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FINITE RANK BRATTELI DIAGRAMS: STRUCTURE OF INVARIANT MEASURES

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Abstract. We consider Bratteli diagrams of finite rank (not necessarily simple) and ergodic invariant measures with respect to the cofinal equivalence relation on their path spaces. It is shown that every ergodic invariant measure (finite or "regular" infinite) is obtained by an extension from a simple subdiagram. We further investigate quantitative properties of these measures, which are mainly determined by the asymptotic behavior of products of incidence matrices. A number of sufficient conditions for unique ergodicity are obtained. One of these is a condition of exact finite rank, which parallels a similar notion in measurable dynamics. Several examples illustrate the broad range of possible behavior of finite rank diagrams and invariant measures on them. We then prove that the Vershik map on the path space of an exact finite rank diagram cannot be strongly mixing, independent of the ordering. On the other hand, for the so-called "consecutive" ordering, the Vershik map is not strongly mixing on all finite rank diagrams.

1. Introduction

Bratteli diagrams, which originally appeared in the theory of operator algebras, turned out to be a powerful method for the study of dynamical systems in ergodic theory and Cantor dynamics. Every minimal and even aperiodic homeomorphism of a Cantor set can be represented as a Vershik map acting on the path space of a Bratteli diagram ([HPS92], [Me06]). The main object of our study is the class of finite rank Bratteli diagrams, i.e., the diagrams whose vertex set at each level is uniformly bounded or, equivalently (after an easy reduction), with the same number of vertices at each level. It is worth pointing out that, in contrast to most papers on Cantor dynamics and Bratteli diagrams, our interest is focused on general, not necessarily simple, Bratteli diagrams. In this context, the present paper is a natural continuation of our previous work [BKMS10] devoted to the study of invariant measures and the structure of stationary non-simple Bratteli diagrams.

Our main goal is to describe the structure of invariant (with respect to Vershik maps or, more generally, the cofinal equivalence relation) Borel non-atomic measures on finite rank Bratteli diagrams. One of our motivations is the application

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to the classification theory of Cantor dynamical systems up to orbit equivalence [GPS95]. Namely, the knowledge of supports of invariant measures, the number of minimal components and ergodic measures, and the measure values on clopen sets are useful for distinguishing non-isomorphic or non-orbit equivalent homeomorphisms. Observe also that invariant measures are in one-to-one correspondence with states of dimension groups determined by the diagram. In [GH82], Goodearl and Handelman studied the problem of state extension for extensions of dimension groups. Some structural results on finite rank dimension groups are presented in [ES79] and [ES81].

The choice of finite rank systems is based on their "relative" combinatorial simplicity and, at the same time, on the intriguing properties and non-trivial dynamical behavior; see, for example, [CDHM03], [BDM10], [Du10, Chapter 6], and [BKMS10]. We observe that substitution dynamical systems, minimal interval exchange transformations, and generalized Morse sequences (Example 4.16) belong to this class of systems [DHS99], [BKM09], [GJ02]. Conversely, every Vershik map on a finite rank diagram is either an odometer or a subshift on a finite alphabet ([DM08], [BKM09]). However, to the best of our knowledge, it is still unknown what kind of subshifts can arise on this way.

Every Bratteli diagram is completely determined by a sequence of incidence matrices. The *Vershik map* is defined once we equip the diagram with an appropriate order. However, if one is interested in the properties of the corresponding dynamical system that do not depend on the order of points in its orbits (such as invariant measures, minimal components, etc.), then it suffices to study only the incidence matrices. We show that the structure of the set of invariant measures can be derived from the "growth rate" of entries of incidence matrix products. In our previous work [BKMS10] we applied a similar idea — studying the asymptotic growth of powers of a single matrix (geometric Perron-Frobenius theory) — to describe ergodic invariant measures for stationary diagrams. In the non-stationary case, we consider the non-homogeneous products (see [Sen81] and [Har02] for the essence of the theory) to study the dynamical properties. For example, a simple Bratteli diagram is uniquely ergodic if and only if the rows in backward products of incidence matrices become nearly proportional (see [Fis09] or Theorem 4.11 below). The property of near proportionality can be checked by methods of linear algebra (the technique of a Birkhoff contraction coefficient), which gives a purely algebraic criterion of unique ergodicity for Vershik maps.

Our main results and the paper organization are as follows.

In Section 2 we give the definition and necessary notation of Bratteli diagrams and Vershik maps. We explain the relation between invariant measures and products of incidence matrixes. We also show that every finite rank diagram can be transformed into a "canonical" block-triangular form which is convenient for describing the structure of invariant measures.

In Section 3 we establish general structural properties of invariant measures on finite rank Bratteli diagrams. We prove that any finite rank Bratteli diagram admits only a finite number of ergodic (both finite and "regular" infinite) measures and every ergodic measure is, in fact, an extension of a finite ergodic measure from a simple subdiagram (Theorem 3.3). This subdiagram has the property that the measures of towers specified by the vertices from the subdiagram are bounded away from zero. We note that this condition on a subdiagram corresponds to the

definition of exact finite rank in measurable dynamics [F97]. As a corollary, we prove that all diagrams of exact finite rank (Definition 3.5) are uniquely ergodic. This fact can be considered as a version of Boshernitzan's theorem [Bos92] proved in the context of symbolic dynamics. It is interesting to note that Boshernitzan's condition for symbolic systems has been used recently to prove uniform convergence in the multiplicative ergodic theorem, which has applications to the spectral properties of Schrödinger operators [DL06].

Section 4 collects results, which are mostly known but scattered in the literature, on unique ergodicity for Vershik maps on simple (finite rank) Bratteli diagrams. In particular, we prove a criterion for unique ergodicity, which appeared (in a slightly different form and with a different proof) in the work of A. Fisher [Fis09] and can be considered as "folklore". We also list several easily computable sufficient conditions of unique ergodicity. As an example, we show how these conditions can be reformulated in symbolic terms when applied to generalized Morse sequences. We note that algebraic conditions of (non-)unique ergodicity were also considered in the paper [FFT09] for diagrams with two and three vertices at each level. All necessary results concerning matrix products and, especially, the notion of a Birkhoff contraction coefficient are also presented in this section.

In Section 5 we study the asymptotic growth rate of tower heights and measures of tower bases. We show that for exact finite rank diagrams the measures of tower bases are (asymptotically) reciprocal to the tower heights. In the case when the tower heights have the same asymptotic behavior, this growth can be estimated by the norm of the product of incidence matrices. These results can be viewed as "adic" counterparts of some results in quantitative recurrence theory; see, for example, [Bos93] and [GK07]. We present an example of a diagram showing that the exact finite rank does not guarantee the same asymptotic growth for tower heights. On the other hand, if a diagram determined by matrices $\{F_n\}$ satisfies the "compactness" condition, $m_n/M_n \geq c > 0$, where m_n and M_n are the smallest and the largest entries of F_n respectively, then the diagram has exact finite rank and the tower heights have the same asymptotic growth.

In Section 6 we focus on non-simple diagrams and further study the construction of the extension of invariant measures from a simple subdiagram developed in Section 3. Our main question here is how to determine (in algebraic terms) when such an extension remains a finite measure. We provide several sufficient conditions for that and give illustrative examples. In the last part of the section we consider such an extension for the diagrams that have only a finite number of distinct incidence matrices (we call such diagrams "diagrams of finite complexity"). For such diagrams the question of finiteness of the extension can be reduced to the comparison of two numbers.

In Section 7, we apply the properties of invariant measures to prove that any Vershik map on a diagram of exact finite rank (for any order) is not strongly mixing. This result generalizes the corresponding facts on linearly recurrent systems [CDHM03] and substitution systems [DK78], [BKMS10]. We then show that the exactness requirement can be dropped if we have a "consecutive ordering" on the diagram. Note that Bratteli diagrams corresponding to minimal interval exchange transformations have consecutive orderings [GJ02]. The absence of mixing for interval exchanges was proved by A. Katok [K80], and our methods have some common features with those of [K80].

2. Bratteli diagrams: Basic constructions

In this section we collect the notation and basic definitions that are used throughout the paper. Since the notion of Bratteli diagrams and the related notion of Vershik transformation have been discussed in numerous recent papers, they might be considered as almost classical nowadays, so we avoid giving detailed definitions. An interested reader may consult the papers [HPS92], [GPS95], [DHS99], [Me06], [BKM09], [BKMS10], and the references therein for all details concerning Bratteli diagrams and Vershik maps. Here we only give some basic definitions in order to fix our notation.

2.1. **Bratteli diagrams.**

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the i-th level of the diagram B. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. We write $e(v, v')$ to denote a path e such that $s(e) = v$ and $r(e) = v'$. For a Bratteli diagram B, we denote by X_B the set of infinite paths starting at the vertex v_0. We endow X_B with the topology generated by cylinder sets $U(e_1, \dots, e_n) = \{x \in X_B : x_i = e_i,\ i = 1, \dots, n\}$, where $(e_1, \dots, e_n)$ is a finite path from B. Then X_B is a 0-dimensional compact metric space with respect to this topology.

Given a Bratteli diagram $B = (V, E)$, the incidence matrix $F_n = (f^{(n)}_{v,w}),\ n \geq 1$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f^{(n)}_{v,w}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.,

$$f^{(n)}_{v,w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

(Here and thereafter $|A|$ denotes the cardinality of the set A.) We notice that F_0 is a vector. We usually assume that $F_0 = (1, ..., 1)^T$.

Observe that every vertex $v \in V$ is connected to v_0 by a finite path and the set $E(v_0, v)$ of all such paths is finite. Set $h^{(n)}_v = |E(v_0, v)|$, where $v \in V_n$. Then

$$h^{(n+1)}_v = \sum_{w\in V_n} f^{(n)}_{v,w} h^{(n)}_w \tag{2.1}$$

or

$$h^{(n+1)} = F_n h^{(n)}, \tag{2.2}$$

where $h^{(n)} = (h^{(n)}_w)_{w\in V_n}$.

Together with the sequence of incidence matrices $\{F_n\}$ we will use the sequence of matrices $\{Q_n\}$ where the entries $q^{(n)}_{v,w}$ of Q_n are defined by the formula

$$q^{(n)}_{v,w} = f^{(n)}_{v,w} \frac{h^{(n)}_w}{h^{(n+1)}_v}, \quad n \geq 1. \tag{2.3}$$

It follows from (2.1) that every Q_n is a *stochastic* matrix.

It is not hard to show that for a given sequence of non-negative rational stochastic $d \times d$ matrices $\{Q_n\}$ there exists a Bratteli diagram B with incidence matrices $\{F_n\}$ whose entries satisfy (2.3). The sequence $\{F_n\}$ is not uniquely determined: matrices F_n and $pF_n,\ p \in \mathbb{N}$, correspond to the same stochastic matrix Q_n.

For $w \in V_n$, the set $E(v_0, w)$ defines the clopen subset

$$X_w^{(n)} = \{x = (x_i) \in X_B : r(x_n) = w\}.$$

The sets $\{X_w^{(n)} : w \in V_n\}$ form a clopen partition of $X_B,\ n \geq 1$. Analogously, each finite path $\overline{e} = (e_1, \ldots, e_n) \in E(v_0, w)$ determines the clopen subset

$$X_w^{(n)}(\overline{e}) = \{x = (x_i) \in X_B : x_i = e_i,\ i = 1, \ldots, n\}.$$

These sets form a clopen partition of $X_w^{(n)}$. We will also use the notation $[\overline{e}]$ for the clopen set $X_w^{(n)}(\overline{e})$ when it does not lead to confusion. The base of the tower $X_w^{(n)}$ is denoted by $B_n(w)$. (In fact, this means that an order is specified on $E(v_0, w)$. But since, in most cases, order is inessential for us, the subset $B_n(w)$ may be represented by any finite path from $E(v_0, w)$.)

Definition 2.2. A Bratteli diagram $B = (V, E)$ is called *simple* if for any level n there is $m > n$ such that each pair of vertices $(v, w) \in (V_n, V_m)$ is connected by a finite path.

Definition 2.3. For a Bratteli diagram B, the *tail (cofinal) equivalence* relation $\mathcal{E}$ on the path space X_B is defined as $x\mathcal{E}y$ if $x_n = y_n$ for all n sufficiently large.

Remark 2.4. Given a dynamical system (X, T), a Bratteli diagram is constructed by a sequence of Kakutani-Rokhlin partitions generated by (X, T) (see [HPS92] and [Me06]). The n-th level of the diagram corresponds to the n-th Kakutani-Rokhlin partition, and the number $h_w^{(n)}$ is the height of the T-tower labeled by the symbol w from that partition.

Throughout the paper we will use the telescoping procedure for a Bratteli diagram. Roughly speaking, in order to telescope a Bratteli diagram, one takes a subsequence of levels $\{n_k\}$ and considers the set of all finite paths between the consecutive levels $\{n_k\}$ and $\{n_{k+1}\}$ as new edges. A rigorous definition of telescoping can be found in many papers on Bratteli diagrams, for example, in [GPS95].

Telescoping, together with the obvious level-preserving graph isomorphism, generate an equivalence relation on the Bratteli diagrams. Two diagrams in the same class are called *isomorphic*.

2.2. **Finite rank Bratteli diagrams.**

Definition 2.5. A Bratteli diagram that has a uniformly bounded number of vertices at each level is called a diagram of *finite rank*.

The next theorem shows that each finite rank Bratteli diagram can be isomorphically transformed into a canonical block-triangular form, which gives a natural decomposition of X_B into a finite number of tail-invariant subsets.

Theorem 2.6. *Any Bratteli diagram of finite rank is isomorphic to a diagram whose incidence matrices $\{F_n\}_{n\geq 1}$ are as follows:*

$$F_n = \begin{pmatrix} F_1^{(n)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & F_2^{(n)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_s^{(n)} & 0 & \cdots & 0 \\ X_{s+1,1}^{(n)} & X_{s+1,2}^{(n)} & \cdots & X_{s+1,s}^{(n)} & F_{s+1}^{(n)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ X_{m,1}^{(n)} & X_{m,2}^{(n)} & \cdots & X_{m,s}^{(n)} & X_{m,s+1}^{(n)} & \cdots & F_m^{(n)} \end{pmatrix}. \tag{2.4}$$

For every $n \geq 1$, the matrices $F_i^{(n)}$, $i = 1, ..., s$, have strictly positive entries and the matrices $F_i^{(n)}$, $i = s+1, ..., m$, have either all strictly positive or all zero entries. For every fixed $j = s+1, ..., m$, there is at least one non-zero matrix $X_{j,k}^{(n)}$, $k = 1, ..., j-1$.

Proof. Let B be a finite rank Bratteli diagram. By telescoping, we obtain that $|V_n| = d$ for all $n \geq 1$. It follows from Proposition 4.6 of [BKM09] that B has finitely many minimal components with respect to the tail equivalence relation, say they are $Z_1, ..., Z_s$. Denote

$$W_n(i) = \{r(x_n) \in V_n : x = (x_n) \in Z_i\}, \quad i = 1, ..., s.$$

Claim. For any $i_1 \neq i_2$, there exists N such that for all $n \geq N$,

$$W_n(i_1) \cap W_n(i_2) = \emptyset, \quad i_1, i_2 = 1, ..., s.$$

To prove the Claim, we fix Z_i and consider the subdiagram B_i of B which is formed by the vertex set $W(i) = \bigcup_{n\geq 1} W_n(i)$ and the edges induced by all paths from Z_i. Then B_i is a simple Bratteli diagram.

Now suppose that the contrary holds, i.e., there exist distinct i_1 and i_2 and a sequence $\{n_k\}$ such that $W_{n_k}(i_1) \cap W_{n_k}(i_2) \neq \emptyset$. Let $\{v_{n_k}\}$ be a sequence of vertices which is chosen from $W_{n_k}(i_1) \cap W_{n_k}(i_2)$. Without loss of generality, we may assume that $n_{k+1} - n_k > 2$. By simplicity of subdiagrams B_{i_1} and B_{i_2}, there are finite paths $\overline{e}_k(1)$ and $\overline{e}_k(2)$ connecting the vertices v_{n_k} and $v_{n_{k+1}}$ and such that $\overline{e}_k(1)$ and $\overline{e}_k(2)$ belong to B_{i_1} and B_{i_2}, respectively. Therefore, there exist infinite paths $x \in Z_{i_1}$ (obtained as a concatenation of $\overline{e}_1(k)$) and $y \in Z_{i_2}$ (obtained as a concatenation of $\overline{e}_2(k)$) which go through the vertices v_{n_k} for every $k \geq 1$. Thus, for every $k \geq 1$, there exists a path $x_k \in Z_{i_1}$ cofinal to x which coincides with the first n_k edges of y. This implies that $\mathrm{dist}(x_k, y) \to 0$ as $k \to \infty$. Hence $\mathrm{dist}(Z_{i_1}, Z_{i_2}) = 0$, which is impossible. To complete the proof of the claim, we use a standard argument based on finiteness of the set of minimal components.

