TRACKING A MOVING POINT IN THE PLANE

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Abstract. The Teichmüller theory of any hyperbolic Riemann surface $R$ induces two closely related metrics on $R$ in the following way. From a theorem of Bers, the fiber $K = \Psi^{-1}(\text{identity})$ of the forgetful map $\Psi$ from the Teichmüller space $\text{Teich}(R \setminus \{p\})$ onto the Teichmüller space $\text{Teich}(R)$ is conformal to a disc and the evaluation map $K \ni [f] \mapsto f(p) \in R$ is a universal covering of $R$. There are two infinitesimal metrics on $K$ coming from Kobayashi’s construction:

(1) $\text{Teich}_K$ is the restriction of the Teichmüller infinitesimal metric on $\text{Teich}(R \setminus \{p\})$ to the submanifold $K$, and

(2) $\text{Kob}_K$ is the Kobayashi metric on $K$.

We show these metrics, respectively, are the lifts via the evaluation map of infinitesimal forms $\lambda$ and $\rho$ on $R$, where $\lambda$ and $\rho$ are the Teichmüller and Poincaré densities. $\lambda$ and $\rho$ have very different descriptions. For plane domains

$\lambda(p) = \inf\{||\bar{\partial}V||_\infty\}$,

where the infimum is taken over all continuous functions $V$ for which $V(p) = 1$ and $V$ vanishes on the boundary of $R$, and

$\rho(p) = \inf\{1/|f'(0)|\}$,

where the infimum is taken over all holomorphic functions $f$ mapping the unit disc into $R$ with $f(0) = p$. We also show

\[(1/2)\text{Kob}_K \leq \text{Teich}_K \leq \text{Kob}_K \quad \text{and} \quad (1/2)\rho \leq \lambda \leq \rho,\]

and $\lambda/\rho = 1/2$ when $R$ is simply connected, $\lambda/\rho = 1$ when $R$ is a thrice punctured sphere, and in all other cases these inequalities are strict.

Introduction

A small movement of a point $p$ to another point on any hyperbolic Riemann surface $R$ changes the conformal structure of $R \setminus \{p\}$ but not the conformal structure of $R$. This idea can be expressed formally by observing that there is a natural projection between Teichmüller spaces,

$\Psi : T(R \setminus \{p\}) \to T(R)$.

Those deformations represented by points in $T(R \setminus \{p\})$ that do not alter the corresponding points in $T(R)$ are realized by homotopy classes of motions of $p$ on $R$. The map $\Psi$ is called the forgetful map because, for any deformation in $T(R \setminus \{p\})$, it forgets the motion of $p$ and remembers only how $R$ was deformed. The fiber $K = \Psi^{-1}(\text{id})$ consists of the deformations caused by moving $p$ in $R$ while leaving
the marked conformal structure of $R$ fixed. This fiber is the main topic of study by Bers in [3] where he shows that when $R$ is of finite analytic type, $K$ is conformal to the universal covering of $R$. Then $T(R \setminus \{p\})$ factored by the natural realization of the fundamental group of $R$ as a subgroup of the mapping class group of $R \setminus \{p\}$ is the universal curve of Riemann surfaces with a given topological type. That is, $T(R \setminus \{p\})$ factored by this group is a complex fiber space over $T(R)$ with the property that the fiber over each point $\tau$ in $T(R)$ is a Riemann surface with marked complex structure equal to the complex structure of $R_\tau$.

This paper concerns a similar setup but with the following differences. We assume $R$ is an open set in the extended complex plane $\mathbb{C}$, possibly disconnected, and that its complement $E = \mathbb{C} \setminus R$ contains three or more points. In this setting we define Teichmüller equivalence by isotopy and post composition by Möbius transformations. After normalizing at three points of $E$, the deformations of $R$ in $\mathbb{C}$ determine sets $f(E)$ complementary to $f(R)$. Thus we may also think of the points of the Teichmüller space as marked deformations of closed subsets $E$ of $\mathbb{C}$. This Teichmüller space is denoted by $T(E)$, so the forgetful map is

$$\Psi : T(E \cup \{p\}) \to T(E).$$

The fiber $K = \Psi^{-1}\{[id]\}$ is a one-dimensional marked deformation space that tracks the motion of $p$ in the component $R_j$ of $\mathbb{C} - E$ where $p$ is located and for deformations in $K$ the only point of $E \cup \{p\}$ that moves is $p$ itself.

Any infinitesimal form for a metric on $T(E \cup \{p\})$ restricts to an infinitesimal form on $K$. We focus attention on two special cases, namely, the Teichmüller and Kobayashi infinitesimal metrics, $Kob(\tau,V)$ and $Teich(\tau,V)$, which are defined in [6] and (7) in Section 2. These metrics are infima over the families of holomorphic functions from the unit disc into $K$ and into $T(E \cup \{p\})$ with derivatives mapping to a given tangent direction. Since $K \subset T(E \cup \{p\})$, the family of such functions with image in $K$ is smaller and so it turns out that

$$Teich(\tau,V) \leq Kob(\tau,V).$$

In [8] for Riemann surfaces a reverse inequality is shown, namely,

$$Kob(\tau,V) \leq 2 \, Teich(\tau,V),$$

where $\tau$ is a point of $K \subset T(E \cup \{p\})$ and $V$ is a tangent vector at $\tau$.

The metric densities $Kob(\tau,V)$ and $Teich(\tau,V)$ on $K$ induce metric densities on $R$, where the former is the Poincaré metric $\rho(z)[dz]$ on the component of $R$ that contains $p$ and the latter is another metric density $\lambda(z)[dz]$, which we have called the Teichmüller density in [8]. Several authors have studied the comparison between $\rho$ and $\lambda$ (see [3] [8] [10] [12] [13] [16]), but in all of these $p$ is viewed as a variable point on a connected Riemann surface. We give two ways of constructing $\lambda$ on $R = \mathbb{C} - E$ and show that the ratio $\lambda/\rho$ is a function defined on $T(R \setminus \{p\})$ bounded above and below by $1$ and $1/2$. In addition, we show that the ratio is strictly less than $1$ unless $R$ has only one component and that component is equal to the Riemann sphere with three points removed. We also show that $\lambda(p)/\rho(p)$ is strictly larger than $1/2$ unless $p$ belongs to a simply connected component of $R$.

1. Teichmüller spaces

Let $E$ be a closed subset of the extended complex plane $\mathbb{C}$ and consider the family $\mathcal{F}$ of all quasiconformal self-maps $f$ of $\mathbb{C}$. Let two elements $f_0$ and $f_1$ in $\mathcal{F}$
be equivalent if there is a Möbius transformation $A$ that carries $f_0(E)$ to $f_1(E)$ and an isotopy through quasiconformal self-maps $g_t$, $0 \leq t \leq 1$, of $\mathbb{C}$ such that

(i) $g_0(z) = A \circ f_0(z)$ for all $z$ in $\mathbb{C}$,
(ii) $g_1(z) = f_1(z)$ for all $z$ in $\mathbb{C}$, and
(iii) $A \circ f_0(p) = g_t(p) = f_1(p)$ for all $t$ with $0 \leq t \leq 1$ and for all $p$ in $E$.

By definition the Teichmüller space $T(E)$ is the space $\mathcal{F}$ factored by this equivalence relation. If $E$ contains zero, one, two or three points, there are no nontrivial quasiconformal deformations and the Teichmüller space $T(E)$ consists of only one point. To avoid this trivial situation we always assume $E$ contains three or more points, say $z_1$, $z_2$ and $z_3$. Since the definition of $T(E)$ is independent of post composing quasiconformal self-maps of $\mathbb{C}$ by Möbius transformations, we also assume that these quasiconformal maps fix the three points $z_1$, $z_2$ and $z_3$. This means that in part (iii) of the definition of equivalence of elements of $\mathcal{F}$ the Möbius transformation $A$ is the identity. Since pullback by Möbius transformations induces isomorphisms of Teichmüller spaces, without loss of generality we may also assume that these three points are $0$, $1$ and $\infty$.

