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THE NON-LINEAR PLATEAU PROBLEM IN NON-POSITIVELY CURVED MANIFOLDS

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ABSTRACT. Using the Perron method, we prove the existence of hypersurfaces of prescribed special Lagrangian curvature with prescribed boundary inside complete Riemannian manifolds of non-positive curvature.

1. INTRODUCTION

In this paper we prove the existence of solutions of the Plateau problem for hypersurfaces of prescribed curvature and prescribed boundary in manifolds of non-positive sectional curvature. The curvature notion used here (special Lagrangian curvature) was introduced by the author in [7] and constitutes a higher dimensional generalisation of two dimensional extrinsic curvature. Its interest is two-fold: firstly, it possesses strong regularity properties that translate into very simple geometric behaviour of limits of sequences of hypersurfaces of prescribed special Lagrangian curvature; secondly, in low dimensions, it reduces to certain well-known notions of curvature. Explicitly, let M^{n+1} be a Riemannian manifold and let N^n be a locally strictly convex, smooth, immersed hypersurface in M. The special Lagrangian curvature, $R_\theta(N)$, is a function of the second fundamental form of N which depends on an angle parameter, $\theta \in [0, n\pi/2[$ (see Section 2 for more details). In low dimensions, we have:

- (i) when the ambient manifold is 3-dimensional, and thus when N is a surface:
$$R_{\pi/2} = K_e^{1/2},$$
where K_e is the extrinsic curvature of N;
- (ii) when the ambient manifold is 4-dimensional:
$$R_\pi = (K_e/H)^{1/2},$$
where H is the mean curvature of N. Moreover:
$$R_{\pi/2} = (R^N - R^M + \mathrm{Ric}^M(\mathsf{N}, \mathsf{N}))^{1/2},$$
where R^N and R^M are the scalar curvatures of N and M, respectively, Ric^M is the Ricci curvature of M and N is the unit exterior normal vector over N.

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Any existence result for hypersurfaces of prescribed special Lagrangian curvature thus translates in particular into existence results for the three notions of curvature given above.

Theorem 1.1. *Let M^{n+1} be a complete $(n+1)$-dimensional Riemannian manifold of non-positive sectional curvature. Let $\phi : M \to]0,\infty[$ be a strictly positive smooth function. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle, and let $(N^n, \partial N^n)$ be a compact locally convex immersed hypersurface in M such that, for all $p \in N$:*

$$R_\theta(N)(p) \geqslant \phi(p),$$

in the weak sense. If $\theta > (n-1)\pi/2$, then there exists a compact, locally convex immersed hypersurface, $(\hat{N}^n, \partial\hat{N}_n)$ in M such that:

(i) *$\hat{N}$ has smooth interior and is $C^{0,1}$ up to its boundary;*
(ii) *$\partial\hat{N} = \partial N$ and $\hat{N}$ is bounded by N;*
(iii) *for all $p \in \hat{N}$:*

$$R_\theta(\hat{N})(p) = \phi(p).$$

In fact, $(\hat{N}, \partial\hat{N})$ is isotopic by locally convex immersions to $(N, \partial N)$.

If $\theta = (n-1)\pi/2$, then the same result holds provided that, in addition, the shape operator of N is everywhere bounded below by $\epsilon\mathrm{Id}$ in the weak sense, for some $\epsilon > 0$.

Remark. See Sections 2 and 3 for terminology.

Remark. This generalises Theorem 1.1 of [9] to more general manifolds and also to the case of submanifolds with boundary.

Remark. When the ambient manifold is 3-dimensional, and when N is thus a surface, Theorem 1.1 constitutes a generalisation of the existence part of Proposition 5.0.3 of [4], which itself constitutes the analytic core of that paper (see also [11]).

Remark. When the ambient manifold is 4-dimensional, although this result proves existence of hypersurfaces for the case where $R_\pi = (K_e/H)^{1/2}$ is prescribed, it says nothing about the case of $R_{\pi/2}$. However, when M is hyperbolic, we can canonically associate to it a de Sitter manifold, M', which is dual to M in a certain sense. This duality has the effect of interchanging R_π and $R_{\pi/2}$. Moreover, since $R^{M'}$ and $\mathrm{Ric}^{M'}$ are constant in this case, this yields an existence result for 3-dimensional hypersurfaces of prescribed scalar curvature and prescribed boundary in 4-dimensional de Sitter manifolds (see also [1]).

Remark. In general, the Plateau problem is stated as follows: given a Riemannian manifold, M, and a finite family $\Gamma_1, ..., \Gamma_n$ of smooth, codimension 2 submanifolds in M, when does there exist an immersed hypersurface $N \subseteq M$ of constant curvature spanning $\Gamma_1 \cup ... \cup \Gamma_n$, i.e. such that

$$\partial N = \Gamma_1 \cup ... \cup \Gamma_n?$$

This result therefore reduces the Plateau problem for prescribed special Lagrangian curvature to the weaker problem of determining when there exists a locally convex immersed hypersurface of M spanning $\Gamma_1 \cup ... \cup \Gamma_n$. This problem is addressed, for example, by Alexander, Ghomi and Wong in [2] and Rosenberg in [6].

The proof of this result uses the Perron method, first applied to the study of hypersurfaces in two independent and simultaneous papers by Guan and Spruck

[3], and Trudinger and Wang [12], where they prove the existence of hypersurfaces of constant extrinsic curvature and prescribed boundary in $\mathbb{R}^{n+1}$. The main new ingredients used here involve various recent results by the author ([7], [9], [10] and [11]) of which the most significant are the regularity properties for special Lagrangian curvature (see [7]), and a regularity result near the boundary for locally convex immersions in Riemannian manifolds (see [11]). Here these results are combined around the following compactness result for locally strictly convex immersions in Riemannian manifolds, which generalises the Main Lemma of [12] and is of independent interest:

Lemma 6.1. *Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Choose $\epsilon > 0$ and let $(\Sigma_n, \partial\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of compact, locally convex immersed hypersurfaces such that:*

(i) $\Gamma_n := \partial\Sigma_n$ *is* C^∞*;*
(ii) *the shape operator of* Σ_n *is greater than* $\epsilon\mathrm{Id}$ *in the weak sense.*

Let Γ_0 be a smooth, compact, codimension 2 submanifold of M and suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ converges to Γ_0 in the C^∞ sense. If there exists a compact subset $K \subseteq M$ and a real number $B > 0$ such that, for all n:

$$\Sigma_n \subseteq K, \qquad \mathrm{Vol}(\Sigma_n) \leqslant B,$$

then there exists a $C^{0,1}$ locally convex immersed hypersurface Σ_0 in M such that:

(i) $\Gamma_0 = \partial\Sigma_0$*;*
(ii) *the shape operator of* Σ_0 *is greater than* $\epsilon\mathrm{Id}$ *in the weak sense;*
(iii) $(\Sigma_n)_{n\in\mathbb{N}}$ *subconverges to* Σ_0.

