TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 365, Number 1, January 2013, Pages 319–360
S 0002-9947(2012)05638-8
Article electronically published on July 24, 2012

MULTIPARAMETER HARDY SPACE THEORY ON CARNOT-CARATHÉODORY SPACES AND PRODUCT SPACES OF HOMOGENEOUS TYPE

YONGSHENG HAN, JI LI, AND GUOZHEN LU

Abstract. This paper is inspired by the work of Nagel and Stein in which the L^p $(1 < p < \infty)$ theory has been developed in the setting of the product Carnot-Carathéodory spaces $\widetilde{M} = M_1 \times \cdots \times M_n$ formed by vector fields satisfying Hörmander's finite rank condition. The main purpose of this paper is to provide a unified approach to develop the multiparameter Hardy space theory on product spaces of homogeneous type. This theory includes the product Hardy space, its dual, the product BMO space, the boundedness of singular integral operators and Calderón-Zygmund decomposition and interpolation of operators. As a consequence, we obtain the endpoint estimates for those singular integral operators considered by Nagel and Stein (2004). In fact, we will develop most of our theory in the framework of product spaces of homogeneous type which only satisfy the doubling condition and some regularity assumption on the metric. All of our results are established by introducing certain Banach spaces of test functions and distributions, developing discrete Calderón identity and discrete Littlewood-Paley-Stein theory. Our methods do not rely on the Journé-type covering lemma which was the main tool to prove the boundedness of singular integrals on the classical product Hardy spaces.

1. Introduction

The main purpose of this paper is to develop a satisfactory theory of multiparameter Hardy spaces in the framework of the product spaces of homogeneous type under only the doubling condition and some regularity assumption of the underlying metric. Such a metric space of homogeneous type includes the model case of Carnot-Carathédory spaces intrinsic to a family of vector fields satisfying Hörmander's condition of finite rank.

This paper is inspired by the works of Nagel and Stein [46, 48, 49] in which the L^p $(1 < p < \infty)$ theory has been developed in the setting of the product Carnot-Carathéodory spaces $\widetilde{M} = M_1 \times \cdots \times M_n$ formed by vector fields satisfying Hörmander's finite rank condition. The crucial geometric properties of the underlying Carnot-Carathéodory space used in [49] to establish Nagel-Stein's product L^p theory are: (a) the doubling property (1.1) of the Lebesgue measure with

Received by the editors May 13, 2010 and, in revised form, February 21, 2011.

2010 *Mathematics Subject Classification.* Primary 42B35; Secondary 32T25, 32W30.

Key words and phrases. Kohn-Laplacian, heat equation, Shilov boundary, finite type domains, multiparameter Hardy space, Carleson measure space, Calderón's reproducing formula, almost orthogonality estimate, sup-inf-type inequality, sequence spaces, duality, singular integral operators.

The first two authors were supported by NNSF of China (Grant No. 11001275).

The third author is the corresponding author and was partly supported by US National Science Foundation grants DMS0500853 and DMS0901761.

respect to the Carnot-Carathédory metric together with the conditions of (1.6); (b) the regularity of the control distance. The main tool they used is the continuous Littlewood-Paley-Stein theory on $L^p, 1 < p < \infty$, where the construction and estimates of the heat kernel $H(s, x, y)$ of the sub-Laplacian play an important role.

The work of Nagel-Stein suggests the following very natural questions: Is the underlying geometry of the Carnot-Carathéodory spaces necessary to develop their product L^p theory? Are the heat kernel estimates of the sub-Laplacian essential in establishing the Littlewood-Paley-Stein theory in the product Carnot-Carathédory spaces? With this in mind, we ponder what minimal assumption would be required to develop a multiparameter Littlewood-Paley-Stein theory and thus the multiparameter Hardy space theory in a general product homogeneous space which includes the product Carnot-Carathéodory spaces considered by Nagel and Stein as a special case.

To address this issue, we recall that (M, ρ, μ) is a space of homogeneous type in the sense of Coifman and Weiss [9] if ρ is a quasi-metric, that is, (i) $\rho(x, y) = 0$ iff $x = y$; (ii) $\rho(x, y) = \rho(y, x)$; (iii) $\rho(x, z) \leq A[\rho(x, y) + \rho(y, z)]$ for some $A \geq 1$, and μ is a nonnegative measure satisfying the following doubling property:

$$|B(x, 2\delta)| \leq C|B(x, \delta)| \quad \text{for all } \ \delta > 0 \text{ and some constant } C, \tag{1.1}$$

where $B(x, \delta) = \{y : \rho(x, y) < \delta\}$ is the metric ball centered at x with radius δ. In [42], Macias and Segovia have proved that one can replace the quasi-metric ρ by another quasi-metric $d \approx \rho$ such that d yields the same topology on M as ρ and, moreover,

$$\mu(B(x, r)) \sim r, \tag{1.2}$$

where $B(x, r) = \{y \in M : d(y, x) < r\}$ and d has the regularity property

$$|d(x, y) - d(x', y)| \leq C_0 d(x, x')^{\vartheta}[d(x, y) + d(x', y)]^{1-\vartheta} \tag{1.3}$$

for some regularity exponent $\vartheta : 0 < \vartheta < 1, 0 < r < \infty$ and all $x,\ x',\ y \in M$. Under these additional conditions of (1.2) and (1.3), the Littlewood-Paley-Stein theory and the Hardy space on (M, d, μ) have been established; see [10] for more details and the references therein.

Nevertheless, the assumption (1.2) is rather restrictive, and many important examples, such as the Carnot-Carathédory spaces, do not generally satisfy such a condition. This motivates us to consider a more general type of homogeneous spaces without assuming (1.2). The main concern of this paper is to develop the product theory on $\widetilde{M} = M_1 \times \cdots \times M_n$, where for $1 \leq i \leq n$, each (M_i, d_i, μ_i) is a space of homogeneous type satisfying the doubling property (1.1) and the regularity assumption (1.3).

To familiarize the reader with the motivation of this paper, we will first recall the background of the product L^p theory of the Carnot-Carathédory spaces developed by Nagel and Stein. In the works of [46], [48] and [49], Nagel and Stein developed the $L^p, 1 < p < \infty$, theory on the product Carnot-Carathéodory spaces $\widetilde{M} = M_1 \times \cdots \times M_n$, where each $M_i, 1 \leq i \leq n$, is a connected smooth manifold and $\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}$ are k given smooth real vector fields on $M_i, 1 \leq i \leq n$, satisfying the Hörmander condition of order m, i.e., these vector fields together with their commutators of order $\leq m$ span the tangent space to $M_i, 1 \leq i \leq n$, at each point. The most important geometric objects used in their work are: (i) a class of equivalent control distances constructed on $M_i, 1 \leq i \leq n$, via the vector fields

$\{\mathbb{X}_1, \dots, \mathbb{X}_k\}$; (ii) the volumes of balls satisfying the doubling property and the certain low bound estimates. More precisely, one variant of the control distance on M is defined as follows. For each $x, y \in M$, let $AC(x, y, \delta)$ denote the collection of absolutely continuous mapping $\varphi: [0,1] \to M$ with $\varphi(0) = x$, $\varphi(1) = y$, and for almost every $t \in [0,1]$, $\varphi'(t) = \sum_{j=1}^{k} a_j \mathbb{X}_j(\varphi(t))$ with $|a_j| \leq \delta$. The control distance $\rho(x,y)$ from x to y is the infimum of the set of $\delta > 0$ such that $AC(x,y,\delta) \neq \emptyset$. See [49] and [50] for more details. It was shown in [49] that there is a pseudo-metric[1] $d \approx \rho$ such that $d(x,y)$ is C^∞ on $M \times M \backslash \{\text{diagonal}\}$, and for $x \neq y$

$$|\partial_X^K \partial_Y^L d(x,y)| \lesssim d(x,y)^{1-K-L}. \tag{1.4}$$

Here ∂_X^K are products of K vector fields $\{X_1, \dots, X_k\}$ acting as derivatives on the x variable, and ∂_Y^L are corresponding L vector fields acting on the y variable.

The volume measure on M is defined as follows. When M is compact, we take any fixed smooth measure on M with strictly positive density. In the unbounded case we take Lebesgue measure and denote the measure of a set E by $|E|$. The ball is defined by $B(x,\delta) = \{y \in M, d(x,y) < \delta\}$, with $0 < \delta \leq 1$ in the compact case and $0 < \delta < \infty$ in the unbounded case, and the volume function is defined by $V(x,y) = |B(x, d(x,y))|$. Nagel, Stein and Wainger [50] proved that the volumes of the balls $B(x,\delta)$ are essentially polynomials in δ with coefficients that depend on x and satisfy the doubling property (1.1) (see [50] for the details).

We point out that the doubling condition (1.1) implies that there exist positive constants C and Q such that for all $x \in M$ and $\lambda \geq 1$,

$$|B(x, \lambda r)| \leq C\lambda^Q |B(x,r)|. \tag{1.5}$$

One example with unbounded total measure studied in [49] is that M arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. Let $\Omega = \{(z,w) \in \mathbb{C}^2 : \text{Im}(w) > P(z)\}$, where P is a real, subharmonic, nonharmonic polynomial of degree m. Then $M = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z,t) : z \in \mathbb{C}, t \in \mathbb{R}\}$. The basic $(0,1)$ Levi vector field is then $\bar{Z} = \frac{\partial}{\partial \bar{z}} - i \frac{\partial P}{\partial \bar{z}} \frac{\partial}{\partial t}$, and we write $\bar{Z} = \mathbb{X}_1 + i\mathbb{X}_2$. The real vector fields $\{\mathbb{X}_1, \mathbb{X}_2\}$ and their commutators of order $\leq m$ span the tangent space to M at each point. In this example, the metric ball satisfies the following conditions: for $s \geq 1$,

$$|B(x, s\delta)| \approx s^{m+2} |B(x,\delta)|. \tag{1.6}$$

To develop the $L^p, 1 < p < \infty$, theory on a product space $\widetilde{M} = M_1 \times \cdots \times M_n$, the key tool used by Nagel and Stein is the Littlewood-Paley-Stein theory for each factor $M_i, 1 \leq i \leq n$, and then passing to the corresponding product theory. To do this, Nagel and Stein consider the sub-Laplacian $\mathcal{L}$ on M in self-adjoint form, given by

$$\mathcal{L} = \sum_{j=1}^{k} \mathbb{X}_j^* \mathbb{X}_j.$$

Here $(\mathbb{X}_j^* \varphi, \psi) = (\varphi, \mathbb{X}_j \psi)$, where $(\varphi, \psi) = \int_M \varphi(x) \bar{\psi}(x) d\mu(x)$, and $\varphi, \psi \in C_0^\infty(M)$, the space of C^∞ functions on M with compact support. In general, $\mathbb{X}_j^* = -\mathbb{X}_j + a_j$,

[1] Here, and throughout the paper, $A \approx B$ means that the ratio A/B is bounded and bounded away from zero by constants that do not depend on the relevant variables in A and B. $A \lesssim B$ means that the ratio A/B is bounded by a constant independent of the relevant variables.

where $a_j \in C^\infty(M)$. The solution of the following initial value problem for the heat equation

$$\frac{\partial u}{\partial s}(x,s) + \mathcal{L}_x u(x,s) = 0$$

with $u(x,0) = f(x)$ is given by $u(x,s) = H_s(f)(x)$, where H_s is the operator given via the spectral theorem by $H_s = e^{-s\mathcal{L}}$ and an appropriate self-adjoint extension of the nonnegative operator $\mathcal{L}$ initially defined on $C_0^\infty(M)$. Nagel and Stein proved that for $f \in L^2(X)$,

$$H_s(f)(x) = \int_M H(s,x,y)f(y)d\mu(y)$$

and $H(s,x,y)$ satisfy the following properties (see Proposition 2.3.1 in [49] and Theorem 2.3.1 in [46]):

(1) For every integer $N \geq 0$,

$$|\partial_s^j \partial_X^L \partial_Y^K H(s,x,y)| \lesssim \frac{1}{(d(x,y)+\sqrt{s})^{2j+K+L}} \frac{1}{V(x,y)+V_{\sqrt{s}}(x)+V_{\sqrt{s}}(y)} \left(\frac{\sqrt{s}}{d(x,y)+\sqrt{s}}\right)^{\frac{N}{2}}.$$

(2) For each integer $L \geq 0$ there exist an integer N_L and a constant C_L so that if $\varphi \in C_0^\infty(B(x_0,\delta))$, then for all $s \in (0,\infty)$

$$|\partial_X^L H_s[\varphi](x_0)| \leq C_L \delta^{-L} \sup_x \sum_{|J| \leq N_L} \delta^{|J|} |\partial_X^J \varphi(x)|.$$

(3) For all $s \in (0,\infty)$, $\int H(s,x,y)dy = \int H(s,x,y)dx = 1$.

We remark that the condition of (1.6) is used in the proof of estimate (1). Set $Q_s = 2s\frac{\partial H_s}{\partial s}$, $s > 0$. The Littlewood-Paley-Stein square function $S(f)$ is defined by

$$\big(S[f](x)\big)^2 = \int_0^\infty |Q_s[f](x)|^2 \frac{ds}{s}.$$

Applying the abstract Littlewood-Paley-Stein theory in [52], Nagel and Stein obtained

Proposition 1.1 ([49]). *For $1 < p < \infty$, if $f \in L^p(M)$, then* $\|S[f]\|_{L^p(M)} \approx \|f\|_{L^p(M)}$.

The above L^p estimates of the Littlewood-Paley-Stein square function are immediately used to study the boundedness of singular integral operators on M. To state the singular integral operators on M studied by Nagel and Stein, we first recall that φ is a bump function associated to a ball $B(x_0,\delta)$ if φ is supported in this ball and satisfies the differential inequalities $|\partial_X^a \varphi| \lesssim \delta^{-a}$ for all monomials ∂_X in $X_1,\ldots,X_k$ of degree a and all $a \geq 0$.

A class of singular integral operators T is initially given as a mapping from $C_0^\infty(M)$ to $C^\infty(M)$ with a distribution kernel $K(x,y)$ which is C^∞ away from the diagonal of $M \times M$, and we suppose the following properties hold:

(I-1) If $\varphi, \psi \in C_0^\infty(M)$ have disjoint supports, then

$$\langle T\varphi, \psi\rangle = \int_{M\times M} K(x,y)\varphi(y)\psi(x)dydx.$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $|\partial_X^a T\varphi| \lesssim r^{-a}$ for each integer $a \geq 0$.

(I-3) If $x \neq y$, then for every integer $a \geq 0$,

$$|\partial_{X,Y}^a K(x,y)| \lesssim d(x,y)^{-a} V(x,y)^{-1}.$$

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator T^t defined by

$$\langle T^t\varphi, \psi\rangle = \langle T\psi, \varphi\rangle.$$

Nagel and Stein proved the following

Theorem 1.2 ([49]). *Each singular integral T satisfying* (I-1) *through* (I-4) *extends to be a bounded operator on $L^p(M)$ whenever $1 < p < \infty$.*

An important example of such operators is a class of NIS operators introduced in [45]. See also [37] and [46].

To pass the above one parameter theory to the product theory, one only needs to consider the two factor case, that is, $\widetilde{M} = M_1 \times M_2$. For each M_i we have a heat operator $H_{s_i}^i$ and a corresponding $Q_{s_i}^i$, $i = 1, 2$. If f is a function on $\widetilde{M}$ we denote $Q_{s_1}^1 \cdot Q_{s_2}^2 = Q_{s_1}^1 \otimes Q_{s_2}^2$ with Q^1 acting on the M_1 and Q^2 acting on the M_2. The product square function $\widetilde{S}$ is then given by

$$\big(\widetilde{S}(f)(x)\big)^2 = \int_0^\infty \int_0^\infty |Q_{s_1}^1 \cdot Q_{s_2}^2(f)(x)|^2 \frac{ds_1 ds_2}{s_1 s_2}.$$

Nagel and Stein showed

Proposition 1.3 ([49]). *For $1 < p < \infty$, $\|\widetilde{S}(f)\|_{L^p(\widetilde{M})} \approx \|f\|_{L^p(\widetilde{M})}$.*

Similar to the single factor case, these product L^p estimates of the Littlewood-Paley-Stein square function provide a main tool for the study of the boundedness of the product singular integral operator on $\widetilde{M} = M_1 \times M_2$. Here the operator T is initially defined from $C_0^\infty(\widetilde{M})$ to $C^\infty(\widetilde{M})$. $K(x_1, y_1, x_2, y_2)$, the distribution kernel of T, is a C^∞ function away from the "cross"$= \{(x,y) : x_1 = y_1 \text{ and } x_2 = y_2;\ x = (x_1, x_2), y = (y_1, y_2)\}$ and satisfies the following additional properties:

(II-1) $\big\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2\big\rangle = \int K(x_1, y_1, x_2, y_2)\varphi_1(y_1)\varphi_2(y_2)\psi_1(x_1)\psi_2(x_2)dydx$

$$\text{whenever} \begin{cases} \varphi_1, \psi_1 \in C_0^\infty(M_1) & \text{and have disjoint support,} \\ \varphi_2, \psi_2 \in C_0^\infty(M_2) & \text{and have disjoint support.} \end{cases}$$

(II-2) For each bump function φ_2 on M_2 and each $x_2 \in M_2$, there exists a singular integral operator T^{φ_2, x_2} (of one parameter) on M_1 so that

$$\big\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2\big\rangle = \int_{M_2} \big\langle T^{\varphi_2, x_2}\varphi_1, \psi_1\big\rangle \psi_2(x_2) dx_2.$$

Moreover, $x_2 \mapsto T^{\varphi_2, x_2}$ is smooth and uniform in the sense that T^{φ_2, x_2}, as well as $\rho_2^L \partial_{X_2}^L (T^{\varphi_2, x_2})$ for each $L \geq 0$, satisfy the conditions (I-1) to (I-4) uniformly.

(II-3) If φ_i is a bump function on a ball $B^i(r_i)$ in M_i, then

$$\big|\partial_{X_1}^{a_1} \partial_{X_2}^{a_2} T(\varphi_1 \otimes \varphi_2)\big| \lesssim r_1^{-a_1} r_2^{-a_2}.$$

In (II-2) and (II-3), both inequalities are taken in the sense of (I-2) whenever φ_2 is a bump function for $B^2(r_2)$ in M_2.

(II-4) $\left|\partial^{a_1}_{X_1,Y_1}\partial^{a_2}_{X_2,Y_2}K(x_1,y_1;x_2,y_2)\right| \lesssim \dfrac{d_1(x_1,y_1)^{-a_1}d_2(x_2,y_2)^{-a_2}}{V_1(x_1,y_1)V_2(x_2,y_2)}$.

(II-5) The same conditions hold when index 1 and 2 are interchanged, that is, whenever the roles of M_1 and M_2 are interchanged.

(II-6) The same properties are assumed to hold for the 3 "transposes" of T, i.e. those operators which arise by interchanging x_1 and y_1, or interchanging x_2 and y_2, or doing both interchanges.

The main result in [49] is the following.

Theorem 1.4 ([49]). *For $1 < p < \infty$, each product singular integral satisfying conditions* (II-1) *to* (II-6) *extends to be a bounded operator on $L^p(\widetilde{M})$.*

Thanks to the earlier works of Nagel-Stein on the L^p theory for $p > 1$ on the product Carnot-Carathéodory spaces, we are naturally led to establish the corresponding product Hardy space space theory. The main concern of this paper is to develop the multiparameter Hardy space theory on the product space $\widetilde{M} = M_1 \times \cdots \times M_n$, where for $1 \le i \le n$, each (M_i, d_i, μ_i) is a space of homogeneous type satisfying only (1.1) and (1.3). The product theory focused in this paper is the following.

Question 1. Can one develop the product Hardy spaces $H^p(\widetilde{M})$ for $p_0 < p < \infty$ with some $0 < p_0 < 1$?

Question 2. Motivated by duality result of between product Hardy space H^1 and product BMO on $\mathbb{R}^n \times \mathbb{R}^m$ of Chang and R. Fefferman, can we establish the duality theory of the $H^p(\widetilde{M})$ for $p_0 < p \le 1$?