The class of the identity map $id(z) = z$ in $T(E)$ denoted by $[id]_E$ represents a special point which we call the base point. If $g$ is a quasiconformal self-map of $\mathbb{C}$, then there is the natural isomorphism $g^*$ from $T(g(E))$ to $T(E)$ that carries the class $[f]_{g(E)}$ to the class of $[f \circ g]_E$. In particular, it carries the base point of $T(g(E))$ to $[g]_E$ in $T(E)$, and the point $[g^{-1}]_{g(E)}$ in $T(g(E))$ to the base point of $T(E)$.

2. $\mathbb{K}$ as a holomorphic motion

If we take a point $p$ in any component $R_j$ of $R = \mathbb{C} \setminus E$, then $E \cup \{p\}$ is also a closed set and there is a natural map $\Psi : T(E \cup \{p\}) \to T(E)$, called the forgetful map, defined by

$$\Psi([f]_{E \cup \{p\}}) = [f]_E.$$ 

It is well defined because quasiconformal maps in an isotopy joining two representatives of $[f]_{E \cup \{p\}}$ fix all of the points of $E \cup \{p\}$. These quasiconformal maps automatically extend continuously to $p$, and the same isotopy is an isotopy of maps that fix the points of $f(E)$.

By a theorem of Bers and Lakic, [2], [11] and [7, page 65], the cotangent space to $T(E)$ for any closed set $E$ in $\mathbb{C}$ containing the point $\infty$ is equal to the $L_1$-closure of the rational meromorphic functions on $\mathbb{C}$ that are holomorphic on $\mathbb{C} \setminus E$ and have at most simple poles on $E$. We shall denote this space by $A(E)$; it is the space of integrable holomorphic quadratic differentials on $\mathbb{C} \setminus E$. The derivative of $\Psi$ at any point $\tau = [f]_{E \cup \{p\}}$ in $T(E \cup \{p\})$ is the linear map that is dual to the inclusion $A(f(E)) \subset A(f(E) \cup \{f(p)\})$. It follows from Theorem 8 which we prove later, that there always is a quadratic differential in $A(f(E) \cup \{f(p)\})$ and not in $A(f(E))$, so this derivative is surjective at every point of $T(E \cup \{p\})$. If furthermore $[f] \in \mathbb{K}$, then $f$ is an orientation-preserving map that fixes the boundary of $R_j$, and so $f(p)$ belongs to $R_j$. If $g$ is any map in the equivalence class of $[f]_{E \cup \{p\}}$ in $T(E \cup \{p\})$, then by part (iii) in the definition of equivalence for $T(E \cup \{p\})$, $g(p) = f(p)$. Therefore, there is an evaluation map $\hat{F}$ from $T(E \cup \{p\})$ into $\mathbb{C}$ defined by

$$\hat{F}([f]_{E \cup \{p\}}) = f(p).$$
If we denote by $F$ the restriction of $\tilde{F}$ to $\mathbb{K}$, then $F$ maps $\mathbb{K}$ to $R_j$. Moreover, because $R_j$ is open and connected one can find an isotopy of quasiconformal maps $f_t$ that equals the identity outside of $R_j$ and carries $p$ to a point $f_t(p) = p_t$ in $R_j$ such that $p_1 = f_1(p)$ is an arbitrary point of $R_j$. The equivalence class $[f_t]_{E \cup \{p\}}$ is a curve in $\mathbb{K}$ connecting $[id]$ to $[f_t]_{E \cup \{p\}}$ and $F([f_t]_{E \cup \{p\}}) = p_1$. This shows that $F$ maps $\mathbb{K}$ onto $R_j$.

Let $\Pi$ be a universal covering from the unit disc $\Delta$ onto $R_j$ such that $\Pi(0) = p$. The map $F$ is also a covering mapping onto $R_j$ in the sense that for any $q$ in $R_j$ there is a disc $D$ in $R_j$ containing $q$ such that $F^{-1}(D)$ is a disjoint union of conformal discs $D_j$ in $\mathbb{K}$ such that $F$ restricted to $D_j$ is a homeomorphism of $D_j$ onto $D$. Since $\Pi$ is a universal covering, there is a unique lift $\tilde{\Pi}$ mapping from $\Delta$ to $\mathbb{K}$ such that $F \circ \tilde{\Pi} = \Pi$ and $\tilde{\Pi}(0)$ is the base point in $T(E \cup \{p\})$.

**Theorem 1.** The lift $\tilde{\Pi}$ of $\Pi$ is bijective and conformal from $\Delta$ onto $\mathbb{K}$.

**Proof.** We have already observed that $F$ maps $\mathbb{K}$ onto $R_j$ and that it is a surjective covering of $R_j$ and so $F$ lifts to a covering $\tilde{\Pi}$ from $\Delta$ onto $\mathbb{K}$. If we can show $\mathbb{K}$ is simply connected, then $\tilde{\Pi}$ must be an isomorphism. By a theorem of Lieb (see [5]) the Teichmüller space $T(E)$ is isomorphic to

$$(\Pi_k T(R_k)) \times M(E),$$

where $R_k$ runs over all the components of the complement of $E$ in $\mathbb{C}$ and $M(E)$ is the open ball of measurable complex-valued functions $\mu$ supported on $E$ with $||\mu||_\infty < 1$. The isomorphism carries the Beltrami coefficients representing the points in each of the Teichmüller spaces $T(R_j)$ to Beltrami coefficients defined independently on the domains $R_j$. The Teichmüller metric is induced by the $L_\infty$ metric on Beltrami coefficients. Now assume $p$ is in a particular component $R_j$ of the complement of $E$. By Bers’ theorem [3] the fiber over the identity of the forgetful map from $T(R_j - \{p\})$ to $T(R_j)$ is simply connected, and by Lieb’s theorem it is isomorphic to the fiber over the identity of

$$\Psi : (\Pi_{k \neq j} T(R_k)) \times M(E) \times T(R_j - \{p\}) \rightarrow (\Pi_{k \neq j} T(R_k)) \times M(E) \times T(R_j).$$

So the points of the fiber over the identity in this projection are represented precisely by the points of the fiber over the identity of $T(R_j - \{p\}) \rightarrow T(R_j)$. \qed

For the next result we first give some basic facts about the Kobayashi metric on a complex manifold $M$. Its infinitesimal form $Kob_M(\tau, V)$ is a positive number assigned to each tangent vector $V$ at a point $\tau$ in $M$ by the following extremal problem:

$$Kob_M(\tau, V) = \inf \{ |k| \mid k g'(0) = V \},$$

where the infimum is taken over all nonnegative numbers $k$ and all holomorphic functions $g$ mapping the unit disc $\Delta$ into $M$.

Because it is known that for any Teichmüller space $T$, $Kob_T(\tau, V) = Teich_T(\tau, V)$, [6] and [14], the infinitesimal norm $Teich_T(\tau, V)$ is given by

$$Kob_T(\tau, V) = \inf \{ |k| \mid k g'(0) = V \} = Teich_T(\tau, V) = \inf \{ ||\partial V||_\infty \},$$

where the first infimum is taken over all $k$ and all holomorphic functions $g$ mapping $\Delta$ into $T$ with $g(0) = \tau$ and the second infimum is over all continuous vector fields $\tilde{V}$ whose restriction to $f^\mu(E \cup \{p\})$ is equal to $V$ and where $f^\mu$ represents the point $\tau \in T(E \cup \{p\})$. However, for complex submanifolds $M$ of $T$, Kobayashi’s
Figure 1

infinitesimal form for $M$ may differ from Teichmüller’s infinitesimal form for $T$ restricted to $M$ since the extremal problem defining $Kob_M(\tau, V)$ involves the further requirement that the holomorphic function $g$ must have image contained in $M$. Thus, obviously,

\[(8) \quad Teich_M(\tau, V) \leq Kob_M(\tau, V).\]

We will see that this is usually a strict inequality when $M = \mathbb{K}$.