By telescoping the diagram B, we may assume that $W_n(i_1) \cap W_n(i_2) = \emptyset$ $(i_1 \neq i_2)$ for all $n \geq 1$. One can also regroup the vertices at each level so that the sets $W_n(1), ..., W_n(s)$ are enumerated from left to right.

Choose a positive constant δ so that $\mathrm{dist}(Z_i, Z_j) \geq \delta, i \neq j$. Again using the method of telescoping, we can easily reduce the general case to that when no edges between vertices from different minimal components exist. Hence we have constructed the collection of simple subdiagrams B_i with incidence matrices

$\{F_i^{(n)}\}$, $i = 1, \dots, s$. Further telescoping the diagram we may ensure that each matrix $F_i^{(n)}$ has strictly positive entries.

Next, we consider the subdiagram B' of B whose vertex set V' is $\bigcup_n V'_n$, where $V'_n = V_n \setminus \bigcup_{i=1}^s W_n(i)$, and the edge set E' consists of the edges that connect vertices from V' only. In other words, we temporarily ignore the set of edges that link vertices from B' and those from B_i, $i = 1, \dots, s$. Then B' is a finite rank Bratteli diagram whose rank is strictly less than the rank of B. We can apply the procedure described above to find all minimal components of B'. In a finite number of such steps, we obtain all simple subdiagrams of B that correspond to non-zero matrices from the set $\{F_j^{(n)}\}$, $j = s+1, \dots, m$. It may happen that there will be some vertices at infinitely many levels that do not belong to the constructed simple subdiagrams. This means that after appropriate telescoping the corresponding incidence matrices $F_j^{(n)}$ must be either zero or strictly positive.

To finish the proof, we return to B and restore all edges that have been temporarily removed. They will now connect some vertices from different subdiagrams $B'_j, j = s+1, \dots, m$, and also connect them with some vertices from $B_i, i = 1, \dots, s$. This set of edges determines the matrices $X_{i,j}^{(n)}$. Certainly, some of these matrices may be zero. But if one fixes a row $i \in \{s+1, \dots, m\}$, then at least one matrix from the collection $\{X_{i,j}^{(n)}\}$ is non-zero. □

2.3. **Invariant measures.**

Definition 2.7. Let B be a Bratteli diagram. By a *finite measure* on B we always mean a Borel non-atomic (not necessarily probability) measure on X_B. For an *infinite σ-finite measure* μ on X_B, we assume that μ takes finite (non-zero) values on some clopen sets.

Definition 2.8. Given a Bratteli diagram $B = (V, E)$, a measure μ on X_B is called *invariant* if $\mu([\overline{e}]) = \mu([\overline{e}'])$ for any two finite paths $\overline{e}$ and $\overline{e}'$ with the same range. In other words, $\mu(X_w^{(n)}(\overline{e})) = \mu(X_w^{(n)}(\overline{e}'))$ for any $n \geq 1$ and $w \in V_n$.

Remark 2.9. The measure μ is invariant on B if and only if it is invariant with respect to the cofinal equivalence relation $\mathcal{E}$.

Definition 2.10. An invariant measure μ is *ergodic* for the diagram B (or *B-ergodic*) if it is ergodic with respect to the cofinal equivalence relation $\mathcal{E}$.

If a Bratteli diagram B admits a unique invariant probability measure, then B is called *uniquely ergodic*.

The next theorem which was proved in [BKMS10] shows that the simplex of invariant measures is completely determined by the sequence of incidence matrices of the diagram. To state the theorem, we will need to introduce the following notation.

For $x = (x_1, \dots, x_N)^T \in \mathbb{R}^N$, we will write $x \geq 0$ if $x_i \geq 0$ for all i, and consider the positive cone $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x \geq 0\}$. Let

$$C_k^{(n)} := F_k^T \cdots F_n^T \left(\mathbb{R}_+^{|V_{n+1}|}\right), \quad 1 \leq k \leq n.$$

Clearly, $\mathbb{R}_+^{|V_k|} \supset C_k^{(n)} \supset C_k^{(n+1)}$ for all $n \geq 1$. Let

$$C_k^\infty = \bigcap_{n \geq k} C_k^{(n)}, \ k \geq 1.$$

Observe that C_k^∞ is a closed non-empty convex subcone of $\mathbb{R}_+^{|V_k|}$. It also follows from these definitions that

$$F_k^T C_{k+1}^\infty = C_k^\infty. \tag{2.5}$$

Theorem 2.11 ([BKMS10, Theorem 2.9])**.** *Let $B = (V, E)$ be a Bratteli diagram such that the tail equivalence relation $\mathcal{E}$ on X_B is aperiodic. If μ is an invariant measure with respect to the tail equivalence relation $\mathcal{E}$, then the vectors $p^{(n)} = (\mu(X_w^{(n)}(\overline{e})))_{w\in V_n}$, $\overline{e} \in E(v_0, w)$, satisfy the following conditions for $n \geq 1$:*

(i) $p^{(n)} \in C_n^\infty$,

(ii) $F_n^T p^{(n+1)} = p^{(n)}$.

Conversely, if a sequence of vectors $\{p^{(n)}\}$ from $\mathbb{R}_+^{|V_n|}$ satisfies condition (ii), then there exists a non-atomic finite Borel $\mathcal{E}$-invariant measure μ on X_B with $p_w^{(n)} = \mu(X_w^{(n)}(\overline{e}))$ for all $n \geq 1$ and $w \in V_n$.

The $\mathcal{E}$-invariant measure μ is a probability measure if and only if

(iii) $\sum_{w\in V_n} h_w^{(n)} p_w^{(n)} = 1$ for $n = 1$,

in which case this equality holds for all $n \geq 1$.

Remark 2.12. It was also proved in [BKMS10, Theorem 3.8] that for stationary Bratteli diagrams the sequence of vectors $\{p^{(n)}\}$ which determines an invariant measure can be completely restored by the initial distribution vector $p^{(1)}$. One can construct an example when this result fails for general diagrams. However, for diagrams of finite rank we can still telescope the diagram in such a way that any two different invariant measures μ and ν can already be distinguished on the first level, i.e., the corresponding vectors $p^{(1)}$ are distinct.

Indeed, it follows from the proof of Proposition 2.13 (see below) that the number of extreme rays of C_k^∞ stabilizes to the number of ergodic measures as $k \to \infty$. By telescoping we may assume that this already holds for every k. By (2.5), we see that the linear map F_k^T sends extreme rays onto extreme rays. Thus, F_k^T is a bijection of the cones C_{k+1}^∞ and C_k^∞ for all k, proving the claim.

In the next result we apply Theorem 2.11 to a finite rank Bratteli diagram to show that any such diagram has a finite number of ergodic measures. This result may be considered "folklore": it was stated in [BDM10] for simple diagrams without a proof, and was certainly known to A. Vershik much earlier [D. Handelman, personal communication]. Since we are not aware of a reference containing the proof, we sketch the proof for the reader's convenience.

Proposition 2.13. *Let B be a Bratteli diagram of finite rank. Suppose that the number of vertices at each level is bounded by d. Then B has no more than d invariant ergodic probability measures.*

Proof. We will use Theorem 2.1 from [Pul71]:

Let $\{C_n\}$ be a sequence of finitely generated cones such that $C_n \supset C_{n+1}$ for all $n \geq 1$. If for all sufficiently large n the cone C_n is finitely generated by at most d rays, then $C = \bigcap_n C_n$ is also a finitely generated cone by at most d rays (the number of generating rays is called the size of the cone).

We can apply Pullman's theorem to the sequence of cones $\{C_k^{(n)}\}_n$ for all $k \geq 1$ and conclude that the cones C_k^∞ are finitely generated of size not greater than d. It follows from (2.5) that $\mathrm{size}(C_k^\infty) \leq \mathrm{size}(C_{k+1}^\infty)$. Hence the sizes must stabilize: $\mathrm{size}(C_k^\infty) = m$ for all $k \geq N_0$. Then B has m ergodic invariant probability measures.

In fact, it easily follows from Theorem 2.11 that there is a one-to-one correspondence between $\mathcal{E}$-invariant measures and $C^{\infty}_{N_0}$ such that the extreme rays correspond to the ergodic measures. □

Remark 2.14. We note that minimal dynamical systems have no infinite invariant measures that take a finite value on a clopen set. For an aperiodic dynamical system (and, in particular, for finite rank non-simple diagrams) such measures can occur; see [BKMS10].

2.4. **Vershik map.** By definition, a Bratteli diagram $B = (V, E)$ is called *ordered* if every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$, is linearly ordered; see [HPS92]. Thus, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographical order. We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$. Notice that, for $v \in V_i,\ i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}$ and $X_{\min}$ the sets of all maximal and minimal infinite paths from X_B, respectively. It is not hard to see that $X_{\max}$ and $X_{\min}$ are finite sets for finite rank Bratteli diagrams (Proposition 6.2 in [BKM09]). Let X^*_B be the $\mathcal{E}$-invariant set of all infinite paths which are cofinal neither to a maximal path nor to a minimal one. Then the set $X_B \setminus X^*_B$ is at most countable for any finite rank diagram.

Definition 2.15. Define a map $T : X^*_B \to X^*_B$ by setting

$$T(x_1, x_2, \ldots) = (x^0_1, \ldots, x^0_{k-1}, \overline{x_k}, x_{k+1}, x_{k+2}, \ldots),$$

where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x^0_1, \ldots, x^0_{k-1})$ is the minimal path in $E(v_0, s(\overline{x_k}))$. In this paper, we will refer to the map T as the *Vershik map* on the ordered Bratteli diagram B.

Remark 2.16. (i) It is still unknown under what conditions the Vershik map can be extended to a homeomorphism of X_B for non-simple Bratteli diagrams. We note only that it is not always possible [Me06].

(ii) Since all orbits of T coincide with classes of $\mathcal{E}$, perhaps except for at most a countable collection of orbits, any $\mathcal{E}$-invariant measure is also T-invariant and vice versa.

Throughout the paper we will always assume that each Bratteli diagram of finite rank meets the following conditions:

(i) The path space X_B has no isolated points, i.e., X_B is a Cantor set.
(ii) The diagram has the same number of vertices at each level, say d. So, each incidence matrix is a $d \times d$ matrix.
(iii) The diagram has simple edges between the top vertex v_0 and the vertices of the first level, i.e., the vector F_0 consists of 1's. (This assumption is not restrictive because any diagram can be isomorphically transformed into a diagram with simple edges on the first level, as in [DHS99, Lemma 9].)
(iv) The cofinal equivalence relation is aperiodic, i.e., it has no finite classes. This assumption is needed to exclude atomic invariant measures from consideration.

3. Structure of invariant measures

In this section we describe the structure of the set of invariant measures. A key observation made here is that ergodic measures occur as extensions of measures

from simple pairwise disjoint subdiagrams (Theorem 3.3). We begin our study by describing the process of measure extension from a subdiagram, which is central for the paper.

Consider a Bratteli diagram $B = (V, E)$ where the vertex set $V = \bigcup_n V_n$ and the edge set $E = \bigcup E_n$ are as in Definition 2.1.

Definition 3.1. By a *subdiagram* of B, we mean a Bratteli diagram $S = (W, R)$ constructed by taking some vertices at each level n of the diagram B and then considering *all* the edges of B that connect these vertices.

Remark 3.2. We notice that our definition of a subdiagram is not, in general, invariant under the telescoping, that is, the telescoping can add additional edges not originally present.

Let $S = (W, R)$ be a subdiagram of B. Consider the set $Y = Y_S$ of all infinite paths of the subdiagram S. Then the set Y is naturally seen as a subset of X_B. Let μ be a finite invariant (with respect to the tail equivalence relation $\mathcal{E}$) measure on Y. Let X_S be the saturation of Y with respect to $\mathcal{E}$. In other words, a path $x \in X_B$ belongs to X_S if it is $\mathcal{E}$-equivalent to a path $y \in Y$. Then X_S is $\mathcal{E}$-invariant and Y is a complete section for $\mathcal{E}$ on X_S. By the *extension of measure μ to X_S* we mean the $\mathcal{E}$-invariant measure $\widehat{\mu}$ on X_S (finite or infinite) such that $\widehat{\mu}$ induced on Y coincides with μ.

Although the procedure of the measure extension with respect to an equivalence relation is well known, the geometric nature of the tail equivalence relation makes this construction more illuminating.

Specifically, take a finite path $\overline{e} \in E_S(v_0, v)$ from the top vertex to a vertex v of level n that belongs to the subdiagram S. Let $[\overline{e}]_S$ be the set of all paths in Y that coincide with $\overline{e}$ in the first n edges. Then $[\overline{e}]_S$ is a cylinder subset of Y. For any finite path $\overline{e}'$ from the diagram B with the same range v, we set $\widehat{\mu}([\overline{e}']) = \mu([\overline{e}]_S)$. In such a way, the measure $\widehat{\mu}$ is extended to the σ-algebra of Borel subsets of X_B generated by all clopen sets of the form $[\overline{x}]$ where a finite path $\overline{x}$ has the range in a vertex from S. Using the properties of tail equivalence relations, one can show that such an extension is well defined. Furthermore, the *support* of $\widehat{\mu}$ is, by definition, the set X_S of all paths which are cofinal to paths from Y. We observe that $\widehat{\mu}(X_S)$ may be either finite or infinite. In fact, one can use the following formula for computing $\widehat{\mu}(X_S)$. Let $W_n = W \cap V_n$ and set $X_S(n) = \{x = (x_i) \in X_B : r(x_i) \in W_i,\ \forall i \geq n\}$. Clearly, $X_S(n) \subset X_S(n+1)$. Then we have the following formula for the measure $\widehat{\mu}(X_S)$:

$$\widehat{\mu}(X_S) = \lim_{n\to\infty} \widehat{\mu}(X_S(n)) = \lim_{n\to\infty} \sum_{w \in W_n} \widehat{h}_w^{(n)} \mu([e_S(v_0, w)]), \tag{3.1}$$

where $\widehat{h}_w^{(n)}$ is the height of the tower $X_w^{(n)}$ in the diagram B and $e_S(v_0, w)$ is a finite path from v_0 to w that belongs to S.

From now on, we may assume that a finite rank Bratteli diagram is reduced by Theorem 2.6 to the form (2.4) when it is convenient for us. Denote by Λ the subset of $\{1, \ldots, m\}$ such that the corresponding incidence matrices are non-zero in (2.4). For $\alpha \in \Lambda$, denote by B_α the subdiagram of B whose incidence matrices are $\{F_\alpha^{(n)}\}$. The fact that the matrix $F_\alpha^{(n)}$ is strictly positive implies that the subdiagram B_α is simple.

Let Y_α be the path space of the Bratteli diagram B_α, $\alpha \in \Lambda$. Denote by $X_\alpha = \mathcal{E}(Y_\alpha)$ the saturation of Y_α with respect to the tail equivalence relation. It is clear that $\{X_\alpha : \alpha \in \Lambda\}$ is a partition of X_B into Borel invariant subsets.

In the next theorem, we describe the structure of the supports of ergodic invariant measures. The support of each ergodic measure turns out to be the set of all paths that stabilize in some subdiagram, which geometrically can be seen as "vertical". Furthermore, these subdiagrams are pairwise disjoint for different ergodic measures. Everywhere below the term "measure" stands for an $\mathcal{E}$-invariant measure. Recall that by an infinite measure we mean any σ-finite non-atomic measure which is finite (non-zero) on some clopen set.

Theorem 3.3. *Let B be a Bratteli diagram of finite rank.*

(1) Each finite ergodic measure on Y_α extends to an ergodic measure on X_α. The extension can be a finite or an infinite measure.

(2) Each ergodic measure (both finite and infinite) on X_B is obtained as an extension of a finite ergodic measure from some Y_α.

(3) The number of finite and infinite (up to scalar multiple) ergodic measures is not greater than d.

(4) We may telescope the diagram B in such a way that for every probability ergodic measure μ there exists a subset W_μ of vertices from $\{1, \dots, d\}$ such that the support of μ consists of all infinite paths that eventually go along the vertices of W_μ only. Furthermore,

(4-i) $W_\mu \cap W_\nu = \emptyset$ for different ergodic measures μ and ν;

(4-ii) given a probability ergodic measure μ, there exists a constant $\delta > 0$ such that for any $v \in W_\mu$ and any level n,

$$\mu(X_v^{(n)}) \geq \delta,$$

where $X_v^{(n)}$ is the set of all paths that go through the vertex v at level n;

(4-iii) the subdiagram generated by W_μ is simple and uniquely ergodic. The only ergodic measure on the path space of the subdiagram is the restriction of measure μ.