Question 3. In consideration of the L^p boundededness of singular integral operators on $\widetilde{M}$, what is the analogous endpoint estimates of singular integral operators when $p = 1$ and $p = \infty$ or more generally for $p_0 < p \le 1$?

These questions will be answered affirmatively. More precisely, we assume that each M_i is space of homogeneous type in the sense of Coifman and Weiss. To answer the first question, we only require that each quasi-metric d_i satisfies condition (1.3) with the regularity ϑ_i for $i = 1, 2, ..., n$. For the second question, one more additional condition is required, that is, for $1 \le i \le n$, there exist positive constants and Q_i such that for all $x \in M_i, i = 1, 2, ..., n$, and $\lambda \ge 1$,

$$\mu_i(B(x, \lambda r)) \approx \lambda^{Q_i}\mu_i(B(x, r)), \tag{1.7}$$

where the implicit constants are independent of x and r. This condition is satisfied by the earlier example of Nagel and Stein, when M arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. For instance, this is the case when $\Omega = \{(z, w) \in \mathbb{C}^2 : \ \mathrm{Im}(w) > P(z)\}$, where P is a real, subharmonic, nonharmonic polynomial of degree m.

Therefore, the results in this paper generalize those in [49] and, moreover, we obtain the endpoint estimates for singular integral operators considered in [49]. We will employ a unified approach to answer these questions. This approach is achieved by the following steps:

1. We first introduce the new Banach spaces, more precisely, spaces of product test functions and distributions in our framework. These spaces on the one-parameter space $\mathbb{R}^n$ were first introduced in [21] and on the one-parameter spaces

of homogeneous type with the additional conditions (1.2) and (1.3) in [33]. In [31] and [32] these spaces were defined on the one-parameter Carnot-Carathéodory-type spaces, where the volume of the ball satisfies upper and lower bounds. In this paper, we will introduce spaces of test functions and distributions on product spaces of homogeneous type in the sense of Coifman and Weiss only satisfying the doubling condition (1.1) and the regularity assumption (1.3). These include the Carnot-Carathédory spaces considered by Nagel and Stein [49].

2. We then establish discrete Calderón identity on such product test function spaces. The classical Calderón identity was first used by Calderón in [2]. Such an identity is a very powerful tool, in particular, in the theory of wavelet analysis. See [41] for more details. Using Coifman's decomposition of the identity operator, David, Journé and Semmes in [11] provided a Calderón-type identity which is a key tool to prove the $T1$ theorems on space of homogeneous type with the conditions (1.2) and (1.3) and the Tb theorem on $\mathbb{R}^n$. The continuous and discrete versions of Calderón's identities on one-parameter spaces of homogeneous type with the conditions (1.2) and (1.3) were developed in [33] and [23] (the same kind of identity was established in [31] and [32] on the one-parameter Carnot-Carathéodory-type space). In this paper, we provide discrete Caldeonón identity on the product spaces of homogeneous type in the sense of Coifman and Weiss with the condition (1.3). This identity will be the main tool for our establishing the whole product theory.

3. We next establish the Plancherel-Pôlya-type (or Sup-Inf) inequality in the multiparameter setting. The classical Plancherel-Pôlya (Sup-Inf) inequality says that the L^p norm of f whose Fourier transform has compact support is equivalent to the ℓ^p norm of the restrictions of f at appropriate lattices. This kind of inequality was first proved in [22] on a one-parameter space of homogeneous type with conditions (1.2) and (1.3), and in [31, 32] on the one-parameter Carnot-Carathéodory-type space. In this paper we prove such inequalities on product spaces of homogeneous type with condition (1.3). As an immediate consequence of the Plancherel-Pôlya inequalities, the product Hardy space on $\widetilde{M}$ is well defined. Moreover, the proofs of these inequalities can be applied to the boundedness of singular integrals on the product Hardy spaces.

4. We then develop the theory of generalized Carleson measure in the multiparameter setting. It was well known that in the classical one-parameter case, the space BMO, as the dual of H^1, can be characterized by the Carleson measure. Moreover, Chang and Fefferman in [7] proved that the dual of the product H^1 is characterized by the product Carleson measure. The generalized Carleson measure space CMO^p was first introduced in the work of the first and third authors [26]. In [24] it was further proved that the space CMO^p is the dual of the product H^p for $0 < p_0 < p \leq 1$ for some p_0 on spaces of homogeneous type with conditions (1.2) and (1.3). In this paper we introduce CMO^p on a product space of homogeneous type with conditions (1.3) and (1.7). We prove that the dual of the product Hardy space H^p studied in this paper can be characterized by CMO^p. In particular, $CMO^1 = BMO(\widetilde{M})$, the dual of $H^1(\widetilde{M})$. Note that condition (1.7) is weaker than (1.2). Therefore, our results in this paper improve those in [24].

5. We prove the $H^p - L^p$ boundedness without using atomic decomposition and Journé's covering lemma. It is well known that in the classical one-parameter case, the atomic decomposition is the main tool in proving the $H^p - L^p$ boundedness of singular integral operators. However, a deep Journé covering lemma ([35], [36])

is needed to apply this atomic decomposition for the product Hardy space on $\widetilde{M}$ to provide the boundedness of operators. (See [13] and [51] for more details.) We will employ a new method without using atomic decomposition, and hence Journé's covering lemma is not required. This is achieved by showing a general estimate, that is, $\|f\|_{L^p} \leq C\|f\|_{H^p}$ for all $f \in L^2 \cap H^p$, where C is the constant independent of the L^2 norm of f. This kind of estimate was first proved in [26] in the multiparameter setting associated with the flag singular integrals and then in [25] in the pure product spaces of homogeneous type satisfying (1.2) and (1.3).

6. We finally establish Calderón-Zygmund's decomposition on product spaces of homogeneous type satisfying (1.3). The Calderón-Zygmund decomposition was a crucial tool in developing Calderón-Zygmund operator theory. This decomposition has many applications in harmonic analysis and partial differential equations. Such a decomposition for the product Euclidean spaces was first provided by Chang and Fefferman in [8]. The main tool used in [8] is the atomic decomposition. Applying discrete Calderón identity established in this paper, Calderón-Zygmund decomposition is proved. As a consequence, we obtain the interpolation results of operators between the product Hardy spaces H^p $(0 < p \leq 1)$ and L^p $(1 < p < \infty)$.

We should point out that the method used in this paper is the discrete Littlewood-Paley theory and discrete Calderón reproducing formula. This method was first developed in the multiparameter setting in the work by the first and third authors in the multiparameter Hardy space theory associated with the flag singular integrals [26], [27]. We have followed the approach employed in [26], [27] closely. This method allows us to avoid the deep Journé covering lemma in various multiparameter settings to establish the boundedness of singular integrals on associated multiparameter Hardy spaces. These multiparameter settings also include the important Zygmund dilations [28] and implicit multiparameter structure on the Heisenberg group [29].

This paper is organized as follows. In Section 2, we develop the Littlewood-Paley-Stein theory on $\widetilde{M} = M_1 \times M_2$. To do this, we first recall some basic definitions, notions and known results established in the one-parameter case and then introduce the spaces of test functions and distributions on $\widetilde{M}$. We prove discrete Calderón identity and the Plancherel-Pôlya (Sup-Inf) inequalities (Definition 2.8 and Theorem 2.9). We introduce the Littlewood-Paley-Stein square function and define the Hardy space $H^p(\widetilde{M})$. Some important properties of $H^p(\widetilde{M})$, particularly the property bounding the L^p norm by the H^p norm for $p \leq 1$ (Theorem 2.18), are also proved in this section. In Section 3, we introduce the generalized Carleson measure space $CMO^p(\widetilde{M})$, particularly $BMO(\widetilde{M}) = CMO^1(\widetilde{M})$, and prove the Plancherel-Pôlya inequalities for $CMO^p(\widetilde{M})$ (Theorem 3.2). We also show that $CMO^p(\widetilde{M})$ is the dual space of $H^p(\widetilde{M})$ (Theorem 3.3). Section 4 deals with the boundedness of singular integral operators in the setting of two parameters. Namely, we prove that singular integral operators are bounded on the Hardy spaces H^p (Theorem 4.3) and from Hardy H^p to L^p (Corollary 4.4), and on the boundedness on $BMO(\widetilde{M})$ (Corollary 4.5). Finally, we prove the Calderón-Zygmund decomposition theorem on multiparameter Hardy spaces (Theorem 4.6) and interpolation theorem (Theorem 4.7).

We would like to point out that all the results in this paper can be easily generalized to the general product spaces of n-factors, namely, $\widetilde{M} = M_1 \times \cdots \times M_n$.

2. The Littlewood-Paley-Stein theory on product space $\widetilde{M} = M_1 \times M_2$

In this section, we consider the two factors case, that is, the underlying space is $\widetilde{M} = M_1 \times M_2$, where each $M_i, i = 1, 2$, is space of homogeneous type with the additional condition (1.7). We also shall suppose that $\mu(M_i) = \infty, \mu(\{x\}) = 0$ and $0 < V_r(x) < \infty$ for all $r > 0, x \in M_i, i = 1, 2$. Denote by Q_i the homogeneous dimension of M_i as in (1.5) for $i = 1, 2$. These hypotheses allow us to construct an approximation to the identity; see the definition below. We first recall the Littlewood-Paley-Stein theory on one single factor M.

2.1. **The Littlewood-Paley-Stein theory on M.** We begin by recalling the definition of an approximation to the identity, which plays the same role as the heat kernel $H(s, x, y)$ does in Nagel-Stein's product theory ([49]).

Definition 2.1. Let ϑ be the regularity exponent on M. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of operators is said to be an approximation to the identity if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in M$, $S_k(x, y)$, the kernel of S_k satisfies the following conditions:

$$(2.1)\quad (i)\quad S_k(x,y) = 0 \text{ if } d(x,y) \geq C2^{-k} \text{ and } |S_k(x,y)| \leq C\frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

$$(2.2)\quad (ii)\quad |S_k(x,y) - S_k(x',y)| \leq C2^{k\vartheta} d(x,x')^{\vartheta} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

(2.3) (iii) property (ii) also holds with x and y interchanged;

$$(2.4)\quad (iv)\quad [S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \leq C2^{2k\vartheta} d(x,x')^{\vartheta} d(y,y')^{\vartheta} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

$$(2.5)\quad (v)\quad \int_M S_k(x,y)d\mu(y) = \int_M s_k(x,y)d\mu(x) = 1.$$

We remark that the existence of such an approximation to the identity follows from Coifman's construction which first appeared in [11] on a space of homogeneous type with conditions (1.2) and (1.3). See also [32] for more details on M.

To define the Littlewood-Paley-Stein square function, we also need to recall the spaces of test functions and distributions on M.

Definition 2.2 ([31])**.** Let $0 < \gamma, \beta \leq \vartheta$, where ϑ is the regularity exponent on M given in (1.3) and $r > 0$. A function f defined on M is said to be a test function of type (x_0, r, β, γ) centered at $x_0 \in M$ if f satisfies the following conditions:

(i) $|f(x)| \leq C\frac{1}{V_r(x_0) + V(x,x_0)}\Big(\frac{r}{r + d(x,x_0)}\Big)^{\gamma}$;

(ii) $|f(x) - f(y)| \leq C\Big(\frac{d(x,y)}{r + d(x,x_0)}\Big)^{\beta}\frac{1}{V_r(x_0) + V(x,x_0)}\Big(\frac{r}{r + d(x,x_0)}\Big)^{\gamma}$ for all $x, y \in M$ with $d(x,y) < \frac{1}{2A}(r + d(x,x_0))$.

If f is a test function of type (x_0, r, β, γ), we write $f \in G(x_0, r, \beta, \gamma)$, and the norm of $f \in G(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{G(x_0,r,\beta,\gamma)} = \inf\{C > 0 :\ (i) \text{ and } (ii) \text{ hold}\}.$$

Now fix $x_0 \in M$; we denote $G(\beta,\gamma) = G(x_0, 1, \beta, \gamma)$ and by $G_0(\beta,\gamma)$ the collection of all test functions in $G(\beta,\gamma)$ with $\int_M f(x)dx = 0$. It is easy to check that $G(x_1, r, \beta, \gamma) = G(\beta,\gamma)$ with equivalent norms for all $x_1 \in M$ and $r > 0$. Furthermore, it is also easy to see that $G(\beta,\gamma)$ is a Banach space with respect to the norm in $G(\beta,\gamma)$.

Let $\overset{\circ}{G}_\vartheta(\beta,\gamma)$ be the completion of the space $G_0(\vartheta,\vartheta)$ in the norm of $G(\beta,\gamma)$ when $0 < \beta, \gamma < \vartheta$. If $f \in \overset{\circ}{G}_\vartheta(\beta,\gamma)$, we then define $\|f\|_{\overset{\circ}{G}_\vartheta(\beta,\gamma)} = \|f\|_{G(\beta,\gamma)}$. $(\overset{\circ}{G}_\vartheta(\beta,\gamma))'$, the distribution space, is defined by the set of all linear functionals L from $\overset{\circ}{G}_\vartheta(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{G}_\vartheta(\beta,\gamma)$,

$$|L(f)| \leq C\|f\|_{\overset{\circ}{G}_\vartheta(\beta,\gamma)}.$$

Let $D_k = S_k - S_{k-1}$, where S_k is an approximation to the identity on M with the regularity exponent ϑ. The Littlewood-Paley-Stein square function is defined by

Definition 2.3 ([31])**.** For each $f \in (\overset{\circ}{G}_\vartheta(\beta,\gamma))'$ with $0 < \beta, \gamma < \vartheta$, $S(f)$, the Littlewood-Paley-Stein square function of f is defined by

$$S(f)(x) = \Big\{ \sum_k |D_k(f)(x)|^2 \Big\}^{\frac{1}{2}}.$$

The above Littlewood-Paley-Stein square function allows one to formally define the Hardy space on M.

Definition 2.4 ([31])**.**

$$H^p(M) = \{f \in (\overset{\circ}{G}_\vartheta(\beta,\gamma))', 0 < \beta, \gamma < \vartheta : S(f) \in L^p(M)\},$$

and if $f \in H^p(M)$, the norm of f is defined by $\|f\|_{H^p(M)} = \|S(f)\|_{L^p}$.

To show that $H^p(M)$ is well defined, we need the following Plancherel-Pôlya inequalities.

Theorem 2.5 ([31])**.** *Let $\{S_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ be two approximations to the identity with regularity exponent θ. For $k \in \mathbb{Z}$, set $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$. For a fixed large integer N and all $f \in \big(\overset{\circ}{G}_\vartheta(\beta,\gamma)\big)'$ with $0 < \beta, \gamma < \vartheta$, $\frac{Q}{Q+\vartheta} < p < \infty$ where Q is the homogeneous dimension given in* (1.5),

$$\Big\| \Big\{ \sum_{k=-\infty}^{\infty} \sum_\tau \sup_{z \in I_\tau^{k+N}} \big|D_k(f)(z)\big|^2 \chi_{I_\tau^{k+N}}(\cdot) \Big\}^{1/2} \Big\|_{L^p(M)} \tag{2.6}$$
$$\approx \Big\| \Big\{ \sum_{k=-\infty}^{\infty} \sum_\tau \inf_{z \in I_\tau^{k+N}} \big|E_k(f)(z)\big|^2 \chi_{I_\tau^{k+N}}(\cdot) \Big\}^{1/2} \Big\|_{L^p(M)}.$$

Here I_τ^k are "dyadic cubes" on M given independently by Christ [4] and Sawyer and Wheeden [53]. The following is taken from [4].

Theorem 2.6 ([4])**.** *Let (M, ρ, μ) be a space of homogeneous type. Then, there exists a collection $\{I_\alpha^k \subset M : k \in \mathbb{Z}, \alpha \in I^k\}$ of open subsets, where I^k is some* *index set and* $C_1, C_2 > 0$*, such that:*

(i) $\mu(M \setminus \bigcup_\alpha I_\alpha^k) = 0$ *for each fixed* k *and* $I_\alpha^k \cap I_\beta^k =$ *if* $\alpha \neq \beta$*;*

(ii) *for any* α, β, k, l *with* $l \geq k$*, either* $I_\beta^l \subset I_\alpha^k$ *or* $I_\beta^l \cap I_\alpha^k = \emptyset$*;*

(iii) *for each* (k, α) *and each* $l < k$ *there is a unique* β *such that* $I_\alpha^k \subset I_\beta^l$*;*

(iv) $\mathrm{diam}(I_\alpha^k) \leq C_1 2^{-k}$*;*

(v) *each* I_α^k *contains some ball* $B(z_\alpha^k, C_2 2^{-k})$*, where* $z_\alpha^k \in M$*.*

Therefore, we can think of I_α^k as being a dyadic cube with diameter roughly 2^{-k} centered at z_α^k. As a result, we consider CI_α^k to be the cube with the same center as I_α^k and diameter $C\mathrm{diam}(I_\alpha^k)$. To simplify notation, we will call I dyadic cubes and denote the side length of I by $\ell(I)$.

As an immediate consequence of the Plancherel-Pôlya inequalities, the Hardy space $H^p(M)$ is well defined. The key tool of the proof of the Plancherel-Pôlya inequalities is the Calderón identity. It is well known that this kind of identity is a powerful tool in classical harmonic analysis. See [2] and [41] for the classical case and [10] for spaces of homogeneous type. Here we concentrate on the following discrete Calderón identity.

Theorem 2.7 ([31, 32])**.** *Let* $\{S_k\}_{k\in\mathbb{Z}}$ *be an approximation to the identity with regularity exponent* ϑ*. Set* $D_k = S_k - S_{k-1}, k \in \mathbb{Z}$*. Then there exist families of linear operators* $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ *and* $\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$ *such that for any fixed* $y_\tau^{k+N} \in I_\tau^{k+N}$*, where* N *is the same as in Theorem* 2.6*, and all* $f \in \overset{\circ}{G}_\vartheta(\beta, \gamma)$ *with* $0 < \beta, \gamma < \vartheta$*,*

$$
\begin{aligned}
(2.7) \qquad f(x) &= \sum_{k=-\infty}^{\infty} \sum_\tau \mu(I_\tau^{k+N}) \widetilde{D}_k(x, y_\tau^{k+N}) D_k(f)(y_\tau^{k+N}) \\
&= \sum_{k=-\infty}^{\infty} \sum_\tau \mu(I_\tau^{k+N}) D_k(x, y_\tau^{k+N}) \widetilde{\widetilde{D}}_k(f)(y_\tau^{k+N}),
\end{aligned}
$$

where the series converges in both the norm of $\overset{\circ}{G}_\vartheta(\beta', \gamma')$ *with* $0 < \beta' < \beta < \vartheta, \gamma' < \gamma < \vartheta$ *and the norm of* $L^p(X)$ *with* $1 < p < \infty$*. Moreover,* $\widetilde{D}_k(x, y)$ *and* $\widetilde{\widetilde{D}}_k(x, y)$*, the kernels of* $\widetilde{D}_k$ *and* $\widetilde{\widetilde{D}}_k$*, satisfy the similar estimates but with* x *and* y *interchanged in (ii): for* $0 < \epsilon < \vartheta$*,*

(2.8)

$$
(i) \quad |\widetilde{D}_k(x, y)| \leq C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon};
$$

(2.9)

$$
\begin{aligned}
(ii) \quad |\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| &\leq C\Big(\frac{d(x, x')}{2^{-k} + d(x, y)}\Big)^\epsilon \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \\
&\quad \times \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} \text{ for } d(x, x') \leq (2^{-k} + d(x, y))/2A;
\end{aligned}
$$

(2.10)

$$
(iii) \quad \int_M \widetilde{D}_k(x, y) d\mu(y) = \int_M \widetilde{D}_k(x, y) d\mu(x) = 0.
$$

We now return to the Littlewood-Paley-Stein theory on $\widetilde{M} = M_1 \times M_2$.

2.2. **Test functions and distributions on $\widetilde{M}$.** We first introduce the space of test functions and distributions on $\widetilde{M}$.

