TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 365, Number 2, February 2013, Pages 861–883
S 0002-9947(2012)05654-6
Article electronically published on September 25, 2012

LIE COALGEBRAS AND RATIONAL HOMOTOPY THEORY II: HOPF INVARIANTS

DEV SINHA AND BEN WALTER

Abstract. We develop a new framework which resolves the homotopy periods problem. We start with integer-valued homotopy periods defined explicitly from the classic bar construction. We then work rationally, where we use the Lie coalgebraic bar construction to get a sharp model for $\mathrm{Hom}(\pi_* X, \mathbb{Q})$ for simply connected X. We establish geometric interpretations of these homotopy periods, to go along with the good formal properties coming from the Koszul-Moore duality framework. We give calculations, applications, and relationships with the numerous previous approaches.

We give a new solution of the "homotopy periods" problem, as highlighted by Sullivan [25], which places explicit geometrically meaningful formulae first dating back to Whitehead [28] in the context of Quillen's formalism for rational homotopy theory and Koszul-Moore duality [20].

We build on [24], which uses graph coalgebras to breathe combinatorial life into the category of differential graded Lie coalgebras. We use that framework to construct a new isomorphism of Lie coalgebras $\eta : H_{*-1}(\mathcal{E}(A^*(X))) \to \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ for X simply connected. Here $A^*(X)$ denotes a model for commutative rational-valued cochains on X, and $\mathcal{E}$ is isomorphic to the Harrison complex. While the existence of such an isomorphism follows from Quillen's seminal work in rational homotopy theory, giving a direct, explicit isomorphism has benefits in both theory and applications.

On the calculational side, we are able to evaluate Hopf invariants on iterated Whitehead products in terms of the "configuration pairing". We can use this, for example, to take the well-known calculation of the rational homotopy groups of a wedge of spheres as a free graded Lie algebra and give a geometric algorithm to determine which element of that algebra a given homotopy class would correspond to. On the formal side, we are able to understand the naturality of these maps in the long exact sequence of a fibration. For applications, we can show for example that the rational homotopy groups of homogeneous spaces are detected by classical linking numbers. Ultimately all of these Hopf invariants are essentially generalized linking invariants, as we explain in Section 1.2.

We proceed in two steps, first using the classical bar complex to define integer-valued homotopy functionals which coincide with evaluation of the cohomology of

Received by the editors August 7, 2010 and, in revised form, April 28, 2011.
2010 *Mathematics Subject Classification.* Primary 55P62; Secondary 16E40, 55P48.
Key words and phrases. Hopf invariants, Lie coalgebras, rational homotopy theory, graph cohomology.

ΩX on the looping of a map from S^n to X. This was also the starting point of Hain's work [12] using Chen integrals, but our definition of functionals is clearly distinct from his. We establish basic properties and give examples using the classical bar complex. In the second part, we use the Harrison complex on commutative cochains, and thus must switch to rational coefficients. Using our graph coalgebraic presentation, we show that a product-coproduct formula established geometrically in the bar complex descends to the duality predicted by Koszul-Moore theory.

Our basic, apparently new, observation is that calculations in bar complexes yield the Hopf invariant formula of Whitehead [28], as well as those of Haefliger, Novikov and Sullivan. This observation could have been made fifty years ago. Our approach incorporates a modern viewpoint by directly using Harrison-André-Quillen homology, the standard algebraic bridge from commutative algebras to Lie coalgebras, with the new graphical presentation essential for a self-contained development. One direction we plan to pursue further is the use of Hopf invariants to realize Koszul-Moore duality isomorphisms in general. A second direction we plan to pursue is that of spaces which are not simply connected, where our graph coalgebra models seem relevant even for $K(\pi, 1)$ spaces.

The problem of finding "homotopy periods" has been addressed before by Boardman-Steer [3], Sullivan [25], Haefliger [11], Hain [12] and Novikov [18], using a wide range of tools. We relate and compare our approach with these at the end of the paper. In summary, we view the cofree Lie coalgebra functor as the best for unifying formalism and geometry, showing how Koszul duality governs homotopy groups through "linking" of cochain data as explained in Section 1.2.

1. Hopf invariants from the bar complex

In these first sections when dealing exclusively with the bar complex, we only need associative cochains and so work integrally. Later when dealing with the Lie coalgebra model of a space, we switch to the exclusive use of commutative cochains and work over the rationals. For consistency with historical practice we use $C^*(X)$ to denote the usual cochains with cup product on a simplicial set X, or equivalently any subalgebra whose inclusion induces an isomorphism on cohomology. Similarly, we let $A^*(X)$ denote the PL forms on X or equivalently a subalgebra model. It is an unfortunate accident of notational history that A^* is commutative while C^* is associative but not commutative (except in an E_∞ sense).

Definition 1.1. Let $B(R)$ denote the bar complex on an associative differential graded algebra R, defined in the standard way as the total complex of a bicomplex spanned by monomials $x_1|\cdots|x_n$, where the x_i have positive degree, multilinear in each variable. The "internal" differential d_R is given by extending that of R by the Leibniz rule, and the "external" differential d_μ is defined by removing bars and multiplying; we write d_B for the total differential in the bar complex when it is not otherwise clear by the context. If x is a monomial in the bar complex, the internal degree of x is the sum of the degrees of its component elements, and the weight of x is its number of component elements. The total degree of x is its internal degree minus its weight.

Let $B(X)$ denote $B(C^*(X))$, and let $H^*_B(X)$ denote $H_*(B(C^*(X)))$.

Throughout the paper, we generally suppress the suspension and desuspension operators s and s^{-1} which are used in the definition of the bar complex and related complexes, as where they need to appear is always determined by the context. We may include them when for example they facilitate computing signs. Also, we will assume throughout that X is a simply connected space. We leave it to a sequel to address the non-simply-connected case, which requires new foundational understanding of Lie coalgebras. Those foundations are being worked out by the second author. We have done some preliminary calculations, in which our techniques work beyond the nilpotent setting.

The classical work of Adams-Hilton and Eilenberg-Moore established that $H^*_B(X)$ is isomorphic additively to the cohomology of the based loopspace of X. We now show that the homology of the bar complex is also the natural setting for Hopf invariants for an arbitrary X, extending the invariants for spheres and other suspensions. Topologically, we are passing from a map $f : S^n \to X$ to its looping $\Omega f : \Omega S^n \to \Omega X$, on which we evaluate cohomology classes from the bar complex. But the way in which we do the evaluation, and the properties we derive, have not to our knowledge been previously considered. We start with a standard calculation in the bar complex for the sphere, including a proof because the central ingredient – weight reduction – yields a method for explicit computation of Hopf invariants.

Lemma 1.2. *$H^{n-1}_B(S^n)$ is rank one, generated by an element of weight one corresponding to the generator of $H^n(S^n)$.*

Proof. Suppose $\alpha \in B^{n-1}(S^n)$ is a cycle. Since α has finitely many terms, its terms have maximal weight k. Write $\alpha|_k$ for the weight k terms of α. If $k > 1$, then its internal differential $d_{C^*}(\alpha|_k) = 0$, so $\alpha|_k$ gives a cocycle in $\otimes_k \bar{C}^*(S^n)$. By the Künneth theorem, $\otimes_k \bar{C}^*(S^n)$ has no homology in degree n, so $\alpha|_k$ is exact in $\otimes_k \bar{C}^*(S^n)$. Any choice of cobounding expression will determine a $\beta \in B(S^n)$ with $d_{C^*}\beta = \alpha|_k$. Therefore $\alpha - d_B\beta$ is a lower weight expression in $B^{n-1}(S^n)$ homologous to α. Inductively, we have that α is homologous to a cycle of weight one, so the map from $H^n(S^n)$ to $H^{n-1}_B(S^n)$ including the weight-one cocycles is surjective.

Applying this weight-reduction argument to a cochain $\beta \in B^n(S^n)$ with $d\beta = x$ for x of weight one, we see that the map from $H^n(S^n)$ to $H^{n-1}_B(S^n)$ is injective as well. $\square$

Definition 1.3. Let $\gamma \in B^{n-1}(S^n)$ be a cocycle. Define $\tau(\gamma) \simeq \gamma$ to be a choice of weight-one cocycle to which γ is cohomologous.

Define $\int_{B(S^n)}$ to be the map from cocyles in $B^{n-1}(S^n)$ to $\mathbb{Z}$ given by $\int_{B(S^n)} \gamma = \int_{S^n} \tau(\gamma)$, where $\int_{S^n}$ denotes evaluation on the fundamental class of S^n.

From Lemma 1.2 it is immediate that the map $\int_{B(S^n)}$ is well defined and induces an isomorphism $H^{n-1}_B(S^n) \cong \mathbb{Z}$.

The standard way to use cohomology to define homotopy functionals is to pull back and evaluate. This is essentially how we define our generalized Hopf invariants, allowing for a homology between the cocycle we pull back and one which we know how to evaluate.

Definition 1.4. Define the Hopf pairing $\langle\ ,\ \rangle_\eta : H^{n-1}_B(X) \times \pi_n(X) \to \mathbb{Z}$ by sending $[\gamma] \times [f]$ to $\int_{B(S^n)} f^*(\gamma)$.

We call $\tau(f^*(\gamma))$ the Hopf cochain (or form) of γ pulled back by f. We name the associated maps $\eta : H^{*-1}_B(X) \to \mathrm{Hom}(\pi_*(X), \mathbb{Z})$ and $\eta^\dagger : \pi_*(X) \to \mathrm{Hom}(H^{*-1}_B(X), \mathbb{Z})$. We say $\eta(\gamma) \in \mathrm{Hom}(\pi_*(X), \mathbb{Z})$ is the Hopf invariant associated to γ.

A choice of Hopf cochain is not unique, but the corresponding Hopf invariant is well defined. It is immediate that the Hopf invariants are functorial. Moreover, the definitions hold with any ring coefficients. Topologically we have the following interpretation.

Proposition 1.5. *The value of the Hopf invariant associated to a cocycle γ in the bar complex on some map f, namely $\eta(\gamma)(f)$, is equal to $[\gamma]\big(\Omega f_*[\Omega S^n]\big)$, the value of the cohomology class given by γ in $H^{n-1}(\Omega X)$ on the image under Ωf of the fundamental class of $H_{n-1}(\Omega S^n)$.*

1.1. **Examples.**

Example 1.6. A cocycle of weight one in $B(X)$ is just a closed cochain on X, which may be pulled back and immediately evaluated. Decomposable elements of weight one in $B(X)$ are null-homologous, consistent with the fact that products evaluate trivially on the Hurewicz homomorphism.