(5) If a probability ergodic measure μ is the extension of a measure from the vertical subdiagram determined by a proper subset $W \subset \{1, \dots, d\}$, then

$$\lim_{n\to\infty} \mu(X_v^{(n)}) = 0 \text{ for all } v \notin W.$$

Remark 3.4. The recent paper [BDM10] contains a notion of a "clean diagram" for simple Bratteli-Vershik diagrams of finite rank, which has some similarities with our description of the measure supports.

Proof. (I) Statements (1), (2), and (3) are similar to Lemma 4.2 from [BKMS10], so we give a sketch of the proof only.

Let μ be a finite or infinite ergodic measure on the path-space X_B. Then there exists α such that μ is supported on X_α. As Y_α is a complete section of X_α, the restriction of μ to Y_α determines an ergodic measure μ_0 on Y_α. Thus, to define a measure on X_α we need to take any finite ergodic measure on Y_α (due to Proposition 2.13 we have finitely many of them up to a normalization) and extend it by invariance to X_α. This process was described at the beginning of this section; see equation (3.1). We note that if the extended measure μ is infinite, but finite on a clopen set, then the minimality of the tail equivalence relation on Y_α implies that the restriction μ_0 is a finite measure. This proves (1), (2), and (3).

(II) To prove (4), we enumerate probability ergodic measures on X_B as $\mu_1, \dots, \mu_p$. In view of (I), we may assume, without loss of generality, that each measure μ_i is restricted to a simple subdiagram B_{α_i}. We start with the measure μ_1. Then

$$\sum_v \limsup_{n\to\infty} \mu_1(X_v^{(n)}) \geq \limsup_{n\to\infty} \sum_v \mu_1(X_v^{(n)}) = 1.$$

Therefore, there exists a vertex v_1 with

$$\limsup_{n\to\infty} \mu_1(X_{v_1}^{(n)}) = \delta_1 > 0.$$

This means that we can telescope the diagram so that $\mu_1(X_{v_1}^{(n)}) > \delta_1/2$ for all levels n. Considering the set of vertices $\{1, \dots, d\} \setminus \{v_1\}$, choose a vertex v_2 (if possible) such that for some positive number δ_2,

$$\limsup_{n\to\infty} \mu_1(X_{v_2}^{(n)}) = \delta_2 > 0.$$

Telescope the diagram so that $\mu_1(X_{v_2}^{(n)}) > \delta_2/2$ for all levels n. Repeating this procedure finitely many times, we will end up with a set of vertices W_1 such that

$$\mu_1(X_v^{(n)}) > \delta > 0$$

for all levels n and any vertex $v \in W_1$ (here $\delta = \frac{1}{2}\min_i \delta_i$) and such that

$$\limsup_n \mu_1(X_v^{(n)}) = 0 \text{ for all } v \notin W_1.$$

We will further telescope the diagram to ensure that

$$\sum_{k=n}^{\infty} \mu_1(\bigsqcup_{v\notin W_1} X_v^{(k)}) < \frac{1}{n} \text{ for any } n.$$

Consider the set S_1 of all paths that eventually go only through the vertices from W_1. We claim that the measure μ_1 is supported on S_1. Indeed, consider the set

$$R_1 = X_B \setminus S_1 = \bigcap_{n\geq 1} \bigcup_{k\geq n} \bigsqcup_{v\notin W_1} X_v^{(k)}.$$

Then

$$\mu_1(R_1) = \lim_{n\to\infty} \mu_1(\bigcup_{k\geq n} \bigsqcup_{v\notin W_1} X_v^{(k)}) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu_1(\bigsqcup_{v\notin W_1} X_v^{(k)}) = 0,$$

which proves the claim.

As soon as W_1 is constructed, we may repeat the arguments above to find the corresponding sets $W_2, \dots, W_p$ for the rest of the ergodic measures.

We claim that $W_i \cap W_j = \emptyset$ for all $i \neq j$. Assume the converse, i.e., that there are two probability ergodic measures μ and ν and a vertex w such that

$$\mu(X_w^{(n)}) \geq \gamma \text{ and } \nu(X_w^{(n)}) \geq \gamma$$

for all n, where $\gamma = \frac{1}{2}\min(\delta(\mu), \delta(\nu)) > 0$.

Set $C = \bigcap_{k\geq 1} \bigcup_{n\geq k} X_w^{(n)}$. It follows that $\mu(C) \geq \gamma$ and $\nu(C) \geq \gamma$. Note that C is exactly the set of all paths that visit the vertex w infinitely many times, which is an $\mathcal{E}$-invariant set. By ergodicity of μ and ν, we see that $\mu(C) = \nu(C) = 1$.

Since μ and ν are mutually singular as distinct ergodic measures, the Radon-Nikodym derivative satisfies

$$\frac{d\mu}{d(\mu+\nu)}(x) \equiv 0$$

for ν-a.e. $x \in X_B$.

For every $x \in X_B$, let $v_n(x)$ denote the vertex of level n the path x goes through. Set $[x]_n = \{y \in X_B : y_j = x_j,\ j = 1, ..., n\}$. We observe that $h^{(n)}_{v_n(x)}\mu([x]_n) = \mu(X^{(n)}_{v_n(x)})$, where $h^{(n)}_{v_n(x)}$ is the number of paths from the vertex $v_n(x)$ to the top vertex.

As $\nu(C) = 1$, we have that for ν-a.e. $x \in C$,

$$\begin{aligned} 0 &= \frac{d\mu}{d(\mu+\nu)}(x) \\ &= \lim_{n\to\infty} \frac{\mu([x]_n)}{(\mu+\nu)([x]_n)} \\ &= \lim_{n\to\infty} \frac{h^{(n)}_{v_n(x)}\mu([x]_n)}{h^{(n)}_{v_n(x)}(\mu+\nu)([x]_n)} \\ &= \lim_{\{n:v_n(x)=w\}} \frac{\mu(X^{(n)}_w)}{\mu(X^{(n)}_w)+\nu(X^{(n)}_w)} \\ &\geq \frac{\gamma}{2} > 0, \end{aligned}$$

which is a contradiction. Thus, statements (4-i) and (4-ii) are proved.

(III) For each ergodic measure $\mu \in \{\mu_1, \ldots, \mu_p\}$, denote by B_μ the subdiagram generated by the vertices W_μ. We note that the diagram B_μ is a subdiagram of the corresponding simple diagram B_{α_i}. Thus, we can telescope the original diagram in such a way that there is at least one edge between any pair of vertices of W_μ at consecutive levels. This will ensure that B_μ is a simple subdiagram.

Now assume that the diagram B_μ admits another probability ergodic measure, say ν. Denote by Y_μ the path space of B_μ. Then by the arguments above the measure ν and the restriction of μ to the path-space of B_μ are extended from (proper) disjoint subdiagrams of B_μ. Hence, there is a vertex $w \in W_\mu$ such that $\limsup_n \mu(X^{(n)}_w) = 0$, which is a contradiction.

(IV) Assume now that statement (5) does not hold. Then there is $v_0 \notin W$ such that $\limsup_n \mu(X^{(n)}_{v_0}) > 0$. Then the set $C = \bigcap_{k\geq 1}\bigcup_{n\geq k} X^{(n)}_{v_0}$ has μ-measure one. Since the set C consists of paths that visit the vertex v_0 infinitely many times, this contradicts the construction of the extension. □

Motivated by the definition of exact rank measure-preserving transformations [F97], we give the following definition.

Definition 3.5. We say that a Bratteli diagram of a finite rank is of *exact finite rank* if there is a finite invariant measure μ and a constant $\delta > 0$ such that after telescoping, $\mu(X^{(n)}_v) \geq \delta$ for all levels n and vertices v.

We would like to emphasize that we apply the definition of exact finite rank only to Bratteli diagrams and not to the dynamical systems that might be modeled by diagrams of exact finite rank. We note that there are Cantor minimal systems that

have different Bratteli-Vershik realizations: one with the exact finite rank property and one without it; see Remark 6.9.

As a corollary of Theorem 3.3, we immediately obtain the following version of Boshernitzan's theorem [Bos92].

Corollary 3.6. *All Bratteli diagrams of exact finite rank are uniquely ergodic.*

Interestingly, the condition of Boshernitzan for symbolic systems has been used to prove uniform convergence in the multiplicative ergodic theorem, with applications to the spectral properties of the corresponding Schrödinger operators [DL06].

4. Unique ergodicity of simple diagrams

In this section, we will use the machinery of a Birkhoff contraction coefficient to answer the question when a simple Bratteli diagram is uniquely ergodic. Most of the results in this section are not new, but they are scattered in the literature, often with terminology different from ours. We provide some (short) proofs for the reader's convenience.

The Birkhoff contraction coefficient method is widely used in matrix theory and the theory of Markov chains as the way to understand asymptotic behavior of non-negative matrix products. The Birkhoff coefficient shows how matrix products "squeeze" the orthant of positive vectors. The first results in the area appeared in Birkhoff's fundamental works [Bir57] and [Bir67]. We refer the reader to the books [Har02] and [Sen81] where a detailed exposition of the material as well as an extensive reference list are presented. For the reader's convenience we include some results from [Har02].

Definition 4.1. For two positive vectors $x, y \in \mathbb{R}^d$ define the *projective metric (Hilbert metric)* as

$$D(x,y) = \ln \max_{i,j} \frac{x_i y_j}{x_j y_i} = \ln \frac{\max_i \frac{x_i}{y_i}}{\min_j \frac{x_j}{y_j}},$$

where (x_i) and (y_i) are entries of the vectors x and y.

Denote by Δ the set of all positive probability vectors of $\mathbb{R}^d$. Note that (Δ, D) is a complete metric space (Theorem 2.5 in [Har02]).

The next proposition says that all non-negative matrices act as (weak) contractions on the orthant of positive vectors. For the proof, see Lemma 2.1 in [Har02].

Proposition 4.2. *Let A be a non-negative $d \times d$ matrix. Then for any positive vectors $x, y \in \mathbb{R}^d$ we have $D(Ax, Ay) \le D(x, y)$.*

Definition 4.3. For a non-negative matrix A, we set

$$\tau(A) = \sup_{x,y>0} \frac{D(Ax, Ay)}{D(x,y)}.$$

The coefficient $\tau(A)$ is called the *Birkhoff contraction coefficient.*

It follows from the definition that $D(Ax, Ay) \le \tau(A) D(x, y)$. Proposition 4.2 implies that $0 \le \tau(A) \le 1$. Note that the Birkhoff contraction coefficient has the property $\tau(AB) \le \tau(A)\tau(B)$.

For a positive matrix $A = (a_{i,j})$, set

$$\phi(A) = \min_{i,j,r,s} \frac{a_{i,j} a_{r,s}}{a_{r,j} a_{i,s}}.$$

If A has a zero entry, then, by definition, we put $\phi(A) = 0$. The next theorem gives the formula for computing the Birkhoff contraction coefficient.

Proposition 4.4 (Theorem 2.6, [Har02]). *Suppose that a matrix A has a non-zero entry in each row. Then*

$$\tau(A) = \frac{1 - \sqrt{\phi(A)}}{1 + \sqrt{\phi(A)}}.$$

In particular, if A is positive, then $\tau(A) < 1$.

Let $\{A_k\}_{k\geq 1}$ be a sequence of $d \times d$ matrices. Denote by P_m^n the forward product $A_m A_{m+1} \cdots A_n,\ n > m$.

Definition 4.5. The products $P_m^n = (p_{i,j}^{(m,n)})$ are said to tend to *row proportionality* if for all k, s the sequence $\frac{p_{k,i}^{(m,n)}}{p_{s,i}^{(m,n)}}$ converges (as $n \to \infty$) to some constant $a = a(k, s, m) > 0$ which does not depend on the column index i.

Similarly, changing column indexes to row indexes, we can define the notion of *column proportionality* (see [Har02, Chapter 5] for details).

Remark 4.6. We note that if P_m^n tends to row proportionality as $n \to \infty$, then its transpose, which is the backward product of $\{A_n^T\}$, tends to column proportionality. Proposition 4.4 also implies that $\tau(A_1 \cdots A_n) = \tau(F_n \cdots F_1)$, where $F_i = A_i^T$.

Lemma 4.7 (Lemma 3.4, [Sen81]). *If $\{A_k\}$ is a sequence of positive matrices, then $\tau(P_m^n) \to 0$ as $n \to \infty$ if and only if the products $\{P_m^n\}$ tend to row proportionality.*

Definition 4.8. For any positive $d \times d$ matrix A denote by $\Theta(A)$ the D-diameter (in the projective metric) of the image of $\mathbb{R}_+^d$ under the action of A.

The next lemma, which was proved by A. Fisher (see Proposition 6.13 and Corollary 6.4 of [Fis09]), is crucial for our study.

Lemma 4.9. *Let $A = (a_{i,j})$ be a positive matrix. Then $\Theta(A) = \Theta(A^T)$. Furthermore,*

$$\Theta(A) = \max_{i,j,k,l} \log \frac{a_{i,k} a_{j,l}}{a_{j,k} a_{i,l}}.$$

As a corollary of this result we deduce the following simple fact which says that the image of the cone of positive vectors under P_m^n has sufficiently small diameter in the projective metric D when n is large enough if and only if the Birkhoff contraction coefficient of P_m^n tends to zero, as $n \to \infty$.

Lemma 4.10. *Suppose that all matrices $\{A_k\}_{k\geq 1}$ are positive. Then $\tau(P_m^n) \to 0$ as $n \to \infty$ if and only if for given $\varepsilon > 0,\ m \in \mathbb{N}$, and any non-negative vectors x, y, there exists $N \in \mathbb{N}$ such that $D(P_m^n x, P_m^n y) < \varepsilon$ for $n \geq N$.*

Proof. Set $F_k = A_k^T$. Suppose that $\tau(P_m^n) \to 0$ as $n \to \infty$. It follows from Remark 4.6 that $\tau(F_n \cdots F_m) \to 0$. Hence the backward product $(P_m^n)^T = F_n \cdots F_m$ tends to column proportionality.

Denote by e_i the i-th column vector from the standard basis. Consider $x = \sum x_i e_i$ and $y = \sum y_j e_j$, where the summation is over indices with $x_i > 0$ and $y_j > 0$, respectively. Then we get that

$$D((P_m^n)^T x, (P_m^n)^T y) \leq \sum_{i,j=1}^{d} D((P_m^n)^T e_i, (P_m^n)^T e_j).$$

Thus it suffices to estimate the distance between the images of basis vectors. Set $v_n = (P_m^n)^T e_i$ and $w_n = (P_m^n)^T e_j$. Then v_n and w_n are exactly the i-th and j-th columns of the matrix $(P_m^n)^T$. Using the definition of projective metric D and the property of column proportionality of $(P_m^n)^T$, we get that $D(v_n, w_n) \to 0$ as $n \to \infty$. Thus, we obtain that $\Theta(P_m^n) = \Theta((P_m^n)^T) \to 0$.

Conversely, using the equality $\Theta(P_m^n) = \Theta((P_m^n)^T)$, we get that $D(v_n, w_n) \to 0$, where v_n and w_n are the columns of $(P_m^n)^T$. It follows from the definition of the metric D that

$$\frac{v_n(i)}{w_n(i)} \cdot \frac{w_n(j)}{v_n(j)} \to 1 \text{ for all } i, j.$$

This implies precisely that the matrices $\{(P_m^n)^T\}$ tend to column proportionality as $n \to \infty$. □

Appropriate matrix norms may serve as numerical characteristics of growth rate for matrix products. For a vector $v \in \mathbb{R}^d$ denote by $||v||_1$ the norm given by

$$||v||_1 = \sum_i |v_i|.$$

Similarly, for a square matrix $A = (a_{i,j})_{i,j}$ we denote by $||A||_1$ the entrywise 1-norm

$$||A||_1 = \sum_{i,j} |a_{i,j}|. \tag{4.1}$$

Note that this is not the operator norm arising from the vector 1-norm. However, it is easy to check that

$$||AB||_1 \le ||A||_1 ||B||_1 \quad \text{and} \quad ||Ax||_1 \le ||A||_1 ||x||_1,$$

whenever the products are defined. Note also that

$$||A||_1 = ||A\overline{1}||_1 = ||\overline{1}^T A||_1 \tag{4.2}$$

for any non-negative matrix A, where $\overline{1} = (1, \dots, 1)^T$.

Now we are ready to give the criterion of unique ergodicity for a simple Bratteli diagram in terms of Birkhoff contraction coefficients. We mention that this result has been previously known to the specialists and can be considered "folklore". For example, the statement of Theorem 4.11 and a part of Proposition 4.12 can be found in [Han99] and [Fis09, Theorem 1.3], but with a somewhat different terminology and approach. We notice that these cited papers also base their proofs on ideas from [Sen81].