Definition 2.8. Let $(x_0, y_0) \in \widetilde{M}$, $0 < \gamma_1, \gamma_2, \beta_1, \beta_2 \leq \vartheta$ and $r_1, r_2 > 0$. A function $f(x, y)$ defined on $\widetilde{M}$ is said to be a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ if for any fixed $y, y' \in M_2$, $f(x, y)$, as a function of the variable of x, is a test function in $G(x_0, r_1, \beta_1, \gamma_1)$ on M_1. Similarly, for any fixed $x, x' \in M_1$, $f(x, y)$, as a function of the variable of y, is a test function in $G(y_0, r_2, \beta_2, \gamma_2)$ on M_2. Moreover, the following conditions are satisfied:

(i) $\|f(\cdot, y)\|_{G(x_0,r_1,\beta_1,\gamma_1)} \leq C \frac{1}{V_{r_2}(y_0) + V(y_0, y)} \Big(\frac{r_2}{r_2 + d(y, y_0)}\Big)^{\gamma_2}$;

(ii) $\|f(\cdot, y) - f(\cdot, y')\|_{G(x_0,r_1,\beta_1,\gamma_1)}$
$$\leq C \Big(\frac{d(y, y')}{r_2 + d(y, y_0)}\Big)^{\beta_2} \frac{1}{V_{r_2}(y_0) + V(y_0, y)} \Big(\frac{r_2}{r_2 + d(y, y_0)}\Big)^{\gamma_2}$$
for all $y, y' \in M_2$ with $d(y, y') \leq (r_2 + d(y, y_0))/2A$;

(iii) properties $(i) - (ii)$ also hold with x and y interchanged.

If f is a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$, we write
$$f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2),$$
and the norm of f is defined by
$$\|f\|_{G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \ (i), (ii) \text{ and } (iii) \ \text{ hold}\}.$$

Similarly, we denote by $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $G(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$ for any fixed $(x_0, y_0) \in \widetilde{M}$. Set that $f(x, y) \in G_0(\beta_1, \beta_2; \gamma_1, \gamma_2)$ if $\int\limits_{M_1} f(x, y)dx = \int\limits_{M_2} f(x, y)dy = 0$. We can check that $G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with equivalent norms for all $(x_0, y_0) \in \widetilde{M}$ and $r_1, r_2 > 0$. Furthermore, it is easy to see that $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

Let $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ be the completion of the space $G_0(\vartheta_1, \vartheta_2; \vartheta_1, \vartheta_2)$ in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $0 < \beta_i, \gamma_i < \vartheta_i$, where ϑ_i is the regularity exponent on $M_i, i = 1, 2$. If $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, we then define $\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} = \|f\|_{G(\beta_1,\beta_2;\gamma_1,\gamma_2)}$.

We define the distribution space $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$ by all linear functionals L from $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,
$$|L(f)| \leq C\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

2.3. **Discrete Calderón identity on $\widetilde{M}$.** We now provide discrete Calderón identity on $\widetilde{M}$.

Theorem 2.9. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be given in Theorem 2.7 on each $M_i, i = 1, 2,$ respectively. Then*

$$f(x,y) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2}\mu_1(I_{\tau_1}^{k_1+N_1})\mu_2(I_{\tau_2}^{k_2+N_2}) \times \widetilde{D}_{k_1}(x,y_{\tau_1}^{k_1+N_1})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2+N_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1+N_1},y_{\tau_2}^{k_2+N_2}), \tag{2.11}$$

where the series converges in both the norm of $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1',\beta_2',\gamma_1',\gamma_2')$ with $0<\beta_i'<\beta_i<\vartheta_i, \gamma_i',\gamma_i<\vartheta_i, i=1,2,$ and the norm of $L^p(M_1\times M_2), 1<p<\infty$.

Proof. The proof of this theorem is based on the method of iteration and some known estimates on one single factor M. We first show the $L^p, 1<p<\infty$, convergence. Denote

$$\begin{aligned} g(x,y) = & \sum_{|k_1|\le L_1}\sum_{|k_2|\le L_2}\sum_{\tau_1}\sum_{\tau_2}\mu_1(Q_{\tau_1}^{k_1+N_1})\mu_2(Q_{\tau_2}^{k_2+N_2})\widetilde{D}_{k_1}(x,y_{\tau_1}^{k_1+N_1}) \\ & \times\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2+N_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1+N_1},y_{\tau_2}^{k_2+N_2}) - f(x,y) \\ =: & \ g_1(x,y)+g_2(x,y), \end{aligned}$$

where

$$\begin{aligned} g_1(x,y) = & \sum_{|k_1|\le L_1}\sum_{\tau_1}\mu_1(Q_{\tau_1}^{k_1+N_1})\widetilde{D}_{k_1}(x,y_{\tau_1}^{k_1+N_1})D_{k_1}\Big(\sum_{|k_2|\le L_2}\sum_{\tau_2}\mu_2(Q_{\tau_2}^{k_2+N_2}) \\ & \times\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2+N_2})D_{k_2}(f(\cdot,y_{\tau_2}^{k_2+N_2}))\Big)(y_{\tau_1}^{k_1+N_1}) \\ & -\sum_{|k_2|\le L_2}\sum_{\tau_2}\mu_2(Q_{\tau_2}^{k_2+N_2})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2+N_2})D_{k_2}(f)(x,y_{\tau_2}^{k_2+N_2}) \end{aligned}$$

and

$$g_2(x,y) = \sum_{|k_2|\le L_2}\sum_{\tau_2}\mu_2(Q_{\tau_2}^{k_2+N_2})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2+N_2})D_{k_2}(f)(x,y_{\tau_2}^{k_2+N_2}) - f(x,y).$$

We now need the following estimates on one single factor M: There exists a constant C such that for $f\in L^p(M), 1<p<\infty$, and any integers L,

$$\Big\|\sum_{|k|\le L}\sum_{\tau}\mu(Q_\tau^{k+N})\widetilde{D}_k(x,y_\tau^{k+N})D_k(f)(y_\tau^{k+N})\Big\|_p \le C\|f\|_p \tag{2.12}$$

and

$$\Big\|\sum_{|k|\le L}\sum_{\tau}\mu(Q_\tau^{k+N})\widetilde{D}_k(x,y_\tau^{k+N})D_k(f)(y_\tau^{k+N}) - f\Big\|_p \le C\Big\|\Big\{\sum_{|k|\ge L}\sum_{\tau}|D_k(f)|^2\Big\}^{\frac12}\Big\|_p. \tag{2.13}$$

Using (2.13) first and then (2.12) yields

$$\|g_1(x,y)\|_{L^p} \le C\Big\|\Big\{\sum_{|k_1|\ge L_1}\sum_{|k_2|\le L_2}|D_{k_1}D_{k_2}(f)|^2\Big\}^{\frac12}\Big\|_p,$$

where the last term goes to zero as L_1 goes to infinity. $\|g_2(x,y)\|_p$ can be handled similarly. This implies the convergence in $L^p(\widetilde{M}), 1<p<\infty$.

To see the convergence in the space of test functions, we need the following estimates on one single factor M: For $f \in \overset{\circ}{G}_\vartheta(\beta,\gamma)$ and any integers L,

$$\| \sum_{|k|\le L} \sum_\tau \mu(Q_\tau^{k+N}) \widetilde{D}_k(x, y_\tau^{k+N}) D_k(f)(y_\tau^{k+N}) \|_{\overset{\circ}{G}_\vartheta(\beta,\gamma)} \le C\|f\|_{\overset{\circ}{G}_\vartheta(\beta,;\gamma)} \tag{2.14}$$

and

$$\| \sum_{|k|\le L} \sum_\tau \mu(Q_\tau^{k+N}) \widetilde{D}_k(x, y_\tau^{k+N}) D_k(f)(y_\tau^{k+N}) - f \|_{\overset{\circ}{G}_\vartheta(\beta',\gamma')} \le C2^{-L\delta}\|f\|_{\overset{\circ}{G}_\vartheta(\beta,\gamma)}, \tag{2.15}$$

where C is a constant, $0 < \beta' < \beta, 0 < \gamma' < \gamma$ and $\delta > 0$.

We observe that if $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$, then $\|f(\cdot,y)\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1,\gamma_1)}$, as a function of the variable y, is in $\overset{\circ}{G}_{\vartheta_2}(\beta_2,\gamma_2)$ and

$$\big\| \|f(\cdot,\cdot)\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1,\gamma_1)} \big\|_{\overset{\circ}{G}_{\vartheta_2}(\beta_2,\gamma_2)} \le \|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

Similarly, $\big\| \|f(\cdot,\cdot)\|_{\overset{\circ}{G}_{\vartheta_2}(\beta_2,\gamma_2)} \big\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1,\gamma_1)} \le \|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)}$. Therefore, we obtain

$$\begin{aligned}
&\|g_1(\cdot,y)\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1',\gamma_1')} \\
&\le\ C2^{-L_1\delta}\big\| \sum_{|k_2|\le L_2} \sum_{\tau_2} \mu_2(Q_{\tau_2}^{k_2+N_2}) \widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2+N_2}) D_{k_2}(f(\cdot, y_{\tau_2}^{k_2+N_2})) \big\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1,\gamma_1)} \\
&\le\ C2^{-L_1\delta}\big\| \|f(\cdot,\cdot)\|_{\overset{\circ}{G}_{\vartheta_2}(\beta_2,\gamma_2)} \frac{1}{V_{r_2}(y_0)+V(y_0,y)} \Big(\frac{r_2}{r_2+d(y,y_0)}\Big)^{\gamma_2} \big\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1,\gamma_1)} \\
&\le\ C2^{-L_1\delta}\|f(\cdot,\cdot)\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} \frac{1}{V_{r_2}(y_0)+V(y_0,y)} \Big(\frac{r_2}{r_2+d(y,y_0)}\Big)^{\gamma_2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|g_2(x,y)\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1',\gamma_1')} \\
&\qquad \le C2^{-L_2\delta}\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} \frac{1}{V_{r_2}(y_0)+V(y_0,y)} \Big(\frac{r_2}{r_2+d(y,y_0)}\Big)^{\gamma_2}.
\end{aligned}$$

Noting that $g(x,y) - g(x,y') = [g_1(x,y) - g_1(x,y')] + [g_2(x,y) - g_2(x,y')]$ and repeating the same estimates imply

$$\begin{aligned}
&\|g(\cdot,y) - g(\cdot,y')\|_{\overset{\circ}{G}_{\vartheta_1}(\beta_1',\gamma_1')} \\
&\qquad \le\ C(2^{-L_1\delta} + 2^{-L_2\delta})\|f(\cdot,\cdot)\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} \Big(\frac{d(y,y')}{r_2+d(y,y_0)}\Big)^{\beta_2} \\
&\qquad\quad \times \frac{1}{V_{r_2}(y_0)+V(y_0,y)} \Big(\frac{r_2}{r_2+d(y,y_0)}\Big)^{\gamma_2},
\end{aligned}$$

where $d(y,y') \le (r_2 + d(y,y_0))/2A$.

The same proof can be carried out to the estimates with x and y interchanged. This implies that

$$\|g(x,y)\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1',\beta_2';\gamma_1',\gamma_2')} \le C(2^{-L_1\delta}+2^{-L_2\delta})\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)},$$

which yields the convergence in $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1',\beta_2';\gamma_1',\gamma_2')$. □

2.4. **Plancherel-Pôlya inequalities on $\widetilde{M}$.** Using discrete Calderón identity we prove the following Plancherel-Pôlya inequalities on $\widetilde{M}$.

Theorem 2.10. *Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{P_{k_i}\}_{k_i\in\mathbb{Z}}$ be two approximations to the identity on $M_i, i=1,2$. For $k_i\in\mathbb{Z}$, set $D_{k_i}=S_{k_i}-S_{k_i-1}$ and $E_{k_i}=P_{k_i}-P_{k_i-1}$. For all $f\in\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$ with $0<\beta_i,\gamma_i<\vartheta_i$, $\frac{Q_i}{Q_i+\vartheta_i}<p<\infty, i=1,2,$*

(2.16)

$$\begin{aligned}
&\Big\|\Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2}\\
&\qquad \sup_{(z_1,z_2)\in I_{\tau_1}^{k_1+N_1}\times I_{\tau_2}^{k_2+N_2}}\big|D_{k_1}D_{k_2}(f)(z_1,z_2)\big|^2\chi_{I_{\tau_1}^{k_1+N_1}}(\cdot)\chi_{I_{\tau_2}^{k_2+N_2}}(\cdot)\Big\}^{1/2}\Big\|_{L^p(\widetilde{M})}\\
&\approx\Big\|\Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2}\\
&\qquad \inf_{(z_1,z_2)\in I_{\tau_1}^{k_1+N_1}\times I_{\tau_2}^{k_2+N_2}}\big|E_{k_1}E_{k_2}(f)(z_1,z_2)\big|^2\chi_{I_{\tau_1}^{k_1+N_1}}(\cdot)\chi_{I_{\tau_2}^{k_2+N_2}}(\cdot)\Big\}^{1/2}\Big\|_{L^p(\widetilde{M})}.
\end{aligned}$$

The proof of the theorem follows from discrete Calderón identity and the orthogonality argument. We first state the orthogonality argument as the following lemma and skip the proof because it is similar to the case of one single factor given in [31, 32].

Lemma 2.11. *Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{P_{k_i}\}_{k_i\in\mathbb{Z}}$ be two approximations to the identity with regularity exponent ϑ_i and $D_{k_i}=S_{k_i}-S_{k_i-1}, E_{k_i}=P_{k_i}-P_{k_i-1}, i=1,2$. Then for any $\epsilon\in(0,\vartheta_1\wedge\vartheta_2)$, there exist positive constants C depending only on ϵ such that $D_{l_1}D_{l_2}E_{k_1}E_{k_2}(x_1,x_2,y_1,y_2)$, the kernel of $D_{l_1}D_{l_2}E_{k_1}E_{k_2}$, satisfies the following estimate:*

$$\begin{aligned}
(2.17)\ |D_{l_1}D_{l_2}E_{k_1}E_{k_2}(x_1,x_2,y_1,y_2)| &\le C2^{-|k_1-l_1|\epsilon}2^{-|k_2-l_2|\epsilon}\\
&\times\frac{1}{V_{2^{-(k_1\wedge l_1)}}(x_1)+V_{2^{-(k_1\wedge l_1)}}(y_1)+V(x_1,y_1)}\frac{2^{-(k_1\wedge l_1)\epsilon}}{(2^{-(k_1\wedge l_1)}+d(x_1,y_1))^{\epsilon}}\\
&\times\frac{1}{V_{2^{-(k_2\wedge l_2)}}(x_2)+V_{2^{-(k_2\wedge l_2)}}(y_2)+V(x_2,y_2)}\frac{2^{-(k_2\wedge l_2)\epsilon}}{(2^{-(k_2\wedge l_2)}+d(x_2,y_2))^{\epsilon}}.
\end{aligned}$$

We also need the discrete version of the Hardy-Littlewood maximal function estimate on one single factor M, which is an analogue to a result of [15] in the Euclidean space. See also [31, 32].

Lemma 2.12. *Let $\epsilon>0$, $k,k'\in\mathbb{Z}$ and y_τ^k be any point in I_τ^k for $\tau\in I_k$. If $\frac{Q}{Q+\epsilon}<r<p\le 1$, then there exists a constant $C>0$ depending only on r such that* *for all* $a_\tau^k \in \mathbb{C}$ *and all* $x \in M$,

$$\sum_{\tau\in I_k} \mu(I_\tau^k)\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y_\tau^k)}\frac{2^{-(k\wedge k')\epsilon}}{\left(2^{-(k\wedge k')}+d(x,y_\tau^k)\right)^{\epsilon}}|a_\tau^k|$$
$$\le C2^{[(k\wedge k')-k]Q(1-1/r)}\Big\{\mathcal{M}\Big(\sum_{\tau\in I_k}|a_\tau^k|^r\chi_{Q_\tau^k}(\cdot)\Big)(x)\Big\}^{1/r},$$

where $\mathcal{M}$ *is the Hardy-Littlewood maximal function on* M.

We now return to the proof of Theorem 2.10.

Proof of Theorem 2.10. We choose r such that $\max\left(\frac{Q_1}{Q_1+\vartheta_1},\frac{Q_2}{Q_2+\vartheta_2}\right) < r < p \le 1$. For any $f \in \left(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\right)'$, we rewrite Theorem 2.9 by

$$f(x,y)=\sum_{k_1',k_2'}\sum_{I',J'}|I'||J'|\widetilde{D}_{k_1'}(x,x_{I'})\widetilde{D}_{k_2'}(y,y_{J'})D_{k_1'}D_{k_2'}(f)(x_{I'},y_{J'}), \tag{2.18}$$

where $|I'|=\mu_1(Q_{\tau_1'}^{k_1'+N_1'}), |J'|=\mu_2(Q_{\tau_2'}^{k_2'+N_2'}), x_{I'}=y_{\tau_1'}^{k_1'+N_1'}$ and $y_{J'}=y_{\tau_2'}^{k_2'+N_2'}$.

By the orthogonality argument we obtain

$$|Q_{k_i}\widetilde{D}_{k_i'}(x,y)| \le C2^{-|k_i-k_i'|\epsilon_i}\frac{1}{V(x,y)+V_{2^{-(k_i\wedge k_i')}}(x)+V_{2^{-(k_i\wedge k_i')}}(y)} \times\Big(\frac{2^{-(k_i\wedge k_i')}}{2^{-(k_i\wedge k_i')}+d(x,y)}\Big)^{\epsilon_i}, \tag{2.19}$$

where $\epsilon_i<\vartheta_i, i=1,2$. From (2.18) and (2.19), for any $k_1,k_2\in\mathbb{Z}$, we have

(2.20)

$$\begin{aligned}
&|Q_{k_1}Q_{k_2}(f)(x,y)|\\
&\quad= \Big|\sum_{k_1',k_2'}\sum_{I',J'}|I'||J'|Q_{k_1}\widetilde{D}_{k_1'}(x_1,x_{I'})Q_{k_2}\widetilde{D}_{k_2'}(x_2,y_{J'})D_{k_1'}D_{k_2'}(f)(x_{I'},y_{J'})\Big|\\
&\quad\le C\sum_{k_1',k_2'}\sum_{I',J'}|I'||J'|2^{-|k_1-k_1'|\epsilon_1}2^{-|k_-k_2'|\epsilon_2}\quad |Q_{k_1'}Q_{k_2'}(f)(x_{I'},y_{J'})|\\
&\qquad\times\frac{1}{V(x_I,x_{I'})+V_{2^{-(k_1\wedge k_1')}}(x_I)+V_{2^{-(k_1\wedge k_1')}}(x_{I'})}\Big(\frac{2^{-(k_1\wedge k_1')}}{2^{-(k_1\wedge k_1')}+d(x_I,x_{I'})}\Big)^{\epsilon_1}\\
&\qquad\times\frac{1}{V(x_J,x_{J'})+V_{2^{-(k_2\wedge k_2')}}(x_J)+V_{2^{-(k_2\wedge k_2')}}(x_{J'})}\Big(\frac{2^{-(k_2\wedge k_2')}}{2^{-(k_2\wedge k_2')}+d(x_J,x_{J'})}\Big)^{\epsilon_2}.
\end{aligned}$$

Therefore,

$$\sum_{k_1,k_2}\sum_{I,J}\sup_{u\in I,v\in J}|Q_{k_1}Q_{k_2}(f)(u,v)|^2\ \chi_I(x)\chi_J(y)$$
$$\leq C\sum_{k_1,k_2}\sum_{I,J}\Bigg(\sum_{k_1',k_2'}\sum_{I',J'}|I'||J'|2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}\quad|Q_{k_1'}Q_{k_2'}(f)(x_{I'},y_{J'})|$$
$$\times\frac{1}{V(x_I,x_{I'})+V_{2^{-(k_1\wedge k_1')}}(x_I)+V_{2^{-(k_1\wedge k_1')}}(x_{I'})}\Big(\frac{2^{-(k_1\wedge k_1')}}{2^{-(k_1\wedge k_1')}+d(x_I,x_{I'})}\Big)^{\epsilon_1}$$
$$\times\frac{1}{V(x_J,x_{J'})+V_{2^{-(k_2\wedge k_2')}}(x_J)+V_{2^{-(k_2\wedge k_2')}}(x_{J'})}\Big(\frac{2^{-(k_2\wedge k_2')}}{2^{-(k_2\wedge k_2')}+d(x_J,x_{J'})}\Big)^{\epsilon_2}\Bigg)^2$$
$$\times\chi_I(x)\chi_J(y).$$

