PATH CONNECTED COMPONENTS
IN WEIGHTED COMPOSITION OPERATORS
ON $h^\infty$ AND $H^\infty$ WITH THE OPERATOR NORM

KEI JI IZUCHI, YUKO IZUCHI, AND SH ŪICHI OHNO

ABSTRACT. We consider the component problem on the sets of weighted composition operators on the spaces of bounded harmonic and analytic functions on the open unit disk with the operator norms, respectively. Especially, we shall determine path connected components in the sets of noncompact weighted composition operators.

1. Introduction

Throughout this paper, let $h^\infty$ and $H^\infty$ be the spaces of bounded harmonic and analytic functions on the open unit disk $D$, respectively. For $f \in h^\infty$, we write $\|f\|_\infty = \sup\{|f(z)| : z \in D\}$. We may define $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ a.e. on the boundary $\partial D$ of $D$. It is known that $\{f^* : f \in h^\infty \} = L^\infty(\partial D)$. We denote by $\|f^*\|_{\partial D}$ the essential supremum norm of $f^*$ on $\partial D$. Then $\|f^*\|_{\partial D} = \|f\|_\infty$. For $u \in L^\infty(\partial D)$, let $\hat{u}(z)$ be the harmonic extension of $u$ defined by

$$\hat{u}(z) = \int_{\partial D} u(e^{i\theta}) P_z(e^{i\theta}) \, dm(e^{i\theta}) \quad \text{for} \quad z \in D,$$

where $P_z$ is the Poisson kernel for the point $z \in D$ and $m$ is the normalized Lebesgue measure on $\partial D$. Besides, for $f \in h^\infty$, $\hat{f}^* = f$ on $D$.

Let $S(D)$ be the set of analytic self-maps of $D$. For each $\varphi \in S(D)$, we may define the composition operator $C_\varphi$ by $C_\varphi f = f \circ \varphi$, where $f$ is analytic on $D$. Composition operators on various analytic function spaces have been studied extensively during the past few decades. See [2, 14] for an overview of these results. Presently, some of the open questions in this field are related to the topological structure of the set of composition operators. For even $f \in h^\infty$, $C_\varphi f = f \circ \varphi$ is also harmonic on $D$. So the question of properties of $C_\varphi$ on $h^\infty$ naturally arises. In [1], Choa and the first and third authors first considered properties of composition operators on $h^\infty$ and, although $h^\infty$ is not algebra, introduced the concept of weighted composition operators $M_uC_\varphi$ on $h^\infty$ defined by

$$(M_uC_\varphi f)(z) = \int_{\partial D} u(e^{i\theta})(f \circ \varphi)^*(e^{i\theta}) P_z(e^{i\theta}) \, dm(e^{i\theta})$$

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for \( u \in L^\infty(\partial \mathbb{D}) \) and \( z \in \mathbb{D} \). We denote by \( C_w(h^\infty) \) the set of nonzero weighted composition operators on \( h^\infty \) with the operator norm \( \| M_u C_\varphi \|_{h^\infty} \). More precisely, \( C_w(h^\infty) = \{ M_u C_\varphi : u \in L^\infty(\partial \mathbb{D}), u \neq 0, \varphi \in \mathcal{S}(\mathbb{D}) \} \). Also, \( C(h^\infty) \) is the set of composition operators on \( h^\infty \) with the operator norm.

Similarly we may define \( C(H^\infty), C_w(H^\infty) \) and \( \| M_u C_\varphi \|_{H^\infty} \). Indeed, \( C_w(H^\infty) = \{ M_u C_\varphi : u \in H^\infty, u \neq 0, \varphi \in \mathcal{S}(\mathbb{D}) \} \). Let \( M_u C_\varphi \in C_w(H^\infty) \). Then \( u \in H^\infty, u \neq 0 \) and \( M_u C_\varphi f = u(f \circ \varphi) \) for \( f \in H^\infty \). In this paper, we investigate path connected components in \( C_w(h^\infty), C_w(H^\infty) \) and in other spaces.

In [12], MacCluer, Zhao and the third author determined path connected components in \( C(H^\infty) \) (see also [8]). We know that \( C(h^\infty) = C(H^\infty) \) as sets, but the topology of \( C(h^\infty) \) is stronger than the one of \( C(H^\infty) \). In [1] Theorem 3.6], Cho and the first and third authors showed that path connected components in \( C(h^\infty) \) and \( C(H^\infty) \) are the same.

Since \( H^\infty \subsetneq h^\infty \), we have \( C_w(H^\infty) \subsetneq C_w(h^\infty) \) as sets. In [7 Theorem 4.1] Hosokawa and the first and third authors showed the following.

**Theorem A.**

(i) Let \( \varphi \in \mathcal{S}(\mathbb{D}) \) and \( m(\{ |\varphi^*| = 1 \}) > 0 \). Then \( \{ M_u C_\varphi : u \in H^\infty, u \neq 0 \} \) is open and closed, and is a path connected component in \( C_w(H^\infty) \).

(ii) The set

\[
\{ M_u C_\varphi : u \in H^\infty, u \neq 0, m(\{ |\varphi^*| = 1 \}) = 0 \}
\]

is closed and is a path connected component in \( C_w(H^\infty) \).

See [6][11] for other subjects on \( C(H^\infty) \).

This paper is organized as follows. In the next section we shall recall precise arguments on \( L^\infty(\partial \mathbb{D}) \) functions investigated in [9] and study the component problem of \( C_w(h^\infty) \) in Section 3. In Theorem 3.6 we shall determine path connected components in \( C_w(h^\infty) \). In proving Theorems A and 3.6 some paths cross the space of compact weighted composition operators. Let \( C_{w,0}(h^\infty) \) and \( C_{w,0}(H^\infty) \) be the spaces of noncompact weighted composition operators on \( h^\infty \) and \( H^\infty \) with the operator norms, respectively. In [7], path connected components in \( C_{w,0}(H^\infty) \) were studied, but there remain some problems ([7 Problem 5.3]). We shall determine path connected components in \( C_{w,0}(h^\infty) \) in Theorem 3.11 In Section 4, we consider path connected components in \( C_w(H^\infty) \) and furthermore characterize path connected components in \( C_{w,0}(H^\infty) \) in Theorem 4.7 which answers problems given in [7]. We may say that the path connected components problem for weighted composition operators is solved completely on \( h^\infty \) and \( H^\infty \) with respect to the operator norms.

The key in this paper is the argument on the boundary \( \partial \mathbb{D} \) as given in [9][10].

2. \( L^\infty(\partial \mathbb{D}) \) functions

In order to study \( L^\infty(\partial \mathbb{D}) \) functions, we recall some elementary facts and notation used in [9]. For \( f \in h^\infty \), let \( R_f \) be the set of \( e^{i\theta} \in \partial \mathbb{D} \) at which \( f \) has a radial limit. We define a function \( f^* \) on \( R_f \) by

\[
f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \quad \text{for} \quad e^{i\theta} \in R_f.
\]

We have \( m(R_f) = 1 \) and \( \| f \|_\infty = \| f^* \|_{\partial \mathbb{D}} = \sup_{e^{i\theta} \in R_f} |f^*(e^{i\theta})| \).
Lemma 2.1. For each $e^{it} \in \partial \mathbb{D}$, let $L_u$ be the Lebesgue set of $u$, that is, $L_u$ is the set of points $e^{it} \in \partial \mathbb{D}$ satisfying
\[
\lim_{m(I) \to 0} \frac{1}{m(I)} \int_I |u(e^{i\theta}) - u(e^{it})| \, dm(e^{i\theta}) = 0,
\]
where $I$ are subarcs of $\partial \mathbb{D}$ centered at $e^{it}$. By [13, p. 138], $m(L_u) = 1$, and by [13] Theorem 11.23, $u(e^{i\theta}) = \lim_{r \to 1} \tilde{u}(re^{i\theta})$ for $e^{i\theta} \in L_u$, that is, $L_u \subset R_u$. By the definition of $L_u$, we have the following.

Lemma 2.2. For each $e^{i\theta} \in L_u$ there are a sequence of open subarcs $\{I_n\}_n$ of $\partial \mathbb{D}$ centered at $e^{i\theta}$ and a sequence of measurable subsets $\{E_n\}_n$ such that $E_n \subset I_n \cap L_u$, $m(I_n) \to 0$, $m(E_n)/m(I_n) \to 1$, and
\[
\lim_{n \to \infty} \sup_{e^{i\theta} \in E_n} |u(e^{i\theta}) - u(e^{it})| = 0.
\]

We note that the value of the function $u$ is well defined on $L_u$ and
\[
\|u\|_{\partial \mathbb{D}} = \sup_{e^{i\theta} \in L_u} |u(e^{i\theta})|.
\]

Lemma 2.3. For $u, v \in L^\infty(\partial \mathbb{D})$, we have the following.
(i) $L_u \cap L_v \subset L_{uv}$ and $\|uv\|_{\partial \mathbb{D}} = \sup_{e^{i\theta} \in L_u \cap L_v} |(uv)(e^{i\theta})|$.
(ii) $L_u \cap L_v \subset L_{u-v}$ and $\|u-v\|_{\partial \mathbb{D}} = \sup_{e^{i\theta} \in L_u \cap L_v} |u(e^{i\theta}) - v(e^{i\theta})|$.
(iii) $L_u \cap L_v \subset L_{|u|+|v|}$ and $\|u|+|v|\|_{\partial \mathbb{D}} = \sup_{e^{i\theta} \in L_u \cap L_v} (|u| + |v|)(e^{i\theta})$.

For $f \in h^\infty$, we define a function $\tilde{f}$ on $\mathbb{D} \cup L_f$ by $\tilde{f}(z) = f(z)$ for $z \in \mathbb{D}$ and $\tilde{f}(e^{i\theta}) = f^*(e^{i\theta})$ for $e^{i\theta} \in L_f$. For $\varphi \in \mathcal{S}(\mathbb{D})$, we have $f \circ \varphi \in h^\infty$ and $(f \circ \varphi)^* = \tilde{f} \circ \varphi$ a.e. on $\partial \mathbb{D}$ (see [9]).

For each number $0 < r < 1$, let
\[
\{r < |\varphi^*| < 1\} = \{e^{i\theta} \in L_{\varphi^*} : r < |\varphi^*(e^{i\theta})| < 1\},
\]
\[
\{|\varphi^*| > r\} = \{e^{i\theta} \in L_{\varphi^*} : |\varphi^*(e^{i\theta})| > r\}
\]
and
\[
\{|\varphi^*| = 1\} = \{e^{i\theta} \in L_{\varphi^*} : |\varphi^*(e^{i\theta})| = 1\}.
\]

Similarly we use $\{|\varphi^*| \leq r\}$ and $\{|\varphi^*| < 1\}$, etc.