Theorem 4.11. *Let B be a simple Bratteli diagram of finite rank with incidence matrices $\{F_n\}_{n\ge1}$. Let $A_n = F_n^T$. Then the diagram B is uniquely ergodic if and only if*

$$\lim_{n\to\infty} \tau(A_m \dots A_n) = 0 \textit{ for every } m.$$

Proof. Set $P_m^n = A_m \cdots A_n$. Denote the cone $\bigcap_{n\ge m} P_m^n \mathbb{R}_+^d$ by C_m. By the compactness argument, $C_m \ne \emptyset$. Furthermore, $A_m C_{m+1} = C_m$. Therefore, for any vector $p^{(1)} \in C_1$ there exists a sequence of non-negative vectors $\{p^{(m)}\}_{m\ge1}$ such that $A_{m-1} p^{(m)} = p^{(m-1)}$. Such a sequence of vectors defines a finite invariant measure. The converse is also true. It follows from Remark 2.12 that in order to

establish the unique ergodicity, it is necessary and sufficient to show that C_1 is a single ray. Now the result follows immediately from Lemma 4.10. □

In the next proposition we collect a number of conditions yielding unique ergodicity that can be easily checked in practice. For the proof, see Corollary 5.1 in [Har02] and Theorem 3.2 in [Sen81].

Proposition 4.12. *Let $\{A_n\}_{n\geq 1}$ be transposes of primitive incidence matrices of a finite rank diagram B.*

(1) The diagram B admits a unique invariant probability measure on X_B if and only if there exists a strictly increasing sequence $\{n_s\}$ such that

$$\sum_{s=1}^{\infty} \sqrt{\phi(P_{n_s}^{n_{s+1}})} = \infty,$$

where $P_{n_s}^{n_{s+1}} = A_{n_s} \cdots A_{n_{s+1}}$. In particular, if

$$\sum_{n=1}^{\infty} \sqrt{\phi(A_n)} = \infty,$$

then B admits a unique invariant probability measure.

(2) If

$$\sum_{n=1}^{\infty} \left(\frac{m_n}{M_n}\right) = \infty,$$

where m_n and M_n are the smallest and the largest entry of A_n respectively, then B admits a unique invariant probability measure.

Example 4.13. Let B be a simple Bratteli diagram with incidence matrices

$$F_n = \begin{pmatrix} f_1^{(n)} & 1 & \cdots & 1 \\ 1 & f_2^{(n)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & f_d^{(n)} \end{pmatrix}.$$

Let $q_n = \max\{f_i^{(n)} f_j^{(n)} : i \neq j\}$. Compute $\phi(F_n) = q_n^{-1}$. For $A_n = F_n^T$, we observe that if

$$\sum_{n=1}^{\infty} \sqrt{\phi(A_n)} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{q}_n} = \infty,$$

then there is a unique invariant probability measure on B. This example generalizes an example considered in [FFT09] for the case of 2×2 matrices.

As a corollary of Proposition 4.12 we immediately obtain that if the incidence matrices do not grow too fast, then the diagram admits a unique invariant measure.

Corollary 4.14. *If a simple Bratteli diagram with incidence matrices $\{F_n\}_{n\geq 1}$ satisfies the condition $||F_n||_1 \leq Cn$ for some $C > 0$ and all sufficiently large n, then the diagram admits a unique invariant probability measure.*

In particular, this result holds if the diagram has only finitely many different incidence matrices.

Proof. Denote by m_n and M_n the smallest and the largest entry of F_n respectively. Using the simplicity of the diagram and an appropriate telescoping, we may assume that $m_n \geq 1$ for all n. By the definition of the entrywise matrix 1-norm, we get that

$$\frac{m_n}{M_n} \geq \frac{1}{||F_n||_1} \geq \frac{1}{Cn}$$

for all n large enough. The result follows from Proposition 4.12. □

Remark 4.15. (1) This corollary gives another proof of the fact that linearly recurrent systems are uniquely ergodic, which was originally established in Proposition 5 of [CDHM03].

(2) It is mentioned in [Haj76, p. 528] that the products of the following sets of positive matrices tend to column proportionality and, in particular, give rise to uniquely ergodic systems:

(i) Any set of primitive incidence matrices which pairwise commute.
(ii) The set Σ of primitive incidence matrices such that if $A \in \Sigma$ and F is primitive, then AF and FA are primitive.

In the next example we show how the technique of Bratteli diagrams can be used to derive a sufficient condition of unique ergodicity for generalized Morse sequences. See the papers [Ke68] and [Ma77] for more information about these systems and a complete characterization of unique ergodicity.

Example 4.16. Let G be a finite abelian group with group operation $+$. Each element $g \in G$ acts on finite words $a = a_0 \cdots a_p$, $a_i \in G$, by $\sigma_g(a)[i] = a_i + g$, $i = 0, \ldots, p$. For two finite words $a = a_0 \cdots a_p$ and $c = c_0 \cdots c_q$ over G, we define

$$a \times c = \sigma_{c_0}(a)\sigma_{c_1}(a) \cdots \sigma_{c_q}(a) \text{ (concatenation of words).}$$

Let $\{b^{(n)}\}_{n\geq 1}$ be a sequence of finite words over G. We assume that the first letter $b^{(n)}[0] = 0$ (group identity), $|b^{(n)}| \geq 2$, and all elements from G occur in every $b^{(n)}$. Define the infinite sequence

$$\omega = b^{(1)} \times b^{(2)} \times \cdots .$$

Consider the symbolic dynamical system (X, T) generated by the shift T on the closure X of the T-orbit of ω. Points from X are represented by bi-infinite sequences. Then (X, T) is called a *generalized Morse dynamical system.* The classical Morse system is included in this scheme.

Denote by $\mathrm{freq}(g, b^{(n)})$ the frequency of an element $g \in G$ in the word $b^{(n)}$. The following fact is "folklore" and was originally established by methods of symbolic dynamics.

Claim. If

$$\sum_{n\geq 1} \min_{g\in G} \mathrm{freq}(g, b^{(n)}) = \infty, \tag{4.3}$$

then the system is uniquely ergodic.

In fact, this is a necessary and sufficient condition for unique ergodicity when G has two elements [Ke68]; the criterion of [Ma77] is stated in different terms.

This result can be proved by using the following approach: find a Bratteli-Vershik model for (X, T) and then show that condition (4.3) allows us to apply Proposition 4.12.

Denote by λ_n the length of the word $c^{(n)} = b^{(1)} \times \cdots \times b^{(n)}$. For each $g \in G$, set

$$B_n(g) = \{x \in X : x[0, \lambda_n - 1] = \sigma_g(c^{(n)})\}.$$

Then the sets $X_g^{(n)} = \{B_g(n), \ldots, T^{\lambda_n - 1} B_g(n)\}$, $g \in G$, are disjoint and $\Xi_n = \{X_g^{(n)} : g \in G\}$ form a Kakutani-Rokhlin partition of X; see [Ma77] for the details. Furthermore, one can check that the sequence $\{\Xi_n\}_{n \geq 1}$ is nested. Thus, we can use the sequence $\{\Xi_n\}_{n \geq 1}$ to construct an ordered finite rank Bratteli diagram B.

Note that the ordering on the diagram B has only finitely many maximal and minimal paths. Denote by X_B^0 a (countable) set of paths which are cofinal either to a minimal path or to a maximal one. Thus, the Vershik map T_B determined by the ordering is well defined and continuous on $X'_B = X_B \setminus X_B^0$.

Set $X_0 = \bigcup_k T^k(\bigcap_n \bigcup_g B_g(n))$ and $X' = X \setminus X_0$. Using the finiteness of G, one can show that X_0 is a countable set. Furthermore, the dynamical systems (X'_B, T_B) and (X', T) are (Borel) isomorphic and share the same set of invariant measures. Therefore, in order to check the unique ergodicity of the generalized Morse system, it is enough to do this for (X'_B, T_B).

Since each tower in a Kakutani-Rokhlin partition Ξ_n is exactly defined by an element $g \in G$, there is a natural correspondence between vertices of level n in the diagram B and elements of G. Using properties of $\{\Xi_n\}_{n \geq 1}$, one can check that the n-th incidence matrix of the diagram B is as $F_n = (f_{g,h}^{(n)})$, where $f_{g,h}^{(n)}$ is the number of occurrences of h in the word $\sigma_g(b^{(n)})$. We observe that $f_{g,h}^{(n)} = f_{0,g-h}^{(n)}$ for any $g, h \in G$. Hence,

$$m_n = \min_{g,h \in G} f_{g,h}^{(n)} = \min_{q \in G} f_{0,q}^{(n)} = \min_{g \in G} \mathrm{freq}(g, b^{(n)}) |b^{(n)}|,$$

where $|b^{(n)}|$ is the length of $b^{(n)}$. Observe that each row in the matrix F_n sums up to $|b^{(n)}|$. Hence $M_n = \max_{g,h} f_{g,h}^{(n)} < |b^{(n)}|$. Thus, we conclude that

$$\min_{g \in G} \mathrm{freq}(g, b^{(n)}) = \frac{m_n}{|b^{(n)}|} \leq \frac{m_n}{M_n}.$$

Now the claim follows from Proposition 4.12.

We observe that by refining the partitions $\{\Xi_n\}_{n \geq 1}$ one can construct a topological (finite rank) Bratteli-Vershik model for (X, T).

5. Quantitative analysis of measures

Throughout this section, we assume that the Bratteli diagram B is simple and uniquely ergodic. Our goal in the section is to study the asymptotic behavior of tower heights and of measures of tower bases. Since the heights of towers determine the recurrence time for points from the bases of towers, our study can be viewed as an "adic version" of the quantitative recurrence analysis.

We start by translating the ergodic theorem into the language of Bratteli diagrams. Let B be a simple Bratteli diagram of finite rank with a unique ergodic probability measure μ. Without loss of generality (after telescoping) we can assume that all vertices of consecutive levels of B are connected by an edge. Then it is easy to enumerate the edges of the Bratteli diagram so that this ordering defines a continuous Vershik map $T = T_B$ (see Section 3 of [HPS92] for details).

Fix an integer $m > 0$. For each infinite path $x \in X_B$, denote by $v_m(x)$ the vertex of level m that the path x goes through. Also denote by $e(v_0, v_m(x))$ the

finite segment of the path x between the vertices v_0 and $v_m(x)$. Let $i_m(x)$ be the least integer such that $T^{-i_m(x)}$ maps $e(v_0, v_m(x))$ to the minimal finite path from the set $E(v_0, v_m(x))$. Similarly, let $j_m(x)$ be the least integer such that $T^{j_m(x)}$ maps $e(v_0, v_m(x))$ to the maximal path from $E(v_0, v_m(x))$. Notice that $h^{(m)}_{v_m(x)} = j_m(x) + i_m(x)$.

Then, by the pointwise ergodic theorem and unique ergodicity of (X_B, T_B), we get that

$$\mu(B_n(w)) = \lim_{m\to\infty} \frac{1}{i_m(x) + j_m(x)} \sum_{i=-i_m(x)}^{j_m(x)} 1_{B_n(w)}(T^i(x)) \tag{5.1}$$

for every $x \in X_B$.

The sum on the right-hand side of (5.1) is equal to the number of paths that connect the vertex w of level n to the vertex $v_m(x)$ of level m. Hence, we obtain the following result.

Proposition 5.1. *Let B be a simple uniquely ergodic Bratteli diagram, and let μ be the unique invariant probability measure on X_B. Then for any vertices v, w, and any level n, we have*

$$\mu(B_n(w)) = \lim_{m\to\infty} \frac{(F_{m-1}\cdots F_n)_{v,w}}{h^{(m)}_v}.$$

Remark 5.2. We should note that such an interpretation of the pointwise ergodic theorem first appeared in [VK81, Theorem 2]; see also [Mel06, Lemma 3.4].

Lemma 5.3. *Let B be a simple uniquely ergodic Bratteli diagram of finite rank. The diagram B can be telescoped to a new diagram with incidence matrices $\{F_n\}_{n\geq 1}$ such that the following properties hold:*

(i) there exist a non-negative probability vector ξ and strictly positive vectors $\{\eta^{(n)}\}_{n\geq 1}$ such that for any $n > 0$ and any vector $x \in \mathbb{R}^d_+$ we have

$$\lim_{m\to\infty} \frac{F_m \cdots F_n x}{||F_m \cdots F_n x||_1} = \xi$$

and

$$\lim_{m\to\infty} \frac{x^T F_m \cdots F_n}{||x^T F_m \cdots F_n||_1} = (\eta^{(n)})^T > 0;$$

(ii)

$$\frac{(\eta^{(n+1)})^T F_n}{||(\eta^{(n+1)})^T F_n||_1} = (\eta^{(n)})^T$$

and $\eta^{(n)} \to \eta \geq 0$ as $n \to \infty$.

Proof. (i) Denote by $\{F_n\}_{n\geq 1}$ the incidence matrices of diagram B. Since B is uniquely ergodic, we obtain, by Theorem 4.11, that $\tau(F_n^T \cdots F_m^T) \to 0$ for any fixed n as $m \to \infty$. Applying Lemmas 4.9 and 4.10, we conclude that the D-diameter of the cone $C^{(n)}_m = F_m \cdots F_n \mathbb{R}^d_+$ tends to zero as $m \to \infty$. Hence, by compactness of the simplex of probability vectors, there exists a non-negative probability vector $\xi^{(1)}$ and a subsequence $\{m_k\}_{k\geq 1}$ such that $C^{(1)}_{m_k} \to ray(\xi^{(1)})$ as $k \to \infty$. Telescope the diagram along the sequence $\{m_k\}_{k\geq 1}$. For convenience, we denote the new incidence matrices by the same symbols $\{F_n\}_{n\geq 1}$.

Applying the same arguments for $n = 2, 3, \ldots$, we inductively telescope the diagram to new levels and find non-negative probability vectors $\{\xi^{(n)}\}_{n\geq 1}$ such that $C_m^{(n)} \to ray(\xi^{(n)})$ as $m \to \infty$ for every fixed n.

It follows from the construction that for any non-negative vector x

$$\frac{F_m \cdots F_n x}{||F_m \cdots F_n x||_1} \to \xi^{(n)} \text{ as } m \to \infty. \tag{5.2}$$

Setting $x = F_{n-1}y$ for some non-negative vector y, we see that

$$\frac{F_m \cdots F_n F_{n-1} y}{||F_m \cdots F_n F_{n-1} y||_1} \to \xi^{(n-1)} \text{ as } m \to \infty.$$

Hence $\xi^{(n)} = \xi^{(n-1)} = \cdots = \xi^{(1)} = \xi$.

(ii) To show the existence of a probability vector $\eta^{(n)}$ that satisfies the condition of the lemma, we consider a decreasing sequence of cones $\widetilde{C}_m^{(n)} = F_n^T \cdots F_m^T \mathbb{R}_+^d$. Lemma 4.9 implies that the D-diameter of these cones tends to zero as $m \to \infty$. It follows that $\bigcap_{m\geq n} \widetilde{C}_m^{(n)} = ray(\eta^{(n)})$ for some strictly positive probability vector $\eta^{(n)}$. Clearly, we can further telescope the diagram to ensure that $\eta^{(n)} \to \eta$ for some probability non-negative vector η. Verifying condition (ii) of the lemma is straightforward. □

Remark 5.4. (1) If a Bratteli diagram B is stationary, i.e., $F_n = F$ for all n, then ξ and η are the normalized right and left Perron-Frobenius eigenvectors of F, respectively.

(2) In general, vectors ξ and η may have zero coordinates (see Example 5.10).

(3) The entries of vectors $\{\eta^{(n)}\}$ represent (non-normalized) values of the invariant measure on cylinder sets. Set

$$p^{(n)} = \frac{\eta^{(n)}}{||(\eta^{(2)})^T F_1||_1 \cdots ||(\eta^{(n)})^T F_{n-1}||_1}.$$

It follows from Lemma 5.3 that $F_n^T p^{(n+1)} = p^{(n)}$ for all n. Thus, this sequence defines a probability measure (see Theorem 2.11).

Next we explore these questions for Bratteli diagrams of exact finite rank; see Definition 3.5. Recall that such diagrams are all uniquely ergodic (Corollary 3.6).

Definition 5.5. For any two sequences of real numbers $\{x_n\}$ and $\{y_n\}$, we will write $x_n \sim y_n$ as $n \to \infty$ to indicate that $\lim_{n\to\infty} x_n/y_n = 1$.

The following simple proposition shows that if the measures of tower bases have the same asymptotic growth, then so do the heights of towers.

Proposition 5.6. *Let B be a simple Bratteli diagram of exact finite rank with the probability invariant measure μ.*

(1) Then

$$\inf_{v,w,n} \frac{\mu(B_n(v))}{\mu(B_n(w))} > 0 \textit{ if and only if } \inf_{v,w,n} \frac{h_v^{(n)}}{h_w^{(n)}} > 0,$$

where $B_n(w)$ is the base of the tower $X_w^{(n)}$ and $h^{(n)} = F_{n-1} \cdots F_1 \overline{1}$ is the vector representing the tower heights.