By the Cauchy-Schwartz inequality, the last term above is dominated by

$$C\sum_{k_1,k_2}\sum_{I,J}\Bigg(\sum_{k_1',k_2'}2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}\sum_{I',J'}|I'||J'|\quad|Q_{k_1'}Q_{k_2'}(f)(x_{I'},y_{J'})|^2$$
$$\times\ \frac{1}{V(x_I,x_{I'})+V_{2^{-k_1}\vee 2^{-k_1'}}(x_I)+V_{2^{-k_1}\vee 2^{-k_1'}}(x_{I'})}\Big(\frac{2^{-(k_1\wedge k_1')}}{2^{-(k_1\wedge k_1')}+d(x_I,x_{I'})}\Big)^{\epsilon_1}$$
$$\times\ \frac{1}{V(x_J,x_{J'})+V_{2^{-(k_2\wedge k_2')}}(x_J)+V_{2^{-(k_2\wedge k_2')}}(x_{J'})}\Big(\frac{2^{-(k_2\wedge k_2')}}{2^{-(k_2\wedge k_2')}+d(x_J,x_{J'})}\Big)^{\epsilon_2}\Bigg)^2$$
$$\times\ \chi_I(x)\chi_J(y).$$

Now apply Lemma 2.12 and Hölder's inequality. Then the above term is bounded by

$$\leq C\sum_{k_1,k_2}\sum_{I,J}\Bigg(\sum_{k_1',k_2'}2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}2^{[(k_1\wedge k_1')-k_1']Q_1(1-\frac{1}{r})}2^{[(k_2\wedge k_2')-k_2']Q_2(1-\frac{1}{r})}$$
$$\times\Big[\mathcal{M}_1\Big(\sum_{I'}\mathcal{M}_2\Big(\sum_{J'}\inf_{u\in I',v\in J'}\big||Q_{k_1'}Q_{k_2'}(f)(u,v)\big|^r\chi_{J'}(\cdot)\Big)(y)\chi_{I'}(\cdot)\Big)(x)\Big]^{\frac{1}{r}}\Bigg)^2$$
$$\times\chi_I(x)\chi_J(y)$$
$$\leq C\sum_{k_1,k_2}\sum_{I,J}\Bigg(\Big[\sum_{k_1',k_2'}2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}2^{[(k_1\wedge k_1')-k_1']Q_1(1-\frac{1}{r})}2^{[(k_2\wedge k_2')-k_2']Q_2(1-\frac{1}{r})}\Big]^{1/2}$$
$$\times\Bigg[\sum_{k_1',k_2'}2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}2^{[(k_1\wedge k_1')-k_1']Q_1(1-\frac{1}{r})}2^{[(k_2\wedge k_2')-k_2']Q_2(1-\frac{1}{r})}$$
$$\times\Big[\mathcal{M}_1\Big(\sum_{I'}\mathcal{M}_2\Big(\sum_{J'}\inf_{u\in I',v\in J'}\big||Q_{k_1'}Q_{k_2'}(f)(u,v)\big|^r\chi_{J'}(\cdot)\Big)(y)\chi_{I'}(\cdot)\Big)(x)\Big]^{\frac{2}{r}}\Bigg]^{1/2}\Bigg)^2$$
$$\times\chi_I(x)\chi_J(y).$$

Since $\max\left(\frac{Q_1}{Q_1+\vartheta_1}, \frac{Q_2}{Q_2+\vartheta_2}\right) < r < p \leq 1$,

$$\Big[\sum_{k_1',k_2'} 2^{-|k_1-k_1'|\epsilon_1} 2^{-|k_2-k_2'|\epsilon_2} 2^{[(k_1\wedge k_1')-k_1']Q_1(1-\frac{1}{r})} 2^{[(k_2\wedge k_2')-k_2']Q_2(1-\frac{1}{r})}\Big] \leq C < \infty,$$

which together with the Hölder inequality yields

(2.21)

$$\begin{aligned}
&\Big\{\sum_{k_1,k_2}\sum_{I,J}\sup_{u\in I, v\in J}|Q_{k_1}Q_{k_2}(f)(u,v)|^2\ \chi_I(x)\chi_J(y)\Big\}^{\frac{1}{2}} \\
&\leq C\Big\{\sum_{k_1,k_2}\sum_{I,J}\sum_{k_1',k_2'} 2^{-|k_1-k_1'|\epsilon_1} 2^{-|k_2-k_2'|\epsilon_2} 2^{[(k_1\wedge k_1')-k_1']Q_1(1-\frac{1}{r})} 2^{[(k_2\wedge k_2')-k_2']Q_2(1-\frac{1}{r})} \\
&\quad \times \Big[\mathcal{M}_1\Big(\sum_{I'}\mathcal{M}_2\Big(\sum_{J'}\inf_{u\in I', v\in J'}\big||Q_{k_1'}Q_{k_2'}(f)(u,v)|^r\chi_{J'}(\cdot)\Big)(y)\chi_{I'}(\cdot)\Big)(x)\Big]^{\frac{2}{r}} \\
&\quad \times \chi_I(x)\chi_J(y)\Big\}^{\frac{1}{2}},
\end{aligned}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the Hardy-Littlewood maximal function on M_1 and M_2, respectively. By taking the $L^p(\widetilde{M})$ norm on both sides of (2.21) and using Fefferman-Stein's vector-valued maximal function inequality, the proof of Theorem 2.10 is concluded. □

2.5. The Littlewood-Paley-Stein square function and the Hardy space on $\widetilde{M}$.

We now introduce the Littlewood-Paley-Stein square function.

Definition 2.13. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be approximations to the identity on M_i and $D_{k_i} = S_{k_i} - S_{k_i-1}, i = 1, 2$. For $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$ with $0 < \beta_i, \gamma_i < \vartheta_i, i = 1, 2$, $\widetilde{S}(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$\widetilde{S}(f)(x,y) = \Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\big|D_{k_1}D_{k_2}(f)(x,y)\big|^2\Big\}^{1/2}.$$

By the results on each $M_i, i = 1, 2$, and iteration as given in [14], we immediately obtain

Theorem 2.14. *If $f \in L^p(\widetilde{M}), 1 < p < \infty$, then $\|\widetilde{S}(f)\|_p \approx \|f\|_p$.*

We, however, point out that the following discrete Littlewood-Paley-Stein square function is more convenient for the study of the Hardy space H^p when $p \leq 1$.

Definition 2.15. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be approximations to the identity on M_i and $D_{k_i} = S_{k_i} - S_{k_i-1}, i = 1, 2$. For $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$ with $0 < \beta_i, \gamma_i < \vartheta_i, i = 1, 2$, $\widetilde{S}_d(f)$, the discrete Littlewood-Paley-Stein square function of f, is defined by

$$\widetilde{S}_d(f)(x,y) = \Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2}|D_{k_1}D_{k_2}(f)(x,y)|^2\chi_{I_{\tau_1}^{k_1+N_1}}(x)\chi_{I_{\tau_2}^{k_2+N_2}}(y)\Big\}^{1/2}.$$

By the Plancherel-Pôlya inequalities in Theorem 2.10, it is not difficult to see that the L^p norm of these two kinds of square functions are equivalent. More precisely, we have

Proposition 2.16. *For all* $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$ *with* $0<\beta_i,\gamma_i<\vartheta_i, \frac{Q_i}{Q_i+\vartheta_i} < p < \infty, i=1,2,$ *then* $\|\widetilde{S}(f)\|_p \approx \|\widetilde{S}_d(f)\|_p$.

We are ready to introduce the Hardy spaces on $\widetilde{M}$.

Definition 2.17.

$$H^p(\widetilde{M}) = \big\{ f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)' : 0<\beta_i,\gamma_i<\vartheta_i, i=1,2, \widetilde{S}_d(f) \in L^p(\widetilde{M}) \big\},$$

and if $f \in H^p(\widetilde{M})$, the norm of f is defined by $\|f\|_{H^p(\widetilde{M})} = \|\widetilde{S}_d(f)\|_p$.

Obviously, by the Plancherel-Pôlya inequalities, the Hardy space $H^p(\widetilde{M})$ is well defined. Before ending this section, we prove the following general result which will be used to provide the $H^p - L^p$ boundedness in Section 4. We also would like to mention that the proof of this general result does not use atomic decomposition, and thus Journé's covering lemma is not required.

Theorem 2.18. *Let* $\max\big(\frac{Q_1}{Q_1+\vartheta_1}, \frac{Q_2}{Q_2+\vartheta_2}\big) < p \le 1$. *If* $f \in L^2(\widetilde{M}) \cap H^p(\widetilde{M})$, *then* $f \in L^p(\widetilde{M})$ *and there exists a constant* $C_p>0$ *which is independent of the* L^2 *norm of* f *such that*

$$\|f\|_{L^p(\widetilde{M})} \le C_p \|f\|_{H^p(\widetilde{M})}.$$

To show the theorem above, we need another version of discrete Calderón identity. Indeed, by Theorem 2.7, for $f \in L^2(M)$,

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{\tau} \mu(Q_\tau^{k+N}) D_k(x, y_\tau^{k+N}) \widetilde{\widetilde{D}}_k(f)(y_\tau^{k+N}). \tag{2.22}$$

Note that the kernel of D_k has compact support, but not for the kernel of $\widetilde{\widetilde{D}}_k$. Moreover, by the same proof, for $\max\big(\frac{Q_1}{Q_1+\vartheta_1}, \frac{Q_2}{Q_2+\vartheta_2}\big) < p \le 1$ and $f \in H^p(\widetilde{M})$, we have

$$\begin{aligned}
\|f\|_{H^p(\widetilde{M})} &\approx \Big\| \Big\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1} \sum_{\tau_2} |D_{k_1} D_{k_2}(f)(x,y)|^2 \\
&\qquad \chi_{I_{\tau_1}^{k_1+N_1}}(x) \chi_{I_{\tau_2}^{k_2+N_2}}(y) \Big\}^{1/2} \Big\|_{L^p(\widetilde{M})} \\
&\approx \Big\| \Big\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1} \sum_{\tau_2} |\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x,y)|^2 \\
&\qquad \chi_{I_{\tau_1}^{k_1+N_1}}(x) \chi_{I_{\tau_2}^{k_2+N_2}}(y) \Big\}^{1/2} \Big\|_{L^p(\widetilde{M})}.
\end{aligned}$$

We now return to the proof of Theorem 2.18.

Proof of Theorem 2.18. Set

$$\Omega_i = \Big\{(x,y)\in \widetilde{M} : \Big\{ \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2} |\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x,y)|^2 \chi_{I_{\tau_1}^{k_1+N_1}}(x)\chi_{I_{\tau_2}^{k_2+N_2}}(y)\Big\}^{1/2} > 2^i\Big\}.$$

Denote

$$B_i = \big\{(k_1,k_2,I,J) : |(I\times J)\cap\Omega_i| > \frac{1}{2A}|I\times J|, |(I\times J)\cap\Omega_{i+1}| \le \frac{1}{2A}|I\times J|\big\},$$

where I, J are dyadic cubes on M_1 and M_2, respectively. Since $f\in L^2(\widetilde{M})$, by the discrete Calderón's identity in (2.12), we have

$$\begin{aligned} f(x,y) &= \sum_{k_1,k_2}\sum_{I,J}|I||J|D_{k_1}(x,x_I)D_{k_2}(y,y_J)\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)\\ &= \sum_i \sum_{(k_1,k_2,I,J)\in B_i}|I||J|D_{k_1}(x,x_I)D_{k_2}(y,y_J)\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J), \end{aligned}$$

where the series converges in the L^2 norm, and hence almost everywhere.

We claim

(2.23)
$$\Big\|\sum_{(k_1,k_2,I,J)\in B_i}|I||J|D_{k_1}(x,x_I)D_{k_2}(y,y_J)\widetilde{D}_{k_1}\widetilde{D}_{k_2}(f)(x_I,y_J)\Big\|^p_{L^p(\widetilde{M})} \le C2^{ip}|\Omega_i|,$$

which, together with the fact $\max\big(\frac{Q_1}{Q_1+\vartheta_1},\frac{Q_2}{Q_2+\vartheta_2}\big) < p \le 1$, yields

$$\|f\|^p_{L^p(\widetilde{M})} \le C\sum_i 2^{ip}|\Omega_i| \le C\|f\|^p_{H^p(\widetilde{M})}.$$

This completes the proof of Theorem 2.18. Thus, it suffices to verify claim (2.23).

Note that $D_{k_1}(x,x_I)$ and $D_{k_2}(y,y_I)$ have compact supports. Hence, if $(k_1,k_2,I,J)\in B_i$, then the supports of $D_{k_1}(x,x_I)$ and $D_{k_2}(y,y_I)$ are contained in

$$\widetilde{\Omega}_i = \big\{(x,y) : M_s(\chi_{\Omega_i})(x,y) > \frac{1}{(2A)^{10}}\big\}.$$

Therefore, by the Hölder inequality,

$$\begin{aligned} &\Big\|\sum_{(k_1,k_2,I,J)\in B_i}|I||J|D_{k_1}(x,x_I)D_{k_2}(y,y_I)\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)\Big\|^p_{L^p(\widetilde{M})}\\ &\quad\le |\widetilde{\Omega}_i|^{1-\frac{p}{2}}\Big\|\sum_{(k_1,k_2,I,J)\in B_i}|I||J|D_{k_1}(x,x_I)D_{k_2}(y,y_I)\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)\Big\|^p_{L^2(\widetilde{M})}. \end{aligned}$$

By the duality argument, for all $g \in L^2(\widetilde{M})$ with $\|g\|_{L^2(\widetilde{M})} \le 1$,

$$\begin{aligned}
&\Big|\Big\langle \sum_{(k_1,k_2,I,J)\in B_i} |I||J| D_{k_1}(\cdot, x_I) D_{k_2}(\cdot, y_I) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J), g(\cdot,\cdot)\Big\rangle\Big| \\
&\quad \le \Big| \sum_{(k_1,k_2,I,J)\in B_i} |I||J| D_{k_1} D_{k_2}(g)(x_I, y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\Big| \\
&\quad \le \Big(\sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\big|^2\Big)^{\frac{1}{2}} \\
&\qquad \times \Big(\sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big| D_{k_1} D_{k_2}(g)(x_I, y_J)\big|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big(\sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big| D_{k_1} D_{k_2}(g)(x_I, y_J)\big|^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big| \inf_{(u,v)\in I\times J} M_s\big(D_{k_1} D_{k_2}(g)\big)(u,v)\big|^2 \chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{k_1,k_2} \int_{\widetilde{M}} \big| M_s\big(D_{k_1} D_{k_2}(g)\big)(x,y)\big|^2 dxdy\Big)^{\frac{1}{2}} \quad \le C\|g\|_{L^2(\widetilde{M})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Big\| \sum_{(k_1,k_2,I,J)\in B_i} |I||J| D_{k_1}(x, x_I) D_{k_2}(y, y_I) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\Big\|_{L^2(\widetilde{M})} \\
&\qquad \le C \Big(\sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\big|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Note also that

$$\begin{aligned}
C2^{2i}|\Omega_i| \quad &\ge \quad \int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}} \Big|\Big\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1}\sum_{\tau_2} |\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x,y)|^2 \\
&\qquad\qquad \chi_{I_{\tau_1}^{k_1+N_1}}(x)\chi_{I_{\tau_2}^{k_2+N_2}}(y)\Big\}\Big| dxdy \\
&\ge \quad \sum_{(k_1,k_2,I,J)\in B_i} \big|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\big|^2 |(I\times J)\cap(\widetilde{\Omega}_i\setminus\Omega_{i+1})| \\
&\ge \quad \frac{1}{2A} \sum_{(k_1,k_2,I,J)\in B_i} |I||J| \big|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)\big|^2,
\end{aligned}$$

where the fact that $|(I\times J)\cap(\widetilde{\Omega}_i\setminus\Omega_{i+1})| > \frac{1}{2A}|I\times J|$ when $(k_1,k_2,I,J)\in B_i$ is used in the last inequality. This finishes the proof of the claim, and hence Theorem 2.18 is concluded. □

We would like to point out that the subset $L^2(\widetilde{M}) \cap H^p(\widetilde{M})$ is dense in $H^p(\widetilde{M})$. Indeed, we have the following.

Proposition 2.19. *$G_0(\vartheta_1, \vartheta_2, \vartheta_1, \vartheta_2)$ is dense in $H^p(\widetilde{M})$.*

The proof of this proposition is similar to the proof of the Plancherel-Pôlya inequalities. More precisely, suppose that $\mathcal{R}$ is any set of indexes of k_1, k_2, τ_1, τ_2. Then we have

$$\begin{aligned} &\| \sum_{\mathcal{R}} \mu_1(I_{\tau_1}^{k_1+N_1})\mu_2(I_{\tau_2}^{k_2+N_2})\widetilde{D}_{k_1}(x, y_{\tau_1}^{k_1+N_1})\widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2+N_2}) \\ &\qquad\qquad \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1+N_1}, y_{\tau_2}^{k_2+N_2}) - f\|_{H^p} \\ \lesssim\ &\| \sum_{\mathcal{R}^c} \mu_1(I_{\tau_1}^{k_1+N_1})\mu_2(I_{\tau_2}^{k_2+N_2})\widetilde{D}_{k_1}(x, y_{\tau_1}^{k_1+N_1})\widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2+N_2}) \\ &\qquad\qquad \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1+N_1}, y_{\tau_2}^{k_2+N_2})\|_{H^p} \\ \lesssim\ &\| \{ \sum_{\mathcal{R}^c} |D_{k_1}D_{k_2}(f)(x,y)|^2 \chi_{I_{\tau_1}^{k_1+N_1}}(x)\chi_{I_{\tau_2}^{k_2+N_2}}(y) \}^{\frac12} \|_p, \end{aligned}$$

where $\mathcal{R}^c$ is the complement of $\mathcal{R}$. We leave the details to the reader.

3. Product Carleson measure space and duality

In this section, each $M_i, i = 1, 2$, is a homogeneous space in the sense of Coifman and Weiss and is assumed to satisfy conditions (1.3) and (1.7). We first introduce the generalized Carleson measure space by the following.

Definition 3.1. Let $\max\left(\frac{2Q_1}{2Q_1+\vartheta_1}, \frac{2Q_2}{2Q_2+\vartheta_2}\right) < p \le 1$ and $0 < \beta_i, \gamma_i < \vartheta_i$ for $i = 1, 2$. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be an approximation to the identity on M_i, and for $k_i \in \mathbb{Z}$, set $D_{k_i} = S_{k_i} - S_{k_i-1}$, $i = 1, 2$. The generalized Carleson measure space $CMO^p(\widetilde{M})$ is defined by the set of all $f \in \left(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ such that

(3.1)

$$\begin{aligned} \|f\|_{CMO^p(\widetilde{M})} = \ &\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_\Omega \sum_{k_1,k_2} \sum_{I\times J\subseteq\Omega} \big|D_{k_1}D_{k_2}(f)(x,y)\big|^2 \chi_I(x)\chi_J(y)dxdy \right\}^{\frac12} \\ < \ &\infty, \end{aligned}$$

where Ω ranges over all open sets in $\widetilde{M}$ with finite measures and where for each k_1 and k_2, I, J range over all the dyadic cubes in M_1 and M_2 with length $\ell(I) = 2^{-k_1-N_1}$ and $\ell(J) = 2^{-k_2-N_2}$, respectively.

In order to show that the space $CMO^p(\widetilde{M})$ is well defined, we prove the following Plancherel-Pôlya inequalities.