Example 1.7. Let ω be a generating 2-cocycle on S^2 and $f : S^3 \to S^2$. Then $\gamma = -\omega|\omega$ is a cocycle in $B(S^2)$ which f pulls back to $-f^*\omega|f^*\omega$, a weight-two cocycle of total degree two on S^3. Because $f^*\omega$ is closed and of degree two on S^3, it is exact. Let $d^{-1}f^*\omega$ be a choice of a cobounding cochain. Then

$$d_B\left(d^{-1}f^*\omega|f^*\omega\right) = f^*\omega|f^*\omega \ + \ \left(d^{-1}f^*\omega \smile f^*\omega\right).$$

Thus $f^*\gamma$ is homologous to $\left(d^{-1}f^*\omega \smile f^*\omega\right)$, and the corresponding Hopf invariant is $\int_{S^3} d^{-1}f^*\omega \smile f^*\omega$, which is the classical formula for the Hopf invariant given by Whitehead [28] (and generalized to maps from arbitrary domains by O'Neill [19]).

Expressions involving choices of d^{-1} for some cochains will be a feature of all of our formulae. On the sphere one can make this explicit as in the proof of the Poincaré Lemma, as Sullivan pointed out when defining similar formulae in Section 11 of [25].

Example 1.8. Let $X = S^n \vee S^m$ and let x be a cochain representative for a generator of $H^n(S^n)$ and similarly y on S^m. Then $\gamma = x|y$ is a cocycle in $B(X)$. Let $f : S^{n+m-1} \to S^n \vee S^m$ be the universal Whitehead product. More explicitly let $p_1 : D^n \times D^m \to S^n$ be projection onto D^n followed by the canonical quotient map, and let $p_2 : D^n \times D^m \to S^m$ be defined similarly. Decompose S^{n+m-1} as

$$S^{n+m-1} = \partial(D^n \times D^m) = D^n \times S^{m-1} \ \smile \ S^{n-1} \times D^m.$$

Then $f|_{D^n\times S^{m-1}} = p_1|_{D^n\times S^{m-1}}$ and $f|_{S^{n-1}\times D^m} = p_2|_{S^{n-1}\times D^m}$.

Proceeding as in the previous example, $\langle \gamma, f\rangle_\eta = (-1)^{|x|+1}\int_{S^{n+m-1}} d^{-1}f^*x \smile f^*y$. But these cochains extend to $D^n \times D^m$. Namely, $d^{-1}p_1^*x$ on $D^n \times D^m$ restricts to $d^{-1}f^*x$, and similarly f^*y is the restriction of p_2^*y. We evaluate as follows:

$$\begin{aligned}(-1)^{|x|+1}\int_{S^{n+m-1}} d^{-1}f^*x \smile f^*y &= \int_{\partial(D^n\times D^m)} (d^{-1}p_1^*x \smile p_2^*y)\big|_{\partial(D^n\times D^m)} \\ &= \int_{D^n\times D^m} p_1^*x \smile p_2^*y = \int_{D^n} x \cdot \int_{D^m} y = 1.\end{aligned}$$

The change in sign in the first equality above is due to the change in orientation on the fundamental class induced by the isomorphism $S^{n+m-1} \cong \partial(D^n \times D^m)$.

We conclude that the Hopf invariant of γ detects the Whitehead product, which recovers a theorem from the third page of [11]. This first case of evaluation of a Hopf invariant on a Whitehead product will be generalized below.

Example 1.9. For an arbitrary X and cochains x_i, y_i and θ on X with $dx_i = dy_i = 0$ and $d\theta = \sum(-1)^{|x_i|} x_i \smile y_i$, the cochain $\gamma = \sum x_i|y_i + \theta \in B(X)$ is closed. The possible formulae for the Hopf invariant are all of the form

$$\langle \gamma,\ f\rangle_\eta = \int_{S^n} \left(f^*\theta - \sum \left((-1)^{|x_i|} t \cdot d^{-1} f^* x_i \smile f^* y_i + (1-t) \cdot f^* x_i \smile d^{-1} f^* y_i \right) \right),$$

for some real number t. This generalizes a formula given in the Computations section of [10] and is also present in [11].

By choosing $t = \frac{1}{2}$ we see that reversing the order to consider $\sum y_i|x_i$ will yield the same Hopf invariant, up to sign. Thus $\sum x_i|y_i \mp y_i|x_i$ yields a zero Hopf invariant. Indeed, there are many Hopf invariants which are zero, a defect which will be remedied by using the Lie coalgebraic bar construction.

Example 1.10. In applications, Hopf forms are easily computed using a weight reduction technique introduced in the proof of Lemma 1.2. The bigrading of $B(X)$ is used as in the following example, illustrated in Figure 1.

Suppose there is a weight-three cocycle in $B(X)$ of the form

$$\gamma = x_1|x_2|x_3 - x_{12}|x_3 + x_{123},$$

where $dx_i = 0$, $dx_{12} = x_1 \smile x_2$, $dx_{123} = x_{12} \smile x_3$ and $x_2 \smile x_3 = 0$, x_1 has odd degree and x_2 has even degree. Consider the element

$$\alpha = d^{-1} f^* x_1 | f^* x_2 | f^* x_3 + d^{-1}\left(d^{-1} f^* x_1 \smile f^* x_2 - f^* x_{12} \right) | f^* x_3.$$

Here we observe that $(d^{-1} f^* x_1 \smile f^* x_2 - f^* x_{12})$ is closed and thus exact in order to know that we may find a d^{-1} for it in $C^*(S^n)$. We express $d_B(\alpha) = (d_{C^*X} + d_\mu)(\alpha)$ in $B(S^n)$, which is naturally a second-quadrant bicomplex, as follows.

$$\begin{tikzcd}
-f^*x_1|f^*x_2|f^*x_3 & & \\
d^{-1}f^*x_1|f^*x_2|f^*x_3 \arrow[u, mapsto, "d_C"] \arrow[r, mapsto, "d_\mu"] & \begin{array}{l} (d^{-1}f^*x_1 \smile f^*x_2)|f^*x_3 \\ -(d^{-1}f^*x_1 \smile f^*x_2 - f^*x_{12})|f^*x_3 \end{array} & \\
& d^{-1}(d^{-1}f^*x_1 \smile f^*x_2 - f^*x_{12})|f^*x_3 \arrow[u, mapsto, "d_C"] \arrow[r, mapsto, "d_\mu"] & -d^{-1}(d^{-1}f^*x_1 \smile f^*x_2 - f^*x_{12}) \smile f^*x_3
\end{tikzcd}$$

FIGURE 1. Calculation of $d_B(\alpha)$ in $B(S^n)$ from Example 1.10.

Observe that

$$f^*(\gamma) + d_B(\alpha) = \left(f^* x_{123} - d^{-1}\left(d^{-1} f^* x_1 \smile f^* x_2 - f^* x_{12} \right) \smile f^* x_3 \right).$$

The right-hand side is a weight-one cocycle in $B(S^n)$, meaning that it gives a Hopf cochain of γ pulled back by f.

1.2. **Hopf invariants as generalized linking numbers.** We explain the geometric significance of these formulae, which is one of the main highlights of this approach. The classical Hopf invariant was understood as a linking number, and the present generalization is rightly understood in terms of linking numbers as well.

The story is clearest when X is a manifold with a collection $\{W_i\}_{i=1}^k$ of proper oriented submanifolds of codimension two or greater, disjoint from each other and the basepoint of X. Denote the associated Thom cochains by ω_i. (There are various settings in which this can be done, the simplest being to define a Thom cochain as a "partially defined" cochain whose value on a smooth singular chain is given by an intersection count. We will instead use de Rham theory, where constructions of Thom forms are well documented as in Chapter 1.6 of [5].) Because the W_i are disjoint, the ω_i can be chosen with disjoint support. Then $\gamma = \omega_1|\cdots|\omega_k$ is a cocycle in $B(X)$. If $f : S^d \to X$, where d is the sum of the codimensions of the W_i minus k, then an associated Hopf form is

$$\tau(f^*(\gamma)) = f^*\omega_1 \wedge d^{-1}\left(f^*\omega_2 \wedge d^{-1}\left(\cdots \smile \left(f^*\omega_{k-1} \wedge d^{-1} f^*\omega_k\right)\cdots\right)\right).$$

Because f is based we can choose a diffeomorphism of $S^d \backslash \{\text{basepoint}\}$ with $\mathbb{R}^d$ and consider the submanifolds $f^{-1}W_i \subset \mathbb{R}^d$. In general if $\partial P = Q$, then we can construct d^{-1} of the Thom form of Q as a Thom form of P. In this case we define $d^{-1}f^*\omega_k$ by first taking the Thom form of the manifold E_k given by extending $f^{-1}W_k$ upwards in $\mathbb{R}^d$ (say in the first coordinate) and eventually outside of a fixed ball B containing all of the $f^{-1}W_i$. The Thom form of the translated copy of $f^{-1}W_k \subset \mathbb{R}^d$ can then be cobounded by a form whose support is outside of B, so $d^{-1}f^*\omega_k$ is the sum of this cobounding form outside of B along with a Thom form of E_k.

Because $f^*\omega_{k-1}$ has support in B, its wedge product with $d^{-1}f^*\omega_k$ is equal to its wedge product with the Thom form of E_k. Because the wedge product of Thom forms is a Thom form of the intersection, this will be a Thom form of the submanifold of $f^{-1}W_{k-1}$ of points which lie above (in $\mathbb{R}^d$) some point of $f^{-1}W_k$. Proceeding in this fashion, the Hopf invariant of f will in the end be the generic count of collections of a point in $f^{-1}W_1 \subset \mathbb{R}^d$ which lies above a point in $f^{-1}W_2$ which in turn lies above a point in $f^{-1}W_3$, etc.

The resulting count is a generalized linking number, equal to the degree of the Gauss map from $\prod_k f^{-1}W_i$ to $\prod_{k-1} S^{d-1}$ sending $(x_1, \ldots, x_k)$ to $(\cdots, \frac{x_i - x_{i-1}}{||x_i - x_{i-1}||}, \cdots)$. This Gauss map factors through the universal Gauss map sending $\prod f^{-1}(W_i)$ to the ordered configuration space $\mathrm{Conf}_k(\mathbb{R}^d)$ by sending a collection of points with one point in each $f^{-1}W_i$ to the configuration given by their images in $\mathbb{R}^d$. These degrees are encoding the image in homology of the fundamental class of $\prod f^{-1}(W_i)$ included in $\mathrm{Conf}_k(\mathbb{R}^d)$. As we discuss in Section 4, these Hopf invariants arising from Thom classes of disjoint submanifolds agree with some of those defined by Boardman and Steer [3], as applied by Koschorke [13].