(2) If either condition holds, then the vector ξ found in Lemma 5.3 is strictly positive and (after an appropriate telescoping)

$$h_w^{(n)} \sim \xi_w ||F_{n-1} \cdots F_1||_1$$

and

$$\mu(B_n(w)) \sim \frac{\rho_w}{||F_{n-1} \cdots F_1||_1}$$

for some strictly positive vector $\rho = (\rho_w)$.

Proof. Both statements are immediate from Lemma 5.3 and the fact that $0 < \delta \leq \mu(X_v^{(n)}) = h_v^{(n)} \mu(B_n(v)) \leq 1$ for all v and n. □

The following proposition defines a large class of diagrams of exact finite rank whose towers grow with the same speed. We note that the condition used in the next proposition is sometimes referred to as the "compactness" condition.

Proposition 5.7. *Let B be a simple Bratteli finite rank diagram with the incidence matrices $\{F_n\}_{n\geq 1}$. Suppose that there is a constant $c > 0$ such that $m_n/M_n \geq c$, for all n, where m_n and M_n are the smallest and the largest entry of F_n, respectively. Then:*

(1) the diagram B is of exact finite rank;

(2) $h_w^{(n)}/h_v^{(n)} \geq c$ for all levels n and all vertices v and w.

Proof. The unique ergodicity follows from Proposition 4.12. Denote by $f_{i,j}^{(n,m)}$ the entries of the product matrix $F_m \cdots F_n$. The entries of F_n are denoted by $f_{i,j}^{(n)}$. We claim that $f_{p,v}^{(n,m)}/f_{p,w}^{(n,m)} \geq c$ for every $m \geq n$ and all vertices p, v, w. By induction, we need to show that if this inequality holds for m, then it is true for $m+1$. Indeed,

$$\begin{aligned}
\frac{f_{p,v}^{(n,m+1)}}{f_{p,w}^{(n,m+1)}} &= \frac{\sum_r f_{p,r}^{(m+1)} f_{r,v}^{(n,m)}}{\sum_i f_{p,i}^{(m+1)} f_{i,w}^{(n,m)}} \\
&= \sum_r \frac{f_{p,r}^{(m+1)} f_{r,w}^{(n,m)}}{\sum_i f_{p,i}^{(m+1)} f_{i,w}^{(n,m)}} \frac{f_{r,v}^{(n,m)}}{f_{r,w}^{(n,m)}} \\
&\geq c \sum_r \frac{f_{p,r}^{(m+1)} f_{r,w}^{(n,m)}}{\sum_i f_{p,i}^{(m+1)} f_{i,w}^{(n,m)}} \\
&= c.
\end{aligned}$$

It follows from Proposition 5.1 that

$$\frac{\mu(B_n(v))}{\mu(B_n(w))} = \frac{f_{p,v}^{(n,m)} h_p^{(m)}}{f_{p,w}^{(n,m)} h_p^{(m)}} = \lim_{m\to\infty} f_{p,v}^{(n,m)}/f_{p,w}^{(n,m)} \geq c$$

for all v, w. Note also that

$$h_w^{(n)} = \sum_r f_{v,r}^{(n)} h_r^{(n-1)} \geq c \sum_r M_n h_r^{(n-1)} \geq c h_v^{(n)}$$

for all w and v. Therefore,

$$\frac{\mu(X_v^{(n)})}{\mu(X_w^{(n)})} = \frac{h_v^{(n)} \mu(B_n(v))}{h_w^{(n)} \mu(B_n(w))} \geq c^2$$

for all levels n and vertices v, w. This proves the proposition. □

The following example shows that there are diagrams of exact finite rank whose tower heights obey different asymptotic rates.

Example 5.8. Consider a simple finite rank Bratteli diagram B determined by the sequence of incidence matrices

$$F_n = \begin{pmatrix} 1 & 1 \\ n & 1 \end{pmatrix}.$$

By Corollary 4.14, this diagram is uniquely ergodic. Denote by c_n and d_n the $(1,1)$- and $(2,1)$-entry of $F_n \cdots F_1$, respectively. By induction, one can show that

$$F_n \cdots F_1 = \begin{pmatrix} c_n & c_n \\ d_n & d_n \end{pmatrix}.$$

Hence $h_1^{(n)} = 2c_{n-1}$ and $h_2^{(n)} = 2d_{n-1}$ for all n. Using the recurrence relations $c_n = c_{n-1} + d_{n-1}$ and $d_n = nc_{n-1} + d_{n-1}$, we see that

$$c_n = 2c_{n-1} + (n-2)c_{n-2} \tag{5.3}$$

and

$$d_n = (n+1)c_{n-1} + (n-2)c_{n-2}. \tag{5.4}$$

Denote by $H_n(z)$ the n-th Hermite polynomial, i.e., $H_0(z) = 1$, $H_1(z) = 2z$, and for all $n \geq 1$

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z). \tag{5.5}$$

It follows from (5.3) and (5.5) that

$$c_n = \left(\frac{-i}{\sqrt{2}}\right)^{n-1} H_{n-1}(i\sqrt{2}). \tag{5.6}$$

The asymptotic formula for Hermite polynomials is given by

$$H_n(z) = \sqrt{2}\,\exp(z^2/2)(2n/e)^{n/2}\cos[z\sqrt{2n+1} - (\pi n)/2](1 + q_n(z)), \tag{5.7}$$

where $z \in \mathbb{C} \setminus \mathbb{R}$ and $q_n(z) \to 0$; see [Ru76]. It follows that

$$\frac{H_n(i\sqrt{2})}{H_{n-1}(i\sqrt{2})} \sim i\sqrt{2n} \text{ as } n \to \infty. \tag{5.8}$$

Claim 1. $h_n^{(1)}/h_n^{(2)} \to 0$ when $n \to \infty$.

Indeed, in view of (5.3) and (5.4) it is enough to show that $c_{n-1}/c_n \to 0$ when $n \to \infty$, which immediately follows from (5.6) and (5.8).

Claim 2. Let μ be the probability invariant measure on X_B. Then $\mu(X_1^{(m)}) \to 1/2$ when $m \to \infty$; hence the diagram has exact finite rank.

We will need the second (linearly independent of $H_n(z)$) solution of (5.5) given by

$$Q_n(z) = -\int_{-\infty}^{\infty} \frac{e^{-t^2} H_n(z)}{t - z} dt, \; z \in \mathbb{C} \setminus \mathbb{R};$$

see [Ru76] for the details. The functions $Q_n(z)$ are called the Hermite functions of the second kind. We note that any other solution of (5.3) is a linear combination of $H_n(z)$ and $Q_n(z)$ [Ru76]. (We are thankful to L. Golinskii and P. Nevai for their suggestions to use the functions $Q_n(z)$.)

The following asymptotic formula was also established in [Ru76] for z in the upper half-plane:

$$Q_n(z) = (-i)^{n+1}\pi\sqrt{2}(2n/e)^{n/2}\exp[-z^2/2 + iz\sqrt{2n+1}](1+k_n(z)), \tag{5.9}$$

where $k_n(z) \to 0$. It follows from (5.7) and (5.9) that

$$\frac{Q_n(i\sqrt{2})}{H_n(i\sqrt{2})} \to 0 \text{ as } n \to \infty. \tag{5.10}$$

Also note that

$$\frac{Q_{n-1}(i\sqrt{2})}{Q_n(i\sqrt{2})} \sim \frac{i}{\sqrt{2n}} \text{ as } n \to \infty.$$

Applying the pointwise ergodic theorem (Proposition 5.1), we get that

$$\mu(X_1^{(m)}) = \lim_{n\to\infty} \frac{h_1^{(m)}(F_{n-1}\cdots F_m)_{1,1}}{h_1^{(n)}}. \tag{5.11}$$

Set $R_n^{(m)} = (F_{n-1}\cdots F_m)_{1,1}$. We observe that the sequence $\{R_n^{(m)}\}_{n\geq m}$ satisfies the recurrence relation (5.3) and initial conditions $R_m^{(m)} = 1$ and $R_{m+1}^{(m)} = m+1$. Note that $R_n^{(1)} = c_n$. Thus,

$$R_n^{(m)} = \left(\frac{-i}{\sqrt{2}}\right)^{n-1}\Big(\alpha_m H_{n-1}(i\sqrt{2}) + \beta_m Q_{n-1}(i\sqrt{2})\Big) \text{ for all } n \geq m,$$

where the constants α_m and β_m are uniquely determined by the initial conditions. The asymptotic ratio (5.10) implies that

$$\frac{R_n^{(m)}}{h_1^{(n)}} \to \frac{\alpha_m}{2} \text{ as } n \to \infty.$$

It follows from (5.11) that $\mu(B_m(1)) = \alpha_m/2$ and

$$\mu(X_1^{(m)}) = \left(\frac{-i}{\sqrt{2}}\right)^{m-2}\alpha_m H_{m-2}(i\sqrt{2}).$$

Solving the system of equations $R_m^{(m)} = 1$ and $R_{m+1}^{(m)} = m+1$ for α_m and β_m, we obtain that

$$\alpha_m = \left(\frac{-\sqrt{2}}{i}\right)^m \frac{(i/\sqrt{2})Q_m(i\sqrt{2}) - (m+1)Q_{m-1}(i\sqrt{2})}{H_{m-1}(i\sqrt{2})Q_m(i\sqrt{2}) - H_m(i\sqrt{2})Q_{m-1}(i\sqrt{2})}.$$

Now it is straightforward to check that $\mu(X_1^{(m)}) \to 1/2$ as $m \to \infty$. We skip the computation.

Remark 5.9. We note that the uniform growth of tower heights does not guarantee the unique ergodicity of the diagram. As an example, consider the Bratteli diagram B with incidence matrices

$$F_n = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix}.$$

Setting $h^{(n)} = F_{n-1}\cdots F_1\overline{1}$, we note that $h_1^{(n)} = h_2^{(n)} = 2^{-1}||F_{n-1}\cdots F_1||_1$. However, it was shown in [FFT09, Proposition 3.1] that the diagram B has exactly two finite ergodic invariant measures (see also the more general Example 6.7). We

may also apply the methods of Section 6 to show that each of these measures is obtained as an extension of a unique invariant measure from the left (right) vertical subdiagram.

The following example presents a uniquely ergodic diagram of non-exact finite rank with different growth of tower heights.

Example 5.10. Consider the Bratteli diagram B determined by the incidence matrices

$$F_n = \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix}.$$

By Corollary 4.14, this diagram is uniquely ergodic. However, the following result holds:

Claim. The diagram B is not of exact finite rank.

Indeed, let $h_i^{(n)}$ be the height of the i-th tower at level n, $i = 1, 2$. Clearly, $h_1^{(n)} \leq h_2^{(n)}$. Hence

$$\frac{h_1^{(n+1)}}{h_2^{(n+1)}} = \frac{h_1^{(n)} + h_2^{(n)}}{h_1^{(n)} + nh_2^{(n)}} \leq \frac{2h_2^{(n)}}{nh_2^{(n)}} \to 0 \text{ as } n \to \infty.$$

It follows that

$$\frac{h_1^{(n)}}{h_2^{(n+1)}} \leq \frac{h_1^{(n)}}{nh_2^{(n)}} \leq \frac{2}{n(n-1)}.$$

Now if we take the invariant probability measure μ on the right (vertical) subdiagram, then the convergence of $\sum_n h_1^{(n)}/h_2^{(n+1)}$ and Proposition 6.1 below imply that the extension of μ is a finite invariant measure. Thus, the unique invariant measure is the extension of μ. Hence by Theorem 3.3 we obtain that $\mu(X_1^{(n)})/\mu(X_2^{(n)}) \to 0$ as $n \to \infty$.

6. Extension of measures from subdiagrams

In view of the structural results of Section 3, each invariant measure on a finite rank Bratteli diagram is obtained as an extension of a measure from some subdiagram. In this section we further study this construction by establishing some algebraic conditions for finiteness of the extension. The motivation for this is to obtain some quantitative properties of diagrams. Throughout the section we will assume that Bratteli diagrams are telescoped to satisfy Theorem 3.3.

6.1. **General condition.** Let $\overline{W} = \{W_n\}$ be a sequence of finite subsets of V_n. We will consider the non-trivial case when, for all n, W_n is a proper subset of V_n and every pair of vertices $(v, w) \in W_n \times W_{n+1}$ is connected by at least one edge. Denote $W_n' = V_n \setminus W_n$. Thus, the sequence $\overline{W}$ determines a proper Bratteli subdiagram $B(\overline{W})$ which is formed by the vertices from $\overline{W}$ and the edges that connect them. Let $Y = Y_{B(\overline{W})}$ be the path space of $B(\overline{W})$. Denote by $X_{B(\overline{W})}$ the set of all paths of the diagram B cofinal to paths from Y. In other words, the set $X_{B(\overline{W})}$ is the saturation of Y with respect to the cofinal equivalence relation. Recall that X_B stands for the path-space of the diagram B. When we extend a measure μ from the subdiagram $\overline{W}$ to B, the extended measure is always supported by the set $X_{B(\overline{W})}$; see Section 3 for the details. The following proposition may be viewed as an analogue of the Kac lemma on the first return map in measurable dynamics.

Proposition 6.1. *Let B be a finite rank diagram with incidence matrices $\{F_n = (f_{v,w}^{(n)})\}$, and $B(\overline{W})$ is a subdiagram as above. Let μ be a finite invariant measure on $B(\overline{W})$.*

(1) Suppose the extension $\widehat{\mu}$ of μ on the support $X = X_{B(\overline{W})}$ is finite. Then

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W_n'} q_{v,w}^{(n)} \mu(X_v^{(n+1)}(\overline{W})) < \infty, \tag{6.1}$$

where $q_{v,w}^{(n)} = f_{v,w}^{(n)} h_w^{(n)} / h_v^{(n+1)}$ are the entries of the stochastic matrix Q_n (see (2.3)), $h_v^{(n+1)}$ is the number of paths from the vertex $v \in V_n$ to the root vertex in the diagram B, and $X_v^{(n+1)}(\overline{W})$ is the tower in the subdiagram $B(\overline{W})$ corresponding to the vertex v.

(2) If

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W_n'} q_{v,w}^{(n)} < \infty, \tag{6.2}$$

then any probability measure μ defined on the path space Y of the subdiagram $B(\overline{W})$ extends to a finite measure $\widehat{\mu}$ on X.

Proof. (1) Let $X_w^{(n)}(\overline{W})$ be the tower in $B(\overline{W})$ corresponding to a vertex $w \in W_n$. Denote by $h_w^{(n)}(\overline{W})$ the height of $X_w^{(n)}(\overline{W})$ and by $\overline{B}_n(w)$ its base; then $\mu(X_w^{(n)}(\overline{W})) = h_w^{(n)}(\overline{W})\mu(\overline{B}_n(w))$. Let $h_w^{(n)}$ be the number of all finite paths from v_0 to w contained in B, i.e., $h_w^{(n)}$ is the height of $X_w^{(n)}$. Set

$$Z_n = \{x \in X_B : r(x_m) \in W_m \text{ for all } m \geq n \text{ and } x_k \in W_k' \text{ for some } k < n\}.$$

Then $\widehat{\mu}(X_B) < \infty$ is finite if and only if $\widehat{\mu}(\bigcup_n Z_n) < \infty$. Observe that

$$\widehat{\mu}\left(\bigcup_n Z_n\right) = \sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W_n'} \frac{f_{v,w}^{(n)} h_w^{(n)}}{h_v^{(n+1)}(\overline{W})} \mu(X_v^{(n+1)}(\overline{W})).$$

Since

$$q_{v,w}^{(n)} = \frac{f_{v,w}^{(n)} h_w^{(n)}}{h_v^{(n+1)}} \leq \frac{f_{v,w}^{(n)} h_w^{(n)}}{h_v^{(n+1)}(\overline{W})},$$

the finiteness of $\widehat{\mu}(\bigcup_n Z_n)$ implies (6.1).