Theorem 3.2. *Let all the notation be the same as above. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{P_{k_i}\}_{k_i\in\mathbb{Z}}$ be two approximations to the identity on $M_i, i = 1, 2$. For $k_i \in \mathbb{Z}$, set $D_{k_i} = S_{k_i} - S_{k_i-1}$ and $E_{k_i} = P_{k_i} - P_{k_i-1}$. Then for all $f \in CMO^p(\widetilde{M})$,*

$$\begin{aligned} &\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_\Omega \sum_{k_1,k_2} \sum_{I\times J\subseteq\Omega} \sup_{u\in I, v\in J} \big|D_{k_1}D_{k_2}(f)(u,v)\big|^2 \chi_I(x)\chi_J(y)dxdy \right\}^{\frac12} \\ &\quad \lesssim \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_\Omega \sum_{k_1,k_2} \sum_{I\times J\subseteq\Omega} \inf_{u\in I, v\in J} \big|E_{k_1}E_{k_2}(f)(u,v)\big|^2 \chi_I(x)\chi_J(y)dxdy \right\}^{\frac12}. \end{aligned}$$

The main result in this section is the following.

Theorem 3.3. *For* $\max\big(\frac{2Q_1}{2Q_1+\vartheta_1},\frac{2Q_2}{2Q_2+\vartheta_2}\big)<p\le 1$, $\big(H^p(\widetilde{M})\big)'=CMO^p(\widetilde{M})$. *In particular,* $\big(H^1(\widetilde{M})\big)'=CMO^1(\widetilde{M})=BMO(\widetilde{M})$.

We first prove Theorem 3.2, the Plancherel-Pôlya inequalities for $CMO^p(\widetilde{M})$.

Proof of Theorem 3.2. For each p satisfying $\max\big(\frac{2Q_1}{2Q_1+\vartheta_1},\frac{2Q_2}{2Q_2+\vartheta_2}\big)<p\le 1$, we choose $\varepsilon\in(0,\vartheta_1\wedge\vartheta_2)$ such that $\max\big(\frac{2Q_1}{2Q_1+\vartheta_1},\frac{2Q_2}{2Q_2+\vartheta_2}\big)<\max\big(\frac{2Q_1}{2Q_1+\varepsilon},\frac{2Q_2}{2Q_2+\varepsilon}\big)<p$.

For any $f\in CMO^p(\widetilde{M})$, by the discrete product reproducing identity (2.18), the Hölder inequality and the almost orthogonality estimate (2.19), we have

$$\begin{aligned}
&\sup_{u\in I,v\in J}\big|D_{k_1}D_{k_2}[f](u,v)\big|^2\\
&\lesssim \sum_{k_1',k_2'}2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}\sum_{I',J'}|I'||J'|\\
&\quad\times\frac{1}{V(x_I,x_{I'})+V_{2^{-(k_1\wedge k_1')}}(x_I)+V_{2^{-(k_1\wedge k_1')}}(x_{I'})}\\
&\quad\times\Big(\frac{2^{-(k_1\wedge k_1')}}{2^{-(k_1\wedge k_1')}+d(x_I,x_{I'})}\Big)^{\epsilon_1}\frac{1}{V(y_J,y_{J'})+V_{2^{-(k_2\wedge k_2')}}(y_J)+V_{2^{-(k_2\wedge k_2')}}(y_{J'})}\\
&\quad\times\Big(\frac{2^{-(k_2\wedge k_2')}}{2^{-(k_2\wedge k_2')}+d(y_J,y_{J'})}\Big)^{\epsilon_2}\big|E_{k_1'}E_{k_2'}[f](x_{I'},y_{J'})\big|^2,
\end{aligned}$$

where ϵ_i is chosen to satisfy $\varepsilon<\epsilon_i<\vartheta_i$ for $i=1,2$, $|I'|=\mu_1(Q^{k_1'+N_1'}_{\tau_1'}),|J'|=\mu_2(Q^{k_2'+N_2'}_{\tau_2'}),x_{I'}=y^{k_1'+N_1'}_{\tau_1'}$ and $y_{J'}=y^{k_2'+N_2'}_{\tau_2'}$.

Note that $2^{-|k_1-k_1'|}\approx\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}$, $2^{-(k_1\wedge k_1')}\approx\mathrm{diam}(I)\vee\mathrm{diam}(I')$ and $d(x_I,x_{I'})\ge\mathrm{dist}(I,I')$. Similar results hold for k_2,k_2' and J,J'. Therefore,

$$\begin{aligned}
&\sup_{u\in I,v\in J}\big|D_{k_1}D_{k_2}[f](u,v)\big|^2\\
&\lesssim \sum_{k_1',k_2'}\sum_{I',J'}|I'||J'|\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1}\Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\wedge\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2}\\
&\quad\times\frac{1}{V_{\mathrm{dist}(I,I')}(x_I)+|I|\vee|I'|}\Big(\frac{\mathrm{diam}(I)\vee\mathrm{diam}(I')}{\mathrm{diam}(I)\vee\mathrm{diam}(I')+\mathrm{dist}(I,I')}\Big)^{\epsilon_1}\\
&\quad\times\frac{1}{V_{\mathrm{dist}(J,J')}(y_J)+|J|\vee|J'|}\Big(\frac{\mathrm{diam}(J)\vee\mathrm{diam}(J')}{\mathrm{diam}(J)\vee\mathrm{diam}(J')+\mathrm{dist}(J,J')}\Big)^{\epsilon_2}\\
&\quad\times\big|E_{k_1'}E_{k_2'}[f](x_{I'},y_{J'})\big|^2.
\end{aligned}$$

Applying the above estimate with any arbitrary points $x_{I'}$ and $y_{J'}$ in I' and J', respectively, and the fact that $ab=(a\vee b)^2\big(\frac{a}{b}\wedge\frac{b}{a}\big)$ for any $a,b>0$, we obtain that for any open set $\Omega \subset \widetilde{M}$ with finite measure,

(3.2)

$$\begin{aligned}
&\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k_1,k_2} \sum_{I\times J\subset\Omega} |I||J| \sup_{u\in I, v\in J} \big|D_{k_1}D_{k_2}[f](u,v)\big|^2 \\
&\lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k_1,k_2} \sum_{I\times J\subset\Omega} \sum_{k_1',k_2'} \sum_{I',J'} \Big[\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big]\Big[\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big]\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1} \\
&\quad\times \Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\wedge\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2} \cdot \big(|I|\vee|I'|\big)\big(|J|\vee|J'|\big) \\
&\quad\times \frac{|I|\vee|I'|}{V_{\mathrm{dist}(I,I')}(x_I)+|I|\vee|I'|}\Big(\frac{\mathrm{diam}(I)\vee\mathrm{diam}(I')}{\mathrm{diam}(I)\vee\mathrm{diam}(I')+\mathrm{dist}(I,I')}\Big)^{\epsilon_1} \\
&\quad\times \frac{|J|\vee|J'|}{V_{\mathrm{dist}(J,J')}(y_J)+|J|\vee|J'|}\Big(\frac{\mathrm{diam}(J)\vee\mathrm{diam}(J')}{\mathrm{diam}(J)\vee\mathrm{diam}(J')+\mathrm{dist}(J,J')}\Big)^{\epsilon_2} \\
&\quad\times \inf_{u\in I', v\in J'} \big|E_{k_1'}E_{k_2'}[f](u,v)\big|^2.
\end{aligned}$$

For convenience, let $R = I\times J$ and $R' = I'\times J'$, where I, J, I' and J' run over all dyadic cubes. Also, set

$$\begin{aligned}
&\textstyle\sum_{k_1,k_2}\sum_{I\times J\subset\Omega} = \sum_{R\subset\Omega}, \quad \sum_{k_1',k_2'}\sum_{I',J'} = \sum_{R'}; \\
&|R| = |I|\times|J|,\ |R'| = |I'|\times|J'|; \\
&r(R,R') = \Big(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big)\Big(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big)\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1} \\
&\qquad\qquad \times\Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\wedge\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2}; \\
&v(R,R') = \big(|I|\vee|I'|\big)\big(|J|\vee|J'|\big); \\
&P(R,R') = \frac{|I|\vee|I'|}{V_{\mathrm{dist}(I,I')}(x_I)+|I|\vee|I'|}\Big(\frac{\mathrm{diam}(I)\vee\mathrm{diam}(I')}{\mathrm{diam}(I)\vee\mathrm{diam}(I')+\mathrm{dist}(I,I')}\Big)^{\epsilon_1} \\
&\qquad\qquad \times\frac{|J|\vee|J'|}{V_{\mathrm{dist}(J,J')}(y_J)+|J|\vee|J'|}\Big(\frac{\mathrm{diam}(J)\vee\mathrm{diam}(J')}{\mathrm{diam}(J)\vee\mathrm{diam}(J')+\mathrm{dist}(J,J')}\Big)^{\epsilon_2}; \\
&S_R = \sup_{u\in I, v\in J}\big|D_{k_1}D_{k_2}[f](u,v)\big|^2; \\
&T_{R'} = \inf_{u\in I', v\in J'}\big|E_{k_1'}E_{k_2'}[f](u,v)\big|^2.
\end{aligned}$$

Thus, (3.2) can be rewritten as

$$\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}|R|S_R \lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'} r(R,R')v(R,R')P(R,R')T_{R'}. \tag{3.3}$$

To complete the proof the theorem, we need to prove that the right-hand side of (3.3) can be controlled by

$$\sup_{\overline{\Omega}} \frac{1}{|\overline{\Omega}|^{\frac{2}{p}-1}} \sum_{R'\subset\overline{\Omega}} |R'|T_{R'},$$

where $\overline{\Omega}$ ranges over all open sets in $\widetilde{M}$ with finite measures.

Similar to the proof of Theorem 3.2 in [24], we point out that the estimates of $v(R,R^{'})$ and $P(R,R^{'})$ are based on the geometrical properties between R and $R^{'}$. More precisely, when the difference of the sizes and the distance of R and $R^{'}$ get larger, then $v(R,R^{'})$ and $P(R,R^{'})$ then become smaller, respectively. Therefore, to estimate $v(R,R^{'})$ and $P(R,R^{'})$, for each $R\subset\Omega$ we group all dyadic cubes R' in $\widetilde{M}$ according to the distances and sizes of R and $R^{'}$ as follows:

Define

$$\Omega^0 =: \bigcup_{R=I\times J\subset\Omega} 3(I\times J).$$

Then, for any $R\subset\Omega$, let

$$A_{0,0}(R)=\{R':\ \mathrm{dist}(I,I^{'})\le \mathrm{diam}(I)\vee\mathrm{diam}(I^{'}),\ \mathrm{dist}(J,J^{'})\le\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\};$$

$$A_{j,0}(R)=\{R':\ 2^{j-1}\big(\mathrm{diam}(I)\vee\mathrm{diam}(I^{'})\big)<\mathrm{dist}(I,I^{'})\le 2^{j}\big(\mathrm{diam}(I)\vee\mathrm{diam}(I^{'})\big),\ \mathrm{dist}(J,J^{'})\le\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\};$$

$$A_{0,k}(R)=\{R':\ \mathrm{dist}(I,I^{'})\le\mathrm{diam}(I)\vee\mathrm{diam}(I^{'}),\ 2^{k-1}\big(\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\big)<\mathrm{dist}(J,J^{'})\le 2^{k}\big(\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\big)\};$$

$$A_{j,k}(R)=\{R':\ 2^{j-1}\big(\mathrm{diam}(I)\vee\mathrm{diam}(I^{'})\big)<\mathrm{dist}(I,I^{'})\le 2^{j}\big(\mathrm{diam}(I)\vee\mathrm{diam}(I^{'})\big);\ 2^{k-1}\big(\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\big)<\mathrm{dist}(J,J^{'})\le 2^{k}\big(\mathrm{diam}(J)\vee\mathrm{diam}(J^{'})\big)\},$$

where $j,k\ge 1$.

Note that for each dyadic rectangle $R^{'}=I^{'}\times J^{'}$, we have $\lim\limits_{j,k\to\infty}2^jI^{'}\times 2^kJ^{'}=\widetilde{M}$. Hence, for any dyadic rectangle $R\subset\Omega$, there exist some j and k such that $R'\in A_{j,k}(R)$.

Consequently, we have

$$\begin{aligned}
(3.3)\quad \le\ & \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{0,0}(R)} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\
&+\sum_{j\ge1}\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{j,0}(R)} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\
&+\sum_{k\ge1}\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{0,k}(R)} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\
&+\sum_{j,k\ge1}\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{j,k}(R)} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\
=:\ & \mathbb{I}+\mathbb{II}+\mathbb{III}+\mathbb{IV}.
\end{aligned}$$

We first consider $\mathbb{I}$. Define

$$B_{0,0}=\{R':\ 3R'\cap\Omega^0\neq\emptyset\}.$$

Then we claim that

$$(3.4)\qquad \mathbb{I}\le\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R'\in B_{0,0}}\ \sum_{\{R:\ R\subset\Omega,R'\in A_{0,0}(R)\}} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}.$$

In fact, for each $R'\notin B_{0,0}$, according to the definition of $B_{0,0}$, we have $3R'\cap\Omega^0=\emptyset$. Hence, for any $R\subset\Omega$, we have $3R'\cap 3R=\emptyset$, which implies that $R'\notin A_{0,0}(R)$.

Therefore, we have $\bigcup_{R\subset\Omega} A_{0,0}(R) \subset B_{0,0}$. As a consequence, we can obtain that claim (3.4) holds.

We make a further decomposition of $B_{0,0}$. First, for each $h \ge 1$, we define $\mathcal{F}_h^{0,0} = \{R^{'} : |R^{'} \bigcap \Omega^0| > \frac{1}{2^h}|3R^{'}|\}$, $\mathcal{D}_h^{0,0} = \mathcal{F}_h^{0,0} \setminus \mathcal{F}_{h-1}^{0,0}$, $\mathcal{F}_0^{0,0} = \emptyset$, and $\Omega_h^{0,0} = \bigcup_{R^{'}\in\mathcal{D}_h^{0,0}} R^{'}$. From the above definitions, we have

$$B_{0,0} = \bigcup_{h\ge 1} \mathcal{D}_h^{0,0}.$$

Therefore, (3.4) can be rewritten as

$$\mathbb{I} \le \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{h\ge 1} \sum_{R^{'}\in\mathcal{D}_h^{0,0}} \sum_{\{R:\ R\subset\Omega, R^{'}\in A_{0,0}(R)\}} v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R^{'}}. \tag{3.5}$$

From the definition of $\mathcal{D}_h^{0,0}$ we can see that for any $R^{'} \in \mathcal{D}_h^{0,0}$ and any R satisfying $R^{'} \in A_{0,0}(R)$, we have $P(R,R^{'}) \le 1$. Hence, to estimate (3.5), we only need to consider the following:

$$\sum_{\{R:\ R\subset\Omega, R^{'}\in A_{0,0}(R)\}} v(R,R^{'})r(R,R^{'}). \tag{3.6}$$

In what follows, we use a simple geometrical argument, which is a generalization of Chang and R. Fefferman's idea; see more details in [7], [24].

Note that $R^{'} \in A_{0,0}(R)$; we have $3R \cap 3R^{'} \neq \emptyset$. We split (3.6) into four cases:

Case 1. $\mathrm{diam}(I^{'}) \ge \mathrm{diam}(I)$, $\mathrm{diam}(J^{'}) \le \mathrm{diam}(J)$. First, it is easy to see that $|I \times 3J^{'}| \lesssim |3R \cap 3R^{'}|$. So we have

$$\frac{|I|}{|3I^{'}|}|3R^{'}| \lesssim |3R \cap 3R^{'}| \lesssim |3R^{'} \cap \Omega^{0,0}| \lesssim \frac{1}{2^{h-1}}|3R^{'}|,$$

which yields that $2^{h-1} \lesssim |3I^{'}|/|I|$, i.e., $2^h|I| \lesssim |I^{'}|$. Since I and $I^{'}$ are all dyadic cubes with measure equivalent to $2^{-a}Q_1$ for some $a \in \mathbb{Z}$ according to the assumption in (1.8), we have $2^h 2^{n_1 Q_1}|I| \approx |I^{'}|$ for some nonnegative integer n_1. Also, for each fixed n_1, the numbers of such I's must be $\lesssim 2^{n_1 Q_1}$.

Denote by x_I and $x_{I^{'}}$ the centers of I and $I^{'}$, respectively. Since $3R \cap 3R^{'} \neq \emptyset$, we have $3I \cap 3I^{'} \neq \emptyset$, which implies that $d(x_I, x_{I^{'}}) \le 6\mathrm{diam}(I^{'})$, and hence $V(x_I, 6\mathrm{diam}(I^{'})) \approx V(x_{I^{'}}, 6\mathrm{diam}(I^{'})) \approx |6I^{'}| \approx |I^{'}|$. Thus,

$$\frac{|I^{'}|}{|I|} \approx \frac{V(x_I, 6\mathrm{diam}(I^{'}))}{V(x_I, \mathrm{diam}(I))} \lesssim \Big(\frac{\mathrm{diam}(I^{'})}{\mathrm{diam}(I)}\Big)^{Q_1}.$$

It follows that for each fix $n_1 \ge 0$,

$$\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})} \lesssim \Big(\frac{|I^{'}|}{|I|}\Big)^{\frac{1}{Q_1}} \lesssim 2^{-\frac{h}{Q_1}} 2^{-n_1}.$$

As for J, similarly, we have $|J| \approx 2^{n_2 Q_2}|J^{'}|$ for some $n_2 \ge 0$. For each fixed n_2, the number of such $J^{'}$'s is less than a constant independent of n_2, since $3J \cap 3J^{'} \neq \emptyset$ and $|J| \gtrsim |J^{'}|$. Moreover, we have $\frac{\mathrm{diam}(J^{'})}{\mathrm{diam}(J)} \lesssim 2^{-n_2}$.

Thus

$$\begin{aligned}
&\sum_{R\in \text{Case 1}} r(R,R^{'})v(R,R^{'})\\
&\quad= \sum_{R\in \text{Case 1}} \left(\frac{|I|}{|I'|}\right)\left(\frac{|J'|}{|J|}\right)\left(\frac{\operatorname{diam}(I)}{\operatorname{diam}(I^{'})}\right)^{\epsilon_1}\left(\frac{\operatorname{diam}(J^{'})}{\operatorname{diam}(J)}\right)^{\epsilon_2}|I'||J|\\
&\quad\lesssim \sum_{n_1,n_2\geq 0} 2^{-(h+n_1Q_1)}2^{-\epsilon_1 n_1-\epsilon_1\frac{h}{Q_1}}2^{-n_2\epsilon_2}2^{n_1Q_1}|I'||J'|\\
&\quad\lesssim 2^{-h(1+\frac{\epsilon_1}{Q_1})}|R^{'}|.
\end{aligned}$$

Case 2. $\operatorname{diam}(I')\leq \operatorname{diam}(I)$, $\operatorname{diam}(J')\geq \operatorname{diam}(J)$. This can be handled similar to Case 1. We have

$$\sum_{R\in \text{Case 2}} r(R,R^{'})v(R,R^{'}) \quad\lesssim\quad 2^{-h(1+\frac{\epsilon_2}{Q_2})}|R^{'}|.$$

Case 3. $\operatorname{diam}(I')\geq \operatorname{diam}(I)$, $\operatorname{diam}(J')\geq \operatorname{diam}(J)$. First, it is easy to see that

$$|R|\lesssim |3R^{'}\cap 3R|\leq |3R^{'}\cap \Omega_{0,0}|\leq \frac{1}{2^{h-1}}|3R^{'}|.$$

Thus $2^{h-1}|R|\lesssim |R^{'}|$. Hence $|R^{'}|\approx 2^{h+n_1Q_1+n_2Q_2}|R|$ for some $n_1,n_2\geq 0$. For each fixed n_1 and n_2, the number of such R's is $\lesssim 2^{n_1Q_1}2^{n_2Q_2}$.