We denote by ball($h^\infty$) and ball($H^\infty$) the closed unit balls of $h^\infty$ and $H^\infty$, respectively. For $z, w \in \mathbb{D},$ let
\[
\rho(z, w) = \sup \{|f(w)| : f \in \text{ball}(H^\infty), f(z) = 0\}.
\]

It is well known that $\rho(z, w) = |(z - w)/(1 - \overline{w}z)|$. For a function $F$ in $L^1(\partial \mathbb{D})$, we denote by $\|F\|_1$ the norm of $F$ in $L^1(\partial \mathbb{D})$. We have
\[
\|P_z - P_w\|_1 = \sup \{|f(z) - f(w)| : f \in \text{ball}(h^\infty)\}.
\]

By [4] p. 42, we have the following.

Lemma 2.3. For $z, w \in \mathbb{D}$, we have
\[
\|P_z - P_w\|_1 = 2 - \frac{4}{\pi} \arccos \rho(z, w).
\]

Hence $\rho(z, w) \leq \|P_z - P_w\|_1 \leq 2\rho(z, w)$. 
3. Weighted composition operators on $h^\infty$

In this section, we shall study path connected components in $C_w(h^\infty)$. For $M_uC_\varphi, M_vC_\psi \in C_w(h^\infty)$, we write $M_uC_\varphi \sim M_vC_\psi$ if there is a continuous map $[0, 1] \ni t \rightarrow M_uC_\varphi \in C_w(h^\infty)$ such that $M_u0C_\varphi = M_uC_\varphi$ and $M_uC_\varphi1 = M_vC_\psi$, that is, $M_vC_\psi$ is contained in the path connected component in $C_w(h^\infty)$ containing $M_uC_\varphi$. We note that $M_uC_\varphi \in C_w(h^\infty)$ if and only if $u \neq 0$.

The following is proved in [9, Theorem 3.2].

Lemma 3.1. Let $u \in L^\infty(\partial \mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $M_uC_\varphi$ is compact on $h^\infty$ if and only if $\|u\chi(|\varphi| > r)\|_{\partial \mathbb{D}} \rightarrow 0$ as $r \rightarrow 1$.

Lemma 3.2. Let $M_uC_\varphi, M_vC_\psi \in C_w(h^\infty)$. Then $M_uC_\varphi \sim M_vC_\psi$ in $C_w(h^\infty)$.

Proof. We may assume that $u \neq v$. It is not difficult to find a continuous function $\gamma : [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = 0$, $\gamma(1) = 1$, and $||(1 - \gamma(t))u + \gamma(t)v||_{\partial \mathbb{D}} \neq 0$ for every $0 \leq t \leq 1$. Let $u_t = (1 - \gamma(t))u + \gamma(t)v$. Then $\|M_uu_tC_\varphi\|_{h^\infty} \neq 0$ and $M_uu_tC_\varphi \rightarrow M_uC_\varphi$ as $t \rightarrow t_0$. Hence $M_uC_\varphi \sim M_uC_\varphi$.

Lemma 3.3. Let $M_uC_\varphi, M_vC_\psi \in C_w(h^\infty)$. If

$$m\{\{|\varphi^*| = 1\} \cup \{\{|\psi^*| = 1\}\} < 1,$$

then $M_uC_\varphi \sim M_vC_\psi$ in $C_w(h^\infty)$.

Proof. By the assumption, there is a constant $r, 0 < r < 1$, such that

$$m\{\{|\varphi^*| > r\} \cup \{\{|\psi^*| > r\}\} < 1.$$

Take $p \in L^\infty(\partial \mathbb{D})$ satisfying that $p = 0 \text{ a.e. on } \{\{|\varphi^*| > r\} \cup \{\{|\psi^*| > r\}\} \text{ and } p \neq 0$. By Lemma 3.2 $M_uC_\varphi \sim M_pC_\varphi$ and $M_vC_\psi \sim M_pC_\psi$. We note that by Lemma 3.1 $M_pC_\varphi$ and $M_pC_\psi$ are compact. Let $\varphi_t = (1 - t)\varphi + t\psi$ for $0 \leq t \leq 1$. Then $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_t \in S(\mathbb{D})$. Let $t, t_0 \in [0, 1]$ and $t \neq t_0$. By Lemma 2.2 we have $L_{\varphi^*} \cap L_{\psi^*} \subset L_{\varphi^*}$ for $0 \leq t \leq 1$. We have

$$\|M_pC_{\varphi_t} - M_pC_{\varphi_{t_0}}\|_{h^\infty} = \sup_{f \in \text{ball}(h^\infty)} \|p((f \circ \varphi_t)^* - (f \circ \varphi_{t_0})^*)\|_{\partial \mathbb{D}}.$$

Let $e^{i\theta} \in \{\{|\varphi^*| \leq r\} \cap \{\{|\psi^*| \leq r\}\}$. Then $|\varphi^*(e^{i\theta})| \leq r$, $|\psi^*(e^{i\theta})| \leq r$, $|\varphi_t^*(e^{i\theta})| \leq r$ and $|\varphi_{t_0}^*(e^{i\theta})| \leq r$. Hence

$$\|M_pC_{\varphi_t} - M_pC_{\varphi_{t_0}}\|_{h^\infty} = \sup_{f \in \text{ball}(h^\infty)} \|p\|_{\partial \mathbb{D}} \|f(\varphi_t^*(e^{i\theta})) - f(\varphi_{t_0}^*(e^{i\theta}))\|_{\partial \mathbb{D}}.$$

By Lemma 2.2

$$\|P_{\varphi_t^*(e^{i\theta})} - P_{\varphi_{t_0}^*(e^{i\theta})}\|_1 = 2 - \frac{4}{\pi} \arccos \rho(\varphi_t^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta})),$$

where $\rho(\cdot, \cdot)$ is the pseudo-distance.
We have
\[ |\varphi_t^*(e^{it\theta}) - \varphi_{t_0}^*(e^{it\theta})| = |t - t_0||\varphi^*(e^{it\theta}) - \psi^*(e^{it\theta})| \leq 2r|t - t_0|. \]

Hence
\[ \rho(\varphi_t^*(e^{it\theta}), \varphi_{t_0}^*(e^{it\theta})) \leq \frac{2r|t - t_0|}{1 - r^2}. \]

Therefore
\[ \|M_pC_{\varphi_t} - M_pC_{\varphi_{t_0}}\|_{h^\infty} \leq \|p\|_{\bar{\Omega}} \left(2 - \frac{4\pi}{\arccos \frac{2r|t - t_0|}{1 - r^2}}\right) \to 0 \quad \text{as } t \to t_0. \]

Thus we get \(M_pC_{\varphi_t} \sim M_pC_{\varphi_{t_0}}\). Consequently we have \(M_uC_{\varphi} \sim M_vC_{\psi}\). 

\[ \square \]

Let \(C_{w,e}(h^\infty)\) be the set of compact weighted composition operators in \(C_w(h^\infty)\).

**Corollary 3.4.** The set \(C_{w,e}(h^\infty)\) is a path connected set in \(C_w(h^\infty)\).

**Proof.** Let \(M_uC_{\varphi}, M_vC_{\psi} \in C_{w,e}(h^\infty)\). By Lemma 3.3 \(\|uX\{|\varphi^*| > 1\}\|_{\partial \bar{D}} \to 0\) and \(\|vX\{|\psi^*| > 1\}\|_{\partial \bar{D}} \to 0\) as \(r \to 1\). In the proof of Lemma 3.3 we have \(M_pC_{\varphi}, M_pC_{\psi} \in C_{w,e}(h^\infty)\). By the proof of Lemma 3.2 both paths which connect “\(M_uC_{\varphi}\) and \(M_pC_{\varphi}\)” and “\(M_vC_{\psi}\) and \(M_pC_{\psi}\)” are contained in \(C_{w,e}(h^\infty)\). By the proof of Lemma 3.3 again, the path which connects \(M_pC_{\varphi}\) and \(M_pC_{\psi}\) is contained in \(C_{w,e}(h^\infty)\). Thus we get the assertion. \(\square\)

The following is proved in [9] Theorem 4.1].

**Lemma 3.5.** Let \(u, v \in L^\infty(\partial \bar{D})\) and \(\varphi, \psi \in \mathcal{S}(\bar{D})\) with \(\varphi \neq \psi\). We put \(\Omega = L_u \cap L_v \cap L_{\varphi^*} \cap L_{\psi^*}\). Then
\[ \|M_uC_{\varphi} - M_vC_{\psi}\|_{h^\infty} = \max \left\{ \|(|u| + |v|)X\{|\varphi^*| = 1\} \right\|_{\partial \bar{D}}, \sup_{e^{it}\in\{|\varphi^*| < 1\} \cap \{|\psi^*| < 1\}} \|u(e^{it})P_{\varphi^*(e^{it})} - v(e^{it})P_{\psi^*(e^{it})}\|_1 \right\}. \]

**Theorem 3.6.** Let \(\varphi \in \mathcal{S}(\bar{D})\). Then we have the following.

(i) If \(m(\{|\varphi^*| = 1\}) = 1\), then \(\{M_uC_{\varphi} \in C_w(h^\infty) : u \in L^\infty(\partial \bar{D}), u \neq 0\}\) is open and closed, and is a path connected component in \(C_w(h^\infty)\) containing \(C_{\varphi}\).

(ii) Suppose that \(m(\{|\varphi^*| = 1\}) < 1\). Let
\[ X = \{M_uC_{\psi} \in C_w(h^\infty) : m(\{|\psi^*| = 1\}) < 1, u \in L^\infty(\partial \bar{D}), u \neq 0\}. \]
Then \(X\) is open and closed, and is a path connected component in \(C_w(h^\infty)\) containing \(C_{\varphi}\).

**Proof.** By Lemma 3.2 \(C_{\varphi} \sim M_uC_{\varphi}\) for \(u \in L^\infty(\partial \bar{D})\) with \(u \neq 0\).

(i) Suppose that \(m(\{|\varphi^*| = 1\}) = 1\). Let \(u \in L^\infty(\partial \bar{D}), u \neq 0\), and \(M_vC_{\psi} \in C_w(h^\infty)\). If \(\varphi \neq \psi\), then by Lemma 3.5 we have
\[ \|M_uC_{\varphi} - M_vC_{\psi}\|_{h^\infty} \geq \|u + v\|_{\partial \bar{D}} \geq \|u\|_{\partial \bar{D}} > 0. \]
Then \(\{M_uC_{\varphi} \in C_w(h^\infty) : u \in L^\infty(\partial \bar{D}), u \neq 0\}\) is open and is a path connected subset in \(C_w(h^\infty)\).

Suppose that there is a sequence \(\{u_n\}\) in \(L^\infty(\partial \bar{D})\) such that \(u_n \neq 0\) for every \(n \geq 1\) and \(M_{u_n}C_{\varphi} \to M_vC_{\psi} \in C_w(h^\infty)\). If \(\varphi \neq \psi\), then by Lemma 3.5 we have
\(u_n \to 0\) as \(n \to \infty\), so \(M_{u_n}C_\varphi \to 0\). Hence \(M_uC_\psi = 0 \notin C_w(h^\infty)\). This is a contradiction. Thus we get \(\psi = \varphi\). Hence \(\{M_uC_\varphi \in C_w(h^\infty) : u \in L^\infty(\partial \mathbb{D}), u \neq 0\}\) is open and closed in \(C_w(h^\infty)\).