(2) Suppose (6.2) holds. Denote

$$I_n = \sum_{w \in W_n} h_w^{(n)} \mu(\overline{B}_n(w)).$$

To prove the finiteness of $\widehat{\mu}(X)$, it suffices to show that the sequence $\{I_n\}$ is bounded since $\lim_n I_n = \widehat{\mu}(X_B)$. We have

$$I_n = \sum_{w \in W_n} \frac{h_w^{(n)}}{h_w^{(n)}(\overline{W})} h_w^{(n)}(\overline{W}) \mu(\overline{B}_n(w)) = \sum_{w \in W_n} \frac{h_w^{(n)}}{h_w^{(n)}(\overline{W})} \mu(X_w^{(n)}(\overline{W})). \tag{6.3}$$

Next, if we show that there exists M such that for all n and $w \in W_n$

$$\frac{h_w^{(n)}}{h_w^{(n)}(\overline{W})} \leq M, \tag{6.4}$$

then we obtain that

$$I_n \leq M \sum_{w \in W_n} \mu(X_w^{(n)}(\overline{B})) \leq M.$$

Let

$$M_n = \max\{\frac{h_w^{(n)}}{h_w^{(n)}(\overline{W})} : w \in W_n\}.$$

Fix a vertex $v \in W_{n+1}$ and consider

$$\begin{aligned}
\frac{h_v^{(n+1)}}{h_v^{(n+1)}(\overline{W})} &= \frac{1}{h_v^{(n+1)}(\overline{W})} \left(\sum_{w \in W_n} f_{v,w}^{(n)} h_w^{(n)} + \sum_{w \in W_n'} f_{v,w}^{(n)} h_w^{(n)} \right) \\
&\leq \frac{M_n}{h_v^{(n+1)}(\overline{W})} \sum_{w \in W_n} f_{v,w}^{(n)} h_w^{(n)}(\overline{W}) + \frac{1}{h_v^{(n+1)}(\overline{W})} \sum_{w \in W_n'} f_{v,w}^{(n)} h_w^{(n)} \\
&= M_n + \frac{h_v^{(n+1)}}{h_v^{(n+1)}(\overline{W})} \sum_{w \in W_n'} f_{v,w}^{(n)} \frac{h_w^{(n)}}{h_v^{(n+1)}} \\
&= M_n + \frac{h_v^{(n+1)}}{h_v^{(n+1)}(\overline{W})} \sum_{w \in W_n'} q_{v,w}^{(n)} \\
&\leq M_n + \frac{h_v^{(n+1)}}{h_v^{(n+1)}(\overline{W})} \varepsilon_n,
\end{aligned}$$

where

$$\varepsilon_n = \sum_{v \in W_{n+1}} \sum_{w \in W_n'} q_{v,w}^{(n)}.$$

It follows from the above inequalities that

$$\frac{h_v^{(n+1)}}{h_v^{(n+1)}(\overline{W})}(1 - \varepsilon_n) \leq M_n \text{ and } M_{n+1} \leq \frac{M_n}{1 - \varepsilon_n}.$$

Finally,

$$M_n \leq \frac{M_1}{\prod_{k=1}^{\infty}(1 - \varepsilon_n)},$$

where the product is convergent in view of (6.2). □

Corollary 6.2. *In the setting of Proposition* 6.1*, if the subdiagram* $B(\overline{W})$ *has exact finite rank, then* (6.2) *is necessary and sufficient for the finiteness of the extension* $\widehat{\mu}$*.*

Proof. This is immediate from Proposition 6.1 and the definition of exact finite rank. □

In the remaining part of this section, we consider finite rank Bratteli diagrams B with incidence matrices of the form

$$F_n = \begin{pmatrix} D_n & 0 \\ A_n & C_n \end{pmatrix}, \quad n \geq 1, \tag{6.5}$$

where matrices D_n and C_n are primitive and A_n is non-zero for all n. Then the subdiagrams $B(D)$ and $B(C)$, with the incidence matrices D_n and C_n, are simple.

By construction, the minimal component of B corresponds to $B(D)$ and the non-minimal one is determined by $B(C)$. Suppose μ is a probability invariant measure on $B(C)$. Denote by $\widehat{\mu}$ the extension of μ to X_B. Let $A_i = (a^{(i)}_{v,u})$ and set

$$\alpha_i = \max\{a^{(i)}_{v,u} : v \in V_{i+1}(C),\ u \in V_i(D)\},$$
$$\beta_i = \min\{a^{(i)}_{v,u} : v \in V_{i+1}(C),\ u \in V_i(D)\}.$$

Using Propositions 5.6 and 6.1, we can establish the following result.

Theorem 6.3. *Let the Bratteli diagram B be as above. Suppose that the Bratteli subdiagrams $B(C)$ and $B(D)$ are of exact finite rank. Assume further that in each of the subdiagrams the heights of towers have the same asymptotic growth. More precisely,* $\inf_{n,v,w} h^{(n)}_v / h^{(n)}_w > 0$, *where v and w run over the vertices of $B(D)$, and also over the vertices of $B(C)$, and $h^{(n)}_v, h^{(n)}_w$ denote the heights of the towers* within *the corresponding subdiagram.*

(i) If

$$\sum_{i=1}^{\infty} \alpha_i \frac{||D_{i-1}\cdots D_1||_1}{||C_i \cdots C_1||_1} < \infty,$$

then the measure $\widehat{\mu}(X_B)$ is finite.

(ii) If $\widehat{\mu}(X_B)$ is finite, then

$$\sum_{i=1}^{\infty} \beta_i \frac{||D_{i-1}\cdots D_1||_1}{||C_i \cdots C_1||_1} < \infty.$$

Proof. To prove the theorem it is enough to check the convergence of the series

$$\sum_{i=1}^{\infty} \sum_{v\in W_{i+1}} \sum_{w\in W'_i} q^{(i)}_{v,w}, \tag{6.6}$$

where $W_i = V(C) \cap V_i$ and $W'_i = V(D) \cap V_i$.

We observe that it follows from the form of the diagram B that the heights $h^{(i)}_w$, for $w \in V(D)$, are completely determined by the products of the matrices $D_{i-1}\cdots D_1$. In view of Proposition 5.6 we see that there are positive constants k_1 and k_2 such that

$$k_1 \le \frac{h^{(i)}_w}{||D_{i-1}\cdots D_1||_1} \le k_2$$

for all levels $i \ge 1$ and all $w \in V(D)$. (Although Proposition 5.6(2) says "after appropriate telescoping", we only need the weaker property that there are two-sided estimates. In that proposition, we have that $||F_{n-1}\cdots F_1||_1$ is the sum of heights. Since the ratios between heights are bounded away from zero, $h^{(n)}_v/||F_{n-1}\cdots F_1||_1$ is bounded from zero and infinity.)

On the other hand, the finiteness of the extension $\widehat{\mu}$ is equivalent to the fact that there exist positive constants r_1 and r_2 such that for all $i \ge 1$ and $v \in V(C)$,

$$r_1 \le \frac{h^{(i)}_v}{||C_{i-1}\cdots C_1||_1} \le r_2.$$

Then, for all $i \geq 1$ we have that

$$\begin{aligned}\sum_{v\in W_{i+1}}\sum_{w\in W_i'} q_{v,w}^{(i)} &= \sum_{v\in W_{i+1}}\sum_{w\in W_i'} f_{v,w}^{(i)}\frac{h_w^{(i)}}{h_v^{(i+1)}}\\ &\leq \alpha_i\frac{k_2|W_i'|\cdot|W_{i+1}|}{r_1}\frac{||D_{i-1}\cdots D_1||_1}{||C_i\cdots C_1||_1}.\end{aligned}$$

Thus, statement (i) implies the convergence of (6.6) and, therefore, establishes the finiteness of the extension.

Statement (ii) is proved analogously from the lower bound for the sum $\sum_{v\in W_{i+1}}\sum_{w\in W_i'} q_{v,w}^{(i)}$. □

Corollary 6.4. *Let B be as in Theorem 6.3. If there are positive integers N_1 and N_2 such that $N_1 \leq \beta_i \leq \alpha_i \leq N_2$ for all $i \geq 1$, then*

$$\mu(X_B) < \infty \Longleftrightarrow \sum_{i=1}^{\infty}\frac{||D_{i-1}\cdots D_1||_1}{||C_i\cdots C_1||_1} < \infty. \tag{6.7}$$

Remark 6.5. (1) The condition $N_1 \leq \beta_i \leq \alpha_i \leq N_2$ ($i \geq 1$) is equivalent to the property of finiteness of the set $\{A_i : i \geq 1\}$ (recall that we consider Bratteli diagrams with incidence matrices (6.5)). In particular, this is the case when the matrices F_i are taken from a finite set of matrices (diagrams of finite complexity, which are discussed below).

(2) For any fixed sequences $\{D_i\}$ and $\{C_i\}$, the condition $\mu(X_B) = \infty$ can be obtained by an appropriate choice of matrices A_i.

(3) In the case of stationary diagrams, Corollary 6.4 is a generalization of the fact that the measure extension is finite if and only if the spectral radius of $C = C_n$ is strictly greater than that of $D = D_n$; see Theorem 4.3 in [BKMS10].

6.2. **Extension from odometers.** We consider an important special case of Proposition 6.1. Let B be a finite rank Bratteli diagram with incidence matrices F_n. Take a sequence $\overline{v} = (v_0, v_1, ...)$ of vertices in B such that $v_i \in V_i$ and denote by $Y_{\overline{v}}$ the corresponding "odometer", i.e., $Y_{\overline{v}}$ is the set of paths $x = (x_i)$ such that $r(x_i) = v_i$ for all i. Let $\mu_{\overline{v}}$ be the ergodic measure on $Y_{\overline{v}}$ such that

$$\mu_{\overline{v}}([e(v_0, v_n]) = \left(\prod_{i=1}^{n-1} f_{v_{i+1},v_i}^{(i)}\right)^{-1}.$$

Let $\widehat{\mu}_{\overline{v}}$ be the extension of $\mu_{\overline{v}}$. Any odometer is trivially of exact finite rank (since it has rank one!), so it follows from Corollary 6.2 that

$$\widehat{\mu}_{\overline{v}}(X_B) < \infty \Longleftrightarrow \sum_{i=1}^{\infty}(1 - q_{v_{i+1},v_i}^{(i)}) < \infty, \tag{6.8}$$

where $q_{v_{i+1},v_i}^{(i)}$ are the entries of the corresponding stochastic matrix (2.3) taken along the sequence $\overline{v}$.

Corollary 6.6. *Let $\overline{v} = (v_0, v_1, ...)$ and $\overline{w} = (w_0, w_1, ...)$ be two sequences of vertices of a finite rank diagram B such that the corresponding measures $\widehat{\mu}_{\overline{w}}$ and $\widehat{\mu}_{\overline{v}}$ are finite. Then there exists a level n_0 such that for all $n \geq n_0$ either $w_n = v_n$ or $w_n \neq v_n$.*

Proof. Indeed, it follows from (6.8) that, without loss of generality, one can assume that for all n the inequality $q^{(n)}_{v_{n+1},v_n} > 1/2$ holds. Since the vector $(q^{(n)}_{v,w})_w$ is probability, there exists at most one vertex $w \in V_n$ such that, for a given $v \in V_{n+1}$, the entry $q^{(n)}_{v,w}$ is greater than $1/2$. □

Now we consider several examples which illustrate different cases of the proved theorems. In particular, one of the examples shows that if a component Y_α of a Bratteli diagram B supports several ergodic probability measures, then some of them might give rise to finite measures and some to infinite ones on $\mathcal{E}(Y_\alpha)$. We observe that our examples have some similarities with the examples constructed in [FFT09], but we use a completely different approach here. In all the examples below we extend ergodic measures from subdiagrams which have the simplest form possible, i.e., they have only one vertex at each level. We should note that not every measure can be obtained as an extension from such an elementary subdiagram.

Example 6.7. Let B be the Bratteli diagram with incidence matrices

$$F_n = \begin{pmatrix} b_n & 1 \\ 1 & c_n \end{pmatrix}, \quad n \geq 1.$$

Then B contains two natural subdiagrams B_1 and B_2 defined by odometers $\{b_n\}$ and $\{c_n\}$ "sitting" on left and right vertices v_1 and v_2, respectively. Let μ_1 and μ_2 be the two invariant probability measures on B_1 and B_2, respectively. Consider the extensions $\widehat{\mu}_1$ and $\widehat{\mu}_2$ of measures μ_1 and μ_2 on $X_1 = \mathcal{E}(Y_1)$ and $X_2 = \mathcal{E}(Y_2)$. To compute $\widehat{\mu}_1(X_1)$, we use the relation (for $\widehat{\mu}_2(X_2)$ we have similar formulas)

$$\widehat{\mu}_1(X_1) = \lim_{n\to\infty} \widehat{\mu}_1(X_1(n)),$$

where $X_1(n) = \{x = (x_i) \in X_B : r(x_i) = v_1,\ i \geq n\}$. Notice that for $n \geq 1$,

$$\begin{aligned} h_1^{(n)} &= b_{n-1}h_1^{(n-1)} + h_2^{(n-1)}, \\ h_2^{(n)} &= c_{n-1}h_2^{(n-1)} + h_1^{(n-1)}. \end{aligned}$$

Then

$$\begin{aligned} \widehat{\mu}_1(X_1(n)) &= \widehat{\mu}_1(X_1(1)) + \sum_{i=2}^{n}(\widehat{\mu}_1(X_1(i)) - \widehat{\mu}_1(X_1(i-1))) \\ &= 1 + \sum_{i=2}^{n}\Big(\frac{h_1^{(i)}}{b_{i-1}\cdots b_1} - \frac{h_1^{(i-1)}}{b_{i-2}\cdots b_1}\Big) \\ &= 1 + \sum_{i=2}^{n}\Big(\frac{b_{i-1}h_1^{(i-1)} + h_2^{(i-1)}}{b_{i-1}\cdots b_1} - \frac{h_1^{(i-1)}}{b_{i-2}\cdots b_1}\Big) \\ &= 1 + \sum_{i=2}^{n}\frac{h_2^{(i-1)}}{b_{i-1}\cdots b_1}. \end{aligned}$$

Finally,

$$\widehat{\mu}_1(X_1) = 1 + \sum_{i=1}^{\infty}\frac{h_2^{(i)}}{b_i\cdots b_1}. \tag{6.9}$$

Thus,

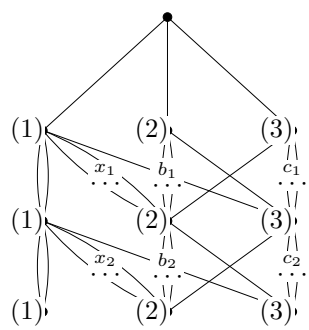
$$\widehat{\mu}_1(X_1) < \infty \Longleftrightarrow \sum_{i=1}^{\infty} \frac{h_2^{(i)}}{b_i \cdots b_1} < \infty.$$

We note that the function $h_2^{(i)}$ depends on $b_1, ..., b_{i-2}$ and $c_1, ..., c_{i-1}$. Based on this observation, we can easily show that the following statement holds:

For any sequence $\{c_n\}$, there exists a sequence $\{b_n\}$ such that $\widehat{\mu}_1(X_1) < \infty$. Similarly, given a sequence $\{b_n\}$, one can find a sequence $\{c_n\}$ such that $\widehat{\mu}_2(X_1) < \infty$. Moreover, one can construct sequences $\{b_n\}$ and $\{c_n\}$ to obtain both measures $\widehat{\mu}_1$ and $\widehat{\mu}_2$ simultaneously, either finite or infinite.

Indeed, formula (6.9) says that, independently of $h_2^{(i)}$, we can always choose b_i to ensure the convergence of the series $\sum_{i=1}^{\infty} h_2^{(i)}(b_i \cdots b_1)^{-1}$. This is possible because b_i is not involved in the formula for $h_2^{(i)}$. Clearly, this kind of argument proves the claim above.

Now we consider the following Bratteli diagram $\overline{B}$:



The incidence matrices of $\overline{B}$ have the form

$$F_n = \begin{pmatrix} 2 & 0 & 0 \\ x_n & b_n & 1 \\ 1 & 1 & c_n \end{pmatrix}.$$

We have proved above that there are sequences $\{b_n\}$ and $\{c_n\}$ such that the subdiagram B of $\overline{B}$ has two finite ergodic measures $\widehat{\mu}_1$ and $\widehat{\mu}_2$. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be extensions of $\widehat{\mu}_1$ and $\widehat{\mu}_2$ from B to $\overline{B}$. In other words, we extend these measures to path spaces $\mathcal{E}(X_i)$, $i = 1, 2$, in the diagram $\overline{B}$. Direct computations, similar to those above, show that one can choose sequences $\{x_n\}$, $\{b_n\}$, and $\{c_n\}$ such that the measure $\overline{\mu}_1$ is infinite and the measure $\overline{\mu}_2$ is finite.

Remark 6.8. (1) One can slightly modify Example 6.7 and consider the sequence of incidence matrices

$$F_n = \begin{pmatrix} b_n & s_n \\ t_n & c_n \end{pmatrix}, \quad n \geq 1,$$

such that the additional condition $b_n + s_n = t_n + c_n = h_n$ holds. Then the corresponding stochastic matrix Q_n has the form

$$Q_n = \begin{pmatrix} \frac{b_n}{h_n} & 1 - \frac{b_n}{h_n} \\ 1 - \frac{c_n}{h_n} & \frac{c_n}{h_n} \end{pmatrix} = \begin{pmatrix} 1 - \varepsilon_n & \varepsilon_n \\ \eta_n & 1 - \eta_n \end{pmatrix}$$

because $h_v^{(n+1)} = h_n h_v^{(n)}$ for any vertex v. It is not hard to show that if

$$\sum_n (\varepsilon_n + \eta_n) < \infty,$$

then there are two finite ergodic invariant measures, and if $\sum_n (\varepsilon_n + \eta_n) = \infty$, then the diagram constructed by $\{F_n\}$ is uniquely ergodic.