Similar to Case 1, since $3R\cap 3R'\neq\emptyset$, we have $\frac{|I'|}{|I|}\lesssim \left(\frac{\operatorname{diam}(I')}{\operatorname{diam}(I)}\right)^{Q_1}$ and $\frac{|J^{'}|}{|J|}\lesssim \left(\frac{\operatorname{diam}(J^{'})}{\operatorname{diam}(J)}\right)^{Q_2}$. Hence, $\frac{\operatorname{diam}(I)\operatorname{diam}(J)}{\operatorname{diam}(I')\operatorname{diam}(J')}\lesssim \left(\frac{|R|}{|R'|}\right)^{\frac{1}{Q_1\vee Q_2}}\lesssim 2^{-\frac{h}{Q_1\vee Q_2}}2^{-\frac{n_1Q_1+n_2Q_2}{Q_1\vee Q_2}}$. As a consequence, we can obtain that

$$\begin{aligned}
\sum_{R\in \text{Case 3}} r(R,R^{'})v(R,R^{'}) \quad&\lesssim \sum_{n_1,n_2\geq 0} 2^{-h-n_1Q_1-n_2Q_2}2^{-\frac{h}{Q_1\vee Q_2}\epsilon_3}\\
&\qquad 2^{-\frac{n_1Q_1+n_2Q_2}{Q_1\vee Q_2}\epsilon_3}2^{n_1Q_1+n_2Q_2}|R^{'}|\\
&\lesssim 2^{-h(1+\frac{\epsilon_3}{Q_3})}|R'|,
\end{aligned}$$

where $\epsilon_3=\epsilon_1\wedge\epsilon_2$ and $Q_3=Q_1\vee Q_2$.

Case 4. $\operatorname{diam}(I^{'})\lesssim \operatorname{diam}(I)$ and $\operatorname{diam}(J^{'})\lesssim \operatorname{diam}(J)$. Similar to Case 3, we have $|R^{'}|\lesssim |3R^{'}\cap 3R|$, which implies that

$$|R^{'}|\lesssim |3R^{'}\cap\Omega_{0,0}|\leq \frac{1}{2^{h-1}}|3R^{'}|.$$

Hence there exists a constant $h_0>0$ independent of R and R' such that $0\leq h\leq h_0$. We obtain that $|R|\approx 2^{h+n_1Q_1+n_2Q_2}|R^{'}|$ for some $n_1,n_2\geq 0$ and that for each fixed n_1 and n_2, the number of such $R^{'}$'s is less than a constant independent of n_1 and n_2. Also, by using the same skills as in Case 3, we have $\frac{\operatorname{diam}(I^{'})}{\operatorname{diam}(I)}\frac{\operatorname{diam}(J^{'})}{\operatorname{diam}(J)}\lesssim 2^{-\frac{h}{Q_3}}2^{-\frac{n_1Q_1+n_2Q_2}{Q_3}}$. Therefore,

$$\sum_{R\in \text{Case 4}} r(R,R^{'})v(R,R^{'}) \quad\lesssim\quad \sum_{n_1\geq 0,n_2\geq 0} 2^{-\frac{h}{Q_3}\epsilon_3}2^{-\frac{n_1Q_1+n_2Q_2}{Q_3}\epsilon_3}|R'|\lesssim 2^{-h\frac{\epsilon_3}{Q_3}}|R^{'}|,$$

where ϵ_3 and Q_3 are the same as in Case 3.

Now let us turn to I_1:

$$\begin{aligned} I_1 &= \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_h\sum_{R'\in\mathcal{D}_h^{0,0}}\Big(\sum_{R\in\text{Case}\,1}+\sum_{R\in\text{Case}\,2}+\sum_{R\in\text{Case}\,3}+\sum_{R\in\text{Case}\,4}\Big)\\ &\quad\times r(R,R^{'})v(R,R^{'})T_{R'}\\ &= I_{11}+I_{12}+I_{13}+I_{14}. \end{aligned}$$

Obviously, combining the fact that $|\Omega_h^{0,0}|\lesssim h2^h|\Omega|$ for $h\ge 1$, $|\Omega_0^{0,0}|\lesssim|\Omega|$, $\epsilon_i\in(\varepsilon,\vartheta_i)$, $i=1,2$, and $\max\big(\frac{2Q_1}{2Q_1+\vartheta_1},\frac{2Q_2}{2Q_2+\vartheta_2}\big)<p$, we have

$$\begin{aligned} I_{11} &\lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_h 2^{-h(1+\frac{\epsilon_1}{Q_1})}|\Omega_h^{0,0}|^{\frac{2}{p}-1}\frac{1}{|\Omega_h^{0,0}|^{\frac{2}{p}-1}}\sum_{R'\subset\Omega_h^{0,0}}|R^{'}|T_{R'}\\ &\lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_h 2^{-h(1+\frac{\epsilon_1}{Q_1})}h^{\frac{2}{p}-1}2^{h(\frac{2}{p}-1)}|\Omega|^{\frac{2}{p}-1}\times\sup_{\bar\Omega}\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R^{'}|T_{R'}\\ &\lesssim \sup_{\bar\Omega}\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R^{'}|T_{R'}. \end{aligned}$$

Similarly we can deal with I_{12} and I_{13}:

$$I_{12},I_{13}\lesssim\sup_{\bar\Omega}\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R^{'}|T_{R'}.$$

For I_{14}, observing that

$$I_{14}=\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h=0}^{h_0}\sum_{R'\in\mathcal{D}_h^{0,0}}\sum_{R\in\text{Case}\,4}r(R,R^{'})v(R,R^{'})T_{R'},$$

we have

$$\begin{aligned} I_{14} &\lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h=0}^{h_0}2^{-h\frac{\epsilon_3}{Q_3}}|\Omega_h^{0,0}|^{\frac{2}{p}-1}\frac{1}{|\Omega_h^{0,0}|^{\frac{2}{p}-1}}\sum_{R'\subset\Omega_h^{0,0}}|R^{'}|T_{R'}\\ &\lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h=0}^{h_0}2^{-h\frac{\epsilon_3}{Q_3}}h^{\frac{2}{p}-1}2^{h(\frac{2}{p}-1)}|\Omega|^{\frac{2}{p}-1}\times\sup_{\bar\Omega}\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}\mu(R^{'})T_{R'}\\ &\lesssim \sup_{\bar\Omega}\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R^{'}|T_{R'}. \end{aligned}$$

Following the same routine and skills as in the proof of Theorem 3.2 in [24], we obtain the estimates of the other three terms in $\mathbb{I}$. Similarly, we can deal with $\mathbb{II}$, $\mathbb{III}$ and $\mathbb{IV}$ with only minor modifications. We leave the details to the reader. This completes the proof of Theorem 3.2. □

To show that the dual of $H^p(\widetilde{M})$ is $CMO^p(\widetilde{M})$ for $\max\big(\frac{2Q_1}{2Q_1+\vartheta_1},\frac{2Q_2}{2Q_2+\vartheta_2}\big)<p\le 1$, we first introduce sequence spaces s^p and c^p as follows.

Definition 3.4. Let $\tilde\chi_R(x)=|R|^{-1/2}\chi_R(x)$ for any dyadic rectangles $R\subset\widetilde{M}$. For $0<p\le 1$, the sequence space s^p is defined by the collection of all complex-value

sequences $s = \{s_R\}_R$ such that

$$\|s\|_{s^p} = \big\|\big\{\sum_R (|s_R|\tilde{\chi}_R(x_1,x_2))^2\big\}^{1/2}\big\|_{L^p}. \tag{3.7}$$

Similarly, for $0 < p \le 1$, the sequence space c^p is defined by the collection of all complex-value sequences $t = \{t_R\}_R$ such that

$$\|t\|_{c^p} = \sup_\Omega \big(\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{R\subseteq\Omega} |t_R|^2\big)^{1/2}, \tag{3.8}$$

where the sup is taken over all open sets $\Omega \subset \widetilde{M}$ with finite measure and R ranges over all the dyadic rectangles in $\widetilde{M}$.

The duality theorem of these sequence spaces is the following:

Theorem 3.5. $\big(s^p\big)^{'} = c^p$.

Proof. First, we prove that for all $t \in c^p$, if

$$L(s) = \sum_R s_R \cdot \bar{t}_R, \qquad \forall s \in s^p, \tag{3.9}$$

then $|L(s)| \lesssim \|s\|_{s^p}\|t\|_{c^p}$. To see this, set

$$\Omega_k = \big\{(x_1,x_2) \in \widetilde{M} \ : \big\{\sum_R \big(|s_R|\tilde{\chi}_R(x_1,x_2)\big)^2\big\}^{1/2} > 2^k\big\},$$

$$B_k = \big\{R : \mu(\Omega_k \bigcap R) > \frac{1}{2A}|R|,\ |\Omega_{k+1}\bigcap R| \le \frac{1}{2A}|R|\big\}$$

and

$$\tilde{\Omega}_k = \big\{(x_1,x_2) \in \widetilde{M} \ : \mathcal{M}_s(\chi_{\Omega_k}) > \frac{1}{2A}\big\},$$

where M_s is the strong maximal function on $\widetilde{M}$. By the Hölder inequality,

$$\begin{aligned} |L(s)| &\le \Big(\sum_k \big(\sum_{R\in B_k} |s_R|^2\big)^{\frac{p}{2}} \big(\sum_{R\in B_k} |t_R|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_k |\tilde{\Omega}_k|^{1-\frac{p}{2}} \big(\sum_{R\in B_k} |s_R|^2\big)^{\frac{p}{2}} \big(\frac{1}{|\tilde{\Omega}_k|^{\frac{2}{p}-1}} \sum_{R\subset\tilde{\Omega}_k} |t_R|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_k |\tilde{\Omega}_k|^{1-\frac{p}{2}} \big(\sum_{R\in B_k} |s_R|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \|t\|_{c^p}, \end{aligned} \tag{3.10}$$

where we have used the fact that if $R \in B_k$, then R is contained in $\tilde{\Omega}_k$. Observing that

$$\int_{\tilde{\Omega}_k\setminus\Omega_{k+1}} \sum_{R\in B_k} \big(|s_R|\tilde{\chi}_R(x)\big)^2 dx \le 2^{2(k+1)}|\tilde{\Omega}_k\setminus\Omega_{k+1}| \le C2^{2k}|\Omega_k|$$

and

$$\begin{aligned}\int_{\tilde{\Omega}_k\backslash\Omega_{k+1}}\sum_{R\in B_k}\big(|s_R|\tilde{\chi}_R(x)\big)^2dx &\geq \sum_{R\in B_k}|s_R|^2|R|^{-1}\big|\tilde{\Omega}_k\backslash\Omega_{k+1}\cap R\big|\\ &\geq \sum_{R\in B_k}|s_R|^2|R|^{-1}\frac{1}{2A}|R|\sum_{R\in B_k}|s_R|^2,\end{aligned}$$

we obtain $\bigg(\sum\limits_{R\in B_k}|s_R|^2\bigg)^{\frac{p}{2}} \lesssim 2^{kp}|\Omega_k|^{\frac{p}{2}}$. Substituting this back into the last term of (3.10) and noting $|\tilde{\Omega}_k| \lesssim |\Omega_k|$ yield that $|L(s)| \lesssim \|s\|_{s^p}\|t\|_{c^p}$.

Conversely, we need to verify that for any $L \in \big(s^p\big)'$, there exists $t \in c^p$ with $\|t\|_{c^p} \lesssim \|L\|$ such that for all $s \in s^p$, $L(s) = \sum\limits_R s_R\bar{t}_R$. Here we adapt a similar idea in the one-parameter case of Frazier and Jawerth in [15] to our multiparameter situation.

For any $L \in \big(s^p\big)'$, then $L(s) = \sum\limits_R s_R\bar{t}_R$. It suffices to show that $\|t\|_{c^p} \leq \|L\|$. To do this, for any open set $\Omega \subset \widetilde{M}$ with finite measure, let $\bar{\mu}$ be a new measure such that $\bar{\mu}(R) = \dfrac{|R|}{|\Omega|^{\frac{p}{2}-1}}$ when $R \subset \Omega$ and $\bar{\mu}(R) = 0$ when $R \not\subseteq \Omega$. Also, let $l^2(\bar{\mu})$ be a sequence space such that when $\{s_R\} \in l^2(\bar{\mu})$, $\big(\sum\limits_{R\subset\Omega}|s_R|^2\dfrac{|R|}{|\Omega|^{\frac{p}{2}-1}}\big)^{1/2} < \infty$. Observe that

$$\begin{aligned}\Big\{\frac{1}{|\Omega|^{\frac{p}{2}-1}}\sum_{R\subset\Omega}|t_R|^2\Big\}^{1/2} &= \big\||R|^{-1/2}|t_R|\big\|_{l^2(\bar{\mu})}\\ &= \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\leq 1}\bigg|\sum_{R\subseteq\Omega}(t_R|R|^{-1/2})\cdot\overline{s}_R\cdot\frac{|R|}{|\Omega|^{\frac{p}{2}-1}}\bigg|\\ &\leq \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\leq 1}\bigg|L\bigg(\chi_{R\subseteq\Omega}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|^{\frac{p}{2}-1}}\bigg)\bigg|\\ &\leq \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\leq 1}\big\|L\big\|\cdot\bigg\|\chi_{R\subseteq\Omega}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|^{\frac{p}{2}-1}}\bigg\|_{s^p}.\end{aligned}$$

By (3.7) and the Hölder inequality, we have

$$\bigg\|\chi_{R\subseteq\Omega}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|^{\frac{p}{2}-1}}\bigg\|_{s^p} \leq \bigg(\sum_{R\subseteq\Omega}|s_R|^2\frac{|R|}{|\Omega|^{\frac{p}{2}-1}}\bigg)^{1/2}.$$

Therefore,

$$\|t\|_{c^p} \leq \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\leq 1}\|L\|\cdot\|s\|_{l^2(\bar{\mu})} \leq \|L\|.$$

□

In order to show Theorem 3.3, we also need to introduce the lifting and projection operators as follows.

Definition 3.6. Suppose $0 < \beta_i, \gamma_i < \vartheta_i$ for $i = 1, 2$. For any

$$f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))',$$

define the lifting operator S by

$$\{(Sf)_R\} = \left\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}D_{k_1}D_{k_2}[f](x_I, y_J)\right\}, \tag{3.11}$$

where $R = I \times J$, I, J are dyadic cubes in M_1 and M_2, with length $\ell(I) = 2^{-k_1-N_1}$ and $\ell(J) = 2^{-k_2-N_2}$, and x_I and y_J are the centers of I and J, respectively.

Definition 3.7. For any complex-value sequence $\lambda = \{\lambda_R\}$ where R are all dyadic cubes in $\widetilde{M}$, define the projection operator T by

$$T(\lambda_R)(x, y) = \sum_{j,k}\sum_{I,J}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\widetilde{D}_{k_1}(x, x_I)\widetilde{D}_{k_2}(y, y_J)\cdot\lambda_R, \tag{3.12}$$

where $\widetilde{D}_{k_1}(x, x_I)\widetilde{D}_{k_2}(y, y_J)$ are the same as in Theorem 2.9. By discrete Calderón identity, we immediately obtain

$$T\circ S(f)(x, y) = \sum_{k_1,k_2}\sum_{I,J}|I||J|\widetilde{D}_{k_1}(x, x_I)\widetilde{D}_{k_2}(y, y_J)D_{k_1}D_{k_2}(f)(x_I, y_J) = f(x, y).$$

This means that $T\circ S$ is the identity operator. Moreover, we also have the following.

Proposition 3.8. *For any $f \in H^p(\widetilde{M})$ with $\max\left(\frac{Q_1}{Q_1+\vartheta_1}, \frac{Q_2}{Q_2+\vartheta_2}\right) < p \le 1$, we have*

$$\|(Sf)_R\|_{s^p} \lesssim \|f\|_{H^p(\widetilde{M})}. \tag{3.13}$$

Conversely, for any $s \in c^p$,

$$\|T(s_R)\|_{H^p(\widetilde{M})} \lesssim \|s_R\|_{s^p}. \tag{3.14}$$

Proposition 3.9. *For any $f \in CMO^p(\widetilde{M})$ with $\max\left(\frac{2Q_1}{2Q_1+\vartheta_1}, \frac{2Q_2}{2Q_2+\vartheta_2}\right) < p \le 1$, we have*

$$\|(Sf)_R\|_{c^p} \lesssim \|f\|_{CMO^p(\widetilde{M})}. \tag{3.15}$$

Conversely, for any $t \in c^p$,

$$\|T(t_R)\|_{CMO^p(\widetilde{M})} \lesssim \|t_R\|_{c^p}. \tag{3.16}$$

The proofs of Propositions 3.8 and 3.9 are similar to the proofs of the Plancherel-Pôlya inequalities in Theorem 2.10 and Theorem 3.2. See also [24] for similar proofs on $\mathbb{R}^n$. We leave these details to the reader. We now return to the proof of Theorem 3.3.

Proof of Theorem 3.3. For $\max\left(\frac{2Q_1}{2Q_1+\vartheta_1}, \frac{2Q_2}{2Q_2+\vartheta_2}\right) < p \le 1$ and any $g \in \overset{\circ}{G}_\vartheta(\vartheta_1, \vartheta_2; \vartheta_1, \vartheta_2)$ and $f \in CMO^p(\widetilde{M})$, by Theorem 2.7,

$$\begin{aligned}\langle f, g\rangle &= \langle\sum_{k_1,k_2}\sum_{I,J}|I||J|\widetilde{D}_{k_1}(\cdot, x_I)\widetilde{D}_{k_2}(\cdot, y_J)D_{k_1}D_{k_2}(f)(x_I, y_J)(f), g\rangle\\ &= \sum_R S_R(f)\tilde{S}_R(g),\end{aligned}$$

where $\tilde{S}_R(g) = \left\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\widetilde{D}_{k_1}\widetilde{D}_{k_2}[f](x_I, y_J)\right\}_{k_1,k_2,I,J}$.

By Propositions 3.8 and 3.9, we obtain

$$|\langle f, g\rangle| \le |\langle S(f), \tilde{S}(g)\rangle| \lesssim \|f\|_{CMO^p(\widetilde{M})}\|g\|_{H^p(\widetilde{M})}.$$

Since $\overset{\circ}{G}_\vartheta(\vartheta_1,\vartheta_2;\vartheta_1,\vartheta_2)$ is dense in $H^p(\widetilde{M})$, it follows that $CMO^p(\widetilde{M}) \subseteq \big(H^p(\widetilde{M})\big)'$.

Conversely, suppose $l \in \big(H^p(\widetilde{M})\big)'$. Then $l_1 \equiv l \circ T \in (s^p)'$ by Proposition 3.8. So by Theorem 3.5, there exists $t \in c^p$ such that $l_1(s) = \langle t, s\rangle$ for all $s \in s^p$, and $\|t\|_{c^p} \approx \|l_1\| \lesssim \|l\|$, since T is bounded. We have $l_1 \circ S = l \circ T \circ S = l$, hence

$$l(g) \quad = \quad l \circ T(S(g)) = \langle t, S(g)\rangle = \langle T(t), g\rangle.$$

By Definition 3.4 and Theorem 3.2, we obtain that $\|T(t)\|_{CMO^p(\widetilde{M})} \lesssim \|t\|_{c^p} \lesssim \|l\|$. Hence $\big(H^p(\widetilde{M})\big)' \subseteq CMO^p(\widetilde{M})$. □

As a consequence of the facts that $(H^1(\widetilde{M}))' = BMO(\widetilde{M})$, $H^1(\widetilde{M}) \cap L^2 \subset L^1(\widetilde{M})$ and $H^1(\widetilde{M}) \cap L^2$ is dense in $H^1(\widetilde{M})$, we obtain

Proposition 3.10. $L^\infty(\widetilde{M}) \subset BMO(\widetilde{M})$.