Suppose that there is an element \(M_uC_\varphi \in C_w(h^\infty)\) such that \(C_\varphi \sim M_uC_\psi\). Let \([0, 1] \ni t \to M_{u_t}C_{\varphi_t}\) be a continuous map in \(C_w(h^\infty)\) such that \(M_{u_0}C_{\varphi_0} = C_\varphi\) and \(M_{u_1}C_{\varphi_1} = M_uC_\psi\). Let \(A = \{t \in [0, 1] : \varphi_t = \varphi\}\). Then \(A \neq \emptyset\). By Lemma 3.3, \(A\) is open, and by the last paragraph \(A\) is closed. So \(A\) is open and closed in \([0, 1]\). Hence \(A = [0, 1]\) and \(\psi = \varphi\). Thus \(\{M_uC_\varphi \in C_w(h^\infty) : u \in L^\infty(\partial \mathbb{D}), u \neq 0\}\) is open and closed, and is a path connected component in \(C_w(h^\infty)\) containing \(C_\varphi\).

(ii) By (i), \(X\) is closed in \(C_w(h^\infty)\). Let \(\eta \in S(\mathbb{D})\) such that \(||\eta||_\infty = 1\) and \(m(\{|\eta^*| = 1\}) = 0\). Let \(M_\eta C_\psi \in X\). By Lemma 3.3, \(C_\varphi \sim M_\eta C_\psi\). Thus \(X\) is a path connected component in \(C_w(h^\infty)\) containing \(C_\varphi\). Let \(\xi \in S(\mathbb{D})\) and \(m(\{|\xi^*| = 1\}) = 1\). By Lemma 3.5 we have

\[
\|M_\nu C_\varphi - M_{\mu \xi} C_\psi\|_{h^\infty} \geq \|v\| + |\mu|\|\partial \|_\partial > 0
\]

for every \(\mu \in L^\infty(\partial \mathbb{D})\) with \(\mu \neq 0\). Then \(X\) is open in \(C_w(h^\infty)\). Thus we get (ii). \(\square\)

Next we will consider the connected component in the set \(C_{w,0}(h^\infty)\) of noncompact weighted composition operators on \(h^\infty\).

The following lemma follows from the proofs of Lemmas 5 and 6 in [12].

**Lemma 3.7.** Let \(z_1, z_2 \in \mathbb{D}\) and \(z_1 \neq z_2\). For \(0 < t < 1\), put \(w_t = (1 - t)z_1 + tz_2\). For \(t_1, t_2 \in [0, 1]\) satisfying

\[
0 < |t_1 - t_2| < \frac{1}{\rho(z_1, z_2)} - 1,
\]

we have

\[
\rho(w_{t_1}, w_{t_2}) \leq \frac{|t_1 - t_2|}{\rho(z_1, z_2) - 1 - |t_1 - t_2|}.
\]

For \(M_uC_\varphi, M_vC_\psi \in C_{w,0}(h^\infty)\), we write \(M_uC_\varphi \sim M_vC_\psi\) if there is a continuous map \([0, 1] \ni t \to M_{u_t}C_{\varphi_t} \in C_{w,0}(h^\infty)\) such that \(M_{u_0}C_{\varphi_0} = M_uC_\varphi\) and \(M_{u_1}C_{\varphi_1} = M_vC_\psi\).

Let \(u \in L^\infty(\partial \mathbb{D})\) and \(\varphi, \psi \in S(\mathbb{D})\) satisfy \(u \neq 0\) and \(\varphi \neq \psi\). Let

\[
\Omega_u = L_u \cap L_{\varphi^*} \cap L_{\psi^*}.
\]

Then \(m(\Omega_u) = 1\). For each \(0 < R < 1\), we put

\[
T_R = \{e^{i\theta} \in \{||\varphi^*| < 1\} \cap \{||\psi^*| < 1\} : \rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \leq R\}.
\]

**Theorem 3.8.** Let \(\varphi, \psi \in S(\mathbb{D}), \|\varphi\|_\infty = \|\psi\|_\infty = 1\) and \(\varphi \neq \psi\). Suppose that there exists a function \(u \in L^\infty(\partial \mathbb{D})\) satisfying the following conditions:

(i) \(M_uC_\varphi \in C_{w,0}(h^\infty)\),

(ii) \(\lim_{R \to 1} \|u\chi_{T_R}\|_{\partial \mathbb{D}} = 0\).

Then \(M_uC_\psi \in C_{w,0}(h^\infty)\) and \(M_uC_\varphi \sim M_uC_\psi\) in \(C_{w,0}(h^\infty)\).

**Proof.** By condition (i) and Lemma 3.1 there exists a positive number \(\varepsilon\) such that \(\|u\chi_{\{||\varphi^*| > r\}|\}_{\partial \mathbb{D}} > \varepsilon\) for every \(0 < r < 1\). Since \(m(\Omega_u) = 1\), we have \(\|u\chi_{\{||\psi^*| > r\}}|\}_{\partial \mathbb{D}} > \varepsilon\). By condition (ii), there is a number \(R_0, 0 < R_0 < 1\), such that

\[
(3.1) \quad \|u\chi_{\Omega_u \setminus T_{R_0}}\|_{\partial \mathbb{D}} < \varepsilon.
\]
Hence
\[ \|u\chi_{\Omega_u \cap \{|\varphi^*| > r\} \cap \mathcal{T}_{R_0}}\|_{\partial\mathcal{D}} > \varepsilon \]
for every 0 < r < 1. Since
\[ \Omega_u \cap \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\} \subset \Omega_u \setminus \mathcal{T}_R \]
for every 0 < R < 1, by condition (ii) again \( u = 0 \) a.e. on \( \Omega_u \cap \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\} \). Thus we get
\begin{equation}
(3.2) \quad m(\Omega_u \cap \{r < |\varphi^*| < 1\} \cap \mathcal{T}_{R_0}) > 0
\end{equation}
and
\begin{equation}
(3.3) \quad \|u\chi_{\Omega_u \cap \{r < |\varphi^*| < 1\} \cap \mathcal{T}_{R_0}}\|_{\partial\mathcal{D}} > \varepsilon
\end{equation}
for every 0 < r < 1. Take
\begin{equation}
(3.4) \quad e^{i\theta_r} \in \Omega_u \cap \{r < |\varphi^*| < 1\} \cap \mathcal{T}_{R_0}.
\end{equation}
Then
\[ r < |\varphi^*(e^{i\theta_r})| < 1 \quad \text{and} \quad \rho(\varphi^*(e^{i\theta_r}), \psi^*(e^{i\theta_r})) \leq R_0. \]
Hence
\begin{equation}
(3.5) \quad \lim_{r \to 1} |\varphi^*(e^{i\theta_r})| = 1 \quad \text{and} \quad \lim_{r \to 1} |\varphi^*(e^{i\theta_r}) - \psi^*(e^{i\theta_r})| = 0.
\end{equation}
By (3.3)–(3.5) and Lemma 3.1, we have \( M_u C_\psi \in C_{w,0}(h^\infty) \).

For each 0 \leq t \leq 1, put \( \varphi_t = (1 - t)\varphi + t\psi \). Then \( \varphi_t \in S(\mathcal{D}) \), so we put \( T_t = M_u C_{\varphi_t} \in C_w(h^\infty) \). By (3.4) and (3.5), we have
\[ \lim_{r \to 1} e^{is} \in \Omega_u \cap \{r < |\varphi^*| < 1\} \cap \mathcal{T}_{R_0} |\varphi^*_t(e^{is})| = 1. \]
By (3.2), \( ||\varphi_t||_\infty = 1 \). By (3.3) and Lemma 3.1, we have \( T_t \in C_{w,0}(h^\infty) \). We note that \( T_0 = M_u C_\varphi \) and \( T_1 = M_u C_\psi \). To show that \( M_u C_\varphi \approx M_u C_\psi \), we need to prove that \( |0, 1| \ni t \to T_t \in C_{w,0}(h^\infty) \) is a continuous map. Let \( \{t_n\}_n \) be a sequence in \( [0, 1] \) satisfying \( t_n \to t_0 \). We may assume that \( t_n \neq t_0 \) for every \( n \). We shall prove that \( ||T_{t_n} - T_{t_0}||_{h^\infty} \to 0 \) as \( n \to \infty \). We have
\[ ||T_{t_n} - T_{t_0}||_{h^\infty} = \sup_{f \in \text{ball}(h^\infty)} ||T_{t_n} f - T_{t_0} f||_{h^\infty} = \sup_{f \in \text{ball}(h^\infty)} \|u((f \circ ((1 - t_n)\varphi + t_n\psi))^* - (f \circ ((1 - t_0)\varphi + t_0\psi))^*)\|_{\partial\mathcal{D}}. \]
Since \( u = 0 \) a.e. on \( \Omega_u \cap \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\} \), we have
\[ ||u((f \circ ((1 - t_n)\varphi + t_n\psi))^* - (f \circ ((1 - t_0)\varphi + t_0\psi))^*)\|_{\partial\mathcal{D}} = \sup_{e^{i\theta} \in \Omega_u \setminus \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\}} |u(e^{i\theta})((f((1 - t_n)\varphi^*(e^{i\theta}) + t_n\psi^*(e^{i\theta}))) - f((1 - t_0)\varphi^*(e^{i\theta}) + t_0\psi^*(e^{i\theta}))))|. \]
We set
\[ A = \Omega_u \setminus \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\} \cup \mathcal{T}_{R_0}. \]
Then
\[ \Omega_u \setminus \{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\} \subset A \cup (\Omega_u \cap \mathcal{T}_{R_0}), \]
and by (3.1) we have \( \|u\chi_A\|_{\partial \Omega} < \varepsilon \). Hence
\[
\|T_{t_n} - T_{t_0}\|_h \leq 2\varepsilon + \frac{|t_n - t_0|}{1 - R_0/R_0} \leq 2\varepsilon + 2\|u\|_{\partial \Omega} \frac{|t_n - t_0|}{1 - R_0/R_0}.
\]

Therefore
\[
\limsup_{n \to \infty} \|T_{t_n} - T_{t_0}\|_h \leq 2\varepsilon.
\]

Since we may take \( \varepsilon \) sufficiently small, we have \( \lim_{n \to \infty} \|T_{t_n} - T_{t_0}\| = 0 \). Thus we get
\[M_u C_\varphi \approx M_u C_\varphi\text{ in } C_{u,0}(h^\infty).\]

**Lemma 3.9.** If \( \varphi \in S(\Omega) \) and \( \|\varphi\|_\infty = 1 \), then \( C_\varphi \approx M_u C_\varphi \) in \( C_{u,0}(h^\infty) \) for every \( u \in L^\infty(\partial \Omega) \) satisfying \( M_u C_\varphi \in C_{u,0}(h^\infty) \).