(2) We also note that the method of Example 6.7 can be applied to construct a simple diagram with d vertices at each level, having exactly k finite ergodic measures, $k \le d$.

Remark 6.9. We observe that there are dynamical systems having both Bratteli diagrams with exact finite rank property and without it. We are grateful to the referee for suggesting the following example. Consider a Bratteli diagram B with single edges between the root vertex and vertices of level one and with incidence matrices

$$F_n = \begin{pmatrix} 1 & m_n \\ 1 & m_n \end{pmatrix}.$$

Let μ be the extension of the (unique) invariant measure from the right subdiagram. We can choose the sequence $\{m_n\}_{n\ge1}$ growing so fast that the measure μ is finite; see equation (6.8). Once the measure μ is finite, we get that $\mu(X_1^{(n)}) \to 0$ as $n \to \infty$, where $X_1^{(n)}$ is the set of all paths coming through the left vertex at level n; see Theorem 3.3 for the details. This implies that the Bratteli diagram B is not of exact finite rank.

Consider the left-to-right ordering '$<$' on the diagram B, i.e., if edges e_1 and e_2 are such that $s(e_1) < s(e_2)$, then $e_1 < e_2$. Let φ_B be the Vershik map determined by this ordering; see Section 2.4. Notice that the diagram B has exactly one minimal path and one maximal path. Therefore, $\varphi_B : X_B \to X_B$ is a minimal homeomorphism. For each level $n \ge 1$ and a vertex $v \in V_n := \{1, 2\}$, denote by $B_v^{(n)}$ the set of all infinite paths $x = (x_1, x_2, \dots,)$ such that the path $(x_1, \dots, x_n)$ is minimal amongst paths connecting v to the root vertex. Let $h_v^{(n)}$ be the number of finite paths between v and the root vertex. Hence

$$\mathcal{P}_n = \{\varphi_B^i(B_v^{(n)}) : i = 0, \dots, h_v^{(n)-1},\ v = 1, 2\}$$

is a Kakutani-Rokhlin partition of the path-space.

Notice that $h_1^{(n)} = h_2^{(n)}$ for every $n \ge 1$. Set $B_n = B_1^{(n)} \cup B_2^{(n)}$. Since the ordering is from left to right, it is not difficult to check that $\mathcal{P}'_n = \{\varphi_B^i(B_n) : i = 0, \dots, h_1^{(n)} - 1\}$ forms a sequence of nested Kakutani-Rokhlin partitions of X_B. Notice that $\bigcap_{n\ge1} B_n$ is a singleton (the left vertical path), $\{\mathcal{P}'_n\}$ separates points of X_B, and the tower $\mathcal{P}'_{n+1}$ intersects $\mathcal{P}'_n$ exactly $m_n + 1$ times. Thus, using the standard cutting and stacking construction (see, for example, [D05]), one can show that the system (X_B, φ_B) is topologically isomorphic to the $\{m_n+1\}_{n=0}^{\infty}$-adic odometer, where $m_0 = 0$. Hence, the system (X_B, φ_B) also has a Bratteli-Vershik model of exact finite rank.

6.3. **Bratteli diagrams of finite complexity.**

Definition 6.10. A Bratteli diagram is said to be *of finite complexity* if it has finitely many different incidence matrices.

Bratteli diagrams of finite complexity have a symbolic counterpart in *S-adic subshifts*; see [Du00]. They arise, in particular, as Bratteli-Vershik models for linearly recurrent Cantor systems; see [DHS99], [Du00], and [CDHM03] for the details. We observe that all substitution minimal systems and all Sturmian systems admit Bratteli-Vershik models of finite complexity [Du00, DDM00], but not all Sturmian systems are linearly recurrent.

We begin our discussion of measures with the following illustrative example.

Example 6.11. Let the diagram B be defined by the incidence matrices

$$F_n = \begin{pmatrix} \tau_n & 0 \\ a_n & \omega_n \end{pmatrix}, \quad n \geq 1,$$

where the entries of F_n are positive integers (greater than one). Let μ be the probability measure defined by the odometer $\{\omega_i\}$. It can be easily shown that

$$\widehat{\mu}(X_B) = 1 + \sum_{i=1}^{\infty} a_i \frac{\tau_{i-1} \cdots \tau_1}{\omega_i \cdots \omega_1} \tag{6.10}$$

(we skipped a routine computation). Then for a particular case when $\omega_n \in \{2, 3\}$, $w_1 = 3$, $a_n = 1$ and $\tau_n = 2$, we obtain

$$\widehat{\mu}(X_B) = 1 + \sum_{i=1}^{\infty} \frac{2^{i-1}}{\omega_i \cdots \omega_1} = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n (i_{n+1} - i_n), \tag{6.11}$$

where $1 = i_1 < i_2 < \cdots < i_n < \cdots$, are all the numbers with $w_{i_n} = 3$. Relation (6.11) yields a number of sufficient conditions for finiteness of $\widehat{\mu}(X_B)$. In particular, suppose that

$$i_{n+1} - i_n \leq K n^c, \quad K, c \in \mathbb{R}_+,$$

for sufficiently large n. Then $\widehat{\mu}(X_B) < \infty$.

Now we will extend this example to the case of diagrams of finite complexity. Let $B = (V, E)$ be a Bratteli diagram of finite complexity with incidence matrices $\{F_n\}_{n \geq 1}$. Denote by $\mathcal{A}$ the set of all different incidence matrices. Then the diagram B naturally defines a sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ with $\omega_i = F_i$. It turns out that the growth rate of the product $||F_n \cdots F_1||_1$ depends heavily on the combinatorial properties of the sequence ω. The next proposition, which was essentially proved in [JB90], is a crucial step for getting estimates for the growth of matrix products.

Let R be a diagonal matrix with positive diagonal entries. Set

$$M(R) = \max_{i,j} R_{i,i} R_{j,j}^{-1}, \qquad m(R) = \min_{i,j} R_{i,i} R_{j,j}^{-1}.$$

Then for any non-negative matrix A, we have the inequalities

$$m(R)||A||_1 \leq ||R^{-1} A R||_1 \leq M(R)||A||_1.$$

For a positive vector x, denote by D_x the diagonal $d \times d$ matrix whose diagonal entries are the entries of x written in the same order. For two positive vectors x and y, denote by x/y their componentwise ratio, i.e., $x/y = (x_1/y_1, ..., x_d/y_d)$. For a vector $x > 0$, let $x_{\max}$ be the maximal entry of x and $x_{\min}$ the minimal one.

Proposition 6.12. *Let $A_1, \ldots, A_n$ be primitive matrices. Let x_i denote a Perron-Frobenius eigenvector for the matrix A_i and $\rho(A_i)$ its spectral radius. Then*

$$\frac{||A_1 A_2 \cdots A_n||_1}{\rho(A_1)\rho(A_2)\cdots\rho(A_n)} \leq \frac{1}{m(D_{x_n})} \left(\frac{x_n}{x_{n-1}}\right)_{\max} \cdots \left(\frac{x_2}{x_1}\right)_{\max} \left(\frac{x_1}{x_n}\right)_{\max}$$

and

$$\frac{||A_1A_2\cdots A_n||_1}{\rho(A_1)\rho(A_2)\cdots\rho(A_n)} \geq \frac{1}{M(D_{x_n})}\left(\frac{x_n}{x_{n-1}}\right)_{\min}\cdots\left(\frac{x_2}{x_1}\right)_{\min}\left(\frac{x_1}{x_n}\right)_{\min}.$$

Proof. It was shown in the proof of Theorem 1 from [JB90] that

$$D_{x_n}^{-1}A_1A_2\cdots A_nD_{x_n}\overline{1}$$

$$\leq \rho(A_1)\rho(A_2)\cdots\rho(A_n)\left(\tfrac{x_n}{x_{n-1}}\right)_{\max}\cdots\left(\tfrac{x_2}{x_1}\right)_{\max}\left(\tfrac{x_1}{x_n}\right)_{\max}\overline{1},$$

where $\overline{1}=(1,\dots,1)^T$. We note that

$$||A_1A_2\cdots A_n||_1 \leq \frac{1}{m(D_{x_n})}||D_{x_n}^{-1}A_1A_2\cdots A_nD_{x_n}||_1$$

and then we use (4.2) to prove the first inequality. The second one follows from the proof of [JB90, Theorem 1] in a similar way by reversing the inequalities. □

Next, consider the sequence $\omega\in\mathcal{A}^{\mathbb{N}}$ defined by a Bratteli diagram B of finite complexity as above. Let $I_A(n)$ be the number of occurrences of the letter A in the word $\omega_1\omega_1\cdots\omega_n$. Let $\mathcal{A}^{(2)}$ be the set of all words of length two from the sequence ω. Denote by $I_{AB}(n)$ the number of occurrences of the pair AB in the word $(\omega_1\omega_2)(\omega_2\omega_3)\cdots(\omega_n\omega_{n+1})$.

Definition 6.13. We will say that the diagram B of finite complexity is *regular* if for every matrix $A\in\mathcal{A}$ and every pair $AB\in\mathcal{A}^{(2)}$ the limits

$$d(A)=\lim_{n\to\infty}\frac{I_A(n)}{n},\quad d(AB)=\lim_{n\to\infty}\frac{I_{AB}(n)}{n}$$

exist. We call $d(A)$ the density of A in ω and $d(AB)$ the density of AB in the sequence $(\omega_1\omega_2)(\omega_2\omega_3)(\omega_3\omega_4)\cdots$.

Let x_A be a Perron-Frobenius eigenvector of $A\in\mathcal{A}$. For any pair of matrices A and B with $AB\in\mathcal{A}^{(2)}$, denote by $\overline{r}(A,B)$ the ratio $(x_B/x_A)_{\max}$. Similarly, we set $\underline{r}(A,B)$ to be the ratio $(x_B/x_A)_{\min}$. Finally, we set

$$\overline{\rho}(\omega)=\prod_{A\in\mathcal{A}}\rho(A)^{d(A)}\times\prod_{AB\in\mathcal{A}^{(2)}}\overline{r}(A,B)^{d(AB)}.$$

We refer to the number $\overline{\rho}(\omega)$ as the *upper spectral radius along the sequence* ω. The number $\underline{\rho}(\omega)$ is defined similarly by using the values $\underline{r}(A,B)$.

The next lemma shows that $\overline{\rho}(\omega)$ and $\underline{\rho}(\omega)$ are well-defined and may serve as the upper and lower bounds for the products of incidence matrices, respectively.

Lemma 6.14. *Let B be a regular diagram of finite complexity with the sequence of primitive incidence matrices $\omega\in\mathcal{A}^{\mathbb{N}}$. Then*

(1) $\overline{\rho}(\omega)$ and $\underline{\rho}(\omega)$ do not depend on the choice of eigenvectors x_A, $A\in\mathcal{A}$;

(2) the following inequalities hold:

$$\liminf_{n\to\infty}\left(||\omega_1\omega_2\cdots\omega_n||_1\right)^{\frac{1}{n}}\geq\underline{\rho}(\omega)$$

and

$$\limsup_{n\to\infty}(||\omega_1\omega_2\cdots\omega_n||_1)^{\frac{1}{n}}\leq\overline{\rho}(\omega).$$

Proof. (1) Let x_A be a Perron-Frobenius eigenvector of A and $x'_A = c_A x_A$, $c_A > 0$. For each n, define

$$\rho_n = \prod_{i=1}^{n} \rho(\omega_i) \cdot \overline{r}(\omega_i, \omega_{i+1}).$$

Let the number ρ'_n be defined similarly to ρ_n, but with the eigenvectors x_A and x_B replaced by x'_A and x'_B. Then, it is not hard to check that

$$\rho'_n = \frac{c_{w_{n+1}}}{c_{w_1}} \rho_n \text{ for all } n.$$

Since the set $\{c_A : A \in \mathcal{A}\}$ is finite, we get that

$$\lim_{n\to\infty} \left(\frac{\rho_n}{\rho'_n}\right)^{\frac{1}{n}} = 1.$$

On the other hand, we see that

$$(\rho_n)^{\frac{1}{n}} = \prod_{A\in\mathcal{A}} \rho(A)^{\frac{I_A(n)}{n}} \times \prod_{AB\in\mathcal{A}^{(2)}} \overline{r}(A,B)^{\frac{I_{AB}(n)}{n}} \to \overline{\rho}(\omega)$$

as $n \to \infty$. This shows that the definition of $\overline{\rho}(\omega)$ does not depend on the choice of Perron-Frobenius eigenvectors. The proof for $\underline{\rho}(\omega)$ is similar and left to the reader.

(2) Using Proposition 6.12 and the fact that the set of matrices is finite, we can find a constant $K > 0$, which does not depend on n, such that

$$\begin{aligned}
(||\omega_1\omega_2\cdots\omega_n||_1)^{\frac{1}{n}} &\leq \left(K \prod_{i=1}^{n} \rho(\omega_i) \cdot \overline{r}(\omega_i, \omega_{i+1})\right)^{\frac{1}{n}} \\
&= K^{\frac{1}{n}} \prod_{A\in\mathcal{A}} \rho(A)^{\frac{I_A(n)}{n}} \times \prod_{AB\in\mathcal{A}^{(2)}} r(A,B)^{\frac{I_{AB}(n)}{n}} \\
&\to \overline{\rho}(\omega)
\end{aligned}$$

as $n \to \infty$. Thus, $\overline{\rho}(\omega) \geq \limsup_{n\to\infty} (||\omega_1\omega_2\cdots\omega_n||_1)^{\frac{1}{n}}$. The other inequality is established in a similar way. □

Let B be a regular Bratteli diagram of finite complexity whose incidence matrices have the form

$$F_n = \begin{pmatrix} D_n & 0 \\ A_n & C_n \end{pmatrix},$$

with D_n and C_n being primitive matrices.

By definition of B, the sequences $\{D_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ have only finitely many different matrices.

The following theorem shows that the spectral radii along the sequences $\{D_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ can distinguish the growth rates of the minimal and non-minimal components of B. This, in particular, answers the question of finiteness of the measure extension from the subdiagram $B(C)$ and allows one to distinguish certain non-orbit equivalent systems.

Theorem 6.15. *Let B be a regular diagram of finite complexity as above.*

(i) If $\underline{\rho}(\{D_n\}_{n\geq 1}) > \overline{\rho}(\{C_n\}_{n\geq 1})$, then the extension of the measure from $B(C)$ is infinite.

(ii) If $\overline{\rho}(\{D_n\}_{n\geq 1}) < \underline{\rho}(\{C_n\}_{n\geq 1})$, then the extension of the measure from $B(C)$ is finite.

Proof. We note that Proposition 5.7 implies that the measure of towers is bounded away from zero and the tower heights grow with the same speed within the subdiagram $B(C)$ and $B(D)$. In view of Corollary 6.4 and Remark 6.5, it is sufficient to verify whether the series

$$\sum_{n=1}^{\infty} \frac{||D_{n-1}\cdots D_1||_1}{||C_n\cdots C_1||_1}$$

is convergent or not. Fix $\varepsilon > 0$ so that $\underline{\rho}(\{D_n\}_{n\geq 1}) - \varepsilon > \overline{\rho}(\{C_n\}_{n\geq 1}) + \varepsilon$. Set

$$r = \limsup_{n\to\infty} \left(\frac{||D_{n-1}\cdots D_1||_1}{||C_n\cdots C_1||_1}\right)^{\frac{1}{n}}.$$

Then, by Lemma 6.14, we get that

$$\begin{aligned}\sup_{n\geq k}\left(\frac{||D_{n-1}\cdots D_1||_1}{||C_n\cdots C_1||_1}\right)^{\frac{1}{n}} &\geq \frac{\inf_{n\geq k}(||D_{n-1}\cdots D_1||_1)^{1/n}}{\sup_{n\geq k}(||C_n\cdots C_1||_1)^{1/n}}\\ &\geq \frac{\underline{\rho}(\{D_n\}_{n\geq 1}) - \varepsilon}{\overline{\rho}(\{C_n\}_{n\geq 1}) + \varepsilon}\\ &> 1\end{aligned}$$

for all k large enough. This implies that $r > 1$ and, hence, the series diverges by the root test. The fact that condition (ii) leads to the convergent series (with $r < 1$) is proved similarly. □

Remark 6.16. (1) We observe that statement (i) in Theorem 6.15 implies that the diagram B has a unique invariant measure supported only by the minimal component. On the other hand, statement (ii) guarantees the existence of a fully supported invariant measure (along with the measure on the minimal component).