If the submanifolds W_i are not disjoint, then their linking numbers need to include "correction terms" given by submanifolds which bound their intersections. The simplest example is that of some $W_1 \cap W_2 = \partial T$. Then there is a cocycle in the bar complex of the form $\omega_1|\omega_2 \pm \theta$, where as before ω_i are Thom cochains of the W_i and now θ is a Thom cochain of T so that $d\theta = \omega_1 \wedge \omega_2$.

This cocyle in the bar complex is a special case of Example 1.9. The associated Hopf invariant is $\int_{S^{d_1+d_2-1}} d^{-1}f^*\omega_1 \wedge f^*\omega_2 \mp f^*\theta$, where the d_i are the codimensions of the W_i. An individual f may be assumed to be transverse to W_1 and W_2, in which case their preimages are disjoint, and the first part of this integral is their linking number. But through a homotopy H from f to another map the preimages of the W_i may intersect at some time.

The change in linking number which occurs because of this intersection is accounted for by a corresponding change in $\int_{S^{d_1+d_2-1}} f^*\theta$, which counts the preimages of T. See Figure 2, in which for f (that is, $H|_{t=0}$) the Hopf invariant has a contribution of ± 1 from $\int_{S^{d_1+d_2-1}} d^{-1}f^*\omega_1 \wedge f^*\omega_2$ and a contribution of ± 1 from $\int_{S^{d_1+d_2-1}} f^*\theta$, but the Hopf invariant of the $H|_{t=1}$ has two contributions of ± 1 of the latter form, counting preimages of T.

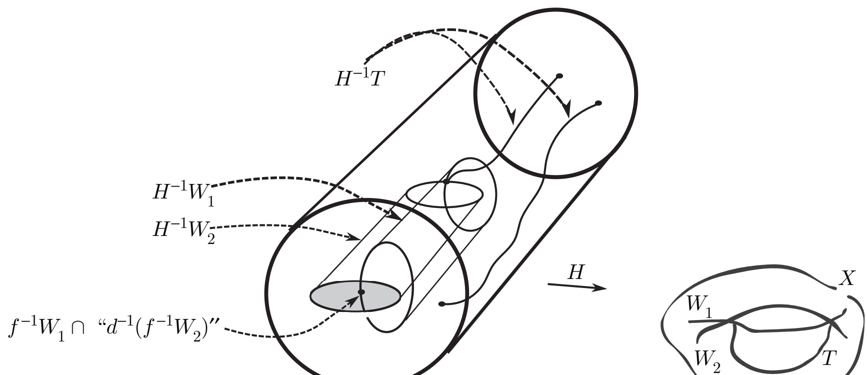


FIGURE 2. An illustration of "linking with correction" remaining constant through a homotopy.

All of the formulae we gave in Section 1.1 may be interpreted in the language of intersection, linking, and boundary behavior of submanifolds, when the classes in question are Thom classes (which they may always be assumed to be if we allow submanifolds with singularities). Indeed, generalizing the definition of Hopf invariants through linking numbers was the starting point for our project. It was in elucidating this geometry that we were led to the formalism of Lie coalgebras, with the cohomology of configuration spaces playing a central role.

1.3. **The generalized Hopf invariant one question.** From Proposition 1.5 and the adjoint of the Milnor-Moore theorem that $\pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X; \mathbb{Q})$ is injective [16], we have the following.

Proposition 1.11. *If X is simply connected, the map $\eta_B : H_B^{*-1}(X;\mathbb{Q}) \to \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ is surjective. In particular, the map $\eta_B : H_B^{*-1}(X) \to \mathrm{Hom}(\pi_*(X), \mathbb{Z})$ is of full rank.*

The question of the kernel of this map, which is large, is addressed in the next sections. That such a large kernel arises is explained from the operadic viewpoint as the fact that we are taking the wrong bar construction. The rational PL cochains on a simplicial set are commutative, so we should be taking a bar construction over the Koszul dual cooperad, namely the Lie cooperad, rather than the associative cooperad. That is, the correct homology theory for a commutative algebra is André-Quillen homology.

The question of the cokernel of the integral map is a natural generalization of one of the most famous questions in the history of topology, namely the Hopf invariant one problem. The cokernel is trivial immediately for X an odd sphere and is $\mathbb{Z}/2$ for X an even sphere other than S^2, S^4 and S^8 by Adams' celebrated result. We will indicate a new line of attack on this problem after we develop the Lie coalgebraic approach.

1.4. **Evaluation on Whitehead products.** Some properties of Hopf invariants, such as naturality, are immediate. A deeper result is to use the standard coproduct of $B(X)$ to compute the value of our generalized Hopf invariants on Whitehead products.

Let $f : S^n \to X$ and $g : S^m \to X$, and recall the definition of the Whitehead product:

$$[f,g] : S^{n+m-1} \xrightarrow{W} S^n \vee S^m \xrightarrow{f \vee g} X.$$

We have $(f \vee g)^* = f^* + g^*$ on $C^*(X)$, so on $B(X)$ the map $(f \vee g)^*$ acts by $(f^* + g^*)$ on each component of a bar expression. For example,

$$\begin{aligned}(f \vee g)^*\big(x_1 \mid x_2\big) &= (f^* + g^*)x_1|(f^* + g^*)x_2 \\ &= f^*x_1|f^*x_2 \;+\; f^*x_1|g^*x_2 \;+\; g^*x_1|f^*x_2 \;+\; g^*x_1|g^*x_2.\end{aligned}$$

We will generally mean for f^* and g^* to be considered as maps $B(X) \to B(S^n \vee S^m)$ as in this example, omitting the inclusions of and projections onto wedge factors.

Theorem 1.12. *Let γ be a cocycle in $B^{n+m-2}(X)$ and $f : S^n \to X$, $g : S^m \to X$. Then*

$$\langle \gamma,\ [f,g] \rangle_\eta = \sum_j \langle \alpha_j,\ f \rangle_\eta \cdot \langle \beta_j,\ g \rangle_\eta \ \mp\ \langle \alpha_j,\ g \rangle_\eta \cdot \langle \beta_j,\ f \rangle_\eta,$$

where $\Delta\gamma \simeq \sum_j \alpha_j \otimes \beta_j$, with all α_j and β_j closed, and $\mp$ is minus the Koszul sign induced by moving $s^{-1}\alpha_j$ past $s^{-1}\beta_j$.

Our convention is for $\langle \gamma,\ f \rangle_\eta = 0$ if $f : S^n \to X$ and $|\gamma| \neq n - 1$. The relation $\Delta\gamma \simeq \sum_j \alpha_j \otimes \beta_j$ with α_i and β_i closed follows from the fact that $\Delta\gamma$ is closed in $B(X) \otimes B(X)$ and an application of the Künneth isomorphism.

In the next section, we will see that Hopf invariants completely determine the homotopy Lie algebra of X rationally. The present theorem recovers some of the integral information as well.

Proof. Because $\langle \gamma,\ [f,g] \rangle_\eta = \langle (f \vee g)^*\gamma,\ W \rangle_\eta$ we may compute the Hopf invariant in S^{n+m-1} by doing intermediate work in $S^n \vee S^m$. Recall $\bar{C}^*(S^n \vee S^m) \cong \bar{C}^*(S^n) \oplus \bar{C}^*(S^m)$, which induces a component-bigrading on $B(S^n \vee S^m)$ as follows. If μ is a monomial in $B(S^n \vee S^m)$, we say that $w \in B(S^n)$ is an S^n component of μ if it is a maximal length subword with support only in S^n; we define S^m components of ω similarly. Bigrade $B(S^n \vee S^m)$ by the number of S^n components and the number of S^m components. If $a, b \in C^*(S^n \vee S^m)$ have disjoint supports, then $a \smile b = 0$, so the differential of $B(S^n \vee S^m)$ preserves component-bigrading.

For example, let $a_i, b_i \in C^*(S^n \vee S^m)$, where the a_i have support in S^n and the b_i in S^m. Then $\mu = b_1|a_2|a_3|b_2|b_3|b_4$ has S^n component $a_2|a_3$ and S^m components b_1 and $b_2|b_3|b_4$, so μ has component-bigrading $(1, 2)$.

Given a cocycle $\gamma \in B^{n+m-2}(X)$, the pullback $(f \vee g)^*\gamma \in B(S^n \vee S^m)$ splits as a sum of cocyles in each component-bigrading. By the Künneth theorem, $B(S^n \vee S^m)$ has no homology in degree $n+m-2$ away from bigrading $(1,1)$ and possibly $(1,0)$ and $(0,1)$. So all terms of $(f \vee g)^*\gamma$ not in these bigradings are exact. The terms in bigrading $(1,0)$ and $(0,1)$ are $f^*\gamma$ and $g^*\gamma$, which are exact after being pulled back to S^{n+m-1} by W. Thus it suffices to focus on just the terms of $(f \vee g)^*\gamma$ in bigrading $(1,1)$.

Bigrading $(1,1)$ splits as the sum of two subcomplexes, one where the S^n component comes first, and the other where the S^m component comes first. The subcomplex of bigrading $(1,1)$ with the S^n component first is isomorphic to $B(S^n) \otimes B(S^m)$. The terms of $(f \vee g)^*\gamma$ in this subcomplex are

$$\sum_{\Delta\gamma = \sum_i a_i \otimes b_i} f^*(a_i) | g^*(b_i).$$

If $\sum_i a_i \otimes b_i \simeq \sum_j \alpha_j \otimes \beta_j$, then the cobounding expression in $B(X) \otimes B(X)$ will determine a homotopy between the above terms and $\sum_j f^*(\alpha_j)|g^*(\beta_j)$. By the Künneth theorem this is cohomologous to $\sum_j \tau(f^*\alpha_j)|\tau(g^*\beta_j)$. By our work in Example 1.8, the Hopf pairing of this is $\sum_j \langle \alpha_j,\ f\rangle_\eta \cdot \langle \beta_j,\ g\rangle_\eta$, as desired.