This paper is structured as follows: Sections 2 and 3 contain definitions and notation; Sections 4, 5 and 6 are devoted to the proof of Lemma 6.1; Section 7 provides a slight generalisation of the solution to the Dirichlet problem studied in [9], which forms the analytic core of the Perron method in this paper; in Section 8 we prove Theorem 1.1.

2. Immersed submanifolds and special Lagrangian curvature

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. An **immersed submanifold** is a pair $\Sigma = (S, i)$, where S is a smooth manifold and $i : S \to M$ is a smooth immersion. An **immersed hypersurface** is an immersed submanifold of codimension 1. We say that an immersed hypersurface is locally (strictly) convex if and only if its shape operator is everywhere positive definite. The special Lagrangian curvature, which is only defined for locally strictly convex immersed hypersurfaces, is defined as follows (see [7] for details): denote by $\mathrm{Symm}(\mathbb{R}^n)$ the space of symmetric matrices over $\mathbb{R}^n$. We define $\Phi : \mathrm{Symm}(\mathbb{R}^n) \to \mathbb{C}^*$ by

$$\Phi(A) = \mathrm{Det}(I + iA).$$

Since Φ never vanishes and $\mathrm{Symm}(\mathbb{R}^n)$ is simply connected, there exists a unique analytic function $\tilde{\Phi} : \mathrm{Symm}(\mathbb{R}^n) \to \mathbb{C}$ such that

$$\tilde{\Phi}(I) = 0, \qquad e^{\tilde{\Phi}(A)} = \Phi(A) \qquad \forall A \in \mathrm{Symm}(\mathbb{R}^n).$$

We define the function $\arctan : \mathrm{Symm}(\mathbb{R}^n) \to (-n\pi/2, n\pi/2)$ by

$$\arctan(A) = \mathrm{Im}(\tilde{\Phi}(A)).$$

This function is trivially invariant under the action of $O(\mathbb{R}^n)$. Moreover, if $\lambda_1, ..., \lambda_n$ are the eigenvalues of A, then

$$\arctan(A) = \sum_{i=1}^{n} \arctan(\lambda_i).$$

For $r > 0$, we define:

$$\mathrm{SL}_r(A) = \arctan(r^{-1}A).$$

If A is positive definite, then SL_r is a strictly decreasing function of r. Moreover, $\mathrm{SL}_\infty = 0$ and $\mathrm{SL}_0 = n\pi/2$. Thus, for all $\theta \in]0, n\pi/2[$, there exists a unique $r > 0$ such that

$$\mathrm{SL}_r(A) = \theta.$$

We define $R_\theta(A) = r$. R_θ is also invariant under the action of $O(n)$ on the space of positive definite, symmetric matrices.

Let M^{n+1} be an oriented Riemannian manifold of dimension $n+1$. Let $\Sigma = (S, i)$ be a locally strictly convex, immersed hypersurface in M. For $\theta \in]0, n\pi/2[$, we define $R_\theta(\Sigma)$ (the *θ-special Lagrangian curvature* of Σ) by

$$R_\theta(\Sigma) = R_\theta(A_\Sigma),$$

where A_Σ is the shape operator of Σ.

3. Locally convex hypersurfaces

Let M^{n+1} be a Riemannian manifold. A **locally convex hypersurface** in M is a pair $\Sigma = (i, S^n)$ where S is an n-dimensional topological manifold and $i : S \to M$ is a continuous map such that, for all $p \in S$, there exists a neighbourhood, U, of p in S, a convex subset $K \subseteq M$ with non-trivial interior, and an open subset $V \subseteq \partial K$ such that i restricts to a homeomorphism from U to V. We refer to such a triplet (U, V, K) as a **convex chart** of Σ. Pulling back the metric on M through i yields a natural length metric on Σ which we denote by d_Σ. Let $(\Sigma_n)_{n\in\mathbb{N}} = (i_n, S_n)_{n\in\mathbb{N}}$ and $S_0 = (i_0, S_0)$ be convex immersions. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ **converges** to Σ_0 if and only if:

(i) $(S_n, d_{\Sigma_n})_{n\in\mathbb{N}}$ converges to (S_0, d_{Σ_0}) in the Gromov-Hausdorff sense;
(ii) $(i_n)_{n\in\mathbb{N}}$ converges to i_0 locally uniformly.

Let $\Sigma = (i, S)$ and $\Sigma' = (i', S')$ be two locally convex hypersurfaces in M. We say that Σ and Σ' are equivalent if and only if there exists a homeomorphism $\phi : S \to S'$ such that

$$i = i' \circ \phi.$$

Example. Let $K \subseteq M$ be a convex subset with non-trivial interior. Then any open subset of ∂K is a locally convex hypersurface. □

Example. Let Σ be a (smooth) hypersurface on M. Σ is a locally convex hypersurface if and only if its second fundamental form is everywhere non-negative definite. □

For $\epsilon > 0$, we say that the second fundamental form of $\Sigma = (i, S)$ is **bounded below** by $\epsilon\mathrm{Id}$ in the **weak sense** if and only if, for every $p \in S$, and for every supporting normal, N_p of S at p, there exists a smooth, embedded hypersurface Σ' such that:

(i) Σ' passes through p;
(ii) the normal to Σ' at p is N_p;
(iii) the shape operator of Σ' at p is equal to $\epsilon\mathrm{Id}$;
(iv) there exists a neighbourhood U of p in S such that $i(U)$ lies entirely to one side of Σ'.

Observe that this definition is stronger than the usual weak notion of curvature bounds (in the Alexandrov, or viscosity, sense), since the latter does not assume the existence of bounding hypersurfaces normal to arbitrary directions at arbitrary points.

We may define the notion of weak lower (and upper) bounds for the special Lagrangian curvature in an analogous way.

Now suppose that M is a Hadamard manifold. Let $K \subseteq M$ be a convex set with non-trivial interior. Let K^o be the interior of K. We define $\pi_K : M \setminus K^o \to \partial K$ to be a projection onto the closest point in ∂K. Let $V \subseteq \partial K$. We call the set $\pi_K^{-1}(V)$ the **end** of V, and we denote it by $\mathcal{E}(V)$. Trivially, $\mathcal{E}(V)$ is foliated by half geodesics leaving points in V in directions normal to K. Let Σ be a locally convex hypersurface. Let (U, V, K) and (U', V', K') be convex charts of Σ. Trivially:

$$\pi_K^{-1}(i(U \cap U')) = \pi_{K'}^{-1}(i(U \cap U')).$$

We thus define the **end** of Σ to be the manifold (with non-smooth, concave boundary) whose coordinate charts are the ends of the convex charts of Σ. We denote this manifold by $\mathcal{E}(\Sigma)$. $\mathcal{E}(\Sigma)$ has the following properties:

(i) Σ naturally embeds as the boundary of $\mathcal{E}(\Sigma)$;
(ii) in the complement of Σ, $\mathcal{E}(\Sigma)$ has the structure of a smooth Riemannian manifold of non-positive curvature;
(iii) $\mathcal{E}(\Sigma)$ is foliated by half geodesics leaving points in Σ in directions normal to Σ;
(iv) there exists a natural embedding $I : \mathcal{E}(\Sigma) \to M$ which restricts to i over Σ and which is a local diffeomorphism over the complement of Σ.