(2) We also note that it is possible to treat the numbers

$$\overline{\lambda}(\omega) = \limsup_{n\to\infty}(||\omega_1\cdots\omega_n||_1)^{\frac{1}{n}} \text{ and } \underline{\lambda}(\omega) = \liminf_{n\to\infty}(||\omega_1\cdots\omega_n||_1)^{\frac{1}{n}}$$

as the growth rate for matrix products. Then Theorem 6.15 still holds if we replace $\overline{\rho}(\omega)$ with $\overline{\lambda}(\omega)$ and $\underline{\rho}(\omega)$ with $\underline{\lambda}(\omega)$.

7. Absence of strong mixing

In this section we study mixing properties of Vershik maps on finite rank Bratteli diagrams. We will prove that if an invariant measure has the property that the measure values of all towers are bounded away from zero (i.e., it has exact finite rank), then any Vershik map on such a diagram is not strongly mixing. This was earlier proved by A. Rosenthal [Ro84] in the context of measure-preserving transformations of exact finite rank by very different methods, in a hard-to-find unpublished manuscript. We then establish the absence of mixing if a Bratteli diagram (not necessarily simple or uniquely ergodic) is equipped with the so-called *consecutive ordering*.

The absence of strong mixing has been earlier established for substitution systems [DK78], [BKMS10], interval exchange transformations [K80], and linearly recurrent systems [CDHM03]. We also mention the Ph.D. thesis of Wargan [War02], devoted to the study of some generalizations of linearly recurrent systems where he proved the absence of strong mixing for such systems. Our methods have some common features with those of [K80].

We start with a preliminary lemma.

Lemma 7.1. *Let $\{(Y_n, \nu_n, S_n)\}_{n\geq 1}$ be a family of probability measure-preserving transformations, where Y_n is a shift-invariant subset of $A^{\mathbb{Z}}, |A| < \infty$, and S_n denotes the left shift. Then there is a word $\omega = \omega_0 \cdots \omega_{r-1}$ from A^+ such that $\omega_0 = \omega_{r-1}$ and $\limsup_n \nu_n([\omega]) > 0$.*

Proof. Set $d = |A|$. Then for every n we have $\sum_{w\in A^{d+1}} \nu_n([w]) = 1$. Therefore,

$$\sum_{w\in A^{d+1}} \limsup_n \nu_n([w]) \geq \limsup_n \sum_{w\in A^{d+1}} \nu_n([w]) = 1.$$

Choose $w \in A^{d+1}$ with $\limsup_n \nu_n([w]) > 0$. Then the word w contains a subword ω starting and ending with the same letter. Hence $\nu_n([\omega]) \geq \nu_n([w])$ for all n, and we are done. □

Theorem 7.2. *Let $B = (V, E, \leq)$ be an ordered simple Bratteli diagram of exact finite rank. Let $T : X_B \to X_B$ be the Vershik map defined by the order $\leq$ on B (T is not necessarily continuous everywhere). Then the dynamical system (X_B, μ, T) is not strongly mixing with respect to the unique invariant measure μ.*

Proof. (I) In the proof we will consider the family $\{X_v^{(n)} : v \in V_n\}$ as a Kakutani-Rokhlin partition of X_B. Then $X_v^{(n)} = \{B_n(v), \ldots, T^{h_v^{(n)}-1} B_n(v)\}$ is a T-tower, where $h_v^{(n)}$ is the number of finite paths from the top vertex v_0 to a vertex v of level n, and $B_n(v)$ is the cylinder set generated by the finite minimal path connecting the vertices v_0 and v.

Set $B_n = \bigsqcup_v B_n(v)$. Consider the induced system (B_n, μ_n, T_n), with $\mu_n = \mu|_{B_n}/\mu(B_n)$, the probability measure invariant with respect to the induced transformation T_n. Set $A = \{1, \ldots, d\}$. Define the map $\pi_n : B_n \to A^{\mathbb{Z}}$ by $\pi_n(x)_i = v$ if and only if $T_n^i(x) \in B_n(v)$. Denote by (Y_n, ν_n, S_n) the factor-system determined by π_n (i.e., $Y_n = \pi_n(B_n)$, and $\nu_n = \pi_n^* \mu_n$).

Applying Lemma 7.1 to the family $\{(Y_n, \nu_n, S_n)\}_{n\geq 1}$, choose a word $\omega = \omega_0 \cdots \omega_{r-1}$ with $\omega_0 = \omega_{r-1}$ and $\limsup_n \nu_n([\omega]) > 0$.

(II) For each infinite path $x \in X_B$, denote by $v_n(x)$ the vertex of level n the path x goes through. Fix a level m and apply the pointwise ergodic theorem to the induced system (B_m, μ_m, T_m) and the set $F_m = \pi_m^{-1}([\omega])$. Then for μ_m-a.e. $x \in B_m$, we have

$$\mu_m(F_m) = \lim_{n\to\infty} \frac{1}{i_n^{(m)}(x) + j_n^{(m)}(x)} \sum_{i=-i_n^{(m)}(x)}^{j_n^{(m)}(x)} 1_{F_m}(T_m^i(x)), \tag{7.1}$$

where $i_n^{(m)}(x)$ is the least integer such that $T_m^{-i_n^{(m)}(x)}$ maps the initial segment of x to the minimal finite path from the set $E(v_0, v_n(x))$. Similarly, $j_n^{(m)}(x)$ is the least integer such that $T_m^{j_n^{(m)}(x)}$ maps the initial segment of x to the maximal path from $E(v_0, v_n(x))$ within B_m. Notice that $j_n^{(m)}(x) + i_n^{(m)}(x)$ is the number of finite paths from the vertices of level m to the vertex $v_n(x)$.

Define the map σ_n from V_n into the set of finite words over V_{n-1} by setting $\sigma_n(v) = s_0 \cdots s_p$, where $s_i \in V_{n-1}$ and $\{s_0, \ldots, s_p\}$ are the sources of the edges terminating at v and taken in the order of $\leq$. In other words, we symbolically

encode the order $\leq$. For $n > m$, set $\sigma^{(m,n)} = \sigma_m \circ \cdots \circ \sigma_{n+1}$. Notice that $|\sigma^{(m,n)}(v)|$ is the number of paths from v to the vertices of level m.

The definition of the set F_m implies that $T_m^i(x) \in F_m$ for some $-i_n^{(m)}(x) \leq i \leq j_n^{(m)}(x)$ if and only if the word ω occurs in $\sigma^{(m,n)}(v_n(x))$ at the position $i + i_n^{(m)}(x)$. Thus, the frequency of ω in $\sigma^{(m,n)}(v_n(x))$ is equal to

$$\mathrm{freq}(\omega, \sigma^{(m,n)}(v_n(x))) = \frac{1}{i_n^{(m)}(x) + j_n^{(m)}(x)} \sum_{i=-i_n^{(m)}(x)}^{j_n^{(m)}(x)} 1_{F_m}(T_m^i(x)).$$

Since $\mu(X_v^{(n)}) \geq \delta > 0$ for all n and v (δ is taken from the definition of exact finite rank), given a vertex v, the set of all paths visiting v infinitely many times has measure one (this follows by ergodicity, as in the proof of Theorem 3.3(II)). Hence (7.1) implies

$$\mu_m(F_m) = \lim_{n_k \to \infty} \mathrm{freq}(\omega, \sigma^{(m,n_k)}(v)), \tag{7.2}$$

for a subsequence n_k, for every vertex v.

(III) Since $\limsup_m \mu_m(F_m) = \limsup_m \nu_m([\omega]) > 0$, equation (7.2) guarantees that there is a telescoping of the diagram such that

$$\mathrm{freq}(\omega, \sigma^{(m,m+1)}(v)) \geq \rho > 0$$

for all m and v. For every level n, define the set S_n of all infinite paths $x \in B_n(\omega_0)$ such that the sources of the first $r-1$ (with respect to $\leq$) successors of the edge x_{n+1} (between levels n and $n+1$) are exactly the vertices $\omega_1, \ldots, \omega_{r-1}$. Also set $C_n = \bigsqcup_{i=0}^{h_{\omega_0}^{(n)}-1} T^i S_n$ (C_n is a subtower of $X_{\omega_0}^{(n)}$). Denote by $(f_{v,w}^{(n)})$ the entries of the n-th incidence matrix. Then

$$\begin{aligned}
\mu(C_n) &= h_{\omega_0}^{(n)} \sum_{v \in V_{n+1}} \mu(B_{n+1}(v)) \mathrm{freq}(\omega_0, \sigma^{(n,n+1)}(v)) f_{v,\omega_0}^{(n)} \\
&\geq \rho h_{\omega_0}^{(n)} \sum_{v \in V_{n+1}} \mu(B_{n+1}(v)) f_{v,\omega_0}^{(n)} \\
&= \rho \mu(X_{\omega_0}^{(n)}).
\end{aligned}$$

It follows that there exists $\gamma > 0$ such that $\mu(C_n) \geq \gamma > 0$ for all n. Set

$$q_n = h_{\omega_0}^{(n)} + \cdots + h_{\omega_{r-2}}^{(n)}.$$

Since $\omega_0 = \omega_{r-1}$, we obtain that for all $n \geq 1$ and $\ell = 0, \ldots, h_{\omega_0}^{(n)} - 1$,

$$T^{q_n + \ell} S_n \subset T^\ell B_n(\omega_0). \tag{7.3}$$

(IV) Choose a level n_0 such that $\mu(B_{n_0}(v)) < \gamma/2$ for all $v = 1, \ldots, d$. For each level $n \geq n_0$, there is a vertex v_n such that $B_n(\omega_0) \subset B_{n_0}(v_n)$. By telescoping we may assume that $v_n = v$ for all n. Set $D_n = C_n \cap B_{n_0}(v)$. We note that $\mu(D_n) > 0$.

Since the Kakutani-Rokhlin partitions $\{X_v^{(n)}\}$ associated to a Bratteli diagram are nested, we obtain that the sets $T^\ell B_n(\omega_0)$, for $0 \leq \ell < h_{\omega_0}^{(n)}$, either lie in the set $B_{n_0}(v)$ or are disjoint from it. Hence, by the definition of D_n, we obtain that if $x \in D_n$, then $x \in T^\ell S_n \subset T^\ell B_n(\omega_0) \subset B_{n_0}(v)$, for some $0 \leq \ell < h_{u_0}^{(n)}$. Condition (7.3) implies that $T^{q_n} x \in T^\ell B_n(\omega_0) \subset B_{n_0}(v)$. Thus,

$$T^{q_n} D_n \subset B_{n_0}(v) \text{ for all } n \geq 1.$$

Hence, $D_n \subset B_{n_0}(v) \cap T^{-q_n} B_{n_0}(v)$. As the Vershik map is aperiodic, we conclude that $q_n \to \infty$ as $n \to \infty$. Thus, the theorem would be proved if we show that

$$\limsup_{n\to\infty} \mu(D_n)/\mu(B_{n_0}(v)) \geq \gamma, \tag{7.4}$$

because then for some $n = n_k \to \infty$ we will have

$$\mu(B_{n_0}(v) \cap T^{-q_n} B_{n_0}(v)) \geq \mu(D_n) \geq (\gamma/2)\mu(B_{n_0}(v)) > \mu(B_{n_0}(v))^2.$$

(V) By the pointwise ergodic theorem we may find a path x that visits the vertex ω_0 infinitely many times and such that

$$\frac{|\{-i_n^{(1)}(x) \leq \ell \leq j_n^{(1)}(x) : T^\ell(x) \in B_{n_0}(v)\}|}{h^{(n)}_{v_n(x)}} \to \mu(B_{n_0}(v)) \text{ as } n \to \infty.$$

Let $\mathcal{N} := \{n : \ v_n(x) = \omega_0\}$, which is infinite by assumption. (Here we use the same notation as in (7.1), which is consistent with (5.1); note that $T_1 = T$.) Then we have for all $n \in \mathcal{N}$:

$$\begin{aligned}
\mu(D_n) &= \frac{|\{\ell = 0, \ldots, h^{(n)}_{\omega_0} - 1 : T^\ell B_n(\omega_0) \subset B_{n_0}(v)\}|}{h^{(n)}_{\omega_0}} \mu(S_n) h^{(n)}_{\omega_0} \\
&= \frac{|\{-i_n^{(1)}(x) \leq \ell \leq j_n^{(1)}(x) : T^\ell(x) \in B_{n_0}(v)\}|}{h^{(n)}_{\omega_0}} \mu(S_n) h^{(n)}_{\omega_0} \\
&\sim \mu(B_{n_0}(v))\mu(C_n) \geq \gamma\mu(B_{n_0}(v)) \text{ as } n \to \infty, \ n \in \mathcal{N}.
\end{aligned}$$

This proves (7.4), and the theorem follows. □

The last theorem holds for any order on the Bratteli diagram. In the next result we show that a somewhat regular ordering allows us to drop the assumption of exact finite rank.

Following [Du10, Chapter 6], by the *consecutive ordering* we mean an ordering on a diagram such that whenever edges e, f, g have the same range, $e \leq f \leq g$, and e and g have the same source, then f has the same source as e and g. We remark that such an ordering is not preserved under the telescoping. Bratteli diagrams with a special case of the consecutive ordering were discussed in [Mel06], [B06], [BP08], and [Du10].

Theorem 7.3. *Let $B = (V, E, \leq)$ be an ordered (not necessarily simple) Bratteli diagram of finite rank, where $\leq$ is a consecutive ordering. Let $T : X_B \to X_B$ be a Vershik map defined by the order $\leq$ on B (T is not necessarily continuous everywhere) and μ a finite T-invariant measure. Assume that if two vertices in consecutive levels are connected by an edge, then there are at least two such edges. Then the dynamical system (X_B, μ, T) is not strongly mixing.*

Proof. To prove the result, we will use the same idea as in the proof of Theorem 7.2. First of all, using Theorem 3.3, we choose a vertex ω_0 such that

$$\mu(X^{(n_k)}_{\omega_0}) \geq \delta > 0 \tag{7.5}$$

along some sequence $n_k \to \infty$ as $k \to \infty$. Set

$$C_n = \{x \in X_B : r(x_n) = \omega_0 \text{ and the source of the successor of } x_{n+1} \text{ is } \omega_0\}.$$

Clearly, C_n is a subtower of $X_{\omega_0}^{(n)}$. Denote by $(f_{w,v}^{(n)})$ the entries of the n-th incidence matrix. Then the definition of the ordering implies that

$$\mu(C_n) = h_{\omega_0}^{(n)} \sum_v (f_{v,\omega_0}^{(n)} - 1)\mu(B_n(v)).$$

Here the summation is taken over all v with $f_{v,\omega_0}^{(n)} > 0$. Since $(f_{v,\omega_0}^{(n)} - 1)/f_{v,\omega_0}^{(n)} \geq 1/2$ whenever $f_{v,\omega_0}^{(n)} > 0$, we get that

$$\mu(C_n) \geq \frac{1}{2} h_{\omega_0}^{(n)} \sum_v f_{v,\omega_0}^{(n)} \mu(B_n(v)) = \frac{1}{2}\mu(X_\omega^{(n)}) \geq \frac{\delta}{2},$$

when n runs along an infinite sequence.

Set $S_n = C_n \cap B_n(\omega_0)$. It follows from the definition of C_n that for all $n \geq 1$ and $\ell = 0, \ldots, h_{\omega_0}^{(n)} - 1$,

$$T^{q_n+\ell} S_n \subset T^\ell B_n(\omega_0),$$

where $q_n = h_{\omega_0}^{(n)}$. To complete the proof, it remains to repeat the arguments from the proof (part (IV)) of Theorem 7.2. We leave this to the reader. □

We do not know if there exist aperiodic Bratteli-Vershik systems of finite rank which are strongly mixing. The well-known Smorodinsky-Adams staircase transformation [A98] is not mixing, but it is constructed using "spacers", which implies that the Bratteli-Vershik model built on its symbolic realization (see [F96]) has a fixed point, and hence not aperiodic (it is also non-simple).

8. Conclusion

In this paper we performed a detailed analysis of invariant measures on finite rank aperiodic Bratteli diagrams, both simple and non-simple. Here are some of the key findings:

- We introduced the notion of a Bratteli diagram of exact finite rank, which parallels the same notion in measurable dynamics.
- Every ergodic measure (finite or infinite σ-finite) is an extension of a finite invariant measure from a simple subdiagram of exact finite rank.
- Exact finite rank implies unique ergodicity.
- Exact finite rank and the identical asymptotic growth of towers imply the identical asymptotic behavior of measures of tower bases.
- Exact finite rank and the identical asymptotic behavior of measures of tower bases imply the identical asymptotic growth of towers.
- The equality of tower heights does not guarantee the unique ergodicity and, as a result, exact finite rank.
- Exact finite rank does not ensure the same asymptotic growth of tower heights and the identical asymptotic behavior of measures of tower bases.
- Exact finite rank implies the absence of strong mixing for the Vershik map for any ordering on the diagram.
- Without the exact finite rank assumption, if the ordering of the Bratteli diagram is consecutive, then the Vershik map is not strongly mixing.

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