The terms of $(f \vee g)^*\gamma$ in bigrading $(1,1)$ with the S^m component first are similar, but with a shift of sign at the final step due to the change in orientation induced by the isomorphism $D^n \times D^m \cong D^m \times D^n$. $\square$

This theorem implies for example that the Hopf invariant of a cocycle of the form $\alpha|\beta \mp \beta|\alpha$ evaluates trivially on all Whitehead products. We saw in Example 1.9 that this Hopf invariant is indeed zero. There are many further classes of Hopf invariants which vanish, though proving this through direct methods quickly becomes difficult. However, Theorem 1.12 suggests a key for identifying bar expressions with vanishing Hopf invariants. If we define in the obvious way an "anti-commutative coproduct" on the bar construction, the submodule where this coproduct vanishes will by Theorem 1.12 have Hopf invariants which evaluate trivially on Whitehead products. Quotienting by this submodule yields the Lie coalgebraic cobar construction, as discussed in Section 3 of [24]. This is our focus for the rest of the paper.

2. Hopf invariants from the Lie coalgebra model of a space, and their completeness

We now switch to the rational, commutative setting and we replace the bar construction B with the Lie coalgebraic bar construction $\mathcal{E}$ from [24].

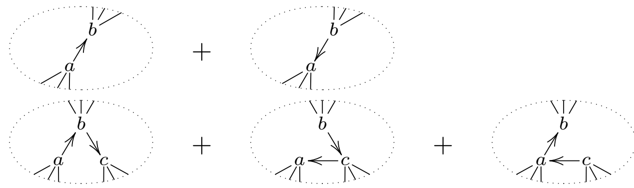
2.1. Lie coalgebraic bar construction.

Definition 2.1. Let V be a vector space. Define $\mathbb{E}(V)$ to be the quotient of $\mathbb{G}(V)$ by $\mathrm{Arn}(V)$, where $\mathbb{G}(V)$ is the span of the set of oriented acyclic graphs with vertices labeled by elements of V modulo multilinearity in the vertices, and where

$\mathrm{Arn}(V)$ is the subspace generated by arrow-reversing and Arnold expressions:

(arrow-reversing)

(Arnold)

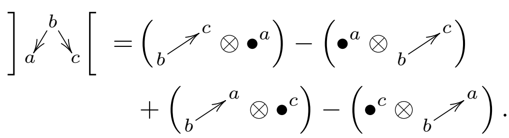


Here a, b, and c are elements of V labeling vertices of a graph which could possibly have edges connecting to other parts of the graph (indicated by the ends of edges abutting a, b, and c), which are not modified in these operations.

A Lie cobracket is defined as

$$]G[= \sum_{e \in G} (G_1^{\hat{e}} \otimes G_2^{\hat{e}} - G_2^{\hat{e}} \otimes G_1^{\hat{e}}),$$

where e ranges over the edges of G, and $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the connected components of the graph obtained by removing e, which points to $G_2^{\hat{e}}$. In [24] we show this is well defined. For example,



We require a graded version of $\mathbb{E}(V)$, which we define in Section 3 of [24]. Our graph-theoretic representation is an explicit model for Lie coalgebras, as first studied in [15].

Proposition 2.2. *If V is graded in positive degrees, then $\mathbb{E}(V)$ is isomorphic to the cofree Lie coalgebra on V.*

Indeed, if V and W are linearly dual, then the *configuration pairing* of [23, 24, 27] can be used to define a perfect pairing between $\mathbb{E}(V)$ and the free Lie algebra on W.

Definition 2.3. Let A be a one-connected commutative differential graded algebra with differential d_A and multiplication μ_A. Define $\mathcal{G}(A)$ to be the total complex of the bicomplex $\big(\mathbb{G}(s^{-1}\bar{A}),\ d_A,\ d_\mu\big)$. Here $s^{-1}\bar{A}$ is the desuspension of the ideal of positive-degree elements of A, the "internal" differential d_A is given by cofreely extending that of A by the Leibniz rule, and the "external" differential $d_\mu(g) = \sum_e \mu_e(g)$, where up to sign $\mu_e(g)$ contracts the edge e in g, multiplying the elements of A labeling its endpoints to obtain the new label. (For the sign convention, see Definition 4.5 of [24].) Let $\mathcal{E}(A)$ be the quotient $\mathcal{G}(A) \big/ \mathrm{Arn}(s^{-1}\bar{A})$.

We define the internal degree and the weight of monomials in $\mathcal{E}(A)$ as the sum of the degrees of component elements and the number of vertices in the graph monomial. As before, total degree is internal degree minus the weight.

Define $\mathcal{E}(X)$ to be $\mathcal{E}(A^*(X))$, where $A^*(X)$ is a model for rational commutative cochains on X, and let $H^*_{\mathcal{E}}(X)$ denote $H^*(\mathcal{E}(X))$.

The complex $\mathcal{E}(A)$ is the "Lie coalgebraic" bar construction on a commutative algebra which computes Harrison homology, constructed presently as a quotient of the "graph coalgebraic" bar construction $\mathcal{G}(A)$.

Harrison homology is a special case of André-Quillen homology [1, 20] for one-connected commutative differential graded algebras. Recall that the Harrison chains $B_H(A)$ are constructed dually to Quillen's functor $\mathcal{L}$. The bar construction of a commutative algebra is a Hopf algebra under the shuffle product and the splitting coproduct. The Harrison complex is given by quotienting to Hopf algebra indecomposables. By a celebrated theorem of Barr [2], the quotient map $p : BA \to B_H A$ has a splitting e when working over a field of characteristic 0.

Proposition 2.4. *There is a short exact sequence of bicomplexes, giving an isomorphism of final terms:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Sh}(A) & \longrightarrow & B(A) & \longrightarrow & B_H(A) & \longrightarrow & 0 \\ & & \downarrow & & \phi\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \mathrm{Arn}(A) & \longrightarrow & \mathcal{G}(A) & \longrightarrow & \mathcal{E}(A) & \longrightarrow & 0, \end{array}$$

where ϕ sends the bar expression $a_1|a_2|\cdots|a_n$ to the graph $a_1 \nearrow^{a_2} \searrow \cdots \nearrow^{a_n}$. *Furthermore this is an isomorphism of Lie coalgebras when the Harrison complex is given the Lie coalgebra structure defined by Schlessinger and Stasheff* [21].

Our presentation of Harrison homology via graphs has a critical advantage over the classical construction. While the set of generators of $\mathcal{G}(A)$ is larger than that of $B(A)$, the set of relations $\mathrm{Arn}(A)$ is simpler to express and is defined locally, unlike the shuffle relations $\mathrm{Sh}(A)$. This greatly simplifies some proofs, constructions, and calculations. By abuse we will sometimes use the bar notation to refer to elements of $\mathcal{E}(A)$, suppressing ϕ.

The significance of Harrison homology in this setting is that it bridges the worlds of cohomology and homotopy. Hopf invariants give a geometric understanding of this bridge. For convenience, we use Barr's splitting and our development of Hopf invariants for the standard bar complex to quickly establish the basic properties. Deeper properties and explicit constructions require the Lie coalgebraic formulation.

2.2. **Hopf invariants.** As we did for the bar complex, we start with the fact that $H^{n-1}_{\mathcal{E}}(S^n)$ is rank one, generated by an element of weight one.

Definition 2.5. Given a cocycle $\gamma \in \mathcal{E}^{n-1}S^n$, we let $\tau(\gamma) \simeq \gamma$ be any cohomologous cocyle of weight one.

Write $\int_{\mathcal{E}(S^n)}$ for the map from cocycles in $\mathcal{E}^{n-1}(S^n)$ to $\mathbb{Q}$ given by $\int_{\mathcal{E}(S^n)} \gamma = \int_{S^n} \tau(\gamma)$.

Lemma 2.6. *The map $\int_{\mathcal{E}(S^n)}$ is well defined and induces the isomorphism $H^{n-1}_{\mathcal{E}}(S^n) \cong \mathbb{Q}$.*

Proof. We observe that $\int_{\mathcal{E}(S^n)} = \int_{B(S^n)} \circ e$, since Barr's splitting e is an isomorphism which is an "identity" on the weight-one parts of these complexes. Thus this lemma follows from Lemma 1.2. $\square$

Following [24], it would be better to use the graph coalgebraic bar construction $\mathcal{G}(A)$ here, but we chose to use the classical bar construction because it is more familiar, has an established Barr splitting, and is directly tied to the cohomology of loopspaces.

We can now take our Definition 1.4 of Hopf invariants and replace the bar complex everywhere by $\mathcal{E}$ (and the integers by the rational numbers). When necessary, we distinguish the Hopf pairings and Hopf invariants by decorations such as $\langle\ ,\ \rangle_\eta^{\mathcal{E}}$ vs. $\langle\ ,\ \rangle_\eta^{B}$, or $\eta^{\mathcal{E}}$ vs. η^B, etc.

Theorem 2.7. *Hopf invariants are compatible with the quotient map p and Barr's splitting e between $B(X)$ and $\mathcal{E}(X)$. That is, $\eta^B = \eta^{\mathcal{E}} \circ p$ and $\eta^{\mathcal{E}} = \eta^B \circ e$.*

Proof. We consider the adjoint maps and fix an $f \in \pi_n(X)$. The images of $(\eta^B)^\dagger(f)$ and $(\eta^{\mathcal{E}})^\dagger(f)$ are then given by the composites going from left to right in the following diagram:

$$\begin{tikzcd}
B(X) \arrow[r, "f^*"] \arrow[dd, shift right, "p"'] & B(S^n) \arrow[dr, "\int_{B(S^n)}"] \arrow[dd, shift right, "p"'] & \\
& & \mathbb{Q}. \\
\mathcal{E}(X) \arrow[r, "f^*"] \arrow[uu, shift right, "e"'] & \mathcal{E}(S^n) \arrow[ur, "\int_{\mathcal{E}(S^n)}"'] \arrow[uu, shift right, "e"'] &
\end{tikzcd}$$

The result follows from commutativity of the two squares (one involving p and one involving e) and two triangles in the above diagram. The squares commute due to the naturality of the quotient map and Barr's splitting, and the triangles commute by our comment in the proof of Lemma 2.6. ☐

We now see that all of the "shuffles" in the kernel of the quotient map $p : B(X) \to \mathcal{E}(X)$ give rise to the zero homotopy functional, constituting a large kernel.

Using this theorem, we can translate some theorems from the bar setting, such as the following, which is immediate from Theorem 1.12.

