0,z_1) - z_1^d G^*(w_0,w_1) = 0,$$

where F^* and G^* are forms of degrees d and h respectively, corresponding to F and G. For $z_1 = 0$ we get

$$H(z_0,0,w_0,w_1) = w_1^h a_d z_0^d = 0,$$

and since $z_0 \neq 0$ we have $w_1 = 0$. It follows from this that we have: $(z, w) \in X(\mathbb{C})$, and then $z = \infty \Leftrightarrow w = \infty$. The equivalence above becomes:

$$\forall\, (z, w) \in X(\mathbb{C}), \quad z \in \mathcal{J}_0(f) \iff w \in \mathcal{J}_0(g).$$

Let $U = \mathcal{J}_0(f)$ and $W = \mathcal{J}_0(g)$. They are both closed, and we can now apply Theorem 4.16 with U, W, $X_0(\mathbb{C})$ and $\varphi = f \times g$. We get:

$$|a_d|\, \gamma(U)^d = |b_h|\, \gamma(W)^h,$$

that is:

$$|a_d|\, \gamma\big(\mathcal{J}_0(f)\big)^d = |b_h|\, \gamma\big(\mathcal{J}_0(g)\big)^h.$$

Finally, from Proposition 4.13, we conclude that both sides are not zero. □

The following proposition will show that if we perturb the morphism (in a very simple manner), for all but one choice of the modulus of one of the parameters, which corresponds to some equilibrium, the set of preperiodic points lying on any curve with equation of type $F(z) - G(w) = 0$ and all morphisms in the family, is finite. Given a rational function f, define the family of conjugates of f:

$$f_{a,b}(z) = h_{a,b} \circ f \circ h_{a,b}^{-1}(z)$$

by affine maps $h_{a,b}(z) = az + b$ with $a, b \in \overline{\mathbb{Q}}$, and given some rational function g let us denote $\varphi_{a,b} = f_{a,b} \times g$. Also let F, G be two polynomials $F = a_d z^d + \cdots$, $G = b_h w^h + \cdots$ of degrees d and h respectively over $\overline{\mathbb{Q}}$. Let us also denote by $\gamma\big(\mathcal{J}_0(f)\big)$ the capacity for the affine Julia set for f (resp. for g). With this, we have the proposition:

Proposition 4.20. *Let $\varphi = f \times g$, $\deg_m(\varphi) \geq 2$, defined over $\overline{\mathbb{Q}}$. Let us assume that $\infty \notin \mathcal{J}(f)$ or $\infty \notin \mathcal{J}(g)$. Then there is a real number δ, given by:*

$$\delta = \frac{1}{d} \log \Big[\frac{|b_h|}{|a_d|} \, \frac{\gamma\big(\mathcal{J}_0(g)\big)^h}{\gamma\big(\mathcal{J}_0(f)\big)^d} \Big]$$

such that, for all rational functions f, and g as above, all curves X given by an affine equation of type $F(z) = G(w)$ over $\overline{\mathbb{Q}}$, and all $a, b \in \overline{\mathbb{Q}}$, if $|a| \neq e^{\delta}$, then $\mathrm{Prep}_{\varphi_{a,b}}(X)$ is finite.

Proof. If $\mathrm{Prep}_{\varphi_{a,b}}(X)$ is infinite for any curve X with affine equation $F(z) = G(w)$, where say

$$F = a_d z^d + \cdots, \quad G = b_h w^h + \cdots$$

are two polynomials over $\overline{\mathbb{Q}}$, then we have from Theorem 4.18:

$$|a_d|\, \gamma\big(\mathcal{J}_0(f_{a,b})\big)^d = |b_h|\, \gamma\big(\mathcal{J}_0(g)\big)^h. \tag{4.7}$$

Now, $\mathcal{J}(f_{a,b}) = h_{a,b}(\mathcal{J}(f))$ (see [2, Thm. 3.1.4] for the Julia set of such a composition). In terms of capacities, we have $\gamma(\mathcal{J}(f_{a,b})) = \gamma(\mathcal{J}_0(f_{a,b}))$, since the two sets differ at most by a point. Since a capacity gets multiplied by the modulus of a number if the set is multiplied by it, we also have $\gamma(\mathcal{J}(f_{a,b})) = |a|\, \gamma(\mathcal{J}(f)) = |a|\, \gamma(\mathcal{J}_0(f))$. We finally get $\gamma(\mathcal{J}_0(f_{a,b})) = |a|\, \gamma(\mathcal{J}_0(f))$. The identity (4.7) then becomes:

$$|a_d| \Big(|a|\, \gamma\big(\mathcal{J}_0(f)\big) \Big)^d = |b_h|\, \gamma\big(\mathcal{J}_0(g)\big)^h. \tag{4.8}$$

We solve for $|a|$ and we find $|a| = e^{\delta}$ with δ as defined in the proposition. □

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