Schwarz’s lemma implies that Kobayashi’s metric for a complex disc coincides with Poincaré’s metric. This observation is expressed in the following proposition.

**Proposition 1.** The Kobayashi density $Kob_{\mathbb{K}}(\tau, V)$ on $\mathbb{K}$ and the Poincaré density $\rho$ on $R_j$ are related by the following formula:

\[(9) \quad \rho(F([f])) |dF(V)| = Kob_{\mathbb{K}}([f], V).\]

**Proof.** Since $\Pi$ is a universal covering of $\Delta$ onto $R_j$ mapping $0$ to $p$, $\frac{1}{|g'(0)|}$ realizes the smallest value of $\frac{1}{|g(0)|}$, where $g(0) = p$ and $g$ is holomorphic mapping $\Delta$ into $R_j$. □

The next theorem identifies the unique lifting $\tilde{\Pi}$ of the previous theorem in a different way. Define a normalized holomorphic motion $h(t, z)$ of $E \cup \{p\}$ by the following formula:

\[(10) \quad h(t, z) = \begin{cases} z & \text{for } z \in E, \\ \Pi(t) & \text{for } z = p. \end{cases}\]

By Slodkowski’s [15] extension theorem, $h$ extends to a normalized holomorphic motion $\tilde{h}$ of the extended complex plane. The map $H$ that sends $t$ to the class of $z \mapsto \tilde{h}(t, z)$ is a holomorphic map from the unit disc into $T(E \cup \{p\})$. Furthermore, since $\tilde{h}(t, z) = h(t, z) = z$ for all $t$ and all $z$ in $E$, the curve $H(t)$ represents the homotopy class that keeps all $H(t)$ in the same class in $T(E)$. Since $H(0)$ is the identity map, the image of $H$ is in $\mathbb{K}$. Since $F(H(t)) = \tilde{h}(t, p) = h(t, p) = \Pi(t)$, the diagram in Figure 1 commutes.

Let $\gamma$ be any path in $\overline{C \setminus E}$ such that $\gamma(0) = p$. Take a lift $\tilde{\gamma}$ of $\gamma$ under the universal covering map $\Pi$ and with the property $\tilde{\gamma}(0) = 0$. Then the path $\tilde{\gamma} = H(\tilde{\gamma})$ is a path in $\mathbb{K}$ that starts at the base point and whose image under $F$ is $\gamma$. Since from the previous theorem $\mathbb{K}$ is simply connected, $H$ is a biholomorphic surjection. Therefore, we have proved the following theorem.
Theorem 2. The map $F$ is a holomorphic covering from $K$ onto $R_j$. Furthermore, the holomorphic motion $h$ in (10) defines the whole fiber $K$. More precisely, for every $\tau$ in $K$, there exists $t$ in $\Delta$ such that $\tau = [z \mapsto \hat{h}(t, z)]$.

3. Comparing metrics on $K$

We now define the infinitesimal form of a metric $\lambda$ naturally associated to any hyperbolic Riemann surface which turns out to be different from but equivalent to the Poincaré metric $p$. In [3] we have called it the Teichmüller density because it is the metric on $R$ induced by $Teich(\tau, V)$ on the fiber $K = \Psi^{-1}(\{id\})$. It will turn out that the ratio $\lambda(p)/p(p)$ is a real-valued functional on domains $E \cup \{p\}$ that takes values always between $1/2$ and $1$, and takes the extreme value $1/2$ only when $p$ is in a simply connected component of $R$ and the extreme value $1$ only when $R$ is the complement of three points in the Riemann sphere.

As a starting point we make the following definitions.

Definition 1. The infinitesimal Teichmüller norm of a vector field $V$ defined on $E \cup \{p\}$ is

$$||V||_T = \inf\{||\partial V||_\infty\},$$

where the infimum is taken over all continuous, complex-valued functions $\tilde{V}$ with distributional derivatives and $\tilde{V}$ equal to $V$ on $E \cup \{p\}$.

Definition 2. The Teichmüller density $\lambda(p)$ evaluated at a point $p$ in $R_j$ is equal to $||V_0||_T$, where $V_0$ is the function defined on $E \cup \{p\}$ which is equal to zero on $E$ and equal to $1$ at $p$.

Proposition 2. The Teichmüller density $Teich_K([f], V)$ on $K$ is related to the Teichmüller density $\lambda$ on $R_j$ by the formula:

$$\lambda(F([f]))|dF(V)| = Teich_K([f], V),$$

where $F : K \to R_j$ is the evaluation map $F([f]) = f(p)$.

Proof. We first prove this equality for $[f] = [id]$. If $[f] = [id]$, then $F([f]) = p$. If $V$ is the zero vector, then $dF(V) = 0$ and (12) reduces to $0 = 0$. Suppose $V$ is a nonzero vector field tangent to $K$. Then $V$ vanishes on $E$, $V(p) \neq 0$ and $\lambda(p) = ||V/V(p)||_T$. Also $dF|_{id}(V) = V(p)$, which follows from [1], page 104. So the left-hand side of (12) is equal to $||V||_T$. From Definition 1 and 7 this is equal to the right-hand side of (12).

Let $[f]$ be any point in $K$ and $G : Teich(E \cup f(p)) \to Teich(E \cup p)$ given by $G([g]) = [g \circ f]$ is an isomorphism. Also, if $\Psi : T(E \cup \{p\}) \to T(E)$ and $\Psi_G : T(E \cup f(p)) \to T(E)$ are the forgetful maps, then $\Psi \circ G = \Psi_G$. If $K'$ is the fiber over the identity of $\Psi_G$ and $K$ is the fiber over the identity of $\Psi$, then $G$ restricted to $K'$ is an isomorphism of $K'$ onto $K$. For $[g] \in K'$, $(F \circ G)([g]) = F([g \circ f]) = g(f(p))$. Thus $F \circ G$ is the evaluation map for $g$ at the point $f(p)$. By what we have just proved,

$$\lambda(FG([id]))|dF(G)|_{[id]}((G^{-1})^*V) = Teich_{K'}([id], (G^{-1})^*V).$$

But $\lambda(F \circ G([id])) = \lambda(f(p))$. Also,

$$Teich_K([id], (G^{-1})^*V) = Teich_K([g], dG_{[id]}((G^{-1})^*V)) = Teich_K([g], V).$$
The chain rule implies
\[ d(FG)_{id}((G^{-1})^*V) = dF_{id}(dG_{id}(G^{-1})^*(V)) = dF_{id}(V), \]
and so (12) follows from (13). \qed

To get a lower bound for the infinitesimal form \( \lambda(p) \) we construct a quadratic differential \( \psi_p \) canonically associated with any Riemann surface \( R \) and a point \( p \) in \( R \). Let \( \Pi \) be a universal covering of a component \( R \) of the complement of \( E \) in \( \overline{C} \). Assume \( \Pi \) is normalized so that \( \Pi(0) = p \) and \( \Pi'(0) \) is positive. Let \( \Gamma \) be the universal covering group for \( \Pi \). \( \Gamma \) is a fixed point free discrete group of Möbius transformations acting on the unit disc \( \Delta \), \( \Pi \circ \gamma = \Pi \) for every \( \gamma \in \Gamma \), and the factor space \( \Delta/\Gamma \) is conformal to \( R \).