We say that a subset $K \subseteq \mathcal{E}(\Sigma)$ is **semi-convex** if and only if for every geodesic segment $\gamma : [0, 1] \to \mathcal{E}(\Sigma)$ contained within $\mathcal{E}(\Sigma)$, if $\gamma(0), \gamma(1) \in K$, then the whole of γ is contained in K. Let K be a semi-convex subset of the end of Σ which contains Σ and coincides with Σ outside a convex set. $(\partial K, I|_{\partial K})$ defines a convex immersion in M which, by abuse of notation, we simply denote by ∂K (see Section 3 of [11] for details). Let Σ and Σ' be two locally convex hypersurfaces in M. We say that Σ is **bounded by** Σ' (and Σ' **bounds** Σ) if and only if there exists a semi-convex subset, $K \subseteq \mathcal{E}(\Sigma)$, which contains Σ and which coincides with Σ outside a compact set such that Σ' is equivalent to ∂K. In this case, we often identify Σ' with ∂K and thus view it as a subset of $\mathcal{E}(\Sigma)$.

Example. Let $K, K' \subseteq M$ be two convex sets. Then ∂K is bounded by $\partial K'$ if and only if $K \subseteq K'$. □

The property of boundedness is preserved by passage to limits. Indeed, if $(\Sigma, \partial\Sigma)$ is a locally convex immersion with boundary, then, for all $r > 0$, we define $B_r(\partial\Sigma)$

to be the set of all points in Σ whose intrinsic distance to $\partial\Sigma$ is less than r. We obtain:

Lemma 3.1. *Let $(\Sigma_n)_{n\in\mathbb{N}}, \Sigma_0$ and $(\Sigma'_n)_{n\in\mathbb{N}}, \Sigma'_0$ be compact, convex immersed hypersurfaces in M such that $(\Sigma_n)_{n\in\mathbb{N}}$ and $(\Sigma'_n)_{n\in\mathbb{N}}$ converge to Σ_0 and Σ'_0, respectively. Suppose that:*

(i) *for all $n > 0$, Σ'_n bounds Σ_n;*
(ii) *there exists $r > 0$ such that, for all $n \in \mathbb{N}$:*

$$B_r(\partial\Sigma_n) = B_r(\partial\Sigma'_n).$$

Then Σ'_0 also bounds Σ_0.

Proof. See Lemma 3.2 of [11]. □

4. Strictly convex hypersurfaces of Euclidean space

Let $B_1(0)$ be the ball of radius 1 about the origin in $\mathbb{R}^{n+1}$, where $n \geqslant 2$. Choose $\epsilon > 0$. Let $(\Sigma^n, \partial\Sigma^n)$ be a compact, locally convex hypersurface with boundary in $\mathbb{R}^{n+1}$, and suppose that the shape operator of Σ is bounded below by $\epsilon\mathrm{Id}$ in the weak sense. Suppose, moreover, that Σ passes through 0. Let H be a supporting hyperplane to Σ at 0. Let d_H be the signed distance to H in $\mathbb{R}^{n+1}$ such that d_H is non-positive over Σ in a neighbourhood of 0. Let $(H_t)_{t\in\mathbb{R}}$ be the foliation of $\mathbb{R}^{n+1}$ by hyperplanes parallel to H. For all $t < 0$, let Σ_t be the connected component of Σ lying above H_t and containing 0.

Proposition 4.1. *Choose $t < 0$. Suppose that, for all $s \in]t, 0[$, Σ_t does not intersect either $\partial\Sigma$ or $\partial B_1(0)$. Then Σ_t is embedded, and (together with H_t), Σ_t bounds a convex set.*

Proof. We prove this by the method of moving planes, as in [12]. In our case, strict convexity allows us to greatly simplify the argument, which we therefore include for the sake of clarity.

We define $T \subseteq]t, 0[$ such that $s \in T$ if and only if Σ_s is embedded and (together with H_s) bounds a convex set. For all s, we denote the convex set bounded by Σ_s and H_s by K_s. Since Σ is strictly convex, all sufficiently small s are elements of T, and T is therefore non-empty. Let t_0 be the infimum of T. The result follows from the fact that all supporting hyperplanes to Σ along $\Sigma_{t_0} \cap H_{t_0}$ are transverse to H_{t_0}, since, in this case, if $t < t_0$, then Σ_{t_0} does not intersect either $\partial\Sigma$ or $\partial B_1(0)$, and T may therefore be extended beyond t_0, which is absurd.

To prove the assertion, we assume the contrary. Thus, choose $p \in \Sigma_{t_0}$ and suppose that the supporting hyperplane to Σ_{t_0} at p is tangent to H_{t_0}. Let V_p be the unit normal vector to H_{t_0} at p which points outwards from K_{t_0}. $V_p = \pm\nabla d_H$. If $V_p = \nabla d_H$, then, since K_{t_0} is convex, it is contained within $d_H^{-1}(]-\infty, t_0[)$, and therefore so is $0 \in \Sigma_{t_0}$, which is absurd.

Now suppose that $V_p = -\nabla d_H$. Let $(S^n, \partial S^n)$ be a compact n-dimensional topological manifold with boundary and $i : S \to \mathbb{R}^{n+1}$ a locally convex immersion such that $\Sigma = (S, i)$. Let $P \in S$ be the inverse image of 0 in S and, for all $t \in T$, let S_t be the connected component of $(d_H \circ i)^{-1}(]t_0, +\infty[)$ containing P. i restricts to a covering map from S_{t_0} to $\partial K_{t_0} \setminus H_{t_0}$. The latter is however homeomorphic to a sphere with a point removed (i.e. a solid ball). Since this is simply connected, i defines a homeomorphism from S_{t_0} to $\partial K_{t_0} \setminus H_{t_0}$.