Corollary 2.8. *Let γ be a cocycle in $\mathcal{E}^{n+m-2}(X)$ and $f : S^n \to X$, $g : S^m \to X$. Then*

$$\langle \gamma,\ [f,g]\rangle_\eta^{\mathcal{E}} = \sum_j \langle \alpha_j,\ f\rangle_\eta^{\mathcal{E}} \cdot \langle \beta_j,\ g\rangle_\eta^{\mathcal{E}},$$

where $]\gamma[\ \simeq \sum_j \alpha_j \otimes \beta_j$, with all α_j and β_j closed.

By recasting our Hopf invariants in terms of Lie coalgebras, we get some immediate applications.

Example 2.9. Consider a wedge of spheres $X = \bigvee_i S^{d_i}$. Let ω_i represent a generator of $H^{d_i}(S^{d_i})$ and W be the differential graded vector space spanned by the ω_i with trivial products and differential. Then W is a model for cochains on X and $\mathcal{E}(W) = \mathbb{E}(W)$ is just the cofree Lie coalgebra on W.

Theorem 1.12 along with Proposition 3.6 of [24] imply that the value of some Hopf invariant η_γ on an iterated Whitehead product P of the basic inclusions $\iota_i = [S^{d_i} \hookrightarrow X] \in \pi_{d_i}(X)$ is given by the configuration pairing (defined in Corollary 3.5

of [24]) of γ and P. This immediately implies that the ι_i generate a copy of a free Lie algebra in the rational homotopy groups of X.

Because this example is universal, a corollary to Theorem 1.12 is that in general the value of a Hopf invariant on an iterated Whitehead product can be calculated through a configuration pairing.

2.3. **Completeness.** Hopf invariants associated to $\mathcal{E}(X)$ give a complete and sharp picture of rational homotopy groups.

Theorem 2.10. *If X is simply connected, then $\eta^{\mathcal{E}} : H_{\mathcal{E}}^{*-1}(X) \to \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ is an isomorphism.*

There are three possible lines of proof. Using Theorem 1.12 and Example 2.9 we could show that η agrees with the isomorphism given by the Quillen equivalences established in [24]. Alternately we could try to start with the surjectivity of this map, which follows from Proposition 1.11, and try to calculate the kernel. Instead we shall prove more directly that η is an isomorphism by developing a long exact sequence of a fibration for $\mathcal{E}$. This approach yields an independent proof and gives a more direct geometric understanding of the isomorphism in Theorem 2.10.

Definition 2.11. Let $\mathcal{E}(E,F)$ be the kernel of the map from $\mathcal{E}(E)$ to $\mathcal{E}(F)$ induced by inclusion. Define relative Hopf invariants for $\pi_n(E,F)$ by pulling back a cycle in $\mathcal{E}(E,F)$ to $\mathcal{E}(D^n, S^{n-1})$ and then reducing to weight one and evaluating.

That $H_{\mathcal{E}}^{n-1}(D^n, S^{n-1})$ is rank one generated by a cycle of weight one follows from a direct calculation. To see that relative Hopf invariants are well defined we may use the same weight-reduction argument.

By construction, there is a long exact sequence for $H^*_{\mathcal{E}}(F)$, $H^*_{\mathcal{E}}(E)$ and $H^*_{\mathcal{E}}(E,F)$. Hopf invariants give a pairing between this long exact sequence and the long exact sequence for relative homotopy groups. For clarity, we will drop the identifier η from our Hopf pairings below and instead subscript them by their ambient space.

Proposition 2.12. *The following long exact sequences pair compatibly:*

$$\begin{array}{c}
\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{j_*} \pi_n(E,F) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \\
\cdots \longleftarrow H_{\mathcal{E}}^{n-1}(F) \xleftarrow{i^*} H_{\mathcal{E}}^{n-1}(E) \xleftarrow{j^*} H_{\mathcal{E}}^{n-1}(E,F) \xleftarrow{\delta} H_{\mathcal{E}}^{n-2}(F) \longleftarrow \cdots
\end{array}$$

That is, for $f \in \pi_n(F)$ and $\gamma \in H_{\mathcal{E}}^{n-1}(E)$, $\langle i^*\gamma, f\rangle_F = \langle \gamma, i_* f\rangle_E$, and similarly elsewhere.

Proof. For i_* and i^* this is immediate from naturality, as both $\langle i^*\gamma,\, f\rangle_F$ and $\langle \gamma,\, i_* f\rangle_E$ are equal to $\int_{S^n} \tau\left((f\circ i)^*\gamma\right)$. The argument for $\langle j^*\gamma,\, f\rangle_E$ and $\langle \gamma,\, j_* f\rangle_{(E,F)}$ is the same once one substitutes $\pi_n(E, pt.)$ for $\pi_n(E)$ and $H^n_{\mathcal{E}}(E, pt.)$ for $H^n_{\mathcal{E}}(E)$. In this case one now sees both pairings as given by $\int_{D^n} \tau\left((f\circ j)^*\gamma\right)$.

Finally we consider the connecting homomorphisms. Recall that $\delta : H_{\mathcal{E}}^{n-2}(F) \to H_{\mathcal{E}}^{n-1}(E,F)$ takes a cocycle $\gamma \in \mathcal{E}^{n-2}(F)$, lifts it at the cochain level to $\mathcal{E}^{n-2}(E)$ and then takes its coboundary. Thus given $f : (D^n, S^{n-1}) \to (E,F)$, we have $\langle \delta(\gamma),\, f\rangle_{(E,F)} = \int_{D^n} \tau(f^* d_{\mathcal{E}}\bar\gamma)$, where $\bar\gamma$ is a lift of γ to E. Using the same lift $\bar\gamma$ we see that

$$\langle \gamma,\, \partial f\rangle_F = \int_{S^{n-1}} \tau\big((\partial f)^*\gamma\big) = \int_{\partial D^n} \tau(f^*\bar\gamma) = \int_{D^n} d\,\tau(f^*\bar\gamma).$$

$\square$

We now connect with the exact sequences of a fibration. Let $p : E \to B$ be a fibration and write $\lambda : \pi_n(B) \to \pi_n(E, F)$ for the standard map defined by lifting.

Proposition 2.13. *The map $p^* : \mathcal{E}(B) \to \mathcal{E}(E, F)$ is a weak equivalence.*

Corollary 2.14. *Let γ be a cycle in $\mathcal{E}(E, F)$. Then γ is homologous to $p^*(\bar{\gamma})$ for some $\bar{\gamma}$ in $\mathcal{E}B$.*

Proof of Proposition 2.13. We use the maps induced by the Sullivan model for the fiber inclusion map i, as given in [9], §15(a):

$$\begin{tikzcd}
A_{PL}F & A_{PL}E \arrow[l, "i^*"'] & A_{PL}B \arrow[l, "p^*"'] \arrow[dl, "\iota"] \\
(\Lambda V_F, \bar{d}) \arrow[u, "m_F", "\simeq"'] & (A_{PL}B \otimes \Lambda V_F, d). \arrow[l, "\varepsilon"'] \arrow[u, "m", "\simeq"'] &
\end{tikzcd}$$

Here $(\Lambda V_F, \bar{d})$ is the Sullivan model for $A_{PL}F$, ι is the map induced by the inclusion of the unit $\mathbb{Q} \to \Lambda V_F$, and ε is the map induced by the augmentation $A_{PL}B \to \mathbb{Q}$.

Write $\hat{\mathcal{E}}(E, F)$ for the kernel of the map $\varepsilon : \mathcal{E}(A_{PL}B \otimes \Lambda V_F, d) \to \mathcal{E}(\Lambda V_F, \bar{d})$. There is an induced quasi-isomorphism $\hat{\mathcal{E}}(E, F) \xrightarrow{\simeq} \mathcal{E}(A_{PL}E, A_{PL}F)$. Our work is completed in the lemmas which follow by establishing that the inclusion maps i_1 and i_2 in the diagram below are both quasi-isomorphisms, and therefore so is the induced map on kernels, namely $\mathcal{E}A_{PL}B \to \hat{\mathcal{E}}(E, F)$:

$$\begin{tikzcd}
\mathcal{E}A_{PL}F & \mathcal{E}A_{PL}E \arrow[l, "i^*"'] & \mathcal{E}(E, F) \arrow[l, "\delta"'] \\
\mathcal{E}(\Lambda V_F) \arrow[u, "\mathcal{E}m_F", "\simeq"'] & \mathcal{E}(A_{PL}B \otimes \Lambda V_F) \arrow[l, "\varepsilon_*"'] \arrow[u, "\mathcal{E}m", "\simeq"'] & \hat{\mathcal{E}}(E, F) \arrow[l, "\bar{\delta}"'] \arrow[u, "\simeq"'] \\
V_F \arrow[u, "i_1", "\simeq"'] & \mathcal{E}A_{PL}B \oplus V_F \arrow[l, "pr"'] \arrow[u, "i_2", "\simeq"'] & \mathcal{E}A_{PL}B, \arrow[l, "\hat{\delta}"'] \arrow[u, "\simeq"'] \arrow[ul, bend left, "\iota_*"] \arrow[uul, bend right, "p^*"']
\end{tikzcd}$$

where the differentials on V_F and ΛV_F are the standard restrictions of that on $A_{PL}B \otimes \Lambda V_F$. $\square$

An alert reader would note that while we have generally steered clear of the traditional approach to rational homotopy through minimal models, we do use the Sullivan model for a fibration here. We only need that the cochain of a total space of a fibration is a twisted tensor product of those on the fiber and base, and we consider Sullivan model theory to be the most convenient and well-known place to reference this fact. We could just as well have used the earlier work of Brown [6] giving this result for simplicial cochains of fibrations, along with a translation to PL cochains and the fact that we can make free models. Thus, our work is still independent from the theory of minimal models.

Lemma 2.15. *The inclusion* $V_F \to \mathcal{E}(\Lambda V_F)$ *sending* v *to* $\overset{v}{\bullet}$ *is a quasi-isomorphism.*

Proof. We proceed in a similar manner as Theorem 19.1 of [9]. The homogeneous elements of $\mathcal{E}\Lambda V_F$ are graphs whose vertices are decorated by non-empty words in V_F, the sum of whose lengths we call the total word length. We define a map of complexes $h : \mathcal{E}\Lambda V_F \to \mathcal{E}\Lambda V_F$ which, away from the image of V_F, gives a chain homotopy between the identity map and a map which increases the weight of a graph while keeping its total word length fixed if possible, or which is zero if that is not possible. Since weight is bounded by total word length, iterated applications of this will eventually produce a null-homology of any cycle not in V_F.