**Theorem 3.** The Poincaré theta series of \( \psi(z) = \frac{1}{\pi z} \),
\[ \sum_{\gamma \in \Gamma} \psi(\gamma(z))\gamma'(z)^2 = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)^2}{\pi \gamma(z)} = \psi_p(z), \]
defines a quadratic differential \( \psi_p \) on \( R \) which is holomorphic on \( R - \{p\} \), which has a simple pole at \( p \) and for which
\[ 1 \leq ||\psi_p|| \leq 2. \]

**Proof.** The tangent vector \( V = H'(0) \) to the holomorphic motion \( H(t, z) \) from Section 2 vanishes on \( E \) and satisfies \( V(p) = \Pi'(0) \). Put
\[ V_0 = \frac{V}{\Pi'(0)}. \]
Since \( 1/||\Pi'(0)|| = \rho(p) \) and since by Schwarz’s lemma \( ||V||_T \leq 1 \), we obtain
\[ ||V_0||_T = \rho(p)||V||_T \leq \rho(p). \]
On the other hand,
\[ ||V_0||_T \geq \frac{|V_0(\hat{\varphi})|}{||\hat{\varphi}||} \]
for every integrable quadratic differential \( \hat{\varphi} \) holomorphic on \( R - \{p\} \). In particular
\[ \rho(p) \geq ||V_0||_T \geq \frac{|V_0(\psi_p)|}{||\psi_p||} \]
for the quadratic differential \( \psi_p \) given by the Poincaré theta series (14).

By the Ahlfors-Bers density theorem, \( \psi_p \) is the \( L_1 \)-limit of integrable holomorphic functions \( \psi_{p_n} \) in the plane with a finite number of simple poles located at \( p \) and on the boundary of \( R_j \). Thus to evaluate \( V(\psi_p) \) we may assume it has only a finite number of poles. But if we let \( \tilde{V_0} \) be a continuous extension of \( V_0 \) with bounded \( \overline{\partial} \)-derivative, then
\[ V_0(\psi_p) = \iint \partial \tilde{V_0}\psi_p dx dy = \iint \tilde{V_0} \psi_p \frac{dz dz}{2i} \]
\[ = \iint d \left( \tilde{V_0} \psi_p \frac{dz}{2i} \right) = -\pi \text{res} (\psi_p, p). \]
The only residue of \( \left( \bar{V}_0 \psi_p dz \right) \) is at \( p \) and this residue is equal to \( 1/(\pi \Pi'(0)) \), where \( \Pi \) is a universal covering of \( R_j \) with \( \Pi(0) = p \). Since \( \rho(p) = 1/|\Pi'(0)| \),

\[
|V_0(\psi_p)| = \rho(p),
\]

and from (17) we get

\[
\rho(p) \geq ||V_0||T \geq \rho(p)||\psi_p|| \geq \rho(p),
\]

which yields \( ||\psi_p|| \geq 1 \).

On the other hand,

\[
||\psi_p|| = \left| \int \int_\omega \sum \gamma'(z)^2 \pi \gamma(z) \right| \leq \int \int_\omega \sum \gamma'(z)^2 \pi \gamma(z) = \int \int_{|z| < 1} \frac{1}{\pi z} dxdy = 2,
\]

where \( \omega \) is a fundamental domain in \( \Delta \) for the covering group. □

Theorem 4 ([8]). The infinitesimal form \( \lambda \) on each domain \( R_j \) is equivalent to the Poincaré infinitesimal form \( \rho \) on that domain. More precisely,

\[
(1/2) \rho(p) \leq \lambda(p) \leq \rho(p).
\]

Proof. The right-hand side of (20) follows from (16). The left-hand side follows by choosing the specific quadratic differential \( \psi_p \) on \( R_j \setminus \{p\} \) given in Theorem 3. The theta series of \( \psi(z) = 1/(\pi z) \) is summed over all covering transformations \( \gamma \) for the covering map \( \Pi \) of \( R_j \). Since \( ||\psi_p|| \leq 2 \), the definition of \( \lambda(p) \) yields

\[
\lambda(p) \geq \frac{|V_0(\psi_p)|}{||\psi_p||} = \rho(p) \frac{||\psi_p||}{||\psi_p||} = \rho(p).
\]

Just as with the Poincaré metric, the integrated form of \( \lambda(p)|dp| \) becomes a metric by the following recipe:

\[
\lambda(p_1, p_2) = \inf_\gamma \int_0^1 \lambda(\gamma(t))|\gamma'(t)|dt|
\]

where the infimum is over all continuous, piecewise differentiable curves \( \gamma \) in \( R \) such that \( \gamma(0) = p_1 \) and \( \gamma(1) = p_2 \).

Corollary 1. The metrics \( \lambda(p_1, p_2) \) and \( \rho(p_1, p_2) \), which are the integrated forms of \( \lambda(p)|dp| \) and \( \rho(p)|dp| \), satisfy the following inequality:

\[
(1/2) \rho(p_1, p_2) \leq \lambda(p_1, p_2) \leq \rho(p_1, p_2).
\]

Proof. To define the integral in (22) it is helpful to know that the integrand is continuous, which is justified by Corollary 2 given in the next section. The inequality of this corollary then follows directly from the corresponding inequality for the infinitesimal metrics. □

4. Residues as densities

Theorem 8 shows the existence of a special quadratic differential with a simple pole at \( p \) for any given point \( p \) in \( R \). We now show that the residue of a meromorphic \( n \)-differential is an \((n-1)\)-differential and, consequently, the residue of a quadratic differential is a one-form. We state this observation as a lemma.
Lemma 1. Suppose \( \varphi = \varphi(z)(dz)^n \) is a meromorphic \( n \)-differential in a neighborhood of \( p \). Then the residue of \( \varphi \) at \( p \) is an \( (n - 1) \)-differential.

Proof. Let \( w \) and \( z \) be charts such that \( z = 0 \) in \( U \) and \( w = 0 \) in \( V \) correspond to the point \( p \). Let \( \varphi_U(z) \) and \( \varphi_V(w) \) be the expressions for \( \varphi \) in terms of the parameters \( z \) and \( w \) so that

\[
\varphi_U(z) = \varphi_V(f(z))f'(z)^n,
\]

where \( w = f(z) \) is the transition mapping. Then

\[
\text{res}(\varphi_V, p)f'(0)^{n-1} = \frac{1}{2\pi i} \int_{w(\gamma)} \varphi_V(w)f'(0)^{n-1} dw = \frac{1}{2\pi i} \int_{\gamma} \varphi_U(z)\frac{f'(0)}{f'(z)^{n-1}} dz.
\]

Here \( \gamma \) is a simple closed curve in the \( z \)-plane winding once around \( z = 0 \) and oriented in the counterclockwise direction and \( w(\gamma) \) has the same properties with respect to \( w = 0 \) in the \( w \)-plane. The integrals depend only on the homotopy class provided \( \gamma \) lies in a sufficiently small neighborhood of 0. Since \( f \) is univalent, the fraction \( f'(0)^{n-1}/f'(z)^{n-1} \) becomes uniformly close to 1 as that neighborhood becomes sufficiently small. Therefore, the integral on the right-hand side is equal to

\[
\frac{1}{2\pi i} \int_{\gamma} \varphi_U(z) dz = \text{res}(\varphi_U, p). \quad \Box
\]

In the case that \( \varphi \) is a quadratic differential, \( \text{res}(\varphi, p)dp \) is a holomorphic one-form and, if \( \varphi \) has at most simple poles, then \( \text{res}(\varphi, p) = 0 \) if and only if \( \varphi \) extends holomorphically at \( p \). Moreover, the absolute value \( |\text{res}(\varphi, p)dp| \) determines a density, that is, an infinitesimal metric. In addition to its residue, any integrable quadratic differential \( \varphi \) holomorphic on \( E \cup \{ p \} \) carries another invariant, namely, its \( L^1 \) norm. Thus we can ask for the largest residue of an integrable quadratic differential holomorphic on \( R \setminus \{ p \} \) subject to the condition that its \( L^1 \) norm is equal to one. Obviously, a differential that realizes this largest value will be identically equal to zero on every component of \( R \) that does not contain \( p \).

Putting these pieces together we obtain the following theorem, which serves as an alternative definition of the Teichmüller density \( \lambda \).