Let $Q \in \partial S_{t_0}$ be such that $i(Q) = p$. Let $U_Q \subseteq S$ be a connected neighbourhood of Q such that the restriction of i to U_Q is a homeomorphism onto an open subset of the boundary of a strictly convex set. Since $V_p = -\nabla d_H$ is the outward pointing supporting normal to Σ at Q, by reducing U_Q if necessary, we may assume that, throughout $U_Q \setminus \{Q\}$, $(d_H \circ i) > t_0$. Since S has dimension at least 2, $U_Q \setminus \{Q\}$ is connected, and therefore, by definition of S_t:

$$\begin{aligned} U \setminus \{Q\} &\subseteq S_t \\ \Rightarrow i(U_Q \setminus \{Q\}) &\subseteq \partial K_{t_0} \\ \Rightarrow i(U_Q) &\subseteq \partial K_{t_0}. \end{aligned}$$

By conservation of the domain, $i(U_Q)$ is an open subset of ∂K_{t_0}. K_{t_0} is therefore strictly convex at p, and thus only meets H_{t_0} at that point. Thus, by continuity, i sends every point of ∂S_{t_0} to p. Let Q' be another point in ∂S_{t_0}. Let $U_{Q'} \subseteq S$ be a connected neighbourhood of Q' such that the restriction of i to $U_{Q'}$ is a homeomorphism onto an open subset of the boundary of a strictly convex set. Suppose, moreover, that

$$U_Q \cap U_{Q'} = \emptyset.$$

Then, since $p \in i(U_Q) \cap i(U_{Q'})$:

$$i(U_Q \setminus \{Q\}) \cap i(U_{Q'} \setminus \{Q'\}) \neq \emptyset.$$

However, the restriction of i to S_{t_0} is a homeomorphism, and it thus follows that ∂S_{t_0} consists of only a single point, and S is homeomorphic to ∂K_{t_0}, and is thus a topological sphere. In particular, ∂S is trivial, which is absurd, and the claim follows. □

Let $\gamma :]-T, T[\to \mathbb{R}^{n+1}$ be a unit speed curve whose geodesic curvature is bounded above by $\epsilon/(2+\epsilon)$. Suppose, moreover, that:

(i) γ is contained within $B_1(0)$;
(ii) γ lies in the exterior of K_{t_0};
(iii) for all $t \in]-T, T[$, $(d_H \circ \gamma)(t) > t_0$.

Proposition 4.2. *If d_K is the distance to K_{t_0} in $\mathbb{R}^{n+1}$, then $(d_K + 2/\epsilon)^2$ restricts to a convex function over γ.*

Proof. Choose $t \in]-T, T[$. Let $P \in \partial K_{t_0}$ be the closest point in K_{t_0} to $\gamma(t)$. Since γ lies above H_{t_0}, $P \in \Sigma_{t_0}$. Let V_P be the outward pointing unit normal to K_{t_0} at P which is tangent to γ. We claim that the shape operator of Σ with respect to V_P is at least $(\epsilon/2)\mathrm{Id}$ in the weak sense. Indeed, if V_P is a supporting normal to Σ, then the assertion follows by definition of Σ. Suppose, therefore, that V_P is not a supporting normal to Σ. In particular, $P \in \partial\Sigma_{t_0} =: \Gamma$, and V_P points upwards from H_{t_0}. Denote $V_1 = -\nabla d_H$, and let V_2 be the unit supporting normal to Σ lying in the plane spanned by V_1 and V_P. V_1 and V_2 define an obtuse, isosceles triangle whose angle at p is 2θ, say. Since V_P points upwards from H_{t_0}, the angle that V_P makes with V_1 and V_2, respectively, is $\theta + \varphi$ and $\theta - \varphi$, for some $0 < \varphi < \theta$. If A_1 and A_2 are the shape operators of Γ with respect to the normals V_1 and V_2, respectively, then

$$A_1 = 0, \qquad A_2 \geqslant \epsilon\mathrm{Id},$$

in the weak sense. Thus, if A_P is the shape operator of Γ with respect to P, then

$$A_P \geqslant \frac{1}{2}(1+\mathrm{Sin}(\varphi)/\mathrm{Sin}(\theta))A_2 > \frac{\epsilon}{2}\mathrm{Id},$$

in the weak sense. We may thus extend Γ near P to a convex hypersurface, Σ_P such that:

(i) Σ_P lies outside K_{t_0};
(ii) Σ_P meets K_{t_0} at P;
(iii) V_P is the outward pointing unit normal to Σ_P at P;
(iv) the shape operator of Σ_P at P is bounded below by $(\epsilon/2)\mathrm{Id}$ in the weak sense.

The claim now follows. Define $r = (2/\epsilon)$. Let $Q \in \mathbb{R}^{n+1}$ be the unique point such that if d_Q is the distance in $\mathbb{R}^{n+1}$ to Q, then

(i) $d_Q(P) = r$;
(ii) $(\nabla d_Q)(P) = V_P$.

In a neighbourhood of $\gamma(t)$, $d_P^2 \leqslant (d_K + (2/\epsilon))^2$. However:

$$\begin{aligned}(\partial_t^2 d_P^2/2) &= (1/2)\mathrm{Hess}(d_P)(\partial_t\gamma,\partial_t\gamma) - d_P\langle\nabla d_P, \nabla_{\partial_t\gamma}\partial_t\gamma\rangle \\ &\geqslant 1-(1+r)\left|\nabla_{\partial_t\gamma}\partial_t\gamma\right|.\end{aligned}$$

Moreover, by definition of the geodesic curvature:

$$|\nabla_{\partial_t\gamma}\partial_t\gamma| \leqslant \frac{\epsilon}{2+\epsilon} = \frac{1}{1+r}.$$

The result follows. □

Proposition 4.3. *There exists $t_0 < 0$, which only depends on ϵ, such that if, for all $s \in]t_0, 0[$, Σ_s does not intersect $\partial\Sigma$, then Σ_{t_0} bounds a convex set K_{t_0} (along with H_{t_0}) and K_{t_0} is strictly contained within $B_1(0)$.*

Proof. Without loss of generality, we may suppose that $\epsilon \leqslant 1$. Let S be the sphere of radius $(4/\epsilon)$ which is tangent to Σ at 0 and locally contains Σ in its interior. For all $t < 0$, let S_t be the connected component of S lying above H_t containing 0. Let t_0 be such that S_t is strictly contained inside $B_1(0)$.

Let T be the set of all $t < 0$ such that, for all $s > t$, Σ_t does not intersect either $\partial\Sigma$ or $\partial B_1(0)$. Since Σ is strictly convex, for all t small, $t \in T$, and t is therefore non-empty. Let t_1 be the infimum of T and suppose that $t_1 > t_0$. By Proposition 4.1, Σ_{t_1} bounds a convex set $K := K_{t_1}$ (along with H_{t_1}). By the hypothesis on Σ, Σ_{t_1} does not intersect $\partial\Sigma$, and thus Σ_{t_1} intersects non-trivially with $\partial B_1(0)$, since otherwise t_1 would not be the infimum of T.