**Proof.** Since \( M_u C_\varphi \in C_{u,0}(h^\infty) \), by Lemma 3.1 there exists a positive constant \( \delta \) such that \( \|u\chi_{\{|\varphi^*| > \tau\}}\|_{\partial \Omega} > \delta \) for every \( 0 < \tau < 1 \). Let \( \{r_n\}_n \) be a sequence of increasing numbers satisfying \( 0 < r_n < 1 \) and \( r_n \to 1 \). By Lemma 2.2 there is a sequence \( \{e^{i\theta_n}\}_n \) in \( L_u \) such that \( |u(e^{i\theta_n})| > \delta \), \( e^{i\theta_n} \in L_{\chi_{\{|\varphi^*| > r_n\}}} \) and \( e^{i\theta_n} \in \{|\varphi^*| > r_n\} \). We may assume that \( u(e^{i\theta_n}) \to \alpha \) and \( |\alpha| \geq \delta \). By Lemma 2.1 there is a sequence of measurable subsets \( \{E_n\}_n \) in \( L_u \) such that \( m(E_n) > 0 \), \( E_n \subset \{|\varphi^*| > r_n\} \) and
\[
\sup_{e^{i\theta} \in E_n} |u(e^{i\theta}) - u(e^{i\theta_n})| \to 0 \text{ as } n \to \infty.
\]

Hence
\[
\sup_{e^{i\theta} \in E_n} |u(e^{i\theta}) - \alpha| \to 0 \text{ as } n \to \infty.
\]
It is easy to see that $M_u C_\varphi \approx M_{\overline{u} u} C_\varphi$. For $0 \leq t \leq 1$, put $u_t = (1-t)\overline{u} u + t$. Since
\[
\sup_{e^{i\theta} \in E_n} |\overline{u} u(e^{i\theta}) - |\alpha|^2| \to 0 \quad \text{as} \quad n \to \infty,
\]
we have
\[
\sup_{e^{i\theta} \in E_n} \left| u_t(e^{i\theta}) - ((1-t)|\alpha|^2 + t) \right| \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence
\[
\|u_t \chi_{\{ |\varphi^*| > r \}} \|_{\partial D} \geq \|u_t \chi_{E_n} \|_{\partial D} \to (1-t)|\alpha|^2 + t \quad \text{as} \quad n \to \infty.
\]
By Lemma 3.1, $M_u C_\varphi \in C_w,0(h^\infty)$ for every $0 \leq t \leq 1$. Also, we have
\[
\|M_u C_\varphi - M_{u_{t_0}} C_\varphi\|_{h^\infty} = \|u_t - u_{t_0}\|_{\partial D} \to 0 \quad \text{as} \quad t \to t_0.
\]
Since $u_0 = \overline{u} u$ and $u_1 = 1$, we have $M_{\overline{u} u} C_\varphi \approx C_\varphi$. Thus we get $C_\varphi \approx M_u C_\varphi$ in $C_w,0(h^\infty)$.

Let $u \in L^\infty(\partial D)$. For $0 < r < 1$, we also write
\[
\{ |u| > r \} = \{ e^{i\theta} \in L_u : |u(e^{i\theta})| > r \}.
\]

**Lemma 3.10.** Let $\{ M_{u_n} C_{\varphi_n} \}_{n}$ be a sequence in $C_w,0(h^\infty)$ such that $M_{u_n} C_{\varphi_n} \to M_u C_\varphi \in C_w,0(h^\infty)$. Then $\|u_n - u\|_{\partial D} \to 0$, $\|M_u (C_{\varphi_n} - C_\varphi)\|_{h^\infty} \to 0$ and $\|e_{\{ |\varphi^*_n - \varphi^*| > r \}}\|_{\partial D} \to 0$ as $n \to \infty$ for every $0 < r < 1$.

**Proof.** Since $\|(M_{u_n} C_{\varphi_n} - M_u C_\varphi)\|_{h^\infty} \to 0$, we have $\|u_n - u\|_{\partial D} \to 0$. Since $\|M_{u_n - u} C_{\varphi_n}\|_{h^\infty} = \|u_n - u\|_{\partial D} \to 0$, we have $\|M_u (C_{\varphi_n} - C_\varphi)\|_{h^\infty} \to 0$. Hence $\|u (\varphi^*_n - \varphi^*)\|_{\partial D} \to 0$. This shows that $\|e_{\{ |\varphi^*_n - \varphi^*| > r \}}\|_{\partial D} \to 0$ for every $0 < r < 1$.

Let $A(\overline{D})$ be the disk algebra on the closure $\overline{D}$ of $D$, that is, $A(\overline{D})$ is the space of continuous functions on $\overline{D}$ which are analytic in $D$. For even $f \in A(\overline{D})$, we use its norm by $\|f\|_{\infty} = \sup\{|f(z)| : z \in \overline{D}\}$. In the following theorem, we determine path connected components in $C_w,0(h^\infty)$. See Theorem 3.6 for path connected components in $C_w(h^\infty)$.

**Theorem 3.11.** (i) Let $\varphi \in S(D)$. If
\[
0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r \})
\]
for $0 < r < 1$ sufficiently close to 1, then $\{ M_u C_\varphi \in C_w,0(h^\infty) : u \in L^\infty(\partial D) \}$ is open and closed, and is a path connected component in $C_w,0(h^\infty)$.

(ii) Let $X := \{ M_u C_\varphi \in C_w,0(h^\infty) : u \in L^\infty(\partial D) \}$, $m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r \})$ for any $r\,0 < r < 1$
\[
\text{is open and closed, and is a path connected component in } C_w,0(h^\infty).
\]

**Proof.** (i) Let $M_u C_\varphi, M_{\overline{u} u} C_\varphi \in C_w,0(h^\infty)$ and $\varphi \neq \psi$. By Lemma 3.1, there exists a positive number $\delta$ satisfying $\|u \chi_{\{|\varphi^*| > r \}}\|_{\partial D} > \delta$ for every $0 < r < 1$. Since $0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r \})$ for $0 < r < 1$ sufficiently close to 1, we have $\|u \chi_{\{|\varphi^*| = 1\}}\|_{\partial D} > \delta$. By Lemma 3.5, we have
\[
\|M_u C_\varphi - M_{\overline{u} u} C_\varphi\|_{h^\infty} \geq \|u \chi_{\{|\varphi^*| = 1\}}\|_{\partial D} > \delta.
\]
Then $\{ M_u C_\varphi \in C_w,0(h^\infty) : u \in L^\infty(\partial D) \}$ is an open subset in $C_w,0(h^\infty)$. 
Suppose that there is a sequence \( \{u_n\}_n \) in \( L^\infty(\partial \mathbb{D}) \) such that \( M_{u_n}C_{\varphi} \in C_{w,0}(h^\infty) \) and \( \|M_{u_n}C_{\varphi} - M_{v}C_{\psi}\|_{h^\infty} \to 0 \). Then \( \|u_n^N\|_{L^\infty(\partial \mathbb{D})} \to 0 \). By Lemma 3.10, \( \|u_n - v\|_{L^\infty(\partial \mathbb{D})} \to 0 \), so we have \( M_{u_n}C_{\varphi} = M_{v}C_{\psi} \) and \( u = v \) on \( \{\|\varphi\| = 1\} \). By Lemma 3.1, \( M_{u_n}C_{\varphi} \) is compact, so \( M_{v}C_{\psi} \notin C_{w,0}(h^\infty) \). This is a contradiction, so \( \{M_{u_n}C_{\varphi} \in C_{w,0}(h^\infty) : u \in L^\infty(\partial \mathbb{D})\} \) is closed. By Lemma 3.9 we get (i).

(ii) We divide the proof into three steps.

**Step 1.** Let \( \varphi \in S(\mathbb{D}) \) with \( \|\varphi\|_\infty = 1 \) satisfying
\[
m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\})
\]
for every \( 0 < r < 1 \). Then there is a sequence of distinct points \( \{e^{i\theta_n}\}_n \) in \( L_{\varphi^*} \) such that \( |\varphi^*(e^{i\theta_n})| < 1 \) and \( |\varphi^*(e^{i\theta_n})| = 1 \). We may assume that \( e^{i\theta_n} \to e^{i\theta_0} \), \( e^{i\theta_n} \neq e^{i\theta_0} \) for every \( n \geq 1 \) and \( \varphi^*(e^{i\theta_n}) \to \alpha \in \partial \mathbb{D} \) as \( n \to \infty \). Since \( \{e^{i\theta_n}\}_{n=0} \) is a peak interpolation set for \( A(\mathbb{D}) \), there is a function \( p \in A(\mathbb{D}) \) such that \( p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n}) \) for every \( n \geq 1 \), \( |p(z)| < 1 \) for \( z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\} \), \( p \neq \varphi \) and \( \|p\|_\infty = 1 \) (see 3). Take a sequence of closed subdiscs \( \{I_n\}_{n \geq 1} \) in \( \partial \mathbb{D} \) such that \( e^{i\theta_n} \) is the center of \( I_n \) and \( I_n \cap I_k = \emptyset \) for \( n \neq k \). We fix \( 0 < r_0 < 1 \). For \( z \in \mathbb{D} \), we write \( \Delta(z) = \{w \in \mathbb{D} : \rho(w, z) < r_0\} \). By Lemma 2.1 for each \( n \geq 1 \) there exists a measurable subset \( E_n \) such that \( E_n \subset I_n \cap L_{\varphi^*} \), \( m(E_n) > 0 \) and
\[
\sup_{e^{i\theta} \in E_n} |\varphi^*(e^{i\theta}) - \varphi^*(e^{i\theta_n})|< \frac{2r_0}{1 + r_0^2}
\]
is sufficiently small. So we may assume that
\[
\{\varphi^*(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).
\]
We have \( \|\varphi^*\chi_{E_n}\|_{\partial \mathbb{D}} < 1 \). Since \( p \in A(\overline{\mathbb{D}}) \) and \( p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n}) \), moreover we may assume that
\[
\{p(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).
\]
By 4 p. 4, we have
\[
\rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq \frac{2r_0}{1 + r_0^2} \quad \text{for} \quad e^{i\theta} \in \bigcup_{n=1}^{\infty} E_n.
\]
We define \( \eta \in L^\infty(\partial \mathbb{D}) \) by \( \eta = 1 \) on \( \bigcup_{n=1}^{\infty} E_n \) and \( \eta = 0 \) on \( \partial \mathbb{D} \setminus \bigcup_{n=1}^{\infty} E_n \). We shall prove that \( M_{q}C_{\varphi} \approx M_{q}C_{p} \) by applying Theorem 3.8. Since \( \|\varphi^*\chi_{E_n}\|_{\partial \mathbb{D}} \to 1 \), by Lemma 3.1 we have \( M_{q}C_{\varphi} \in C_{w,0}(h^\infty) \). Let \( \Omega_\eta = \Omega_\eta \cap L_{\varphi^*} \), and, as before, for each \( 0 < R < 1 \) we set
\[
\mathcal{T}_R = \{e^{i\theta} \in \{|\varphi^*| < 1\} \cap \{|p| < 1\} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq R\}.
\]
Take \( R_0 \) satisfying
\[
\frac{2r_0}{1 + r_0^2} < R_0 < 1.
\]
Then we have
\[
\Omega_\eta \setminus \mathcal{T}_R \subset \{\{|\varphi^*| = 1\} \cup \{|p| = 1\} \cup \{e^{i\theta} \in \Omega_\eta \cap \{|\varphi^*| < 1\} \cap \{|p| < 1\} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) > R_0\}
\]
\[
\subset \partial \mathbb{D} \setminus \bigcup_{n=1}^{\infty} E_n.
\]
Hence $\eta = 0$ a.e. on $\Omega_\eta \setminus T_{R_0}$. Since $\Omega_\eta \setminus T_R$ is decreasing in $R$, we have $\|\eta \chi_{(\Omega_\eta \setminus T_R)}\|_{BD} \to 0$ as $R \to 1$. Thus, by Theorem 3.8 we get that $M_\eta C_p \in C_{w,0}(h^\infty)$ and $M_\eta C_\varphi \approx M_\eta C_p$. By Lemma 3.9 we have