Theorem 5.

\[
\lambda(p) = \pi \sup_{||\varphi||=1} \text{Re} \text{ res}(\varphi, p),
\]

where the supremum is over all quadratic differentials holomorphic in \( R \setminus \{ p \} \) with \( L^1 \) norm one. Moreover, there is a unique quadratic differential \( \varphi_p \) that realizes the supremum in (23).

Proof. Let \( V(p) = 1 \) and \( V \) vanish on \( E \), and let \( A(E \cup \{ p \}) \) be the Banach space of integrable quadratic differentials \( \varphi \) holomorphic on the complement of \( E \cup \{ p \} \) with norm given by

\[
||\varphi|| = \int_R |\varphi| dxdy.
\]

By the Hahn-Banach and Riesz representation theorems, there exists a continuous extension \( \tilde{V} \) of \( V \) with bounded \( \partial \)-derivative such that

\[
||\partial \tilde{V}||_{\infty} = \sup \left\{ \left| \int \int_R \partial \tilde{V} \varphi \right| \right\},
\]

\[
\partial \tilde{V}_\psi = \frac{1}{2\pi i} \int_{\gamma} \varphi_U(z) dz = \text{res}(\varphi_U, p).
\]

\[
||\partial \tilde{V}||_{\infty} \leq \sqrt{\text{Re} \text{ res}(\varphi, p)}.
\]

Since \( \tilde{V}(p) = \tilde{V}_\psi(p) = 1 \), we have

\[
\text{Re} \text{ res}(\varphi, p) = \text{Re} \text{ res}(\tilde{V}, p).
\]

Put \( \varphi = \tilde{V} \) in (23). Then

\[
||\varphi||_{\infty} = \int_R |\tilde{V}| dxdy = \int_R |\varphi| dxdy = ||\varphi||_{\infty}.
\]
where the supremum is over all \( \varphi \in A(R \setminus \{p\}) \) with \( \|\varphi\| = 1 \), and where \( \tilde{V} \) is a continuous extension of \( V \) to \( R \).

By the same residue calculation given in the proof of Theorem 3 we find that

\[
\int \int_R \partial \tilde{V} \varphi = -\pi \text{res}(\varphi, p).
\]

Therefore the infimum in Definition 1 is equal to the supremum in (23).

To prove the second part of the theorem, the evaluation of the residue of \( \varphi \) at a point \( p \) is a linear functional \( V \in A(E \cup \{p\})^* \) given by the formula

\[
(25) V(\varphi) = \frac{1}{2\pi i} \oint_{\gamma} \varphi(z)dz,
\]

where \( \gamma \) is a closed curve with winding number 1 around the point \( p \). Now suppose \( \varphi_n \) is a sequence of differentials in \( A(E \cup \{p\}) \) with \( \|\varphi_n\| = 1 \) such that \( \text{Re} \pi \text{res}(\varphi_n, p) \) approaches the supremum in (23). From equicontinuity of bounded holomorphic functions there must be a subsequence of \( \varphi_n \) which converges uniformly on compact subsets of \( R - \{p\} \). But it is not possible for any subsequence of \( \varphi_n \) to converge uniformly on compact subsets to any quadratic differential \( \varphi \) with \( \|\varphi\| < 1 \), because replacing \( \varphi \) by \( \varphi/\|\varphi\| \) gives a quadratic differential of norm 1 with a larger residue. Therefore, a subsequence of \( \varphi_n \) converges uniformly on compact subsets to \( \varphi \) and \( \|\varphi_n\| = 1 = \|\varphi\| \), which implies \( \|\varphi_n - \varphi\| \) converges to zero and so \( \varphi \) realizes the supremum in (23). The point \( \varphi \) on the unit sphere realizing the supremum is unique because the unit ball of \( A(E \cup \{p\}) \) is strictly convex. Thus, every convergent subsequence of \( \varphi_n \) converges to the same limit, which implies the original sequence itself converges to that limit. \( \square \)

**Corollary 2.** \( \lambda(p) \) is continuous in \( p \).

**Proof.** Assume \( |q - p| < \delta \). Then the Bers’ \( L \)-operator \( [7] \) yields a Banach space isomorphism \( L \) from \( A(E \cup \{q\}) \) onto \( A(E \cup \{p\}) \) such that

\[
\max\{\|L\|, \|L^{-1}\|\} < 1 + \epsilon(\delta),
\]

where \( \epsilon(\delta) \) is bounded by the \( L_{\infty} \)-norm of the Beltrami coefficient of any quasiconformal self-map of \( R \) that carries \( q \) to \( p \) and is equal to the identity outside a fixed disc of radius \( \delta_1 \) with center \( p \) and with \( |q - p| < (1/2)\delta_1 \). \( \square \)

**Theorem 6.** Let \( \rho \) be the Poincaré metric for \( R \). That is, in any component \( R_j \) of \( R \), \( \rho \) is equal to \( \rho_{R_j} \), the Poincaré metric for the component \( R_j \). Then for every quadratic differential \( \varphi \) in \( A(R \setminus \{p\}) \), we have

\[
|\text{res}(\varphi, p)| \leq \frac{\rho(p)}{\pi} \|\varphi\|.
\]

**Proof.** From Theorem 5

\[
\lambda(p) \geq \pi|\text{res}(\varphi, p)|/\|\varphi\|
\]

and we already know \( \rho(p) \geq \lambda(p) \), so the theorem follows. \( \square \)
5. Domain differentials

In the previous two sections we constructed two quadratic differentials $\psi_p$ and $\varphi_p$ in $A(E \cup \{p\})$ canonically associated to a plane domain and a point $p$ in that domain. $\psi_p$ is an element of $A(E \cup \{p\})$ given by a theta series of the function $q(z) = 1/(\pi z)$ summed over the elements of a Fuchsian covering group acting on the unit disc and chosen so that 0 covers $p$ and $\Pi'(0)$ is positive. $\varphi_p$ is an element of $A(E \cup \{p\})$ that maximizes the real part of $V_0(\varphi)$ with $||\varphi|| = 1$, where $V_0$ is a vector field vanishing on $E$ and equal to 1 at $p$. We have already seen in Theorem 5 that $\varphi_p$ realizing this extreme is uniquely determined. These two differentials determine the Teichmüller and Poincaré densities by the following formulæ:

$$\lambda(p) = \pi \text{ Re } \text{res}(\varphi_p, p) \quad \text{and} \quad \rho(p) = \pi \text{ Re } \text{res}(\psi_p, p).$$

The left-hand formula comes from Theorem 5 and the right-hand comes from (18) and (19).

Before stating the next theorem we need to recall the definition of the Zygmund Banach space $Z(E)$ on an arbitrary closed set $E \subset \mathbb{C}$. By definition, $Z(E)$ consists of all continuous complex-valued vector fields $V$ defined on $E$ with the property that $V$ has a continuous extension $\tilde{V}$ to the complex plane $\mathbb{C}$ such that $||\tilde{V}'||_\infty < \infty$. The assumption that the vector field $V(z) \frac{\partial}{\partial z}$ is continuous at $z = \infty$ implies there is a constant $C$ such that $|\tilde{V}(z)| \leq C|z|^2$. Definition 1 defines the Teichmüller norm of $V$, and one easily shows that $||V||_T = 0$ if and only if $V$ is the restriction of a quadratic polynomial to $E$.

There is another natural norm on $Z(E)$, which we call the cross ratio norm, and it is given by a supremum of cross ratios. To define the cross ratio norm, assume $Q = \{a, b, c, d\}$ is an arbitrary quadruple of four points contained in $E$ and form

$$||V||_{cr} = \sup |cr(Q)\rho_{01}(cr(Q))V[Q]|,$$

where

$$V[Q] = \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d},$$

and where $\rho_{01}(p)$ is the Poincaré metric of the domain $\mathbb{C} - \{0, 1, \infty\}$. In [7] page 67 it is shown that the two norms $||V||_T$ and $||V||_{cr}$ are equivalent by universal constants independent of the set $E$. Moreover, $Z(E)$ is a Banach space and the pairing between $A(E)$ and $Z(E)$,

$$(\varphi, V) = \text{ Re } \int \int \varphi \tilde{V} dxdy,$$

induces an isomorphism from $Z(E)$ onto $A(E)^*$.