Let $\gamma : \mathbb{R} \to S$ be a geodesic in S such that $\gamma(0) = 0$. Let s_1 be such that the connected component of $\gamma^{-1}(S_{t_1})$ containing 0 coincides with $]-s_1, s_1[$. If d is the distance in $\mathbb{R}^{n+1}$ to K, then $d\circ\gamma$ achieves a local minimum at 0. Moreover, the geodesic curvature of γ, viewed as a curve in $\mathbb{R}^{n+1}$, is equal to $\epsilon/4 < \epsilon/(\epsilon+2)$. Thus, by Proposition 4.2, the restriction of $d\circ\gamma$ to $]-s_1, s_1[$ is convex, and it follows that $d\circ\gamma$ is strictly positive over $]-s_1, s_1[\setminus\{0\}$. In particular, $\gamma(]-s_1, s_1[)$ lies outside K. Since γ is arbitrary, the whole of S_{t_1} lies outside K and K is therefore contained within the convex set bounded by S_{t_1} and H_{t_1}. In particular, K does not intersect $\partial B_1(0)$, which is absurd, and this completes the proof. □

Proposition 4.4. *For all $l \in [0,1]$, there exists $r > 0$ which only depends on l and ϵ such that, if $K \subseteq \mathbb{R}^{n+1}$ is a convex subset such that:*

(i) *the shape operator of ∂K is everywhere no less than $\epsilon\mathrm{Id}$ in the weak sense;*
(ii) *K contains a geodesic segment of length $2l$,*

then K contains a ball of radius r.

Proof. Let $\gamma : [-l,l] \to K$ be a geodesic segment. Let $r_1 = 4/\epsilon$. Let $\eta : [-s,s] \to \mathbb{R}^{n+1}$ be a circular arc of radius at least r_1 and of angle no more than π such that

$$\eta(-s) = \gamma(-l), \qquad \eta(s) = \gamma(l).$$

Every point in η lies at a distance no greater than 1 from γ, and thus also from K. It follows by Proposition 4.2 that η lies entirely within K. Thus, if $\Omega \subseteq \mathbb{R}^{n+1}$ is the locus of all points traced out by such arcs, then:

$$\Omega \subseteq K.$$

However, if $r > 0$ is such that

$$(r_1 - r)^2 + l^2 = r_1^2,$$

then

$$B_r(\gamma(0)) \subseteq \Omega.$$

The result follows. □

5. Strictly convex hypersurfaces of general manifolds

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $K \subseteq M$ be a compact subset. Choose $\epsilon > 0$, and let $(\Sigma, \partial\Sigma)$ be a compact, locally convex immersed hypersurface in M such that:

(i) Σ is contained in K;
(ii) $\Gamma := \partial\Sigma$ is smooth;
(iii) the shape operator of Σ is everywhere greater than $\epsilon\mathrm{Id}$ in the weak sense.

Lemma 5.1. *There exists $r_1 > r_2 > 0$ (which only depend on ϵ, M, K and Γ) such that, for all $p \in \Sigma$:*

(i) *the connected component of $\Sigma \cap B_{r_1}(p)$ containing p is embedded and lies on the boundary of a convex set;*
(ii) *moreover, this convex set contains an open ball of radius r_2.*

Choose $\epsilon > 0$ and let $(\Sigma', \partial\Sigma')$ be a smooth, strictly convex, immersed hypersurface in $\mathbb{R}^{n+1}$ such that:

(i) the shape operator of Σ' is greater than ϵ in the weak sense;
(ii) $\partial\Sigma'$ consists of two connected components, which we denote by $\partial\Sigma'_1$ and $\partial\Sigma'_2$ respectively.

If $\partial\Sigma'_1 = \Gamma$, then we refer to $(\Sigma', \partial\Sigma')$ as a **thickening** of Γ. Let N be the outward pointing unit normal vector field over Σ'. Choose $P \in \partial\Sigma'_2$. Let $\mathcal{N}_P$ denote the circle of unit vectors normal to $\partial\Sigma'_2$ at P. For any vector, V in $\mathcal{N}_P$, let H_V be the (oriented) hyperplane in T_PM normal to V at P. We identify H_V with its image under the exponential map of M, and we define Σ'_V to be the connected component of Σ' lying above H_V and containing P. We have the following elementary result:

Proposition 5.2. (i) *There exists $\theta_{\Sigma'} > 0$, which only depends on Σ', such that, if the angle between V and $\mathsf{N}(P)$ is less than $\theta_{\Sigma'}$, then Σ'_V does not intersect $\partial\Sigma'_1$;*
(ii) *there exists $h_{\Sigma'} > 0$, which also only depends on Σ' such that, if V makes an angle of at least $\theta_{\Sigma'}$ with $\mathsf{N}(P)$, then the furthest point in Σ'_V from H_V lies at a distance no less than $h_{\Sigma'}$ from H_V.*

Proof. If V coincides with $\mathsf{N}(P)$, then $\Sigma'_V = \{P\}$. The first assertion follows by continuity and compactness of $\partial\Sigma'_2$. The second assertion now follows by compactness of $\partial\Sigma'_2$, and this completes the proof. □

Proof of Lemma 5.1. Let S be a thickening of Γ such that:

(i) S is homeomorphic to $\Gamma \times [0,1]$;
(ii) if $\pi : S \to [0,1]$ is the canonical projection, then $\pi^{-1}(\{0\})$ coincides with Γ;
(iii) the shape operator of S is bounded below by $\epsilon/2$;
(iv) $S \cup \Sigma$ is a locally convex immersed hypersurface.

S may be chosen independant of Σ (despite condition (iv)). Indeed, if S is chosen such that:

(i) its outward pointing normal along Γ points in the same direction as that of Σ;
(ii) the lowest eigenvalue of the shape operator of S along Γ is always in $[\epsilon/4, \epsilon/2]$,

then S satisfies condition (iv) (see Section 4 of [11] for details).

Let g be the Riemannian metric over M. For $q \in M$, identify T_qM with $\mathbb{R}^{n+1}$, and let $\mathrm{Exp}_q : \mathbb{R}^{n+1} \to M$ be the exponential map. Let ∇^0 be the Euclidean covariant derivative over $\mathbb{R}^{n+1}$, let ∇^q be the pull-back through Exp_q of the Levi-Civita covariant derivative of g, and let $\Omega^q = \nabla^q - \nabla^0$ be the connexion 2-form of ∇^q with respect to ∇^0. There exists $r_1 > 0$, which only depends on K such that, for all $q \in K$, $\|\Omega^q\| < \epsilon/4$ over the ball of radius r_1 about q. If we replace g with $r_1^{-1}g$, then the shape operator of $S \cup \Sigma$ is bounded below by $(\epsilon r_1)/2$ in the weak sense, and $\|\Omega^q\| < (\epsilon r_1)/4$ over the ball of radius 1 about q, for all $q \in K$.

Denote $\epsilon_1 = (\epsilon r_1)/4$. Choose $p \in \Sigma$, and let $(\tilde{\Sigma}, \partial\tilde{\Sigma})$ be the connected component of $S \cup \Sigma$ lying in $B_1(p)$ containing p. We identify $(\tilde{\Sigma}, \partial\tilde{\Sigma})$ with its pull-back through Exp_q in $\mathbb{R}^{n+1}$ and observe that the shape operator of $\tilde{\Sigma}$ with respect to the rescaled Euclidean metric over $\mathbb{R}^{n+1}$ is everywhere bounded below by ϵ_1 in the weak sense.