4. The boundedness and endpoint estimates of singular integral operators on product space

In this section, we first introduce singular integral operators on one single factor M and prove the boundedness of such operators on the Hardy space on a homogeneous space M satisfying the doubling condition (1.1) and the regularity assumption on the metric (1.3). We then pass these results to the product space with two factors. For this purpose, we recall that φ is a bump function associated to a ball $B(x_0,\delta)$ if φ is supported in this ball, $|\varphi(x)| \le 1$ and $\|\varphi\|_\eta \le \delta^{-\eta}$, where $C^\eta(M)$ is the collection of all continuous functions f on M satisfying

$$\|f\|_\eta =: \sup_{x\neq y} \frac{|f(x)-f(y)|}{(d(x,y))^\eta} < \infty.$$

We consider a class of singular integral operators T which are initially defined from $C_0^\eta(M)$, C^η functions with compact supports, $0 < \eta \le \vartheta$ to $C^\eta(M)$ with a distribution kernel $K(x,y)$. The following properties hold:

(I-1) If $\varphi, \psi \in C_0^\eta(M)$ have disjoint supports, then

$$\langle T\varphi, \psi\rangle = \int_{M\times M} K(x,y)\varphi(y)\psi(x)dydx.$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $\|T\varphi\|_\infty \lesssim 1$ and $\|T\varphi\|_\epsilon \lesssim r^{-\epsilon}, \epsilon \le \eta$.

(I-3) If $x\neq y$, then $|K(x,y)| \lesssim V(x,y)^{-1}$ and

$$|K(x,y) - K(x,y')| \lesssim \Big(\frac{d(y,y')}{d(x,y)}\Big)^\epsilon V(x,y)^{-1}$$

for $d(y,y') \le \frac{1}{2A} d(x,y)$.

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator T^t defined by

$$\langle T^t\varphi, \psi\rangle = \langle T\psi, \varphi\rangle.$$

We remark that this class of operators is similar to those considered in [49]. We obtain the following boundedness result which generalizes the result in [49].

Theorem 4.1. *Each singular integral T satisfying* (I-1) *through* (I-4) *extends to be a bounded operator on $H^p(M)$ whenever $\frac{\epsilon}{\epsilon+\vartheta} < p < \infty$.*

The strategy to prove Theorem 4.1 is that we first show the L^2 boundedness of T and then use discrete Calderón identity to express $T(f)$ by

$$T(f)(x) = \sum_{k_2,\tau_2} \mu(I_{\tau_2}^{k_2+N}) T(D_{k_2}(\cdot, y_{\tau_2}^{k_2+N}))(x) \widetilde{\widetilde{D}}_{k_2}(f)(y_{\tau_2}^{k_2+N}))(x_{\tau_1}^{k_1}) \chi_{I_{\tau_1}^{k_1}}.$$

The H^p boundedness of T then follows by applying the Littlewood-Paley-Stein square function, the orthogonal estimate and the proof of the Plancherel-Pôlya inequalities. To show the L^2 boundedness of T, we use an idea in [11]. More precisely, we will, as in [11], construct a special operator T_N such that T_N is invertible on $L^2(M)$ and λ^s, where λ^s is the closure of $C_0^\eta(M)$ with respect to the norm of $C_0^s(M), s < \eta$, and, moreover, T_N and T_N^{-1}, the inverse of T_N, are both bounded on λ^s and $L^2(M)$ for any fixed large N. The construction of T_N follows from Coifman's decomposition of the identity. We outline the construction of T_N as follows. See [11] for more details. Suppose that S_k is an approximation to the identity. Set $D_k = S_{k+1} - S_k$. Then for $f \in L^2, f(x) = \sum_{j,k} D_j D_k(f)(x) = \sum_{|j-k|\le N} D_j D_k(x) = \sum_k D_k^N D_k(f)(x_{I_\tau^k}) + R_N(f)(x) = T_N(f)(x) + R_N(f)(x)$, where $D_k^N = \sum_{|j|\le N} D_{k+j}$. It was proved that T_N and R_N are both bounded on $L^2(M)$. Moreover, there exist a constant C and $\delta > 0$ such that $\|R_N(f)\|_2 \le C2^{-\delta N}\|f\|_2$. It was shown in [11] that T_N is bounded on $\lambda^s, s > 0$, and the operator norm of $T_N - I$ on λ^s tends to 0 as N tends to ∞. Therefore, as in [11], we can show that T is bounded on L^2. For this purpose, we need the following orthogonal estimates:

Proposition 4.2. *Each singular integral T satisfies* (I-1) *through* (I-4),

(4.17)

$$|D_j T D_k(x,y)| \lesssim 2^{-|k-j|\epsilon'} \frac{1}{V(x,y) + V_{2^{-(k\wedge j)}}(x) + V_{2^{-(k\wedge j)}}(y)} \Big(\frac{2^{-(k\wedge j)}}{2^{-(k\wedge j)} + d(x,y)}\Big)^{\epsilon'},$$

where D_k are the same as in Theorem 2.7 *and $\epsilon' < \epsilon$.*

We assume Proposition 4.2 for the moment and now prove that T is bounded on L^2. Let $f_0 \in \lambda^s$ have compact support, and set $f_1 = T_N^{-1}(f_0) \in \lambda^s \cap L^2$. Thus, $\sum_{-n<k<n} D_k^N D_k(f_1)$ tends to f_0 as n tends to ∞. For any compact supported $g \in L^2$ we have, by Proposition 4.2,

$$\begin{aligned} &|\langle \sum_{-n_2<k_2<n_2} D_{k_2}^N D_{k_2}(g), T(f_0)\rangle| \\ &= |\lim_{n_1\to\infty} |\langle \sum_{-n_2<k_2<n_2} D_{k_2}^N D_{k_2}(g), T(\sum_{-n<k<n} D_k^N D_k(f_1))\rangle| \\ &\le C\|g\|_2\|f_1\|_2 \le C\|g\|_2\|f_0\|_2. \end{aligned}$$

A similar argument can be carried out to show that if $g_0 \in \lambda^s$ has compact support, then $|\langle g_0, T(f_0)\rangle| \le C\|g_0\|_2\|f_0\|_2$. Thus, T extends to a bounded operator on L^2. Once the L^2 boundedness of T is proved, we show the H^p boundedness of T as follows. For $f \in L^2 \cap H^p$,

$$\begin{aligned}
\|T(f)\|_{H^p} &\lesssim \Big\|\Big\{\sum_{k_1,\tau_1} D_{k_1}(Tf)(y_{\tau_1}^{k_1})\chi_{I_{\tau_1}^{k_1}}(x)\Big\}^{\frac{1}{2}}\Big\|_p \\
&\lesssim \Big\|\Big\{\sum_{k_1,\tau_1} D_{k_1}\big(T\sum_{k_2,\tau_2}\mu(I_{\tau_2}^{k_2+N})D_{k_2}(\cdot,y_{\tau_2}^{k_2+N})\widetilde{\widetilde{D}}_{k_2}(f)(y_{\tau_2}^{k_2+N})\big) \\
&\qquad\qquad\qquad (x_{\tau_1}^{k_1})\chi_{I_{\tau_1}^{k_1}}(x)\Big\}^{\frac{1}{2}}\Big\|_p \\
&\lesssim \Big\|\Big\{\sum_{k_2,\tau_2} |\widetilde{\widetilde{D}}_{k_2}(f)(y_{\tau_2}^{k_2+N})|^2\chi_{I_{\tau_2}^{k_2}}(x)\Big\}^{\frac{1}{2}}\Big\|_p \lesssim \|f\|_{H^p},
\end{aligned}$$

where the L^2 boundedness of T and Calderón's identity are used in the second inequality and where the third inequality follows from the orthogonal estimate given in Proposition 4.2 and the proof of the Plancherel-Pôlya inequalities.

We now return to the proof of Proposition 4.2.

Proof of Proposition 4.2. We only consider the case where $k \geq j$ because the proof for other case is the same. Write

$$\begin{aligned}
D_jTD_k(x,y) &= \int\int D_j(x,u)K(u,v)D_k(v,y)d\mu(u)d\mu(v) \\
&= \int\Big[\int D_j(x,u)K(u,v)d\mu(u) - \int D_j(x,u)K(u,y)d\mu(u)\Big]D_k(v,y)d\mu(v)
\end{aligned}$$

since $\int D_k(v,y)d\mu(v) = 0$. By property (I-2), we obtain

$$(4.18)\quad |D_jTD_k(x,y)| \lesssim \int 2^{j\epsilon'}\frac{1}{V_{2^{-j}}(x)}|v-y|^{\epsilon'}|D_k(v,y)|d\mu(v) \lesssim 2^{(j-k)\epsilon'}\frac{1}{V_{2^{-j}}(x)}$$

because $V_{2^{-j}}(x)D_j(x,u)$, as the function of u, is a bump function with $x_0 = x, \delta = 2^{-j}$. The estimate in (4.18) implies (4.17) when $k \geq j$ and $d(x,y) \leq 10A^2 2^{-j}$, since in this case, $V(x,y) \approx V_{2^{-j}}(x) \approx V_{2^{-j}}(y)$.

If $d(x,y) \geq 10A2^{-j}$, we write

$$\begin{aligned}
D_jTD_k(x,y) &= \int\int D_j(x,u)K(u,v)D_k(v,y)d\mu(u)d\mu(v) \\
&= \int\int D_j(x,u)[K(u,v)-K(u,y)]D_k(v,y)d\mu(u)d\mu(v).
\end{aligned}$$

Note that when $d(x,y) \geq 10A2^{-j}$, then $d(v,y) \leq \frac{1}{2A}d(u,v), d(u,v) \approx d(x,y)$ and $V(u,v) \approx V(x,y)$. By the property (I-3), we have

$$\begin{aligned}
(4.19)\quad |D_jTD_k(x,y)| &\lesssim \int\int |D_j(x,u)|\frac{1}{Vu,v)}\Big(\frac{d(v,y)}{d(u,v)}\Big)^{\epsilon}|D_k(v,y)|d\mu(u)d\mu(v) \\
&\lesssim 2^{(j-k)\epsilon}\frac{1}{V(x,y)}\Big(\frac{2^{-j}}{d(x,y)}\Big)^{\epsilon}.
\end{aligned}$$

The estimate of (4.19) implies (4.17) since $V_{2^{-j}}(x) + V_{2^{-j}}(y) \lesssim V(x,y)$ when $d(x,y) \geq 10A2^{-j}$. Hence Property 4.2 is concluded. $\square$

To pass the above boundedness to the product case $\widetilde{M} = M_1 \times M_2$, again similar to [49], we will consider the operator T initially defined from $C_0^\eta(\widetilde{M})$ to $C^\eta(\widetilde{M})$ associated with a distribution kernel $K(x_1,y_1,x_2,y_2)$ which is a C^ϵ function away from the "cross" $= \{(x,y) : x_1 = y_1 \text{ and } x_2 = y_2;\ x = (x_1,x_2), y = (y_1,y_2)\}$ and satisfies the following additional properties:

(II-1) $\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int K(x_1,y_1,x_2,y_2)\varphi_1(y_1)\varphi_2(y_2)\psi_1(x_1)\psi_2(x_2)dydx$

$$\text{whenever} \begin{cases} \varphi_1, \psi_1 \in C_0^\eta(M_1) & \text{and have disjoint support,} \\ \varphi_2, \psi_2 \in C_0^\eta(M_2) & \text{and have disjoint support.} \end{cases}$$

(II-2) For each bump function φ_2 on M_2 and each $x_2 \in M_2$, there exists a singular integral operator T^{φ_2,x_2} (of one parameter) on M_1 so that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{M_2} \langle T^{\varphi_2,x_2}\varphi_1, \psi_1 \rangle \psi_2(x_2)dx_2.$$

Moreover, $x_2 \mapsto T^{\varphi_2,x_2}$ is in $C^\eta(M_2)$ and uniform in the sense that $T^{\varphi_2,x_2}(x_1,y_1)$, as the kernel on M_1, satisfies conditions (I-1) to (I-4) uniformly.

(II-3) If φ_i is a bump function on a ball $B^i(r_i)$ in M_i, then $\|T(\varphi_1 \otimes \varphi_2)\|_{\eta,\eta} \lesssim r_1^{-a_1} r_2^{-a_2}$, where

$$\|f(x_1,x_2)\|_{\eta,\eta} =: \sup_{x_1 \neq x_1', x_2 \neq x_2'} \frac{|[f(x_1,x_2) - f(x_1',x_2)] - [f(x_1,x_2') - f(x_1',x_2')]|}{|x_1 - x_1'|^\eta |x_2 - x_2'|^\eta}.$$

In (II-2) and (II-3), both inequalities are taken in the sense of (I-2) whenever φ_2 is a bump function for $B^2(r_2)$ in M_2.

(II-4) $|K(x_1,y_1,x_2,y_2)| \lesssim V(x_1,y_1)^{-1}V(x_2,y_2)^{-1}$,

$$|K(x_1,y_1,x_2,y_2) - K(x_1',y_1,x_2,y_2)| \lesssim (\frac{d(x_1,x_1'}{d(x_1,y_1)})^\epsilon V(x_1,y_1)^{-1}V(x_2,y_2)^{-1}$$

for $d(x_1,x_1') \leq \frac{1}{2A}d(x_1,y_1)$, and

$$|[K(x_1,y_1,x_2,y_2) - K(x_1',y_1,x_2,y_2)] - [K(x_1,y_1,x_2',y_2) - K(x_1',y_1,x_2',y_2)]|$$
$$\lesssim (\frac{d(x_1,x_1')}{d(x_1,y_1)})^\epsilon (\frac{d(x_2,x_2')}{d(x_2,y_2)})^\epsilon V(x_1,y_1)^{-1}V(x_2,y_2)^{-1}$$

for $d(x_1,x_1') \leq \frac{1}{2A}d(x_1,y_1)$ and $d(x_2,x_2') \leq \frac{1}{2A}d(x_2,y_2)$.

(II-5) The same conditions hold when the index 1 and 2 are interchanged, that is, whenever the roles of M_1 and M_2 are interchanged.

(II-6) The same properties are assumed to hold for the 3 "transposes" of T, i.e. those operators which arise by interchanging x_1 and y_1, interchanging x_2 and y_2, or doing both interchanges.

The main result in this section is the following.

Theorem 4.3. *Each singular integral T satisfying* (II-1) *through* (II-6) *extends to be a bounded operator on $H^p(\widetilde{M})$ to itself whenever $\frac{\epsilon}{\epsilon+\vartheta_i} < p < \infty, i = 1,2$.*

Note that one can similarly construct operators T_N and prove the orthogonal estimates on $\widetilde{M}$. Applying the iteration and conditions (II-1)-(II-6) will yield the proof of Theorem 4.3. See [49] for a similar disposal. We leave the details to the reader.

As consequences of Theorem 4.3, Definition 2.8, Theorem 3.3 and Proposition 3.10, we obtain the $H^p - L^p$ and BMO boundedness.

Corollary 4.4. *Each singular integral T satisfying* (II-1) *through* (II-6) *extends to be a bounded operator from $H^p(\widetilde{M})$ to $L^p(\widetilde{M})$ whenever $\frac{\epsilon}{\epsilon+\vartheta_i} < p < \infty, i = 1,2$.*

In the next corollary, we assume $\widetilde{M} = M_1 \times M_2$ and require the additional assumption (1.7) on each metric space M_i.

Corollary 4.5. *Each singular integral T satisfying* (II-1) *through* (II-6) *extends to be a bounded operator on $BMO(\widetilde{M})$ and from $L^\infty(\widetilde{M})$ to $BMO(\widetilde{M})$.*

Next we provide the Calderón-Zygmund decomposition and prove interpolation theorems on $H^p(\widetilde{M})$. We note that $H^p(\widetilde{M}) = L^p(\widetilde{M})$ for $1 < p < \infty$.

Theorem 4.6. *Let $\frac{\epsilon}{\epsilon+\vartheta_i} < p_2 \le 1, i = 1, 2, p_2 < p < p_1 < \infty$, $\alpha > 0$ be given and $f \in H^p_{com}$. Then we may write $f = g + b$ where $g \in H^{p_1}(\widetilde{M})$ and $b \in H^{p_2}(\widetilde{M})$ such that $\|g\|^{p_1}_{H^{p_1}} \le C\alpha^{p_1-p}\|f\|^p_{H^p}$ and $\|b\|^{p_2}_{H^{p_2}} \le C\alpha^{p_2-p}\|f\|^p_{H^p}$, where C is an absolute constant.*

Theorem 4.7. *Let $\frac{\epsilon}{\epsilon+\vartheta_i} < p_2 < p_1 < \infty$ and let T be a linear operator which is bounded from $H^{p_2}(\widetilde{M})$ to $L^{p_2}(\widetilde{M})$ and from $H^{p_1}(\widetilde{M})$ to $L^{p_1}(\widetilde{M})$. Then T is bounded from $H^p(\widetilde{M})$ to $L^p(\widetilde{M})$ for all $p_2 < p < p_1$. Similarly, if T is bounded on $H^{p_2}(\widetilde{M})$ and $H^{p_1}(\widetilde{M})$, then T is bounded on $H^p(\widetilde{M})$ for all $p_2 < p < p_1$.*

We first prove Theorem 4.6.

Proof of Theorem 4.6. We assume $f \in L^2 \cap H^p(\widetilde{M})$. Let $\alpha > 0$ and $\Omega_\ell = \{(x, y) \in \widetilde{M} : \widetilde{\widetilde{S}}_d(f)(x, y) > \alpha 2^\ell\}$, where

$$\widetilde{\widetilde{S}}_d(f)(x,y) = \Big\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1} \sum_{\tau_2} |\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x,y)|^2 \chi_{I^{k_1+N_1}_{\tau_1}}(x) \chi_{I^{k_2+N_2}_{\tau_2}}(y) \Big\}^{1/2}.$$

Note that $\|f\|_{H^p(\widetilde{M})} \approx \|\widetilde{\widetilde{S}}_d(f)\|_p$. Let

$$\mathcal{R}_0 = \left\{ R = I \times J \text{ such that } |R \cap \Omega_0| < \frac{1}{2A}|R| \right\},$$

and for $\ell \ge 1$,

$$\mathcal{R}_\ell = \left\{ R = I \times J \text{ such that } |R \cap \Omega_{\ell-1}| \ge \frac{1}{2A}|R| \text{ but } |R \cap \Omega_\ell| < \frac{1}{2A}|R| \right\}.$$

Since $f \in L^2(\widetilde{M})$, by the discrete Calderón's identity in (2.11) we have

$$\begin{aligned} f(x,y) &= \sum_{k_1,k_2} \sum_{I,J} |I||J| D_{k_1}(x, x_I) D_{k_2}(y, y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J) \\ &= \sum_{\ell \ge 1} \sum_{(k_1,k_2,I,J) \in \mathcal{R}_\ell} |I||J| D_{k_1}(x, x_I) D_{k_2}(y, y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J) \\ &\quad + \sum_{(k_1,k_2,I,J) \in \mathcal{R}_0} |I||J| D_{k_1}(x, x_I) D_{k_2}(y, y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J) \\ &=: b(x,y) + g(x,y). \end{aligned}$$

When $p_1 > 1$, using the duality argument, it is easy to show that

$$\|g\|_{p_1} \le C \|\{ \sum_{R = I \times J \in \mathcal{R}_0} |\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I, y_J)|^2 \chi_I \chi_J \}^{\frac{1}{2}} \|_{p_1}.$$

Next, we estimate $\|g\|_{H^{p_1}(\widetilde{M})}$ when $p_1 \le 1$. Clearly, the duality argument will not work here. Nevertheless, we can estimate the $H^{p_1}(\widetilde{M})$ norm directly. To this end, using the proof of the Plancherel-Pôlya inequalities, we observe that

$$\begin{aligned}\|g\|_{H^{p_1}} &\le \|\{\sum_{j',k'}\sum_{I',J'}|D_{j'}D_{k'}(g)(x_{I'},y_{J'})|^2\chi_{I'}(x)\chi_{J'}(y)\}^{\frac12}\|_{L^{p_1}}\\ &\lesssim \|\{\sum_{R=I\times J\in\mathcal{R}_0}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I\chi_J\}^{\frac12}\|_{p_1}.\end{aligned}$$

This shows that for all P_1,

$$\|g\|_{H^{p_1}} \lesssim \|\{\sum_{R=I\times J\in\mathcal{R}_0}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I\chi_J\}^{\frac12}\|_{p_1}.$$

Claim 1.

$$\int_{\widetilde{\widetilde{S}}_d(f)(x,y)\le\alpha}(\widetilde{\widetilde{S}}_d)^{p_1}(f)(x,y)dxdy \ge C\|\{\sum_{R=I\times J\in\mathcal{R}_0}|D_jD_k(f)(x_I,y_J)|^2\chi_I\chi_J\}^{\frac12}\|_{p_1}.$$

This claim implies

$$\begin{aligned}\|g\|_{p_1} &\le C\int_{\widetilde{\widetilde{S}}_d(f)(x,y)\le\alpha}(\widetilde{\widetilde{S}}_d)^{p_1}(f)(x,y)dxdy\\ &\le C\alpha^{p_1-p}\int_{\widetilde{\widetilde{S}}_d(f)(x,y)\le\alpha}(\widetilde{\widetilde{S}}_d)^{p}(f)(x,y)dxdy\\ &\le C\alpha^{p_1-p}\|f\|_p^p.\end{aligned}$$

To show Claim 1, we choose $0<q<p_1$ and note that

$$\begin{aligned}&\int_{\widetilde{\widetilde{S}}_d(f)(x,y)\le\alpha}\widetilde{\widetilde{S}}_d(f)^{p_1}(f)(x,y)dxdy\\ &= \int_{\widetilde{\widetilde{S}}_d(f)(x,y)\le\alpha}\Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1}\sum_{\tau_2}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I(x)\chi_J(y)\Big\}^{\frac{p_1}{2}}dxdy\\ &\ge C\int_{\Omega_0^c}\{\sum_{R\in\mathcal{R}_0}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I(x)\chi_J(y)\Big\}^{\frac{p_1}{2}}dxdy\\ &= C\int_{\widetilde{M}}\{\sum_{R\in\mathcal{R}_0}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_{R\cap\Omega_0^c}(x,y)\Big\}^{\frac{p_1}{2}}dxdy\\ &\ge C\int_{(\widetilde{M})}\{\{\sum_{R\in\mathcal{R}_0}(M_s(|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f))(x_I,y_J)|^q\chi_{R\cap\Omega_0^c}(x,y))^{\frac{2}{q}}\}^{\frac{q}{2}}\}^{\frac{p_1}{q}}dxdy\\ &\ge C\int_{(\widetilde{M})}\{\sum_{R\in\mathcal{R}_0}|\widetilde{\widetilde{D}}_{k_1}\widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_R(x,y)\}^{\frac{p_1}{2}}dxdy,\end{aligned}$$

where in the last inequality we have used the fact that $|\Omega_0^c\cap(I\times J)|\ge\frac12|I\times J|$ for $I\times J\in\mathcal{R}_0$, and thus $\chi_R(x,y)\le 2^{\frac1q}M_s(\chi_{R\cap\Omega_0^c})^{\frac1q}(x,y)$, and in the second to the last inequality we have used the vector-valued Fefferman-Stein inequality for strong maximal functions

$$\|(\sum_{k=1}^{\infty}(M_s(f_k))^r)^{\frac1r}\|_p \le C\|(\sum_{k=1}^{\infty}|f_k|^r)^{\frac1r}\|_p$$

with the exponents $r=2/q>1$ and $p=p_1/q>1$. Thus the claim follows.