In general the one-dimensional spaces in $A(E \cup \{p\})$ spanned by $\varphi_p$ and $\psi_p$ are not equal except, of course, when $A(E \cup \{p\})$ is one dimensional. The only case when $A(E \cup \{p\})$ is one dimensional occurs when $E$ is the smallest possible set, namely, when $E = \{0, 1, \infty\}$. These comments lead to the following theorem.

**Theorem 7.** If $E = \{0, 1, \infty\}$, then $\lambda(p) = \rho(p)$. Furthermore inequality (15) in Theorem 3 becomes $||\psi_p|| = 1$, $\varphi_p = \psi_p$ and

$$\psi_p(w) = \left(\frac{1}{\pi \Pi'(0)}\right) \cdot \frac{p(p-1)}{w(w-1)(w-p)},$$

where $\Pi$ is the covering map described in Theorem 3.
Proof. For every Teichmüller space $T(E \cup \{ p \})$, we know that $||\varphi_p|| = 1$ and $1 \leq ||\psi_p|| \leq 2$. When $E = \{0, 1, \infty\}$, $K = T(E \cup \{ p \})$ and so Teichmüller’s metric equals Kobayashi’s metric because the Teichmüller and Kobayashi metrics coincide on any Teichmüller space. Thus, from Propositions 1 and 2, $\lambda(p) = \rho(p)$. Because of (20) this means $\text{res}(\varphi_p, p) = \text{res}(\psi_p, p)$. Note that $A(\{0, 1, p, \infty\})$ is one dimensional and generated by

$$\frac{p(p-1)}{w(w-1)(w-p)},$$

which has residue at $p$ equal to 1. From the calculation preceding (19),

$$\text{res}(\psi_p(w), p) = \text{res}\left(\frac{1}{z}, 0\right) \cdot \left(\frac{dz}{dw}\big|_{w=p}\right) = \frac{1}{\pi \Pi'(0)},$$

where $w = \Pi(z)$. Thus, after multiplication by $1/((\pi \Pi'(0))$, the formula in (27) has the same residue as $\psi_p$ at $p$ and so must coincide with $\psi_p$. Since both $\varphi_p$ and $\psi_p$ have the same residue at $p$ and are in the same one-dimensional space, $\varphi_p = \psi_p$. □

Corollary 3 (Agard’s formula). The Poincaré metric for $\mathbb{C} - \{0, 1, \infty\}$ is

$$\rho(p) = \left(\frac{1}{\pi} \int \int \frac{p(p-1)}{w(w-1)(w-p)} |dudv\right)^{-1}.$$

Proof. From its definition, $\varphi_p$ has norm equal to 1. Since $\psi_p = \varphi_p$, $\psi_p$ also has norm equal to 1 and (28) is equivalent to the equation $||\psi_p|| = 1$. □

Theorem 8. For $p$ in any simply connected component $R_j$ of $\mathbb{C} - E$, $\lambda(p) = (1/2)\rho(p)$. Moreover, the pullbacks of the quadratic differentials $\varphi_p$ and $\psi_p$ to the unit disc by the Riemann map that maps $p$ to 0 and has positive derivative at $p$ are equal to $1/(2\pi z)$ and $1/(\pi z)$, respectively.

Proof. Since both metrics are invariant under conformal change of coordinates, to show $\lambda(p) = (1/2)\rho(p)$ we only need to show that $\lambda_\Delta(0) = \rho_\Delta(0)/2$, where $\Delta$ is the unit disc. Since the covering group is trivial in this case, from Theorem 3 it is obvious that $\psi_0(z) = 1/(\pi z)$. We claim that $\varphi_0(z) = 1/(2\pi z)$. Since $||\varphi_0(z)|| = 1$, we know from Theorem 5 that $\lambda_\Delta(0) \geq 1/2$. The reverse inequality follows by considering the vector field $V(z) = 1 - |z|$ for $|z| \leq 1$ and $V(z) = 0$ for $|z| \geq 1$. Then $V(0) = 1$ and $\overline{\partial}V(z) = -(1/2)z^{1/2}z^{-1/2}$, so $||\overline{\partial}V||_\infty = 1/2$. This shows that any $\varphi$ for which $\pi\text{res}(\varphi, 0) > 1/2$ must have norm greater than 1. But from Theorem 5 the quadratic differential with positive residue at 0 that realizes the maximum value of $V(\varphi)$ is unique. Therefore, $\varphi_0 = 1/(2\pi z)$. □

6. Kobayashi and Teichmüller extremal domains

In this section we show that the inequalities in (20) in Theorem 4 are sharp except in two special cases. That is, we show $\lambda(p) = \rho(p)$ if and only if $E$ consists of three points and $\lambda(p) = (1/2)\rho$ if and only if $p$ is in a simply connected component $R_j$ of $R$.

First, we make a general remark about notation. If we are given a continuous infinitesimal metric $\alpha(p)|dp|$ defined on a domain and a differentiable curve $\gamma(t)$, $0 \leq t \leq 1$, in that domain, we shall denote the $\alpha$-arclength of $\gamma$ by $\alpha(\gamma)$, that is,

$$\alpha(\gamma) = \int_0^1 \alpha(\gamma(t))|\gamma'(t)||dt|.$$
Just as in formula (22), we define the \( \alpha \)-distance from \( p \) to \( q \) by
\[
\alpha(p, q) = \inf\{\alpha(\gamma)\},
\]
where the infimum is taken over all curves \( \gamma \) in the domain that join \( p \) to \( q \).

**Theorem 9.** If \( E \) has exactly three points, then the map \( H \) in Figure 1 is an isometry from the unit disc \( \Delta \) with the hyperbolic metric onto \( \mathbb{H} \) with the infinitesimal form of Teichmüller’s metric. That is,
\[
\text{Teich}_\mathbb{H}(H(t), H(s)) = \rho_\Delta(t, s).
\]

If \( E \) is connected, or equivalently, if every component of \( R \) is simply connected, then
\[
\text{Teich}_\mathbb{H}(H(t), H(s)) = \frac{1}{2} \rho_\Delta(t, s)
\]
for all \( t \) and \( s \) in \( \Delta \). In all other cases, we have
\[
\frac{1}{2} \rho_\Delta(\gamma) < \text{Teich}_\mathbb{H}(H(\gamma)) < \rho_\Delta(\gamma),
\]
for any non-constant, piecewise differentiable path \( \gamma \) in \( \Delta \).

**Proof.** Since we have already shown that \((1/2)\rho(p) \leq \lambda(p) \leq \rho(p)\), and since \( \rho_\Delta \) is the lift to \( \Delta \) of \( \rho \) and \( \text{Teich}_\mathbb{H} \) is the lift to \( \mathbb{H} \) of \( \lambda \),
\[
(29) \quad \frac{1}{2} \rho_\Delta(\gamma) \leq \text{Teich}_\mathbb{H}(H(\gamma)) \leq \rho_\Delta(\gamma).
\]
Equality in (29) can hold only if the corresponding equality holds in \((30)\)
\[
(1/2)\rho(p) \leq \lambda(p) \leq \rho(p).
\]
We have seen that \((1/2)\rho(p) = \lambda(p)\) if the component \( R_j \) of \( R \) that contains \( p \) is simply connected. Conversely, if \( R_j \) is not simply connected, then \( R_j \) has a nontrivial function covering group \( \Gamma \), and the quadratic differential
\[
\psi_p(z) = \sum_{\Gamma} \frac{\gamma'(z)^2}{\pi \gamma(z)}
\]
has norm strictly less than \( 2 \). From (21) this implies that \( \lambda(p) > (1/2)\rho(p) \).