Let H be a supporting hyperplane to $\tilde{\Sigma}$ at p. Let d be the (signed) distance in $\mathbb{R}^{n+1}$ to H such that, near p, d is non-positive over $\tilde{\Sigma}$. For all $t \in \mathbb{R}$, let H_t be the hyperplane parallel to H lying at distance t from H. For all $t < 0$, let $\tilde{\Sigma}_t$ be the connected component of $\tilde{\Sigma}$ lying above H_t and containing p. Let $t_0 < 0$ be as in Proposition 4.3. Let θ_S and h_S be as in Proposition 5.2, now chosen such that the proposition remains valid with respect to the rescaled Euclidean metric over $\mathbb{R}^{n+1}$ (as opposed to g). Suppose that $|t_0| < h_S$.

Let T be the set of all $t \in]t_0, 0[$ such that for all $s \in]t, 0[$, $\tilde{\Sigma}_t$ does not intersect either $\partial\tilde{\Sigma}$ or $\partial B_1(0)$. For all t sufficiently close to 0, $t \in T$. In particular, T is non-empty. Let t_1 be the infimum of T. By Proposition 4.1, Σ_{t_1} (along with H_{t_1}) bounds a convex set, K, say.

Suppose now that $t_1 > t_0$. By Proposition 4.1, since t_1 is the infimum of T, $\tilde{\Sigma}_{t_1}$ meets either $\partial\tilde{\Sigma}$ or $\partial B_1(p)$, since, otherwise, it could be extended further. By Proposition 4.3, $\tilde{\Sigma}_{t_1}$ does not intersect $\partial B_1(0)$. It follows that $\tilde{\Sigma}_{t_1}$ intersects $\partial\tilde{\Sigma}_{t_1}$ at some point, Q, say. H_{t_1} is tangent to $\partial\tilde{\Sigma}_{t_1}$ at Q. Since $\tilde{\Sigma}_{t_1}$ intersects non-trivially with $\partial\Sigma$, it follows by assertion (i) of Proposition 5.2 that H_{t_1} makes an angle of at least θ_S with the outward pointing normal to H_S at Q. Since the connected component of S lying above H_{t_1} is contained in $\tilde{\Sigma}_{t_1}$, it follows by assertion (ii) of Proposition 5.2 that the supremum of the distance to H_{t_0} over $\tilde{\Sigma}_{t_1}$ is at least h_S. However, by convexity, $\tilde{\Sigma}_{t_1}$ lies below H_0, and so:

$$h_S \leqslant |t_1| < |t_0| < h_S.$$

This is absurd, and it follows that $t_1 = t_0$. Assertion (i) now follows for $r_1 < |t_0|$. Assertion (ii) follows by Proposition 4.4, since $\Sigma_{t_0} \cap \partial B_{r_1}(p) \neq \emptyset$. This completes the proof. $\square$

6. Compactness

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Choose $\epsilon > 0$ and let $(\Sigma_n, \partial\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of compact, locally convex immersed hypersurfaces in M such that, for all n:

(i) $\Gamma_n := \partial\Sigma_n$ is C^∞;
(i) the shape operator of Σ_n is greater than $\epsilon\mathrm{Id}$ in the weak sense.

Let Γ_0 be a smooth, compact, codimension 2 immersed submanifold of M and suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ converges to Γ_0 in the C^∞ sense. In other words, for sufficiently large n, Γ_n is a normal graph over Γ_0 and the (unique) function of which Γ_n is a graph tends to 0 in the C^∞ sense.

For all n, let $\mathrm{Vol}(\Sigma_n)$ be the volume of Σ_n with respect to the intrinsic measure.

Lemma 6.1. *Suppose that there exists a compact subset $K \subseteq M$ and a real number $B > 0$ such that, for all $n \in \mathbb{N}$:*

$$\Sigma_n \subseteq K, \qquad \mathrm{Vol}(\Sigma_n) \leqslant B.$$

Then there exists a locally convex immersed hypersurface Σ_0 in M such that:

(i) $\Gamma_0 = \partial\Sigma_0$;
(ii) *the shape operator of Σ_0 is greater than $\epsilon\mathrm{Id}$ in the weak sense;*
(iii) $(\Sigma_n)_{n\in\mathbb{N}}$ *subconverges to Σ_0.*

Proof. For all n, let Σ_n^c be a thickening of Γ_n such that:

(i) Σ_n^c is homeomorphic to $\Gamma_n \times [0,1]$;
(ii) if $\pi_n : \Sigma_n^c \to [0,1]$ is the canonical projection, then $\pi^{-1}(\{0\})$ coincides with Γ_n;
(iii) the shape operator of Σ_n^c is bounded below by $\epsilon/2$;
(iv) $\hat{\Sigma}_n := \Sigma_n^c \cup \Sigma_n$ is a locally convex immersed hypersurface.

Suppose, moreover, that $(\Sigma_n^c)_{n\in\mathbb{N}}$ converges to Σ_0^c in the C^∞ sense. As in the proof of Lemma 5.1, $(\Sigma_n^c)_{n\in\mathbb{N}}$ may be chosen independant of $(\Sigma_n)_{n\in\mathbb{N}}$ (despite condition (iv)). For all n, let $S_n \subseteq \hat{S}_n$ be abstract manifolds and $i_n : \hat{S}_n \to M$ be a locally convex immersion such that:

$$\Sigma_n = (i_n, S_n), \qquad \hat{\Sigma}_n = (i_n, \hat{S}_n).$$

For all n, we furnish $\hat{S}_n$ with the metric and the measure induced by i_n. For all n, for all $p \in S_n$ and for all $\epsilon > 0$, let $B_{n,\epsilon}(p)$ be the intrinsic ball of radius ϵ about p in $\hat{\Sigma}_n$.

Let $r_1 > r_2 > 0$ be as in Lemma 5.1. Choose $\epsilon \in]0, r_1[$. For all n, let p_n be a point in Σ_n. By Lemma 5.1 for all n, there exists a convex set $K_n \subseteq M$ such that:

(i) K_n contains an open ball of radius r_2;
(ii) the restriction of i_n to $B_{n,r_1}(p_n)$ is a homeomorphism onto the open ball of radius r_1 about $i_n(p_n)$ in ∂K_n.

By compactness of the family of compact, convex sets, there exists a compact, convex set K_0 towards which $(K_n)_{n\in\mathbb{N}}$ subconverges. Moreover, by (i), K_0 contains an open ball of radius r_2, and therefore has non-trivial interior. $i_n(B_{n,\epsilon}(p_n))$ subconverges to an open ball of radius ϵ about some point in ∂K_0. In particular $(\mathrm{Vol}(B_{n,\epsilon}(p_n)))_{n\in\mathbb{N}}$ tends towards the volume of this ball, which is non-zero. Since $(p_n)_{n\in\mathbb{N}}$ was arbitrary, it follows that, for all $\epsilon > 0$, there exists v_ϵ such that, for all n and for all $p \in S_n$:

$$\mathrm{Vol}(B_{n,\epsilon}(p)) \geqslant v_\epsilon.$$

By increasing B if necessary, we may suppose that, for all n, $\mathrm{Vol}(\hat{S}_n) \leqslant B$. Thus, if $N_\epsilon \geqslant B/v_\epsilon$, then, for all n, the maximum number of disjoint balls in $\hat{S}_n$ of radius ϵ with centres in S_n is at most N_ϵ. Thus, the minimum number of balls in $\hat{S}_n$ of radius 2ϵ required to cover S_n is at most N_ϵ. It follows by the fundamental compactness theorem of metric spaces that there exists a metric space S_0 towards which $(S_n)_{n\in\mathbb{N}}$ subconverges in the Gromov/Hausdorff sense (see [5]).