Write a generic long graph $\gamma \in \mathcal{E}^{m+1}\Lambda V$ of total word length n as

$$\gamma = \left(\begin{tikzcd} & v_{k_1}v_{k_1+1}\cdots \arrow[dr] & & v_{k_j}\cdots \arrow[dr] & & v_{k_m}\cdots v_n \\ v_1v_2\cdots \arrow[ur] & & \cdots \arrow[ur] & & \cdots \arrow[ur] & \end{tikzcd} \right).$$

Define $h(\gamma)$ to be zero if the total word length of γ is equal to $m+1$ (in which case $\gamma = v_1|\cdots|v_{m+1}$). Otherwise γ has at least one word of length two decorating some vertex. In this case $h(\gamma)$ is a sum, over the letters in such words, of the graphs obtained by removing each letter in turn and using it to decorate a new vertex attached to the old vertex while fixing the rest of the graph. Explicitly,

$$h(\gamma) \;=\; \frac{1}{n} \sum_{\substack{j \text{ with} \\ k_{j+1}-k_j>1}} \;\sum_{k_j \le p \le k_{j+1}-1} (-1)^{\kappa(p)} \left(\begin{tikzcd} & v_p \arrow[d] & \\ & v_{k_j}\cdots\widehat{v}_p\cdots v_{k_{j+1}-1} \arrow[dr] & \\ \cdots \arrow[ur] & & \cdots \end{tikzcd} \right),$$

where the $(-1)^{\kappa(p)}$ is the Koszul sign coming from moving a degree-one operator to v_p's vertex and moving v_p to the front of its word and across an s^{-1} and n is the total word length.

More generally if γ is any decorated graph, we define h as a sum over the letters in the words of length at least two decorating the vertices of γ, removing each letter in turn and using it to decorate a new vertex as above. The internal ordering of the vertices in the resultant graph is the same as for γ, with the new vertex occurring immediately before the vertex its letter was removed from.

That h is well defined follows easily from the local nature of the anti-symmetry and Arnold relations in the definition of $\mathcal{E}$. Away from V_F, h decreases its degree by one (due to the unwritten s^{-1} in front of v_p) and increases its graph length by one.

It is straightforward to check that $d_{\mathcal{E}}h + hd_{\mathcal{E}} = id +$ (graphs of greater length) outside of V_F. The Arnold and arrow-reversing relations are required in order to establish equality for graphs with internal vertices decorated by singleton words.

For example, using notation from [24] and writing $(-1)^a$ instead of $(-1)^{|a|}$, we have

$$
\begin{aligned}
d_{\mathcal{E}}\, h \left(\begin{smallmatrix} & s^{-1}b & \\ \nearrow & & \searrow \\ s^{-1}a & & s^{-1}c \end{smallmatrix} \right) &= d_{\mathcal{E}}\, 0 = 0, \\
h\, d_{\mathcal{E}} \left(\begin{smallmatrix} & s^{-1}b & \\ \nearrow & & \searrow \\ s^{-1}a & & s^{-1}c \end{smallmatrix} \right) &= h \Bigg((-1)^a \; s^{-1}ab \to s^{-1}c \; + (-1)^{a+1+b} \; s^{-1}a \to s^{-1}bc \\
&\qquad + d_A \Big(\begin{smallmatrix} & s^{-1}b & \\ \nearrow & & \searrow \\ s^{-1}a & & s^{-1}c \end{smallmatrix} \Big) \Bigg) \\
&= (-1)^a \, \frac{1}{3} \Bigg((-1)^a \underset{s^{-1}a\otimes s^{-1}b\otimes s^{-1}c}{\begin{smallmatrix} 1 & \\ \downarrow & \\ 2 & \longrightarrow 3 \end{smallmatrix}} + (-1)^{ab+b} \underset{s^{-1}b\otimes s^{-1}a\otimes s^{-1}c}{\begin{smallmatrix} 1 & \\ \downarrow & \\ 2 & \longrightarrow 3 \end{smallmatrix}} \Bigg) \\
&\quad + (-1)^{a+1+b} \, \frac{1}{3} \Bigg((-1)^{a+1+b} \underset{s^{-1}a\otimes s^{-1}b\otimes s^{-1}c}{\begin{smallmatrix} & 2 \\ & \downarrow \\ 1 \longrightarrow & 3 \end{smallmatrix}} \\
&\qquad + (-1)^{a+1+bc+c} \underset{s^{-1}a\otimes s^{-1}c\otimes s^{-1}b}{\begin{smallmatrix} & 2 \\ & \downarrow \\ 1 \longrightarrow & 3 \end{smallmatrix}} \Bigg) + 0 \\
&= \frac{2}{3} \begin{smallmatrix} & s^{-1}b & \\ \nearrow & & \searrow \\ s^{-1}a & & s^{-1}c \end{smallmatrix} - \frac{1}{3} \Bigg((-1)^{(a+b)(c+1)} \begin{smallmatrix} & s^{-1}a & \\ \nearrow & & \searrow \\ s^{-1}c & & s^{-1}b \end{smallmatrix} \\
&\qquad + (-1)^{(a+1)(b+c)} \begin{smallmatrix} & s^{-1}c & \\ \nearrow & & \searrow \\ s^{-1}b & & s^{-1}a \end{smallmatrix} \Bigg) = \begin{smallmatrix} & s^{-1}b & \\ \nearrow & & \searrow \\ s^{-1}a & & s^{-1}c \end{smallmatrix}
\end{aligned}
$$

The general proof proceeds by "adding whiskers" to the input vertices of these graphs, so that this calculation becomes a local part of a larger graph. The Arnold identity is used an additional time to move whiskers back to their correct vertex.

In the end, on the basic case of graphs whose total word length is $m+1$, h is a chain homotopy between the identity map and the projection onto V_F . ☐

Remark 2.16. Our proof of Lemma 2.15 makes essential use of the graph complex representation of Harrison homology. In particular, the definition of h above does not preserve long graphs, so it is not defined on the bar complex representation of Harrison homology. The standard proof of the bar complex analogue of Lemma 2.15 uses the homotopy $h(v_1 \otimes \cdots \otimes v_{k_1} | \cdots | \cdots \otimes v_n) = (v_1 | v_2 \otimes \cdots \otimes v_{k_1} | \cdots | \cdots \otimes v_n)$ if $k_1 \neq 1$ and 0 otherwise. This does not induce a well-defined map either from commutative algebras (using symmetric tensors) or to bar expressions modulo shuffles. Thus for example [9] Proposition 22.8, which is approximately dual to Lemma 2.15, must be proven by non-constructive methods, detouring through the universal enveloping algebra.

Lemma 2.17. *The inclusion $\mathcal{E}A_{PL}B \oplus V_F \to \mathcal{E}(A_{PL}B \otimes \Lambda V_F)$ is a quasi-isomorphism.*

Sketch of proof. The proof is similar to the previous lemma. Homogeneous elements of $\mathcal{E}(A_{PL}B \otimes \Lambda V_F)$ are graphs whose vertices are decorated by an element of $A_{PL}B$ and a word in V_F. On any vertex, the decorating element from $A_{PL}B$ may be trivial or the V_F word may be empty, but not both at once.

As in the proof of Lemma 2.15, we define a chain homotopy h which evaluates on a graph as a sum over replacements of each vertex "pulling out" V_F letters in turn, connecting them to the vertex by a new edge. As before it is straightforward to calculate that $d_{\mathcal{E}}h + hd_{\mathcal{E}} = id +$ (graphs of greater length) outside of $\mathcal{E}A_{PL}B \oplus V_F$. □

To our knowledge, the dual of this result does not appear in the literature. Attempting to directly prove the dual statement by methods analogous to the proof of Proposition 22.8 in [9] would be difficult; though it could be deduced through more abstract methods, we require an explicit definition of chain homotopies for computations, such as in Section 3.2.

Proposition 2.18. *The evaluation of Hopf invariants is compatible with the isomorphisms $\pi_n(E, F) \cong \pi_n(B)$ and $H^n_{\mathcal{E}}(E, F) \cong H^n_{\mathcal{E}}(B)$. That is, $\langle p^*\gamma,\, f\rangle_{(E,F)} = \langle \gamma,\, p_* f\rangle_B$ and $\langle \gamma,\, \lambda f\rangle_{(E,F)} = \langle \bar{\gamma},\, f\rangle_B$.*

Proof. The first equality is immediate since both $\langle p^*\gamma,\, f\rangle_{(E,F)}$ and $\langle \gamma,\, p_* f\rangle_B$ are equal to $\int_{D^n} \tau\left((p \circ f)^*\gamma\right)$.

For the second equality, we have

$$\begin{aligned}\langle \gamma,\, \lambda f\rangle_{(E,F)} &= \int_{D^n} \tau\left((\lambda f)^*\gamma\right) = \int_{D^n} \tau\left((\lambda f)^* p^*\bar{\gamma}\right) \\ &= \int_{D^n} \tau\left((p \circ \lambda f)^*\bar{\gamma}\right) = \int_{D^n} \tau(f^*\bar{\gamma}) = \langle \bar{\gamma},\, f\rangle_B.\end{aligned}$$

□

We can now quickly prove Theorem 2.10.

Proof of Theorem 2.10. Applying the perfect pairing between the $\mathcal{E}$ and homotopy long exact sequences of a fibration to the Postnikov tower of a space, it is enough to know that $\eta^{\mathcal{E}}$ gives an isomorphism on Eilenberg-MacLane spaces.

Recall that $A_{PL}K(\mathbb{Q}^m, n)$ is quasi-isomorphic to a free graded algebra with m generators in degree n. Lemma 2.15 gives a null homotopy of all elements of $\mathcal{E}(\Lambda s^n \mathbb{Q}^m)$ except for these generators. The Hopf invariants associated to these generators, given by simply pulling back cohomology and evaluating, are linearly dual to $\pi_n(K(\mathbb{Q}^m, n))$ by the Hurewicz theorem. □

We need the existence but not the uniqueness of the Postnikov tower. On the whole, the only theorems from topology which we use to define our functionals and prove they are complete are this existence and the fact that a fibration can be modeled by a twisted tensor product of cochain models. The rest of the basic input has been algebraic, namely our development of the Lie cooperadic bar construction.

3. Applications and open questions

We give some applications and present some questions now opened up to investigation. We have listed one such already, namely the generalized Hopf invariant one question. Our work in the previous section gives rise to a new approach to this problem. Namely one could try to understand how the classical bar construction over the integers, modulo Harrison shuffles, fails to yield an exact sequence of a fibration. Such an approach could give rise to a new, very different proof of Adams' result. Answering the question for general X would yield significant insight into the relationship between homotopy theory in characteristic zero and characteristic p.