It remains to show that if \( R \) is not conformal to \( \bar{\mathbb{C}} \setminus \{0, 1, \infty\} \), then \( \lambda(p) < \rho(p) \).
We will prove this in the remainder of this section. \( \square \)

**Theorem 10.** If there is any point \( p \) in \( \bar{\mathbb{C}} - E \) for which \( \lambda(p) = \rho(p) \), then \( A(E \cup \{p\}) \) is one dimensional and \( E = \{0, 1, \infty\} \).

**Proof.** First we show that the component of \( R \) containing \( p \) cannot be simply connected and it cannot be conformal to a punctured disc. Finally we show that if a component does not fall into one of these two cases, then it must contain a simple closed hyperbolic geodesic and whenever this is the case \( \lambda(p) < \rho(p) \).

The first situation is excluded because we have already observed that if a component of \( R \) is simply connected, then \( \lambda(p) = (1/2)\rho(p) \) on that component.

In the second situation, we can assume the component of \( R \) is equal to the punctured disc \( \Delta' = \{z : 0 < |z| < 1\} \) and that \( 0 < p < 1 \). Consider the vector field \( V(z) \frac{\partial}{\partial z} \) defined on \( \Delta' \) by
\[
(31) \quad V(z) = \frac{1}{p \ln p} (\ln |z|)z = \frac{1}{p \ln p} \left( \frac{\ln z + \ln \pi}{2} \right)z.
\]
Notice that $V$ vanishes at 0 and on $\{ z : |z| = 1 \}$, that $V(p) = 1$ and finally that
\begin{equation}
\mu = \overline{\partial}V = \frac{1}{2p \ln p} \cdot \frac{z}{\bar{z}}.
\end{equation}

Thus
\[ \lambda_{\Delta'}(p) \leq ||\overline{\partial}V||_\infty \leq \frac{1}{|2p \ln p|} = \rho_{\Delta'}(p). \]

But the expression in (32) is the Beltrami coefficient associated to the quadratic differential $q(z) = \frac{(dz)^2}{z^2}$ which over $\Delta'$ is a quadratic differential of infinite norm. By Theorem 5 there is a unique quadratic differential holomorphic in $\Delta'$ with the largest possible real residue at $p$ and with $L_1$-norm equal to one, which we denoted by $\varphi_p$. This linear functional $V$ is represented by the Beltrami coefficient $\lambda(p)|\varphi_p|/\varphi_p$. That is, any other Beltrami differential not of this form that represents $V$ must have larger $L_\infty$-norm. Since $\frac{(dz)^2}{z^2}$ is not integrable on $\Delta'$, we conclude that $\mu$ in (32) does not have the smallest possible $L_\infty$-norm and so $\lambda(p) < \rho(p)$.

If $A(E \cup \{ p \})$ is not one dimensional, then $E$ must contain four or more points.

If the component $R_j$ of $\overline{C} - E$ that contains $p$ is neither simply connected nor conformal to a punctured disk, then $R_j$ contains a simple closed curve homotopic to a Poincaré geodesic of positive length. Suppose that $\lambda(p) = \rho(p)$. Then by (16), $||V||_T = 1$, where $V = H'(0)$ and $H(t,z) = f^{\mu(t)}(z)$ is a holomorphic motion from Section 2. Thus, $||\mu'(0)||_\infty = 1$, and
\[ V_0 = \left[ \frac{\mu'(0)}{\Pi'(0)} \right]. \]

Formula (25) implies that
\[ V_0 = \left[ k \lambda(p) \frac{|\varphi_p|}{\varphi_p} \right], \]
where $|k| = 1$ and $\varphi_p$ is the differential in (26). Therefore,
\[ \frac{\mu'(0)}{\Pi'(0)} = k \lambda(p) \frac{|\varphi_p|}{\varphi_p}. \]

Thus,
\[ \mu'(0) = \tilde{k} \frac{|\varphi_p|}{\varphi_p}, \]
where $|\tilde{k}| = 1$.

Let
\[ f(t) = \iint_R \mu(t)\varphi_p dxdy. \]

Then $f$ maps $\Delta$ to $\Delta$ and
\[ f'(0) = \iint_R \mu'(0)\varphi_p dxdy = \tilde{k}. \]

Thus, by Schwarz’s lemma, $f(t) = \tilde{k}t$. We have
\[ |t| = |\tilde{k}t| = |\iint_R \mu(t)\varphi_p dxdy| \leq \iint_R |\mu(t)||\varphi_p|dxdy \leq |t| \iint_R |\varphi_p|dxdy = |t|. \]
Thus,

$$\mu(t)\varphi_p = \tilde{t}\varphi_p$$

and $\mathbb{K}$ is a Teichmüller disc. The results in [11] or [4] now imply that $E$ is a three-point set.

**Corollary 4.** The ratio

$$G([f]) = \frac{\lambda(f(p))}{\rho(f(p))}$$

is a well-defined real-valued function of $[f]$ in the Teichmüller space $T(E \cup \{p\})$ taking values between 1/2 and 1, and the extreme value 1/2 only when $f(p)$ lies in a simply connected component of $f(R)$ and the extreme value 1 only when $f(R)$ is the complement of three points in the Riemann sphere.

7. **Local and global Teichmüller densities**

In the proof of Theorem 10 we proved that $\lambda_D(p) < \rho_D(p)$ for the case that the domain $D$ is the punctured disc. Here we will give another method that can be applied to any domain. It involves the introduction of another density on $D$ which we will call the *global Teichmüller density*. We have seen that the density $\lambda(p)$ is equal to the Teichmüller norm of a vector field that is equal to zero on the boundary of $D$ and equal to 1 at $p$. In Section 2 we have also seen that the holomorphic motion $h$ in [10] has an extension to the whole complex plane. Now consider all possible extensions $\tilde{h}$ of $h$ to the whole plane and put

$$V = \frac{d}{dt}|_{t=0}\tilde{h}(t, z)$$

and finally put

$$\sigma(p) = \rho(p) \inf\{||\partial V||_{\infty}\},$$

(33)

where the infimum is taken over all such tangent vectors $V$ obtained as tangent vectors to holomorphic motions $\tilde{h}(t, z)$ of this type. Since $\sigma$ is obtained from the infimum over a subset of the vector fields used in the definition of $\lambda$, we obviously have the following inequality:

$$\lambda(p) \leq \sigma(p).$$

The inequality

$$\sigma(p) \leq \rho(p)$$

is obvious because $\sigma$ is given by the infimum in (33) and by Schwarz’s inequality $||V||_T \leq 1$. We are calling $\sigma_D$ and $\lambda_D$ global and local because $\sigma_D$ depends on the class of vector fields that arise from global holomorphic motions defined for $|t| < 1$ that do not alter the conformal structure of $D$, and the tangent vectors that determine $\lambda_D$ do not necessarily arise from such holomorphic motions.

There is also a corresponding metric which we call $\Sigma_K(\tau, V)$ on the fiber $\mathbb{K}$. We can define it in the following way. Let $\tau$ be a point in $\mathbb{K}$ and consider the family $\mathcal{H}$ of all holomorphic maps $H$ from $\Delta$ into $\mathbb{K}$ with $H(0) = \tau, H'(0) = V$ and with $F \circ H = \Pi$ in Figure 1. Let

$$\Sigma_K(\tau, V) = \inf\{||\tilde{V}||_{\infty}\},$$

where the infimum is taken over all continuous extensions $\tilde{H}$ of the map $h$ in (10) with $\tilde{V} = \tilde{H}'(0)$. 