$$C_\varphi \approx M_\eta C_\varphi \approx M_\eta C_p \approx C_p.$$  

Step 2. Suppose that $\psi \in \mathcal{S}(\mathbb{D})$, $\|\psi\|_{\infty} = 1$ and $m(\{|\varphi^*| = 1\}) < m(\{|\psi^*| > r\})$ for every $0 < r < 1$. By Step 1, there exist a function $q \in A(\mathbb{D})$ and a point $e^{it_0} \in \partial \mathbb{D}$ satisfying $C_\psi \approx C_q$, $|q(e^{it_0})| = 1$ and $|q(z)| < 1$ for $z \in \mathbb{D} \setminus \{e^{it_0}\}$.

In this step, we shall show that if $e^{i\theta_0} \neq e^{it_0}$, then $C_p \approx C_q$. Suppose that $e^{i\theta_0} \neq e^{it_0}$. Then there is a sequence of distinct points $\{e^{it_n}\}_n$ in $\partial \mathbb{D}$ such that $e^{it_n} \to e^{it_0}$ and $e^{it_n} \neq e^{it_0}$ for every $n$. Moreover, we may assume that $\{e^{i\theta_0}\}_{n \geq 0} \cap \{e^{it_n}\}_{n \geq 0} = \emptyset$. Then $\{e^{i\theta_n}\}_{n \geq 0} \cup \{e^{it_n}\}_{n \geq 0}$ is a peak interpolation set for $A(\mathbb{D})$, so there is a function $g \in A(\mathbb{D})$ satisfying $g(e^{i\theta_n}) = p(e^{i\theta_n})$, $g(e^{it_n}) = q(e^{it_n})$ for every $n \geq 0$ and $|g(z)| < 1$ for $z \in \mathbb{D} \setminus \{e^{i\theta_0}, e^{it_0}\}$. By the same argument given in Step 1, we have $C_p \approx C_g$ and $C_q \approx C_g$. Hence $C_p \approx C_q$.

Step 3. Let $M_u C_\varphi, M_v C_\psi \in C_{w,0}(h^\infty)$ such that $m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\})$ and $m(\{|\psi^*| = 1\}) < m(\{|\psi^*| > r\})$ for every $0 < r < 1$. If $e^{i\theta_0} \neq e^{it_0}$, then by Step 2 and Lemma 3.9 we have

$$M_u C_\varphi \approx C_\varphi \approx C_p \approx C_q \approx C_\psi \approx M_v C_\psi.$$  

Suppose that $e^{i\theta_0} = e^{it_0}$. Take $e^{i\theta_0} \in \partial \mathbb{D} \setminus \{e^{it_0}\}$ and a function $h \in A(\mathbb{D})$ such that $|h(e^{i\theta_0})| = |h(e^{it_0})| = 1$ and $|h(z)| < 1$ for $z \in \mathbb{D} \setminus \{e^{i\theta_0}, e^{it_0}\}$. Then, by Step 2, we have $C_p \approx C_h \approx C_q$. Hence we have $M_u C_\varphi \approx M_v C_\psi$.

By (i), $X$ is a closed set. To prove that $X$ is open in $C_{w,0}(h^\infty)$, let $M_u C_\varphi \in X$. Suppose that there is a sequence $\{M_{u_n} C_{\varphi_n}\}_n$ in $C_{w,0}(h^\infty)$ such that $M_{u_n} C_{\varphi_n} \to M_u C_\varphi$ as $n \to \infty$. Then $m(\{|\varphi_n^*| = 1\}) > 0$, and for each $n$ there is a number $r_n, 0 < r_n < 1$, such that

$$m(\{r_n < |\varphi_n^*| < 1\}) = 0.$$  

By Lemma 3.10 $\|u_{\chi_{\{|\varphi_n^*| = 1\}}}\|_{h^\infty} \to 0$ as $n \to \infty$. By Lemma 3.5

$$\|u_{\chi_{\{|\varphi_n^*| = 1\}}}|_{\partial \mathbb{D}} \to 0 \quad \text{as} \quad n \to \infty.$$  

This shows that $u = 0$ a.e. on $\{|\varphi^*| = 1\}$ and

$$\|u_{\chi_{\{|\varphi^*| = 1\}}}|_{\partial \mathbb{D}} \to 0 \quad \text{as} \quad n \to \infty.$$  

Since $M_u C_\varphi \in X$, $m(\{r < |\varphi^*| < 1\}) > 0$, and by Lemma 3.1 there exists a positive constant $\delta$ such that $\|u_{\chi_{\{|\varphi^*| < r\}}}|_{\partial \mathbb{D}} > \delta$. Hence

$$\|u_{\chi_{\{|\varphi^*| < 1\}}}|_{\partial \mathbb{D}} > \delta$$  

for every $r$. Let

$$E_r = \{|u| > \delta\} \cap \{r < |\varphi^*| < 1\} \cap \bigcap_{n=1}^\infty L_{\varphi_n^*}.$$  

By (3.8), $m(E_r) > 0$ for every $r$, and by (3.7) there is a positive integer $n_0$ such that $m(E_r \cap \{|\varphi_n^*| = 1\}) = 0$ for $n \geq n_0$. Let

$$G_r = E_r \setminus \bigcup_{n=n_0}^\infty \{|\varphi_n^*| = 1\}.$$
Then \( m(G_r) = m(E_r) > 0 \) and \( \sup_{e^{i\theta} \in G_r} |\varphi^*(e^{i\theta})| = 1 \). By (3.6), \( |\varphi_n^*| \leq r_n \) a.e. on \( G_r \) for \( n \geq n_0 \). For each \( n \geq n_0 \), we have

\[
\| M_u (C_{\varphi_n} - C_\psi) \|_{h^\infty} = \sup_{f \in \text{ball}(h^\infty)} \| u ((f \circ \varphi_n)^* - (f \circ \varphi)^*) \|_{\partial D} \\
\geq \sup_{f \in \text{ball}(h^\infty)} \| u \chi_{G_r} ((f \circ \varphi_n)^* - (f \circ \varphi)^*) \|_{\partial D} \\
\geq \delta \sup_{f \in \text{ball}(h^\infty)} \text{ess sup}_{e^{i\theta} \in G_r} |f(\varphi_n^*(e^{i\theta})) - f(\varphi^*(e^{i\theta}))| \quad \text{by (3.9)} \\
= \delta \text{ess sup}_{e^{i\theta} \in G_r} \| P_{\varphi_n^*(e^{i\theta})} - P_{\varphi^*(e^{i\theta})} \|_1 \\
= 2\delta.
\]

This is a contradiction. Thus \( X \) is open in \( C_{w,0}(h^\infty) \), and we get (ii). \( \square \)

### 4. Weighted composition operators on \( H^\infty \)

For \( M_u C_\varphi, M_u C_\psi \in C_w(H^\infty) \), we write \( M_u C_\varphi \sim M_v C_\psi \) if there is a continuous map \( [0, 1] \ni t \to M_u C_{\varphi_t} \in C_w(H^\infty) \) such that \( M_{u_t} C_{\varphi_0} = M_u C_{\varphi_t} \) and \( M_{u_t} C_{\varphi_1} = M_v C_{\varphi_t} \). We note that \( M_u C_\varphi \in C_w(H^\infty) \) if and only if \( u \in H^\infty \) and \( u \neq 0 \).

In \cite[Theorem 4.1]{10}, Hosokawa and the first and third authors determined path connected components in \( C_w(H^\infty) \) as follows. See Theorem 3.6 for path connected components in \( C_w(h^\infty) \).

**Theorem 4.1.**

(i) Let \( \varphi \in \mathcal{S}(D) \). If \( m(\{|\varphi^*| = 1\}) > 0 \), then \( \{ M_u C_\varphi : u \in H^\infty, u \neq 0 \} \) is open and closed, and is a path connected component in \( C_w(H^\infty) \) containing \( C_\varphi \).

(ii) The set

\[
\{ M_u C_\varphi : u \in H^\infty, u \neq 0, \varphi \in \mathcal{S}(D), m(\{|\varphi^*| = 1\}) = 0 \}
\]

is closed, and is a path connected component in \( C_w(H^\infty) \), but the set is not open.

**Lemma 4.2.** Let \( M_u C_\varphi \in C_w(H^\infty) \). Then \( M_u C_\varphi \) is compact on \( H^\infty \) if and only if \( \| u^* \chi_{\{|\varphi^*| > r\}} \|_{\partial D} \to 0 \) as \( r \to 1 \).

**Proof.** By \cite[Lemma 4.7]{10}, \( M_u C_\varphi \) is compact on \( H^\infty \) if and only if \( M_u C_\varphi \) is compact on \( h^\infty \). By Lemma 3.1 we get the assertion. \( \square \)

We put

\[
d_\infty(z, w) = \sup_{f \in \text{ball}(H^\infty)} |f(z) - f(w)| \quad \text{for} \quad z, w \in \mathbb{D}.
\]

It is known that

\[
d_\infty(z, w) = \frac{2(1 - \sqrt{1 - \rho(z, w)^2})}{\rho(z, w)}
\]

and \( \rho(z, w) \leq d_\infty(z, w) \leq 2\rho(z, w) \) (see \cite{12}).

Let \( C_{w,c}(H^\infty) \) be the set of compact weighted composition operators in \( C_w(H^\infty) \).