We now recall $\widetilde{\Omega}_\ell = \{(x,y) \in \widetilde{M} : M_s(\chi_{\Omega_\ell}) > \frac{1}{2A}\}$.

Claim 2. For $p_2 \le 1$,

$$\|\sum_{I\times J\in\mathcal{R}_\ell} |I||J| D_{k_1}(x,x_I) D_{k_2}(y,y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)\|^{p_2}_{H^{p_2}} \le C(2^\ell\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}|.$$

Claim 2 implies

$$\begin{aligned} ||b||^{p_2}_{H^{p_2}} &\le \sum_{\ell\ge1}(2^\ell\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}| \\ &\le C\sum_{\ell\ge1}(2^\ell\alpha)^{p_2}|\Omega_{\ell-1}| \le C\int_{\widetilde{\widetilde{S}}_d(f)(x,y)>\alpha} \widetilde{\widetilde{S}}_d(f)^{p_2}(f)(x,y)dxdy \\ &\le C\alpha^{p_2-p}\int_{\widetilde{\widetilde{S}}_d(f)(x,y)>\alpha} \widetilde{\widetilde{S}}_d(f)^{p}(f)(x,y)dxdy \le C\alpha^{p_2-p}\|f\|^p_{H^p}. \end{aligned}$$

To show Claim 2, again we have

$$\begin{aligned} &|| \sum_{I\times J\in\mathcal{R}_\ell} |I||J| D_{k_1}(x,x_I) D_{k_2}(y,y_J) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)||^{p_2}_{H^{p_2}} \\ &\le C\|\{\sum_{j'k'}\sum_{I',J'}|\sum_{I\times J\in\mathcal{R}_\ell}|I||J|D_{j'}D_{k_1}(x_{I'},x_I)D_{k'}D_{k_2}(y_{J'},y_J) \\ &\quad\times \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\}^{\frac12}\|_{L^{p_2}} \\ &\le C\|\{\sum_{R=I\times J\in\mathcal{R}_\ell}|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I\chi_J\}^{\frac12}\|_{p_2}, \end{aligned}$$

where the last inequality follows from the orthogonal estimate and the similar proof of the Plancherel-Pôlya inequalities.

However,

$$\begin{aligned} \sum_{\ell=1}^\infty(2^\ell\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}| &\ge \int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell} \widetilde{\widetilde{S}}_d(f)(f)^{p_2}(x,y)dxdy \\ &= \int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell}\{\sum_{j,k}\sum_{I,J}|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I(x)\chi_J(y)\}^{\frac{p_2}{2}}dxdy \\ &= \int_{\widetilde{M}}\{\sum_{j,k}\sum_{I,J}|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)}(x,y)\}^{\frac{p_2}{2}}dxdy \\ &\ge \int_{\widetilde{M}}\{\sum_{I\times J\in\mathcal{R}_\ell}|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)}(x,y)\}^{\frac{p_2}{2}}dxdy \\ &\ge \int_{\widetilde{M}}\{\sum_{I\times J\in\mathcal{R}_\ell}|\widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(x_I,y_J)|^2\chi_I(x)\chi_J(y)\}^{\frac{p_2}{2}}dxdy. \end{aligned}$$

In the above string of inequalities, we have used the fact that if $R \in \mathcal{R}_\ell$, then $R \subset \widetilde{\Omega}_{\ell-1}$. Therefore, $|R \cap (\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)| > \frac12|R|$. Thus the same argument applies here to conclude the last inequality above. Finally, since $L^2(\widetilde{M})$ is dense in $H^p(\widetilde{M})$, Theorem 4.6 is proved. □

We are now ready to prove the interpolation theorem on Hardy spaces $H^p(\widetilde{M})$.

Proof of Theorem 4.7. Suppose that T is bounded from $H^{p_2}(\widetilde{M})$ to L^{p_2} and from $H^{p_1}(\widetilde{M})$ to L^{p_1}. For any given $\lambda > 0$ and $f \in H^p(\widetilde{M})$, by the Calderón-Zygmund decomposition,

$$f(x,y) = g(x,y) + b(x,y)$$

with

$$\|g\|_{H^{p_1}}^{p_1} \le C\lambda^{p_1-p}\|f\|_{H^p}^p \quad \text{and} \quad \|b\|_{H^{p_2}}^{p_2} \le C\lambda^{p_2-p}\|f\|_{H^p}^p.$$

Moreover, we have proved the estimates

$$\|g\|_{H^{p_1}}^{p_1} \le C\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)\le\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_1}(x,y)dxdy$$

and

$$\|b\|_{H^{p_2}}^{p_2} \le C\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)>\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_2}(x,y)dxdy,$$

which implies that

$$\begin{aligned}
\|Tf\|_p^p &= p\int_0^\infty \alpha^{p-1}|\{(x,y) : |Tf(x,y)| > \alpha\}|d\alpha \\
&\le p\int_0^\infty \alpha^{p-1}|\{(x,y) : |Tg(x,y)| > \frac{\alpha}{2}\}|d\alpha \\
&\quad +p\int_0^\infty \alpha^{p-1}|\{(x,y) : |Tb(x,y)| > \frac{\alpha}{2}\}|d\alpha \\
&\le p\int_0^\infty \alpha^{p-p_1-1}\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)\le\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_1}(x,y)dxdyd\alpha \\
&\quad +p\int_0^\infty \alpha^{p-p_2-1}\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)>\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_2}(x,y)dxdyd\alpha \\
&\le C\|f\|_{H^p}^p
\end{aligned}$$

for any $p_2 < p < p_1$. Hence, T is bounded from H^p to L^p.

To prove the second assertion that T is bounded on H^p for $p_2 < p < p_1$, for any given $\lambda > 0$ and $f \in H^p$, by the Calderón-Zygmund decomposition again

$$\begin{aligned}
&|\{(x,y) : |S_d(Tf)(x,y)| > \alpha\}| \\
&\le |\{(x,y) : |S_d(Tg)(x,y)| > \frac{\alpha}{2}\}| + |\{(x,y) : |S_d(Tb)(x,y)| > \frac{\alpha}{2}\}| \\
&\le C\alpha^{-p_1}||Tg||_{H^{p_1}}^{p_1} + C\alpha^{-p_2}||Tb||_{H^{p_2}}^{p_2} \\
&\le C\alpha^{-p_1}||g||_{H^{p_1}}^{p_1} + C\alpha^{-p_2}||b||_{H^{p_2}}^{p_2} \\
&\le C\alpha^{-p_1}\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)\le\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_1}(x,y)dxdy \\
&\quad +C\alpha^{-p_2}\int_{\widetilde{\widetilde{S}}_d(f)(f)(x,y)>\alpha} \widetilde{\widetilde{S}}_d(f)(f)^{p_2}(x,y)dxdy
\end{aligned}$$

which, as above, shows that $\|Tf\|_{H^p} \le C\|S_d(Tf)\|_p \le C\|f\|_{H^p}$ for any $p_2 < p < p_1$. □

Acknowledgements

The authors wish to express their sincere appreciation to the referee for his/her very helpful and constructive comments and suggestions on an early version of this paper. The improvement of our main results to the more general case of product space of homogeneous type only satisfying the doubling condition (1.1) and the regularity condition (1.3) on the underlying metric is largely inspired by the question raised by the referee: what are the minimal assumptions on the underlying metric spaces in order to develop a satisfactory multiparameter Hardy space theory? The authors would also like to thank Alex Nagel and Elias Stein whose work [49] has had a substantial impact on our consideration of the multiparameter Hardy space in the framework of Carnot-Carathédory spaces and their generalization, and they thank Alex Nagel for his encouragement and support during the course of this work and for his patience on their revisions of an early version.

References

[1] A. Boggess, R. Dwilewicz and A. Nagel, *The hull of holomorphy of a nonisotropic ball in a real hypersurface of finite type*, Trans. Amer. Math. Soc. **323** (1991), 209-232. MR1079050 (92a:32017)

[2] A. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113-190. MR0167830 (29:5097)

[3] A. Carbery and A. Seeger, H^p and L^p variants of multiparameter Calderón-Zygmund theory, Transaction of Amer. Math. Soc. **324**(2), 719-747. MR1072104 (93b:42035)

[4] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601-628. MR1096400 (92k:42020)

[5] M. Christ, *Regularity properties of the $\overline{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension* 3, J. Amer. Math. Soc. **1** (1988), 587-646. MR928903 (89e:32027)

[6] M. Christ, *On the $\overline{\partial}_b$ equation for three-dimensional CR manifolds*, Proc. Sympos. Pure Math., Amer. Math. Soc. **52** (3), (1991), 63-82. MR1128584 (92g:32035)

[7] S. Y. Chang and R. Fefferman, A continuous version of the duality of H^1 and BMO on the bidisc, Ann. of Math. **112** (1980), 179-201. MR584078 (82a:32009)

[8] S. A. Chang and R. Fefferman, The Calderón-Zygmund decomposition on product domains, Amer. J. Math. **104** (1982), 455-468. MR658542 (84a:42028)

[9] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. **242**, Springer, Berlin, 1971. MR0499948 (58:17690)

[10] D.G. Deng, Y.S. Han, Harmonic Analysis on Spaces of Homogeneous Type, Lecture Notes in Mathematics, 1966, Springer-Verlag, Berlin, 2009. MR2467074 (2010i:43005).

[11] G. David, J.L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoam. **1** (1985) 1-56. MR850408 (88f:47024)

[12] C. Fefferman and J.J. Kohn, *Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds*, Adv. in Math. **69** (1988), 233-303. MR946264 (89g:32027)

[13] R. Fefferman *Harmonic analysis on product spaces*, Ann. of Math. **126** (1987), 109-130. MR898053 (90e:42030)

[14] R. Fefferman and E.M. Stein, *Singular integrals on product spaces*, Adv. Math. **45** (1982), 117-143. MR664621 (84d:42023)

[15] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. **93** (1990), 34-170. MR1070037 (92a:46042)

[16] S. H. Ferguson and M. Lacey, A characterization of product BMO by commutators, Acta Math. **189** (2002), 143-160. MR1961195 (2004e:42026)

[17] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS-AMS Regional Conf. Ser. **79**, 1991. MR1107300 (92m:42021)

[18] G.B. Folland and E.M. Stein, *Estimate for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group*, Comm. Pure and Appl. Math. **27** (1974), 429-522. MR0367477 (51:3719)

[19] Y. Guivarc'h, *Groissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333-379. MR0369608 (51:5841)

[20] R. Gundy and E.M. Stein, *H^p theory for the polydisc*, Proc. Nat. Acad. Sci. **76** (1979), 1026-1029. MR524328 (80j:32012)

[21] Y.S. Han, Calderón-type reproducing formula and the Tb theorem, Rev. Mat. Iberoamericana **10** (1994), 51-91. MR1271757 (95h:42020)

[22] Y. Han, *Plancherel-Pôlya type inequality on space of homogeneous type and its applications*, Proc. Amer. Math. Soc. **126** (1998), 3315-3327. MR1459123 (99a:42010)

[23] Y. Han, *Discrete Calderón-type reproducing formula*, Acta Math. Sin. (Engl.Ser.), **16** (2000), 277-294. MR1778708 (2001k:42024)

[24] Y. Han, J. Li and G. Lu, *Duality of multiparameter Hardy space H^p on product spaces of homogeneous type*, Ann. Sc. Nor. Super. Pisa Cl. Sc. (5) **9** (2010), no. 4, 645–685. MR2789471

[25] Y. Han, J. Li, G. Lu and P.Y. Wang, *$H^p \to H^p$ boundedness implies $H^p \to L^p$ boundedness*, Forum Math. **23** (2011), no. 4, 729–756.

[26] Y. Han and G. Lu, *Discrete Littlewood-Paley-Stein theory and multi-parameter Hardy spaces associated with the flag singular integrals*, available at: http://arxiv.org/abs/0801.1701.

[27] Y. Han and G. Lu, *Some Recent Works on Multiparameter Hardy Space Theory and Discrete Littlewood-Paley Analysis*, Trends in Partial Differential Equations, ALM 10, 99-191, High Education Press and International Press (2009), Beijing-Boston. MR2648281 (2011e:42043)

[28] Y. Han and G. Lu, *Endpoint estimates for singular integral operators and Hardy spaces associated with Zygmund dilations*, Preprint 2008.

[29] Y. Han, G. Lu and E. Sawyer, *Implicit multiparameter Hardy spaces and Marcinkiewcz multipliers on the Heisenberg group*, Preprint 2010.

[30] Y. Han, G. Lu and D. Yang *Product theory on space of homogeneous type*, unpublished manuscript.

[31] Y. Han, D. Müller and D. Yang, *A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces*, Abstract and Applied Analysis, Vol 2008, Article ID 893409. MR2485404 (2010c:46082)

[32] Y. Han, D. Müller and D. Yang, *Littlewood-Paley-Stein characterizations for Hardy spaces on spaces of homogeneous type*, Math. Nachr. **279** (2006), 1505-1537. MR2269253 (2007g:42035)

[33] Y. Han and E. Sawyer, *Littlewood-Paley-Stein theorem on space of homogeneous type and classical function spaces*, Mem. Amer. Math. Soc. **110** (1994), no. 530, 1-126. MR1214968 (96a:42016)

[34] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Funda. Math. **25** (1935), 217-234.

[35] J.L. Journé, *Calderón-Zygmund operators on product space*, Rev. Mat. Iberoamericana **1** (1985), 55-92. MR836284 (88d:42028)

[36] J.L. Journé, *Two problems of Calderón-Zygmund theory on product spaces*, Ann. Inst. Fourier (Grenoble) **38** (1988), 111-132. MR949001 (90b:42031)

[37] K. Koenig, *On maximal Sobolev and Hölder estimates for the tangential Cauchy-Riemann operator and boundary Laplacian*, Amer. J. Math. **124** (2002), 129-197. MR1879002 (2002m:32061)

[38] J.J. Kohn, *Boundary behaviour of $\overline{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523-542. MR0322365 (48:727)

[39] J.J. Kohn, *Subellipticity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains: Sufficient conditions*, Acta Math. **142** (1979), 79-122. MR512213 (80d:32020)

[40] J.J. Kohn, *Estimate for the $\overline{\partial}_b$ on compact pseudoconvex CR manifolds*, Proc. Symposia Pure Math. **43**, 207-217, A.M.S., Providence, RI, 1985. MR812292 (87c:32025)

[41] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, 1992. MR1228209 (94f:42001)

[42] R.A. Macías and C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 271-309. MR546296 (81c:32017b)

[43] D. Müller, F. Ricci and E.M. Stein, *Marcinkiewicz multiplers and multiparameter structure on Heisenberg (-type) groups*. I, Invent. Math. **119** (1995), 119-233. MR1312498 (96b:43005)

[44] A. Nagel, F. Ricci and E.M. Stein, *Harmonic analysis and the fundamental solutions on nilpotent Lie groups, in: Analysis and partial differential equations*, 249-275, Lecture Notes in Pure and Appl. Math., **122**, Dekker, New York, 1990. MR1044793 (91m:22016)

[45] A. Nagel, J.P. Rosay, E.M. Stein and S. Wainger, *Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$*, Ann. of Math. **129** (1989), 113-149. MR979602 (90g:32028)

[46] A. Nagel and E.M. Stein, *The $\Box_b$-Heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}^2$*, Math. Z. **238** (2001), 37-88. MR1860735 (2002h:32031)

[47] A. Nagel and E.M. Stein, *Differentiable control metrics and scaled bump functions*, J. Differential Geom. **57** (2001), 37-88. MR1882665 (2003i:58003)

[48] A. Nagel and E.M. Stein, *The $\bar{\partial}_b$-complex on decoupled boundarise in $\mathbb{C}^n$*, Ann. of Math. (2) **164** (2006), 649-713. MR2247970 (2007d:32036)

[49] A. Nagel and E.M. Stein, *On the product theory of singular integrals*, Rev. Mat. Iberoamericana **20** (2004), 531-561. MR2073131 (2006i:42023)

[50] A. Nagel, E.M. Stein and S. Wainger, *Balls and metrics defined by vector fields I. Basic properties*, Acta Math. **155** (1985), 103-147. MR793239 (86k:46049)

[51] J. Pipher, *Journé's covering lemma and its extension to higher dimensions*, Duke Mathematical Journal **53** (1986), 683-690. MR860666 (88a:42019)

[52] L.P. Rothschild and E.M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247-320. MR0436223 (55:9171)

[53] E. Sawyer and R. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, Amer. J. Math. **114** (1992), 813-874. MR1175693 (94i:42024)

[54] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ (1993). MR1232192 (95c:42002)

[55] E.M. Stein, *Some geometrical concepts arising in harmonic analysis*, Geom. Func. Anal. 2000, Special Volume, Part I, 434-453. MR1826263 (2002f:42014)

Department of Mathematics, Auburn University, Auburn, Alabama 36849
E-mail address: hanyong@auburn.edu

Department of Mathematics, Zhongshan University, Guangzhou, 510275, People's Republic of China
E-mail address: liji6@mail.sysu.edu.cn

School of Mathematical Science, Beijing Normal University, Beijing, 100875, People's Republic of China – and – Department of Mathematics, Wayne State University, Detroit, Michigan 48202
E-mail address: gzlu@math.wayne.edu