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From this definition it is obvious that for any point $\tau$ in $\mathcal{K}$ and any tangent vector $V$ at $\tau$,

$$Teich_{\mathcal{K}}(\tau, V) \leq \Sigma_{\mathcal{K}}(\tau, V) \leq Kob_{\mathcal{K}}(\tau, V).$$

We also have a proposition for $\sigma$ parallel to Propositions 1 and 2 for $\rho$ and $\lambda$.

**Proposition 3.** The $\Sigma$-density $\Sigma_{\mathcal{K}}([f], V)$ on $\mathcal{K}$ is related to the $\sigma$-density on $R_j$ by the formula:

$$\sigma(F([f])|dF(V)| = \Sigma_{\mathcal{K}}([f], V),$$

where $F : \mathcal{K} \to R_j$ is the evaluation map $F([f]) = f(p)$ in Figure 1.

An alternative technique for showing that $\lambda(p) < \rho(p)$ relies on two observations. First, just as in the proof of Theorem 10, a vector field $V$ equal to 1 at $p$ and 0 on the boundary for which $\lambda(p) = ||\partial V||_\infty$ is unique and its $\partial$-derivative must have Teichmüller form, that is, $\partial V$ must be equal to

$$k\lambda(p)\frac{|\varphi_p|}{\varphi_p},$$

for some integrable holomorphic quadratic differential $\varphi_p$ and some number $k$, with $|k| = 1$. Therefore, if we can find a vector field $V$ with $V(p) = 1$ such that $||\partial V||_\infty \leq \rho(p)$ and such that $\partial V$ is not of Teichmüller form, then necessarily

$$\lambda(p) < \rho(p).$$

The second observation is that to find such a vector field $V$, instead of finding a formula for $V(z)$ on the whole plane as in the proof of Theorem 10, it suffices to find a holomorphic motion $H(t, z)$ that has the following properties:

(a) $H(t, z)$ extends $h(t, z)$,
(b) $H(t, z)$ is holomorphic for $|t| < 1$,
(c) $H(t, z)$ is defined and holomorphic for $z$ in some nonempty open neighborhood $U$ contained in $D$, and
(d) for any $|t| < 1$, $H(t, z)$ is injective in the variable $z$ on its domain of definition.

If $H(t, z)$ has these properties, then by Slodkowski’s theorem it extends to a holomorphic motion $\tilde{H}(t, z)$ of $\mathbb{C}$ and putting

$$V = \frac{\partial}{\partial \overline{z}} \left( \frac{d}{dt} \bigg|_{t=0} \tilde{H}(t, z) \right)$$

by Schwarz’s lemma $||\partial V||_\infty \leq 1$. If we can do this, then since by part (c) $V$ is holomorphic in $U$, $V$ cannot be of Teichmüller form and therefore we conclude that $\lambda(p)$ is strictly less than $\rho(p)$.

To apply this approach to the punctured disc $D = \Delta' = \{z : 0 < |z| < 1\}$ we create the partial extension $H(t, z)$ by decomposing the covering map $\Pi : \Delta \to \Delta'$ in the following way. Let $T(w)$ be the Möbius transformation

$$z = T(w) = \frac{w - i}{w + i}$$

that transforms the upper half-plane onto the unit disc and let $f = \Pi \circ T$.

By normalizing $\Pi$ we may assume that $f$ is the universal covering from the upper half-plane given by the formula

$$f(w) = \exp(2\pi iw/c),$$
where $c > 0$ and the covering group is the cyclic group generated by the translation
$w \mapsto w + c$. Since $f(i) = p$ we have $c = 2\pi / \log(1/p)$. Let $W$ be a small open set
with $p \in W$. Let $g$ be a branch of the inverse of $f$ for which $g(p) = i$ and let $U$ be
the intersection of $W$ with $\{w : |w| \leq p\}$. The formula for $g(z)$ in $U$ is
\[
g(z) = \frac{c}{2\pi i} \log z.
\]

Now we define a partial extension of the motion $h(t, z)$ given in (10) by
\[
h(t, z) = \begin{cases} f(g(z) - i + T^{-1}(t)) & \text{for } z \in U, \\
z & \text{for } |z| \geq 1 \text{ and for } z = 0.
\end{cases}
\]
Then $h(0, w) = f(g(w)) = w$ for all $w \in U$ and the value of $h(t, z)$ remains the
same if we replace $g(z)$ by another inverse branch of $f$. Also $h$ is holomorphic in $t$
and
\[
h(t, p) = f(g(p) - i + T^{-1}(t)) = f(T^{-1}(t)) = \Pi(t).
\]
To check that $h$ is injective in $z$ assume
\[
f(g(z_1) - i + T^{-1}(t)) = f(g(z_2) - i + T^{-1}(t)).
\]
That implies there is an integer $n$ such that
\[
g(z_1) - i + T^{-1}(t) = g(z_2) - i + T^{-1}(t) + nc.
\]
Therefore $g(z_1) = g(z_2)$ for $z_1$ and $z_2$ in $U$ and since $g$ is injective on $U$, that implies
$z_1 = z_2$.

By Slodkowski’s extension theorem this $h$ can be extended to a motion of the whole
plane. Any such extension will have tangent vector $V$ whose $\overline{\partial}$-derivative
vanishes on $U$. Thus $V$ cannot be the extremal vector field that realizes the infimum
in the definition of $\lambda(p)$ and so $\lambda(p)$ is strictly less than $\rho(p)$.

In general we can prove the following theorem.

**Theorem 11.** If $D$ is a simply connected domain, then
\[
\lambda_D(p) = \sigma_D(p) = \rho_D(p)/2,
\]
if $D$ is conformal to $\mathbb{C} \setminus \{0, 1\}$, then
\[
\lambda_D(p) = \sigma_D(p) = \rho_D(p),
\]
and for any other hyperbolic connected plane domain $D$,
\[
(1/2)\rho_D(p) < \lambda_D(p) \leq \sigma_D(p) \leq \rho_D(p).
\]

**Proof.** Since these metrics are conformally invariant, to prove the first statement
we only have to show that $\sigma_\Delta(0) \leq \rho_\Delta(0)/2$, where $\Delta$ is the open unit disc. This
inequality is realized by considering the holomorphic motion defined for $|t| < 1$ by
\[
h(t, z) = \begin{cases} z + t(1 - |z|) & \text{for } |z| \leq 1, \\
z & \text{for } |z| \geq 1.
\end{cases}
\]
(36)

Note that $h(t, z)$ is continuous, normalized to fix three points for all $|t| < 1$ and
fixes points $z$ in the exterior of the disc. To see that $z \mapsto h(t, z)$ is injective, first
observe that for $|z| \leq 1$, $|h(t, z)| \leq |z| + |t|(1 - |z|) \leq |z| + (1 - |z|) = 1$. Secondly,
if $h(t, z_1) = h(t, z_2)$ and $|z_1|$ and $|z_2|$ are less than 1, then
\[
z_1 - z_2 = t(|z_1| - |z_2|).
\]
If \( z_1 \) is not equal to \( z_2 \) and \( |t| < 1 \), this implies
\[
|z_1 - z_2| < ||z_1| - |z_2||,
\]
which is impossible, and we conclude that \( z_1 = z_2 \).

To summarize, \( h(t,z) \) is a holomorphic motion that fixes the boundary of the unit disc and \( h(t,0) = t \).

Finally, for \( z \) in the interior of the unit circle
\[
V = \left. \frac{d}{dt} \right|_{t=0} h(t, z) = 1 - |z| = 1 - z^{1/2} \bar{z}^{1/2},
\]
and so
\[
\partial V = -(1/2) \frac{z^{1/2}}{\bar{z}^{1/2}}
\]
and the maximum value of \( |\partial V| \) in the unit disc is \( 1/2 \), which implies \( \sigma(p) = (1/2) \rho(p) \).

The second statement of the theorem follows since we have already shown that in the case that \( D = \mathbb{C} \setminus \{0,1\} \), \( \lambda(p) = \rho(p) \), and the third statement is a consequence of Theorem 9. \( \square \)

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REFERENCES


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