**Proposition 4.3.** The set \( C_{w,c}(H^\infty) \) is a path connected set in \( C_w(H^\infty) \).
Proof. Let \( M_uC_\varphi, M_\psi C_\psi \in C_{w,c}(H^\infty) \) and let \( \eta \in S(\mathbb{D}) \) satisfy \( \|\eta\|_\infty < 1 \). For \( 0 \leq t \leq 1 \), we write \( \varphi_t = (1-t)\varphi + t\eta \). Then \( \varphi_0 = \varphi \) and \( \varphi_1 = \eta \). Since \( \|\varphi_t\|_\infty < 1 \) for \( 0 < t < 1 \), we have \( M_uC_{\varphi_t} \in C_{w,c}(H^\infty) \) for every \( 0 \leq t \leq 1 \). Let \( \{t_n\}_n \) be a sequence in \([0,1]\) such that \( t_n \to t_0 \). Then \( \|\varphi_{t_n}^* - \varphi_{t_0}^*\|_{\partial\mathbb{D}} \to 0 \). We have

\[
\|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\|
= \sup_{f \in \text{ball}(H^\infty)} \|u(f \circ \varphi_{t_n} - f \circ \varphi_{t_0})\|_\infty
= \sup_{f \in \text{ball}(H^\infty)} \max \{\|u\chi_{\{|\varphi^*| > r\}}((f \circ \varphi_{t_n})^* - (f \circ \varphi_{t_0})^*)\|_{\partial\mathbb{D}}, \|\varphi_{t_n}^* - \varphi_{t_0}^*\|_{\partial\mathbb{D}} \} \quad (0 < r < 1)
\leq \sup_{f \in \text{ball}(H^\infty)} \max \{2\|u\chi_{\{|\varphi^*| > r\}}\|_{\partial\mathbb{D}}, \|\varphi_{t_n}^* - \varphi_{t_0}^*\|_{\partial\mathbb{D}} \} \sup_{e^{i\theta} \in \{||\varphi^*|| \leq r\} \cap L_{n^*}} d_\infty(\varphi_{t_n}^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta}))
= \max \{2\|u\chi_{\{|\varphi^*| > r\}}\|_{\partial\mathbb{D}}, \|\varphi_{t_n}^* - \varphi_{t_0}^*\|_{\partial\mathbb{D}} \} \sup_{e^{i\theta} \in \{||\varphi^*|| \leq r\} \cap L_{n^*}} d_\infty(\varphi_{t_n}^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta}))
\]

Since \( \varphi_{t_n}^* \to \varphi_{t_0}^* \) uniformly on \( \partial\mathbb{D} \) and \( \|\varphi_{t_0}^* \chi_{\{|\varphi^*| \leq r\}}\|_{\partial\mathbb{D}} < 1 \), we have

\[
\lim_{n \to \infty} \sup_{e^{i\theta} \in \{||\varphi^*|| \leq r\} \cap L_{n^*}} d_\infty(\varphi_{t_n}^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta}))
\leq 2 \lim_{n \to \infty} \sup_{e^{i\theta} \in \{||\varphi^*|| \leq r\} \cap L_{n^*}} \rho(\varphi_{t_n}^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta})) = 0.
\]

Hence

\[
\limsup_{n \to \infty} \|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\| \leq 2\|u\chi_{\{|\varphi^*| > r\}}\|_{\partial\mathbb{D}}
\]

for every \( 0 < r < 1 \). Letting \( r \to 1 \), by Lemma \[3.2\] we get

\[
\lim_{n \to \infty} \|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\| = 0.
\]

This shows that \( \{0,1\} \ni t \to M_uC_{\varphi_t} \) is a continuous path in \( C_{w,c}(H^\infty) \). Therefore \( M_uC_{\varphi} \sim M_uC_\eta \) in \( C_{w,c}(H^\infty) \). Similarly we have \( M_\psi C_\psi \sim M_\psi C_\eta \) in \( C_{w,c}(H^\infty) \). In the same way as Lemma \[3.3\] we may prove that \( M_uC_\eta \sim M_\psi C_\eta \) in \( C_{w,c}(H^\infty) \). As a consequence, we get the assertion.

In this section, we determine path connected components in \( C_{w,0}(H^\infty) \) which answer questions in [7, Problem 5.3]. We also write \( M_uC_{\varphi} \approx M_\psi C_\psi \) if there is a continuous path connecting \( M_uC_{\varphi} \) and \( M_\psi C_\psi \) in \( C_{w,0}(H^\infty) \).

In the same way as in the proof of Lemma \[3.3\] we may also prove the following.

**Lemma 4.4.** If \( \varphi \in S(\mathbb{D}) \) and \( \|\varphi\|_\infty = 1 \), then \( C_\varphi \approx M_uC_{\varphi} \) in \( C_{w,0}(H^\infty) \) for every \( u \in H^\infty \) satisfying \( M_uC_{\varphi} \in C_{w,0}(H^\infty) \).

In the same way as in the proof of Theorem \[3.8\] we may prove the following. Recall that for \( 0 < R < 1 \),

\[
\mathcal{T}_R = \{^{i\theta} \in \{||\varphi^*|| < 1\} \cap \{||\psi^*|| < 1\} : \rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) < R\}.
\]
Theorem 4.5. Let $\varphi, \psi \in S(\mathbb{D})$ and $\varphi \neq \psi$. Suppose that there exists a function $u \in H^\infty$ satisfying the following conditions:

(i) $M_u C_\varphi \in C_{w,0}(H^\infty)$,
(ii) $\lim_{r \to 0^+} \|u^*\chi_{\{\varphi^* = 1\}}\|_{\partial \mathbb{D}} = 0$.

Then $M_u C_\psi \in C_{w,0}(H^\infty)$ and $M_u C_\varphi \approx M_u C_\psi$ in $C_{w,0}(H^\infty)$.

In Theorem 4.5, since $M_u C_\varphi \in C_{w,0}(H^\infty)$, we have $u \neq 0$, and by condition (ii) we have $m(\{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\}) = 0$. In [7, Theorem 5.2], we proved a similar theorem using some work on the maximal ideal space of $L^\infty(\partial \mathbb{D})$. Condition (ii) in Theorem 4.5 is much simpler than the corresponding one in [7].

There is a big difference in properties between $C_w(h^\infty)$ and $C_w(H^\infty)$. For a measurable subset $E$ of $\partial \mathbb{D}$ satisfying $0 < m(E) < 1$, there is $u \in h^\infty$ such that $u^* = 0$ a.e. on $E$ and $u \neq 0$, but there are no $u \in H^\infty$ satisfying $u^* = 0$ a.e. on $E$ and $u \neq 0$.

The following follows from [10, Theorem 4.1].

Lemma 4.6. Let $M_u C_\varphi, M_v C_\psi \in C_w(H^\infty)$ and $\varphi \neq \psi$. Then

$$\|M_u C_\varphi - M_v C_\psi\|_{H^\infty} \geq \|(|u^*| + |v^*|)\chi_{\{|\varphi^*| = 1\}}\|_{\partial \mathbb{D}}.$$ 

The following is the main theorem in this section. Compare Theorem 4.7 with Theorems 3.11 and 4.11.

Theorem 4.7. (i) Let $\varphi \in S(\mathbb{D})$. If $m(\{|\varphi^*| = 1\}) > 0$, then $\{M_u C_\varphi : u \in H^\infty, u \neq 0\}$ is open and closed, and is a path connected component in $C_{w,0}(H^\infty)$ containing $C_\varphi$.

(ii) The set

$$X = \{M_u C_\varphi \in C_{w,0}(H^\infty) : u \in H^\infty, \varphi \in S(\mathbb{D}), m(\{|\varphi^*| = 1\}) = 0\}$$

is closed, and is a path connected component in $C_{w,0}(H^\infty)$.

Proof. (i) Let $\varphi \in S(\mathbb{D})$ and $m(\{|\varphi^*| = 1\}) > 0$. Let $M_u C_\varphi, M_v C_\psi \in C_{w,0}(H^\infty)$. We have $\|u^*\chi_{\{|\varphi^*| = 1\}}\|_{\partial \mathbb{D}} > 0$. If $\varphi \neq \psi$, then by Lemma 4.6 we have

$$\|M_u C_\varphi - M_v C_\psi\|_{H^\infty} \geq \|(|u^*| + |v^*|)\chi_{\{|\varphi^*| = 1\}}\|_{\partial \mathbb{D}} \geq \|u^*\chi_{\{|\varphi^*| = 1\}}\|_{\partial \mathbb{D}} > 0.$$ 

Thus $\{M_u C_\varphi : u \in H^\infty, u \neq 0\}$ is open in $C_{w,0}(H^\infty)$.

Let $\{M_{u_n} C_{\varphi_n}\}_n$ be a sequence in $C_{w,0}(H^\infty)$ such that $M_{u_n} C_{\varphi_n} \to M_u C_\varphi \in C_{w,0}(H^\infty)$. Suppose that $\varphi \neq \psi$. Then we have $\|u_n - v\| \to 0$ and $\|u_n^*\chi_{\{|\varphi^*| = 1\}}\|_{\partial \mathbb{D}} \to 0$ as $n \to \infty$. Hence $v = 0$ a.e. on $\{|\varphi^*| = 1\}$. Since $m(\{|\varphi^*| = 1\}) > 0$, we have $v = 0$, so $M_v C_\psi = 0 \notin C_{w,0}(H^\infty)$. This is a contradiction. Therefore $\psi = \varphi$ and $\{M_{u_n} C_{\varphi_n} : u \in H^\infty, u \neq 0\}$ is closed in $C_{w,0}(H^\infty)$.

Let $M_u C_\varphi, M_v C_\psi \in C_{w,0}(H^\infty)$. Suppose that $M_u C_\varphi \approx M_v C_\psi$ in $C_{w,0}(H^\infty)$. Let $[0, 1] \ni t \to M_u C_{\varphi_t} \in C_{w,0}(H^\infty)$ be a continuous map satisfying $M_{u_t} C_{\varphi_0} = M_u C_\varphi$ and $M_{u_t} C_{\varphi_1} = M_v C_\psi$. Let $A = \{t \in [0, 1] : \varphi_t = \varphi\}$. By the first paragraph, $A$ is open in $[0, 1]$. By the last paragraph, $A$ is closed, so $A = [0, 1]$. Hence $\psi = \varphi$. Therefore we get (i).