3.1. **Wedges of spheres.** Revisiting Example 2.9, recall that a model for cochains on a wedge of spheres X is just the graded vector space W with one generator w_i in the appropriate dimension for each sphere, with trivial products and differential. Theorem 2.10 now implies that the linear dual of $\pi_*(X)$ is just the cofree Lie coalgebra on W. We take advantage of Corollary 2.8 and the perfect pairing between free Lie algebras and cofree Lie coalgebras to deduce the Hilton-Milnor Theorem that the homotopy groups of X are just the free Lie algebra on W.

We not only know what the homotopy Lie algebra is, but we see explicitly how to determine whether two maps into wedges of spheres are homotopic. The vector space $\mathcal{E}(W)$ is spanned by cocycles of the form $\gamma_\sigma = \underset{w_{\sigma(1)}}{}\nearrow^{w_{\sigma(2)}}\searrow\cdots\nearrow^{w_{\sigma(k)}}$ as σ varies over the symmetric group. Following Section 1.2, represent w_i by the Thom class associated to a point p_i (away from the basepoint) of each wedge factor S^{d_i}. Given some map $f : S^n \to X$ we let $W_i = f^{-1}(p_i) \subset \mathbb{R}^{d_i}$. The argument in Section 1.2 applies, and the Hopf invariant of γ_σ is a linking number of the W_i, counting the number of times (with signs) that a point in $W_{\sigma(1)}$ lies "directly above" one in $W_{\sigma(2)}$ which in turn lies above one in $W_{\sigma(3)}$, etc. The rational homotopy class of f is determined by such counts and can be named as an element of the free Lie algebra using the configuration pairing.

3.2. **Homogeneous spaces.** Consider the standard Puppe sequence (for fibrations, as in [17])

$$H \xrightarrow{i} G \xrightarrow{p} G/H \xrightarrow{f} BH \xrightarrow{Bi} BG.$$

Here H and G are H-spaces, i is a map of H-spaces, and G/H is defined as the homotopy fiber of Bi, which models the corresponding homogeneous space when i is an inclusion of Lie groups. Rationally an H-space is a generalized Eilenberg-MacLane space, which is formal with free rational cohomology algebra. By Lemma 2.15, we have for example that $\mathcal{E}(G) \simeq V_G$, where V_G is the vector space of indecomposables, represented as weight-one cocycles. The associated Hopf invariants are simply an evaluation of these cohomology classes, as expected in the Eilenberg-MacLane setting.

If we also identify $\mathcal{E}(H) \simeq V_H$, then $\mathrm{Hom}(\pi_*(G/H), \mathbb{Q})$, which is isomorphic to $H^*_{\mathcal{E}}(G/H)$ if G/H is simply connected, is isomorphic to the direct sum of the kernel and suspended cokernel of $i^* : V_G \to V_H$. (Here to suspend and obtain classes in $H^*(BG)$ and $H^*(BH)$ we can for example use the transgression in the Leray-Serre spectral sequence.) Hopf invariants associated to the cokernel of i^* are simple to interpret. A cohomology class in V_H suspends to a cohomology class of BH which can be pulled back to G/H and evaluated on homotopy. This will be trivial if the class was already pulled back from BG.

On the other hand, consider $x \in V_G$, which we identify with a representative weight-one cocycle in $\mathcal{E}(G)$, with $i^*(x) = 0$. In the model $A_{PL}G/H \otimes \Lambda V_H$ for the cochains of G, represent x by $a_0 \otimes 1 + \sum a_i \otimes f_i$ with the a_i closed and $f_i \neq 1$. Then in $\mathcal{E}(A_{PL}G/H \otimes \Lambda V_H)$ we have

$$x - d_{\mathcal{E}}(\sum a_i|f_i) = a_0 + \sum a_i|b_i,$$

where $b_i = df_i$ is in $A_{PL}G/H$ since ΛV_H has trivial differential. Thus x is homologous to the pull-back through p of $a_0 + \sum a_i|b_i \in \mathcal{E}(G/H)$. (A useful exercise here is to consider the Hopf maps.)

The Hopf invariants for G/H (which are complete when G/H is simply connected) are therefore given by either an evaluation of cohomology or by classical linking invariants of the pull-backs of the Poincaré duals of the a_i and b_i as in Section 1.2. When H and G are Lie groups, one can also use de Rham theoretic models of all of these spaces defined through the Lie algebras of H and G (the antecedents of Quillen functors) for explicit calculations.

While weight-one cocycles account for the Hopf invariants of a generalized Eilenberg-MacLane space, weight-two cocycles were required here. In general the highest weight which appears in $H^*_{\mathcal{E}}(X)$ should reflect the number of stages in the smallest tower atop which the rationalization of X sits whose fibers are generalized Eilenberg-MacLane spaces.

3.3. **Configuration spaces.** Next consider the ordered configuration space $\mathrm{Conf}_n(\mathbb{R}^d)$, with $n = 3$. Its cohomology algebra is generated by classes a_{12}, a_{13} and a_{23}, where for example a_{12} is pulled back from the map which sends (x_1, x_2, x_3) to $\frac{x_2 - x_1}{||x_2 - x_1||} \in S^{d-1}$. These cohomology classes satisfy the Arnold identity $a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12} = 0$ (the same identity at the heart of our approach to Lie coalgebras, an overlap which we explain below). Its homotopy Lie algebra is generated by classes whose images in homology are dual, which we denote by b_{ij}, for which "x_i and x_j orbit each other." Their Lie brackets satisfy the identity that $[b_{12}, b_{23}] = [b_{23}, b_{31}] = [b_{31}, b_{12}]$.

The best way to construct Hopf invariants is not with the standard a_{ij} since their products are non-trivial at the cochain level. The simplest cocycle in $\mathcal{E}(\mathrm{Conf}_3(\mathbb{R}^d))$ using these cochains would be

$$ {}_{a_{12}}\!\nearrow^{a_{23}} + {}_{a_{23}}\!\nearrow^{a_{31}} + {}_{a_{31}}\!\nearrow^{a_{12}} + \overset{\theta}{\bullet}, $$

where θ is a cochain cobounding the Arnold identity. Consider instead the submanifold of points (x_1, x_2, x_3) which are collinear. This submanifold has three components, which we label by which x_i is "in the middle". These are proper submanifolds, so we consider their Thom classes, which we denote by Co_i. By intersecting these with the cycles representing the homology generators b_{ij} we see that for example Co_1 is cohomologous to $a_{31} + a_{12}$. Because they are disjoint, we have that any graph ${}_{Co_i}\!\nearrow^{Co_j}$ is a cocycle in $\mathcal{E}(\mathrm{Conf}_n(\mathbb{R}^d))$. We may use Corollary 2.8 to see for example that

$$\begin{aligned} \left\langle {}_{Co_1}\!\nearrow^{Co_2},\ [b_{12}, b_{23}] \right\rangle_\eta &= (a_{31} + a_{12})(b_{12}) \cdot (a_{12} + a_{23})(b_{23}) \\ &\quad + (a_{12} + a_{23})(b_{12}) \cdot (a_{31} + a_{12})(b_{23}) \\ &= 1 \cdot 1 + 1 \cdot 0 = 1. \end{aligned}$$

Thus, to understand an element f of $\pi_{2d-3}(\mathrm{Conf}_3(\mathbb{R}^d))$ it suffices to understand the linking behavior of $f^{-1}(Co_i)$. This calculation is reflected in the results of [7], which used relative Hopf invariants of an evaluation (or Gauss) map for knots to give a new interpretation of the simplest finite-type knot invariant. Understanding the framework of how such homotopy invariants can be defined and evaluated on more complicated Whitehead products is a primary motivation for our development of Hopf invariants.

3.4. **The graph complex.** Though it played only an incidental role in our current development, all of our work in the bar complex and some of our work in the Lie coalgebra complex can be shifted onto the graph complex $\mathcal{G}(X)$. In fact, from the point of view of $\mathcal{E}(X)$, the graph complex is a much more natural home for the development of Hopf invariants than the bar complex.

Constructions and proofs proceed in the same manner as those for the bar complex, with arguments establishing basic properties applying verbatim. Equipping the graph complex with the anti-cocommutative graph-cutting cobracket, the proof of Theorem 1.12 applies to give compatibility with Whitehead products. Note that Theorem 1.12 requires only a component-bigrading and internal differential argument. In the graph complex, components are given by maximal size subgraphs.

Furthermore, since the quotient map $\mathcal{G}(S^n) \to \mathcal{E}(S^n)$ is the identity on weight one, the proof of Theorem 2.7 applies to show that $\eta^{\mathcal{G}} = \eta^{\mathcal{E}} \circ p$. However, if we wished to use $\eta^{\mathcal{G}}$ to show that $\eta^{\mathcal{E}}$ is well defined we would either need an alternate, direct, proof that Hopf invariants vanish on arrow-reversing and Arnold expressions, or else we would need a splitting of the quotient map $\mathcal{G}(A) \to \mathcal{E}(A)$, whose existence is open at the moment. In modern terminology, the splitting of the quotient $B(A) \to \mathcal{E}(A)$ is given by the first Eulerian idempotent $e_*^{(1)}$ ([14], §4.5). Together with the other Eulerian idempotents, this splits the bar complex into the Hodge decomposition. It is tempting to seek a similar decomposition of the graph complex, which might give an alternate way of understanding the Eulerian idempotents.

3.5. **The non-simply-connected setting.** In future work we plan to extend these results beyond the simply connected case. One main reason we consider only simply connected spaces here is that Lie coalgebras are not well understood outside of the (simply) connected setting, unless there is a nilpotence condition in effect. In [26], the first author is putting coalgebras on a sounder footing. We expect results even beyond the nilpotent setting, as we have seen in preliminary calculations that Hopf invariants, while more delicate to define, do distinguish the rational homotopy groups of $S^1 \vee S^2$.

It would be interesting to compare our Hopf invariants with homotopy invariants coming from H^0 of the generalized Eilenberg-Moore spectral sequence for $\mathrm{Map}(X, Y)$. Such a comparison might be useful in extending our techniques to give invariants of homotopy classes of maps from X to Y more generally.