Choose $p \in S_0$. For all n, let p_n be a point in S_n and suppose that $(p_n)_{n\in\mathbb{N}}$ converges to p_0. Constructing $(K_n)_{n\in\mathbb{N}}$ and K_0 as before, we obtain an isometry, i_0, from $B_{0,\epsilon}(p_0)$ to a ball of radius ϵ about a point in ∂K_0. Moreover, the sequence of restrictions of $(i_n)_{n\in\mathbb{N}}$ to the balls of radius ϵ about the $(p_n)_{n\in\mathbb{N}}$ converges locally uniformly to i_0. Using a diagonal argument, we obtain a $C^{0,1}$ mapping i_0 from the whole of S_0 into M which is a locally convex immersion. $\Sigma_0 = (S_0, i_0)$ is the desired hypersurface, and this completes the proof. □

7. The Dirichlet problem

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold of non-positive curvature. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$. Let $H \subseteq M$ be a smooth, locally convex hypersurface. Let $\Omega \subset H$ be a bounded, open subset of H. Let $\hat{\Sigma} \subseteq M$ be a convex hypersurface such that $\partial\hat{\Sigma} = \partial\Omega =: \Gamma$. Let $\phi : M \to]0, \infty[$ be a smooth, positive function such that, for all $p \in H$:

$$R_\theta(H)(p) < \phi(p),$$

and, for all $p \in \hat{\Sigma}$:

$$R_\theta(\hat{\Sigma})(p) > \phi(p),$$

in the weak sense.

Theorem 3.22 of [9] may be adapted to yield:

Proposition 7.1. *Suppose that $\hat{\Sigma}$ is a graph over Ω and that Γ is strictly convex as a subset of M with respect to the outward pointing normal to Γ in $\hat{\Sigma}$. If $\theta > (n-1)\pi/2$, then there exists an immersed hypersurface $\Sigma \subseteq M$ such that:*

(i) *Σ is C^0 and C^∞ in its interior;*
(ii) *$\partial\Sigma = \Gamma$;*
(iii) *Σ is a graph over Ω lying below $\hat{\Sigma}$;*
(iv) *for all $p \in \Sigma$, $\hat{R}_\theta(\Sigma)(p) = \phi(p)$.*

Moreover, the same result holds for $\theta = (n-1)\pi/2$ provided that, in addition, the shape operator of $\hat{\Sigma}$ is everywhere bounded below by $\epsilon\mathrm{Id}$ in the weak sense, for some $\epsilon > 0$.

Remark. In fact, we don't need the result for $\theta = (n-1)\pi/2$.

Remark. The hypotheses of this proposition are satisfied when the norm of the second fundamental form of H is small with respect to that of $\hat{\Sigma}$ and the normal of $\hat{\Sigma}$ is sufficiently bounded away from TH along Γ. Explicitly, if the second fundamental form of $\hat{\Sigma}$ is greater than $\epsilon\mathrm{Id}$ in the weak sense, if the norm of the second fundamental form of H is bounded above by δ, and if the angle between the normal to $\hat{\Sigma}$ and TH is bounded below by θ along Γ, then the hypotheses are satisfied provided that

$$\epsilon\mathrm{Sin}(\theta) - \delta > 0.$$

Sketch of the proof. We use the continuity method, which reduces to showing openness and closedness of a certain interval, from which existence is deduced by connectedness. In [9], openness is shown by proving that the linearisation of the special-Lagrangian curvature operator is always invertible, which follows from the stronger hypotheses used there by Lemma 7.5 of [8]. In our case, the hypotheses on the curvature of the ambient manifold and the special Lagrangian curvature of the immersed hypersurface are weaker, but openness may still be obtained (generically), using the differential topological argument discussed (in the case of Gaussian curvature) in Section 11 of [11]. This approach requires a strengthening of the compactness result (used to prove closedness) to treat the case of hypersurfaces of prescribed (non-constant) curvature. This strengthening is required anyway to obtain the existence result in the generality given here, and is obtained by a fairly trivial adaptation of the reasoning presented in Sections 3.1 to 3.6 of [9] used there to prove compactness in the case of hypersurfaces of constant special Lagrangian curvature. This completes the proof. □

8. The Plateau problem

Let M^{n+1} be a Hadamard manifold. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$. Let $(\hat{N}^n, \partial\hat{N}^n)$ be a locally convex immersed hypersurface in M. Let $\phi : M \to]0,\infty[$ be a strictly positive, smooth function such that, for all $p \in \hat{N}$:

$$R_\theta(N)(p) \geqslant \phi(p),$$

in the weak sense.

Proof of Theorem 1.1. We first consider the case where $\theta > (n-1)\pi/2$. By Lemma 2.2 of [9], there therefore exists $k > 0$ such that, if N' is a smooth, locally convex immersed hypersurface, and if A is the shape operator of N', then:

$$A \geqslant kR_\theta(N').$$

Now let $(N, \partial N)$ be an n-dimensional, locally convex immersed hypersurface in M and suppose that, for all $p \in N$:

$$R_\theta(N)(p) \geqslant \phi(p),$$

in the weak sense. Let $p \in N$ be an interior point. Let N_p be a supporting normal to N at p. By Lemma 4.7 of [9], we may suppose that there exists $\eta > 0$ such that, if N_p' is any other supporting normal to N at p, then

$$\langle \mathsf{N}_p, \mathsf{N}_p' \rangle \geqslant \eta.$$

Choose $\epsilon > 0$ such that $\epsilon < kR_\theta(N)$, and let H be a smooth, strictly convex hypersurface passing through p such that:

(i) the outward pointing unit normal to H at p coincides with N_p;
(ii) the norm of the shape operator of H is always less than ϵ.