(ii) We divide the proof into three steps. Some parts of the proof are similar to those in Theorem 3.11.
Step 1. Let $\varphi \in \mathcal{S}(\mathbb{D})$, $\|\varphi\|_{\infty} = 1$ and $m(\{|\varphi^*| = 1\}) = 0$. Then $m(\{|\varphi^*| > r\}) > 0$ for every $0 < r < 1$. Take a sequence of distinct points $\{e^{i\theta_n}\}_{n \in \mathbb{N}}$ in $L_{\varphi^*}$ such that $|\varphi^*(e^{i\theta_n})| < 1$ and $|\varphi^*(e^{i\theta_0})| \to 1$. We may assume that $e^{i\theta_n} \to e^{i\theta_0}$ and $\varphi^*(e^{i\theta_n}) \to \alpha \in \partial \mathbb{D}$. Since $\{e^{i\theta_n}\}_{n \geq 0}$ is a peak interpolation set for $A(\overline{\mathbb{D}})$, there is a function $p \in A(\overline{\mathbb{D}})$ such that $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$ for every $n \geq 1$, $|p(z)| < 1$ for $z \in \mathbb{D} \setminus \{e^{i\theta_0}\}$, $p \neq \varphi$ and $\|p\|_{\infty} = 1$. We shall show that $C_\varphi \approx C_p$ in $\mathcal{C}_{w,0}(H^\infty)$ by applying Theorem 4.5.

Take a sequence of closed subarcs $\{I_n\}_{n \geq 1}$ in $\partial \mathbb{D}$ such that $e^{i\theta_n}$ is the center of $I_n$ and $I_n \cap I_k = \emptyset$ for $n \neq k$. We fix $0 < R_0 < 1$. For $z \in \mathbb{D}$, we write $\Delta(z) = \{w \in \mathbb{D} : \rho(w, z) < R_0\}$. By Lemma 2.1 for each $n \geq 1$ there exists a measurable subset $E_n$ such that $E_n \subset I_n \cap L_{\varphi^*}$, $m(E_n) > 0$ and

$$\sup_{e^{i\theta} \in E_n} |\varphi^*(e^{i\theta}) - \varphi^*(e^{i\theta_n})|$$

is sufficiently small. So we may assume that

$$\{\varphi^*(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

We have $\|\varphi^* \chi_{E_n}\|_{\partial \mathbb{D}} < 1$ and $\|\varphi^* \chi_{E_n}\|_{\partial \mathbb{D}} \to 1$ as $n \to \infty$. Since $p \in A(\overline{\mathbb{D}})$ and $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$, moreover we may assume that

$$\{p(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

By [4] p. 4, we have

$$\rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq \frac{2R_0}{1 + R_0^2} \quad \text{for} \quad e^{i\theta} \in \bigcup_{n=1}^{\infty} E_n. \quad (4.1)$$

For each $0 < R < 1$, we put

$$\mathcal{T}_R = \{e^{i\theta} : \{e^{i\theta} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq R\}. \quad (4.2)$$

Since $m(\{|\varphi^*| = 1\} \cup \{|p| = 1\}) = 0$, we have $m(\mathcal{T}_R) \to 1$ as $R \to 1$. When $m(\mathcal{T}_R) = 1$ for some $0 < R < 1$, we set $u = 1$. Then we have $M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty)$ and $\|u^* \chi_{\mathcal{T}_R}\|_{\partial \mathbb{D}} = 0$. By Theorem 4.5 we have $C_\varphi \approx C_p$ in $\mathcal{C}_{w,0}(H^\infty)$. So we may assume that $m(\mathcal{T}_R) < 1$ for every $0 < R < 1$.

To show $C_\varphi \approx C_p$, take a sequence of increasing numbers $\{R_k\}_k$ such that $2R_0/(1 + R_0^2) < R_k < 1$ and $R_k \to 1$. By (4.1) and (4.2), we have

$$\bigcup_{n=1}^{\infty} E_n \subset \mathcal{T}_{R_1}. \quad (4.3)$$

Let $A_0 = \mathcal{T}_{R_1}$ and $A_k = \mathcal{T}_{R_{k+1}} \setminus \mathcal{T}_{R_k}$ for $k \geq 1$. Since $\mathcal{T}_{R_k} \subset \mathcal{T}_{R_{k+1}}$, $\{A_k\}_{k \geq 0}$ is a set of mutually disjoint sets and

$$m\left(\bigcup_{k=0}^{\infty} A_k\right) = 1. \quad (4.3)$$

Let $\{a_k\}_{k \geq 1}$ be a sequence of positive numbers such that $0 < a_k < 1$, $\lim_{k \to \infty} a_k = 0$ and

$$\sum_{k=0}^{\infty} m(A_k) \log a_k > -\infty.$$
We define a function \( \eta \in L^\infty(\partial \mathbb{D}) \) by
\[
(4.4) \quad \eta(e^{i\theta}) = \begin{cases} 
1, & e^{i\theta} \in A_0, \\
 a_k, & e^{i\theta} \in A_k, \ k \geq 1.
\end{cases}
\]

Then we have
\[
\int_{\partial \mathbb{D}} \log \eta(e^{i\theta}) \, dm(e^{i\theta}) = \sum_{k=1}^{\infty} m(A_k) \log a_k > -\infty.
\]

By [5, p. 53], there exists a function \( u \in H^\infty \) such that \( |u^*| = \eta \) a.e. on \( \partial \mathbb{D} \).

By (4.3), \( |u^*| = 1 \) a.e. on \( \bigcup_{n=1}^{\infty} E_n \). Since \( \|\varphi \chi_{E_n}\|_{\partial \mathbb{D}} \to 1 \) as \( n \to \infty \), we have \( \|u^* \chi_{\{|\varphi^*| > r\}}\|_{\partial \mathbb{D}} \to 1 \) as \( r \to 1 \). By Lemma 4.2, we get
\[
(4.5) \quad M_u C_\varphi \in C_{w,0}(H^\infty).
\]

We shall prove that
\[
(4.6) \quad \lim_{k \to \infty} \|u^* \chi_{T^*_{R_k}}\|_{\partial \mathbb{D}} = 0.
\]

By (4.4) we have
\[
\|u^* \chi_{T^*_{R_k}}\|_{\partial \mathbb{D}} = \|u^* \chi_{\bigcup_{j=1}^{n_k} A_j}\|_{\partial \mathbb{D}} \leq \max_{k \leq j \leq \infty} a_j \to 0 \ \text{as} \ k \to \infty.
\]

Thus we get (4.6). By (4.5), (4.6) and Theorem 1.5, we have \( M_u C_p \in C_{w,0}(H^\infty) \) and \( M_u C_\varphi \approx M_u C_p \). By Lemma 4.3, we have \( C_\varphi \approx C_p \) in \( C_{w,0}(H^\infty) \).

**Step 2.** Suppose that \( \psi \in S(\mathbb{D}), \|\psi\|_\infty = 1 \) and \( m(\{|\psi^*| = 1\}) = 0 \). By Step 1, there are a point \( e^{i\theta_0} \in \partial \mathbb{D} \) and a function \( q \in A(\mathbb{D}) \) such that \( |q(e^{i\theta_0})| = 1, |q(z)| < 1 \) for \( z \in \mathbb{D} \setminus \{e^{i\theta_0}\} \) and \( \|q\|_\infty = 1 \). By the same argument given in Step 1, we have \( C_p \approx C_q \) and \( C_q \approx C_{\varphi} \). Hence \( C_p \approx C_{\varphi} \) in \( C_{w,0}(H^\infty) \).

**Step 3.** Let \( M_u C_{\varphi}, M_u C_\psi \in X \). Then \( \|\varphi\|_\infty = \|\psi\|_\infty = 1 \) and \( m(\{|\varphi^*| = 1\}) = m(\{|\psi^*| = 1\}) = 0 \). We shall show that \( M_u C_{\varphi} \approx M_u C_\psi \) in \( C_{w,0}(H^\infty) \). By Steps 1 and 2, there are functions \( p, q \in A(\overline{\mathbb{D}}) \) such that \( C_{\varphi} \approx C_p, C_\psi \approx C_q, \|p\|_\infty = |q|_\infty = 1, |p(e^{i\theta})| = |q(e^{i\theta})| = 1, |p(z)| < 1 \) for \( z \in \mathbb{D} \setminus \{e^{i\theta_0}\} \) and \( |q(z)| < 1 \) for \( z \in \mathbb{D} \setminus \{e^{i\theta_0}\} \). By Step 2, if \( e^{i\theta_0} \neq e^{i\theta_0} \), then \( C_p \approx C_q \), so by Lemma 4.3 we have
\[
M_u C_{\varphi} \approx C_{\varphi} \approx C_p \approx C_q \approx C_\psi \approx M_u C_\psi.
\]

Suppose that \( e^{i\theta_0} = e^{i\theta_0} \). Take \( e^{i\theta_0} \in \partial \mathbb{D} \setminus \{e^{i\theta_0}, e^{i\theta_0}\} \) and \( h \in A(\overline{\mathbb{D}}) \) such that \( |h(e^{i\theta_0})| = 1 \) and \( |h(z)| < 1 \) for \( z \in \mathbb{D} \setminus \{e^{i\theta_0}\} \). By Step 2, we have \( C_p \approx C_h \) and \( C_q \approx C_h \), so \( C_p \approx C_q \). Hence we have \( M_u C_{\varphi} \approx M_u C_\psi \). Therefore \( X \) is a path connected component in \( C_{w,0}(H^\infty) \).

By (i), \( X \) is closed in \( C_{w,0}(H^\infty) \). This completes the proof.

In Theorem 1.7, \( X \) is not open. To show this fact, we need the following lemma (for example, see [13, Theorem 15.4]).
Lemma 4.8. Let \( \{M_n\}_n \) be a sequence of positive numbers satisfying \( \sum_{n=1}^{\infty} M_n < \infty \). Let \( \{g_n\}_n \) be a sequence in \( A(\mathbb{D}) \) such that \( \|g_n\|_\infty = 1 \) and \( \|g_n - 1\|_\infty \leq M_n \) for every \( n \). Then \( \prod_{n=1}^{\infty} g_n \) converges uniformly on \( \mathbb{D} \) and \( \prod_{n=1}^{\infty} g_n \in A(\mathbb{D}) \).