4. Comparison with other approaches

Our theorems give a resolution of the "homotopy period" problem of explicitly representing homotopy functionals, a classical question considered many times over the past seventy-five years. A formula for homotopy periods using the same ingredients, namely pulling back forms to the sphere, taking d^{-1} and wedge products, and then integrating, was featured on the first page and then in Section 11 of Sullivan's seminal paper [25]. We have not been able to compare our formula with Sullivan's, since that formula comes from a minimal model for path spaces which uses a chain homotopy between an isomorphism and the zero map which is also a derivation. In some examples, we cannot find such a chain homotopy, and if one relaxes the condition of being a derivation, then the inductive formula is not clear.

In the same vein, some of the lower weight cases of our constructions were treated in work of Haefliger [11] and Novikov [18]. Haefliger in particular gave formulae for

Hopf forms which are special cases of Examples 1.9 and 1.10. He also gives formulae for evaluating these forms on Whitehead products, which of course follow from our Theorem 1.12. Haefliger's comment was that "It is clear that one could continue this way for higher Whitehead products, if one is not afraid of complicated formulas." The bracket-cobracket formalism and the configuration pairing make these formulae simple, at least conceptually. Haefliger uses Hopf invariants associated to cocycles in $\mathcal{E}(X)$ of the form $\omega_1|\cdots|\omega_n$ where $\omega_i\omega_{i-1} = 0$ to show that such X have summands of the corresponding free Lie algebra in their homotopy groups, which is also immediate from our approach.

As mentioned above, Hain's thesis [12] solves the rational homotopy period using Chen integrals. Here one appeals to the Milnor-Moore theorem that identifies rational homotopy within loopspace homology [16], so that dually loopspace cohomology must give all homotopy functionals. The main work is to find an irredundant collection. The geometric heart of Chen integrals is the evaluation maps $\Delta^n \times \Omega X \to X^n$ which evaluate a loop at n points. Thus when writing down these functionals explicitly, the integrals are over $S^n \times \Delta^k$ not S^n itself. Our approach is thus distinct and more intrinsic. Indeed, understanding homotopy solely through loopspace homology is understanding a Lie algebra through its universal enveloping algebra.

As mentioned in Section 1.2, our Hopf invariants also connect with those of Boardman and Steer [3, 4]. Their Hopf invariants are maps $\lambda_n : [\Sigma Y, \Sigma X] \to [\Sigma^n Y, \Sigma X^{\wedge n}]$. In some cases, the latter group can be computed, yielding homotopy functionals. A key intermediate they take is defining a map $\mu_n : [\Sigma Y, \bigvee_{i=1}^n X_i] \to [\Sigma^n Y, \bigwedge_{i=1}^n X_i]$. Comparing the proof of Theorem 6.8 of [4] with our Section 1.2 we see that their functionals agree with ours in this setting. While their technique is more general in the direction of analyzing maps out of any suspension as opposed to only spheres, our approach allows the analysis of homotopy groups of spaces which are not suspensions. Indeed, the behaviour $\mathcal{E}(\Sigma X)$ is fairly trivial, since finite sets of cochains may be assumed to have disjoint support, so the bar complex collapses.

Finally, we follow up on Example 2.9, where we showed that the value of one of our Hopf invariants on a Whitehead product is given by the configuration pairing between free Lie algebras and coalgebras. This pairing appears in the homology and cohomology of ordered configuration spaces. Recall from Section 1.2 that if $\{W_i\}$ is a disjoint collection of proper submanifolds of X and f is a map $S^d \to X$, then our Hopf invariants are encoding the map on homology induced by inclusion of $\prod f^{-1}(W_i)$ in $\mathrm{Conf}_k(\mathbb{R}^d)$. As Fred Cohen often says, "this cannot be a coincidence." In this case the connection is readily explained through Cohen's work on the homology of iterated loopspaces.

Consider $Y = \bigvee_{i=1}^k S^{d_i+2}$ and let $n = d_1 + \cdots + d_k + k - 1$, the dimension in which one can see a Whitehead product P of the inclusion maps ι_i with each map appearing once. Then $\pi_n(Y) \cong \pi_{n-2}(\Omega^2\Sigma^2(\bigvee S^{d_i}))$, which maps to $H_{n-2}(\Omega^2\Sigma^2(\bigvee S^{d_i}))$ under the Hurewicz homomorphism. Loopspace theory identifies this homology with that of $\bigcup_m \mathrm{Conf}_m(\mathbb{R}^2) \wedge_{\Sigma_m} (S^{d_1} \vee \cdots \vee S^{d_k})^{\wedge m}$, which in turn has a summand coming from the homology of $\mathrm{Conf}_k(\mathbb{R}^2) \times_{\Sigma_k} \left(\bigcup_{\sigma \in \Sigma_k} S^{d_{\sigma(1)}} \wedge \cdots \wedge S^{d_{\sigma(k)}}\right)$. But since the Σ_k action is free on the union of products of spheres, we get that this is simply

$$H_{n-2}(\mathrm{Conf}_k(\mathbb{R}^2) \wedge S^{\sum d_i}) \cong H_{k-1}(\mathrm{Conf}_k(\mathbb{R}^2)) \cong \mathcal{L}ie(n).$$

(See [22] for an expository treatment of this last isomorphism.) In Remark 1.2 of [8], established at the end of Section 13, Cohen gives a diagram establishing the compatibility of Whitehead products and Browder brackets, through the Hurewicz homomorphism. This result implies that a Whitehead product P maps to this $\mathcal{L}ie(k)$ summand in the canonical way, going to the homology class with the same name (up to sign).

In Example 2.9, we showed that the Hopf invariant η_γ evaluated on P according to the configuration pairing between free Lie algebras and coalgebras, modeled in the operad/cooperad pair $\mathcal{L}ie$ and $\mathcal{E}il$. Thus a cohomology class $g \in \mathcal{E}il(k) \cong H^{k-1}(\mathrm{Conf}_k(\mathbb{R}^2))$ represented by a graph (see [22]) gets mapped under the linear dual of the Hurewicz homomorphism to essentially the same graph (with each vertex label replaced by a volume form on the corresponding sphere) in $\mathcal{E}(Y)$.

In summary, for wedges of spheres and other double suspensions, our Hopf invariants coincide with applying the Hurewicz homomorphism and then evaluating on the cohomology of configuration spaces, which explains the combinatorial similarity between our Hopf invariants and that cohomology. At one point we outlined a proof of Corollary 2.8 using this kind of loopspace machinery, but we later found the more elementary approach given here.

References

1. Michel André, *Homologie des algèbres commutatives*, Springer-Verlag, Berlin, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 206. MR0352220 (50:4707)
2. Michael Barr, *Harrison homology, Hochschild homology and triples*, J. Algebra **8** (1968), 314–323. MR0220799 (36:3851)
3. J. M. Boardman and B. Steer, *Axioms for Hopf invariants*, Bull. Amer. Math. Soc. **72** (1966), 992–994. MR0202138 (34:2012)
4. ______, *On Hopf invariants*, Comment. Math. Helv. **42** (1967), 180–221. MR0221503 (36:4555)
5. Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982. MR658304 (83i:57016)
6. Edgar H. Brown, Jr., *Twisted tensor products. I*, Ann. of Math. (2) **69** (1959), 223–246. MR0105687 (21:4423)
7. Ryan Budney, James Conant, Kevin P. Scannell, and Dev Sinha, *New perspectives on self-linking*, Adv. Math. **191** (2005), no. 1, 78–113. MR2102844 (2005h:57013)
8. Frederick R. Cohen, Thomas J. Lada, and J. Peter May, *The homology of iterated loop spaces*, Springer-Verlag, Berlin, 1976. MR0436146 (55:9096)
9. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR1802847 (2002d:55014)
10. Phillip A. Griffiths and John W. Morgan, *Rational homotopy theory and differential forms*, Progress in Mathematics, vol. 16, Birkhäuser Boston, Mass., 1981. MR641551 (82m:55014)
11. André Haefliger, *Whitehead products and differential forms*, Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976), Lecture Notes in Math., vol. 652, Springer, Berlin, 1978, pp. 13–24. MR505648 (80a:55012)
12. Richard M. Hain, *Iterated integrals and homotopy periods*, Mem. Amer. Math. Soc. **47** (1984), no. 291, iv+98 pp. MR727818 (85d:55015)
13. Ulrich Koschorke, *A generalization of Milnor's μ-invariants to higher-dimensional link maps*, Topology **36** (1997), no. 2, 301–324. MR1415590 (2000a:57063)
14. Jean-Louis Loday, *Cyclic homology*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998, Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili. MR1600246 (98h:16014)
15. Walter Michaelis, *Lie coalgebras*, Adv. in Math. **38** (1980), no. 1, 1–54. MR594993 (82b:17016)

16. John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR0174052 (30:4259)
17. Yasutoshi Nomura, *On mapping sequences*, Nagoya Math. J. **17** (1960), 111–145. MR0132545 (24 #A2385)
18. S. P. Novikov, *Analytical theory of homotopy groups*, Topology and geometry—Rohlin Seminar, Lecture Notes in Math., vol. 1346, Springer, Berlin, 1988, pp. 99–112. MR970074 (89k:55027)
19. Edward J. O'Neill, *Higher order Massey products and links*, Trans. Amer. Math. Soc. **248** (1979), no. 1, 37–66. MR521692 (80h:55016)
20. Daniel Quillen, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, RI, 1970, pp. 65–87. MR0257068 (41:1722)
21. Michael Schlessinger and James Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra **38** (1985), no. 2-3, 313–322. MR814187 (87e:13019)
22. Dev P. Sinha, *Operads and knot spaces*, J. Amer. Math. Soc. **19** (2006), no. 2, 461–486 (electronic). MR2188133
23. ———, *A pairing between graphs and trees*, math.QA/0502547, 2006.
24. Dev P. Sinha and Ben Walter, *Lie coalgebras and rational homotopy theory, I: graph coalgebras*, math.AT/0610437.
25. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331 (1978). MR0646078 (58:31119)
26. Ben Walter, *Cofree coalgebras over cooperads*, In preparation.
27. ———, *Lie algebra configuration pairing*, arXiv:1010.4732 [math.RA], 2010.
28. J. H. C. Whitehead, *An expression of Hopf's invariant as an integral*, Proc. Nat. Acad. Sci. U. S. A. **33** (1947), 117–123. MR0020255 (8,525h)

Department of Mathematics, University of Oregon, Eugene, Oregon 97403
E-mail address: dps@math.uoregon.edu

Department of Mathematics, Middle East Technical University, Northern Cyprus Campus, Kalkanli, Guzelyurt, KKTC, Mersin 10 Turkey
E-mail address: benjamin@metu.edu.tr