We extend H to a foliation $(H_t)_{t \in]-\delta,\delta[}$ of a neighbourhood of p such that the norm of the shape operator of each leaf of the foliation is always less than ϵ. For all sufficiently small $t < 0$, let N_t be the connected component of N lying above H_t and containing p. For ϵ sufficiently small, there exists $t_0 < 0$, and, for all $t \in]t_0, 0[$, an open subset $\Omega_t \subseteq H_t$ such that N_t is a graph over Ω_t. Moreover, by reducing ϵ further if necessary, for all $t \in]t_0, 0[$, H_t and N_t satisfy the hypotheses of Proposition 7.1. There thus exists, for all such t, a smooth, locally convex immersed hypersurface N_t' lying between H_t and N_t such that, for all $p \in N_t'$:

$$R_\theta(N_t')(p) = \phi(p).$$

Let N' be the locally convex immersed hypersurface obtained by replacing N_{t_0} with N_{t_0}'. For all $p \in N'$ lying in the complement of $\partial N_{t_0}'$:

$$R_\theta(N')(p) \geqslant \phi(p),$$

in the weak sense. By deforming slightly, we may suppose that N_{t_0}' is smooth up to the boundary. By Lemma 2.4 of [9], since R_θ is a concave function, it follows that $R_\theta(N')(p) \geqslant \phi(p)$ in the weak sense also for all $p \in \partial N_{t_0}'$, and therefore over the whole of N'. We observe finally that N' is trivially bounded by N, and we call N' a **Perron Regularisation** of height t_0 of N about p.

Let $\mathcal{F}$ denote the family of all locally strictly convex, immersed hypersurfaces that may be obtained from N by a finite number of iterations of the Perron Regularisation process. For all $N \in \mathcal{F}$, let $\mathrm{Vol}(N)$ be the volume of N. Vol defines a continuous function from $\mathcal{F}$ into $[0, \infty[$. Observe that, by convexity, and since the ambient manifold is non-positively curved, Perron Regularisation always reduces volume. Define V_0 by

$$V_0 = \inf \left\{ \mathrm{Vol}(N) \text{ s.t. } N \in \mathcal{F} \right\}.$$

Let $(N_n)_{n \in \mathbb{N}} \in \mathcal{F}$ be such that:

$$\mathrm{Vol}(N_n)_{n \in \mathbb{N}} \to V_0.$$

By Lemma 6.1, $(N_n)_{n\in\mathbb{N}}$ subconverges to a locally strictly convex $C^{0,1}$ immersed hypersurface, N_0 such that:

$$\mathrm{Vol}(N_0) = V_0.$$

We aim to show that N_0 is smooth away from the boundary, and at every point $p \in N_0$:

$$R_\theta(N_0)(p) = \phi(p).$$

Choose $p_0 \in N_0$. For all n, choose $p_n \in N_n$ such that $(p_n)_{n\in\mathbb{N}}$ converges to p_0. Since N_0 is locally strictly convex, and since $(N_n)_{n\in\mathbb{N}}$ converges to N_0, there exists $t_0 < 0$, and, for all $n \in \mathbb{N}$, a Perron Regularisation N'_n of height t_0 of N_n about p_n. For all n, N'_n is bounded by N_n and $\mathrm{Vol}(N'_n) \leqslant \mathrm{Vol}(N_n)$. By Lemma 6.1, $(N'_n)_{n\in\mathbb{N}}$ subconverges to a locally convex immersed hypersurface, N'_0, say. By continuity, $\mathrm{Vol}(N'_0) \leqslant \mathrm{Vol}(N_0)$. Moreover, by Lemma 3.1, N'_0 is bounded by N_0. However, by definition of V_0:

$$\mathrm{Vol}(N'_0) \geqslant V_0 = \mathrm{Vol}(N_0).$$

Consequently $\mathrm{Vol}(N'_0) = \mathrm{Vol}(N_0)$, and since N'_0 is bounded by N_0, the two hypersurfaces must coincide. For all n, choose $p'_n \in N'_n$ such that $(p'_n)_{n\in\mathbb{N}}$ converges to p_0. For all $r > 0$ and for all $n \in \mathbb{N} \cup \{0\}$, let $B'_r(p'_n)$ be the ball of radius r (with respect to the intrinsic distance) about p'_n in N'_n. Choose $r_0 < |t_0|$. For all n, N'_n is smooth over the ball of radius r_0 about p'_n, and for all $p \in B'_{r_0}(p'_n)$:

$$R_\theta(N'_n)(p) = \phi(p).$$

Taking limits, it follows by Theorem 1.4 of [7] that $N'_0 = N_0$ is also smooth over the ball of radius r about p_0, and that, for all $p \in B_r(p_0)$:

$$R_\theta(N_0)(p) = \phi(p).$$

Since $p_0 \in N_0$ is arbitrary, the result now follows for the case where $\theta > (n-1)\pi/2$.

Now suppose that $\theta = (n-1)\pi/2$. Let $(\delta_n)_{n\in\mathbb{N}} > 0$ be a sequence converging to 0, let $(\theta_n)_{n\in\mathbb{N}} > (n-1)\pi/n$ be a sequence converging to θ and, for all $n \in \mathbb{N}$, let N_n be a locally strictly convex hypersurface with smooth interior such that $\partial N_n = \Gamma$ and, for every interior point $p \in N_n$:

$$R_{\theta_n}(N_n)(p) = \phi(p) - \delta_n.$$

This is possible, since the shape operator of N is everywhere bounded below by $\epsilon\mathrm{Id}$ in the weak sense, for some $\epsilon > 0$.

For $\epsilon > 0$ and $n \in \mathbb{N}$, let $N_{\epsilon,n}$ be the set of all points in N_n whose (intrinsic) distance from ∂N_n is at most ϵ. By Lemma 5.1 of [11], there exists $\epsilon > 0$ such that $N_{\epsilon,n}$ subconverges to a locally strictly convex hypersurface, $N_{\epsilon,0}$, say. Choose $r > 0$ and for all $n \in \mathbb{N}$, let $p_n \in N_n$ be a point whose (intrinsic) distance from ∂N_n is at least r and let $B_{n,r}(p_n)$ be the (intrinsic) ball of radius r about p_n. By Theorem 1.4 of [7], $(B_{n,r}(p_n))_{n\in\mathbb{N}}$ converges either to a smooth immersed hypersurface or to a geodesic segment of length r.

For all n, since N_n is bounded by N, N is (more or less) a graph over N_n, and there exists a canonical projection $\pi_n : N \to N_n$ (see Section 3 of [11] for details). For all n, π_n is 1-Lipschitz and coincides with the identity along $\partial N = \partial N_n$. Thus, by the Arzela-Ascoli Theorem, we may assume that there exists $\pi_0 : N \to M$ towards which $(\pi_n)_{n\in\mathbb{N}}$ converges. The set of all $p \in N$ where $(B_{n,r}(\pi_n(p)))_{n\in\mathbb{N}}$ converges to a smooth immersed hypersurface is open, likewise so is the set where it converges towards a geodesic segment. Since N_ϵ is contained in the former of the two, it follows that this subset is non-trivial, and thus, by connectedness, coincides with the whole of N. Thus every such limit is a smooth immersed hypersurface, and we deduce, as before, that there exists a convex immersed hypersurface N_0 towards which $(N_n)_{n\in\mathbb{N}}$ subconverges. Moreover, by Theorem 1.4 of [7], N_0 is smooth away from the boundary, and for every interior point $p \in N_0$:

$$R_\theta(N_0)(p) = \phi(p).$$

This completes the proof. $\square$

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