Example 4.9. We shall show that \( X \) is not open in \( C_{w,0}(H^n) \). Let \( \{M_n\}_n \) be a sequence of positive numbers satisfying \( \sum_{n=1}^{\infty} M_n < 1 \). Let \( \{\theta_j\}_j \) be a sequence of positive numbers satisfying
\[
\{ z \in \mathbb{D} : |z - e^{i\theta_j}| < |1 - e^{i\theta_j}| \} \subset \{ z \in \mathbb{D} : |z - 1| < M_j \}.
\]
Then \( \theta_j \to 0 \) as \( j \to \infty \). Put
\[
I_j = \{ e^{i\theta} \in \partial \mathbb{D} : |e^{i\theta} - e^{i\theta_j}| < |1 - e^{i\theta_j}| \}.
\]
By the theorems of Riemann mapping and Carathéodory, there is a function \( g_j \in A(\mathbb{D}) \) such that \( g_j \) is a conformal map from \( \mathbb{D} \) onto
\[
\{ z \in \mathbb{D} : |z - e^{i\theta_j}| < |1 - e^{i\theta_j}| \},
\]
\( |g_j| = 1 \) on \( I_j \), \( g_j(I_j) = I_j \), \( g_j(1) = 1 \) and \( |g_j| < 1 \) on \( \mathbb{D} \setminus I_j \). We note that \( |g_j(z) - 1| \leq M_j \) for every \( z \in \mathbb{D} \) and \( j \geq 1 \). Also, there are a function \( h_0 \in A(\mathbb{D}) \) and a sequence of distinct numbers \( \{t_k\}_k \) in \((-1, 0)\) satisfying \( t_k \to 0 \) such that
\[
|h_0| = 1 \quad \text{on} \quad I_1, \quad h_0(I_1) = I_1, \quad h_0(1) = h_0(e^{it_k}) = 1 \quad \text{for every} \quad k, \quad \text{and} \quad |h_0| < 1 \quad \text{on} \quad \mathbb{D} \setminus (I_1 \cup \{e^{it_k}\}_k).
\]
Let
\[
\varphi = g_1 \prod_{j=2}^{\infty} g_j \circ h_0.
\]
Then \( \|g_j \circ h_0 - 1\|_\infty \leq M_j \) for every \( j \geq 2 \), so by Lemma 4.8 we have that
\[
\varphi_n := g_1 \prod_{j=2}^{n} g_j \circ h_0 \to \varphi
\]
uniformly on \( \mathbb{D} \) as \( n \to \infty \), \( \varphi \in A(\mathbb{D}) \), \( \|\varphi\|_\infty = 1 \), \( \{\varphi = 1\} = \{1\} \) and \( \varphi(1) = 1 \). Also, we have that \( |g_1(e^{it_k})| < 1 \), \( g_1(e^{it_k}) \to 1 \) as \( k \to \infty \) and \( (g_j \circ h_0)(e^{it_k}) = g_j(1) = 1 \) for every \( j \geq 2 \) and \( k \geq 1 \). Thus we get that \( \varphi_n(e^{it_k}) = g_1(e^{it_k}) = \varphi(e^{it_k}) \) for every \( k \geq 1 \) and \( n \geq 2 \), \( |\varphi(e^{it_k})| < 1 \) and \( \varphi(e^{it_k}) \to 1 \) as \( k \to \infty \). We have
\[
\{|\varphi_n| = 1\} = \{|g_1| = 1\} \cap \bigcap_{j=2}^{n} \{|g_j \circ h_0| = 1\}
\]
\[
= I_1 \cap \bigcap_{j=2}^{n} \{e^{i\theta} \in I_1 : h_0(e^{i\theta}) \in I_j\}
\]
\[
= \{e^{i\theta} \in I_1 : h_0(e^{i\theta}) \in I_n\}.
\]
Hence \( m(\{|\varphi_n| = 1\}) > 0 \), \( \{|\varphi_n| = 1\} \) is a closed subarc of \( \partial \mathbb{D} \) and \( \{|\varphi_n| = 1\} \) converges to the point \( 1 \). Since \( \varphi_n \to \varphi \) uniformly on \( \mathbb{D} \) and \( \varphi_n(e^{it_k}) = \varphi(e^{it_k}) \), there is a closed subarc \( J_k \) of \( \partial \mathbb{D} \) centered at \( e^{it_k} \) such that for each fixed \( k \),
\[
\lim_{n \to \infty} \sup_{e^{i\theta} \in J_k} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) = 0
\]
and
\[
\lim_{k \to \infty} \sup_{e^{i\theta} \in J_k} \sup_{n \geq 2} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) = 0.
\]
Moreover, we may assume that \( J_k \cap J_\ell = \emptyset \) for \( k \neq \ell \). Then \( 1 \notin J_k \) for every \( k \).
Let
\[ L_1 = \left\{ e^{i\theta} \in \partial \mathbb{D} \setminus \bigcup_{k=1}^{\infty} J_k : |e^{i\theta} - 1| > 1 \right\}, \]
and for \( j \geq 2 \) let
\[ L_j = \left\{ e^{i\theta} \in \partial \mathbb{D} \setminus \bigcup_{k=1}^{\infty} J_k : \frac{1}{j+1} < |e^{i\theta} - 1| \leq \frac{1}{j} \right\}. \]

Take a sequence of numbers \( \{a_j\}_j \) in \( (0,1) \) satisfying that
\[ \sum_{j=1}^{\infty} m(L_j) \log a_j > -\infty. \]

Let
\[ \eta(e^{i\theta}) = \begin{cases} 
1, & e^{i\theta} \in \bigcup_{k=1}^{\infty} J_k, \\
 a_j, & e^{i\theta} \in L_j, \\
0, & e^{i\theta} = 1. 
\end{cases} \]

Then \( \eta \in L^\infty(\partial \mathbb{D}) \) and
\[ \int_{\partial \mathbb{D}} \log \eta(e^{i\theta}) \, dm(e^{i\theta}) = \sum_{j=1}^{\infty} m(L_j) \log a_j > -\infty. \]

By [5] p. 53, there is a function \( u \in H^\infty \) such that \( |u^*| = \eta \) a.e. on \( \partial \mathbb{D} \). We have \( |u^*| = 1 \) a.e. on \( \bigcup_{k=1}^{\infty} J_k \). We note that \( \|u\|_{H^\infty} = 1 \) and \( |u^*| = a_j \) a.e. on \( L_j \). Since \( \varphi(1) = 1 \), we have \( \|\varphi \chi_{J_k}\|_{\partial \mathbb{D}} \to 1 \), so \( \|u^* \chi_{\{\varphi > r\}}\|_{\partial \mathbb{D}} = 1 \) for every \( 0 < r < 1 \). Hence, by Lemma 4.2, \( M_u C_\varphi \in C_{w,0}(H^\infty) \) and \( M_u C_\varphi \in X \). Since \( m(\{ \varphi_n = 1 \}) > 0 \), we have \( M_u C_{\varphi_n} \notin X \).

For each positive integer \( n \geq 2 \), we have
\[ \|M_u C_{\varphi_n} - M_u C_\varphi\| = \sup_{f \in \text{ball}(H^\infty)} \|u(f \circ \varphi_n - f \circ \varphi)\|_{\infty}, \]
\[ = \sup_{f \in \text{ball}(H^\infty)} \max \left\{ \sup_{k \geq 1} \|u^* \chi_{J_k}((f \circ \varphi_n)^* - (f \circ \varphi)^*)\|_{\partial \mathbb{D}}, \right. \]
\[ \left. \sup_{j \geq 1} \|u^* \chi_{L_j}((f \circ \varphi_n)^* - (f \circ \varphi)^*)\|_{\partial \mathbb{D}} \right\}. \]

Since \( |u^*| = 1 \) a.e. on \( J_k \), we have
\[ \sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{J_k}((f \circ \varphi_n)^* - (f \circ \varphi)^*)\|_{\partial \mathbb{D}} \]
\[ = \sup_{f \in \text{ball}(H^\infty)} \|\chi_{J_k}((f \circ \varphi_n)^* - (f \circ \varphi)^*)\|_{\partial \mathbb{D}} \]
\[ = \sup_{f \in \text{ball}(H^\infty)} \sup_{e^{i\theta} \in J_k} |f(\varphi_n(e^{i\theta})) - f(\varphi(e^{i\theta}))| \]
\[ = \sup_{e^{i\theta} \in J_k} d_{\infty}(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) \]
\[ \leq 2 \sup_{e^{i\theta} \in J_k} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})). \]
Take $\varepsilon > 0$ arbitrarily. By (4.8), there exists a positive integer $k_0$ such that
\[
\sup_{e^{i\theta} \in J_k} \sup_{n \geq 2} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) < \varepsilon \quad \text{for} \quad k \geq k_0.
\]
Hence, by (4.10), we have
\[
\sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{J_k} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} < 2\varepsilon \quad \text{for} \quad k \geq k_0.
\]
For $1 \leq k < k_0$, by (4.7) there exists a positive integer $n_0$ such that
\[
\sup_{e^{i\theta} \in J_k} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) < \varepsilon \quad \text{whenever} \quad n \geq n_0.
\]
Hence, for $n \geq n_0$, by (4.10) again we have
\[
\sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{J_k} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} < 2\varepsilon \quad \text{whenever} \quad 1 \leq k < k_0.
\]
Thus
\[
\sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{J_k} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} < 2\varepsilon
\]
for $k \geq 1$ and every $n \geq n_0$. Therefore, by (4.9), for $n \geq n_0$ we have
\[
(4.11) \quad \|M_n C_{\varphi_n} - M_n C_{\varphi}\| \\
\leq \max \left\{ 2\varepsilon, \sup_{f \in \text{ball}(H^\infty)} \sup_{j \geq 1} \|u^* \chi_{L_j} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} \right\}.
\]
Since $a_j \rightarrow 0$, there exists a positive integer $j_0$ such that $a_j < \varepsilon$ for every $j \geq j_0$. Since $|u^*| = a_j$ a.e. on $L_j$, we have
\[
(4.12) \quad \sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{L_j} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} < 2\varepsilon
\]
for $j \geq j_0$ and every $n \geq 1$. Since $\{ |\varphi_n| = 1 \}$ is a closed subarc of $\partial \mathbb{D}$ and converges to $1$ as $n \rightarrow \infty$, we may assume that $L_j \cap \{ |\varphi_n| = 1 \} = \emptyset$ for every $1 \leq j < j_0$ and $n \geq n_0$. For $1 \leq j < j_0$ and $n \geq n_0$, we have
\[
\sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{L_j} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} \\
\leq \sup_{f \in \text{ball}(H^\infty)} \sup_{e^{i\theta} \in L_j} |f(\varphi_n(e^{i\theta})) - f(\varphi(e^{i\theta}))| \\
= \sup_{e^{i\theta} \in L_j} d_{\infty}(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) \\
\leq 2 \sup_{e^{i\theta} \in L_j} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})).
\]
Since $\varphi_n \rightarrow \varphi$ uniformly on $\overline{\mathbb{D}}$ and $\sup_{e^{i\theta} \in L_j} |\varphi(e^{i\theta})| < 1$, we may further assume that
\[
\sup_{e^{i\theta} \in L_j} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) < \varepsilon
\]
for $1 \leq j < j_0$ and every $n \geq n_0$. Thus, by (4.12), we get
\[
\sup_{f \in \text{ball}(H^\infty)} \|u^* \chi_{L_j} \left( (f \circ \varphi_n)^* - (f \circ \varphi)^* \right) \|_{\partial \mathbb{D}} \leq 2\varepsilon \quad \text{for} \quad n \geq n_0.
\]
Combining this with (4.11), we get
\[
\|M_n C_{\varphi_n} - M_n C_{\varphi}\| \leq 2\varepsilon \quad \text{for} \quad n \geq n_0.
\]
Therefore we have $M_u C_{\varphi_n} \to M_u C_{\varphi}$ as $n \to \infty$. Since $M_u C_{\varphi} \in X$ and $M_u C_{\varphi_n} \notin X$, $X$ is not open in $C_{w,0}(H^\infty)$.

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References