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PATH CONNECTED COMPONENTS IN WEIGHTED COMPOSITION OPERATORS ON h^∞ AND H^∞ WITH THE OPERATOR NORM

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Abstract. We consider the component problem on the sets of weighted composition operators on the spaces of bounded harmonic and analytic functions on the open unit disk with the operator norms, respectively. Especially, we shall determine path connected components in the sets of noncompact weighted composition operators.

1. Introduction

Throughout this paper, let h^∞ and H^∞ be the spaces of bounded harmonic and analytic functions on the open unit disk $\mathbb{D}$, respectively. For $f \in h^\infty$, we write $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$. We may define $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. It is known that $\{f^* : f \in h^\infty\} = L^\infty(\partial\mathbb{D})$. We denote by $\|f^*\|_{\partial\mathbb{D}}$ the essential supremum norm of f^* on $\partial\mathbb{D}$. Then $\|f^*\|_{\partial\mathbb{D}} = \|f\|_\infty$. For $u \in L^\infty(\partial\mathbb{D})$, let $\widehat{u}(z)$ be the harmonic extension of u defined by

$$\widehat{u}(z) = \int_{\partial\mathbb{D}} u(e^{i\theta})P_z(e^{i\theta})\, dm(e^{i\theta}) \quad \text{for} \quad z \in \mathbb{D},$$

where P_z is the Poisson kernel for the point $z \in \mathbb{D}$ and m is the normalized Lebesgue measure on $\partial\mathbb{D}$. Then $\widehat{u} \in h^\infty$. Besides, for $f \in h^\infty$, $\widehat{f^*} = f$ on $\mathbb{D}$.

Let $\mathcal{S}(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. For each $\varphi \in \mathcal{S}(\mathbb{D})$, we may define the composition operator C_φ by $C_\varphi f = f\circ\varphi$, where f is analytic on $\mathbb{D}$. Composition operators on various analytic function spaces have been studied extensively during the past few decades. See [2, 14] for an overview of these results. Presently, some of the open questions in this field are related to the topological structure of the set of composition operators. For even $f \in h^\infty$, $C_\varphi f = f \circ \varphi$ is also harmonic on $\mathbb{D}$. So the question of properties of C_φ on h^∞ naturally arises. In [1], Choa and the first and third authors first considered properties of composition operators on h^∞ and, although h^∞ is not algebra, introduced the concept of weighted composition operators M_uC_φ on h^∞ defined by

$$(M_uC_\varphi f)(z) = \int_{\partial\mathbb{D}} u(e^{i\theta})(f \circ \varphi)^*(e^{i\theta})P_z(e^{i\theta})\, dm(e^{i\theta})$$

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for $u \in L^\infty(\partial\mathbb{D})$ and $z \in \mathbb{D}$. We denote by $\mathcal{C}_w(h^\infty)$ the set of nonzero weighted composition operators on h^∞ with the operator norm $\|M_uC_\varphi\|_{h^\infty}$. More precisely, $\mathcal{C}_w(h^\infty) = \{M_uC_\varphi : u \in L^\infty(\partial\mathbb{D}), u \neq 0, \varphi \in \mathcal{S}(\mathbb{D})\}$. Also, $\mathcal{C}(h^\infty)$ is the set of composition operators on h^∞ with the operator norm.

Similarly we may define $\mathcal{C}(H^\infty)$, $\mathcal{C}_w(H^\infty)$ and $\|M_uC_\varphi\|_{H^\infty}$. Indeed, $\mathcal{C}_w(H^\infty) = \{M_uC_\varphi : u \in H^\infty, u \neq 0, \varphi \in \mathcal{S}(\mathbb{D})\}$. Let $M_uC_\varphi \in \mathcal{C}_w(H^\infty)$. Then $u \in H^\infty$, $u \neq 0$ and $M_uC_\varphi f = u(f \circ \varphi)$ for $f \in H^\infty$. In this paper, we investigate path connected components in $\mathcal{C}_w(h^\infty)$, $\mathcal{C}_w(H^\infty)$ and in other spaces.

In [12], MacCluer, Zhao and the third author determined path connected components in $\mathcal{C}(H^\infty)$ (see also [8]). We know that $\mathcal{C}(h^\infty) = \mathcal{C}(H^\infty)$ as sets, but the topology of $\mathcal{C}(h^\infty)$ is stronger than the one of $\mathcal{C}(H^\infty)$. In [1, Theorem 3.6], Choa and the first and third authors showed that path connected components in $\mathcal{C}(h^\infty)$ and $\mathcal{C}(H^\infty)$ are the same.

Since $H^\infty \subsetneqq h^\infty$, we have $\mathcal{C}_w(H^\infty) \subsetneqq \mathcal{C}_w(h^\infty)$ as sets. In [7, Theorem 4.1] Hosokawa and the first and third authors showed the following.

Theorem A. (i) *Let $\varphi \in \mathcal{S}(\mathbb{D})$ and $m(\{|\varphi^*| = 1\}) > 0$. Then $\{M_uC_\varphi : u \in H^\infty, u \neq 0\}$ is open and closed, and is a path connected component in $\mathcal{C}_w(H^\infty)$.*

(ii) *The set*

$$\left\{M_uC_\varphi : u \in H^\infty, u \neq 0, m(\{|\varphi^*| = 1\}) = 0\right\}$$

is closed and is a path connected component in $\mathcal{C}_w(H^\infty)$.

See [6, 11] for other subjects on $\mathcal{C}(H^\infty)$.

This paper is organized as follows. In the next section we shall recall precise arguments on $L^\infty(\partial\mathbb{D})$ functions investigated in [9] and study the component problem of $\mathcal{C}_w(h^\infty)$ in Section 3. In Theorem 3.6, we shall determine path connected components in $\mathcal{C}_w(h^\infty)$. In proving Theorems A and 3.6, some paths cross the space of compact weighted composition operators. Let $\mathcal{C}_{w,0}(h^\infty)$ and $\mathcal{C}_{w,0}(H^\infty)$ be the spaces of noncompact weighted composition operators on h^∞ and H^∞ with the operator norms, respectively. In [7], path connected components in $\mathcal{C}_{w,0}(H^\infty)$ were studied, but there remain some problems ([7, Problem 5.3]). We shall determine path connected components in $\mathcal{C}_{w,0}(h^\infty)$ in Theorem 3.11. In Section 4, we consider path connected components in $\mathcal{C}_w(H^\infty)$ and furthermore characterize path connected components in $\mathcal{C}_{w,0}(H^\infty)$ in Theorem 4.7, which answers problems given in [7]. We may say that the path connected components problem for weighted composition operators is solved completely on h^∞ and H^∞ with respect to the operator norms.

The key in this paper is the argument on the boundary $\partial\mathbb{D}$ as given in [9, 10].

2. $L^\infty(\partial\mathbb{D})$ functions

In order to study $L^\infty(\partial\mathbb{D})$ functions, we recall some elementary facts and notation used in [9]. For $f \in h^\infty$, let R_f be the set of $e^{i\theta} \in \partial\mathbb{D}$ at which f has a radial limit. We define a function f^* on R_f by

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \quad \text{for} \quad e^{i\theta} \in R_f.$$

We have $m(R_f) = 1$ and $\|f\|_\infty = \|f^*\|_{\partial\mathbb{D}} = \sup_{e^{i\theta}\in R_f} |f^*(e^{i\theta})|$.

For $u \in L^\infty(\partial\mathbb{D})$, let L_u be the Lebesgue set of u, that is, L_u is the set of points $e^{it} \in \partial\mathbb{D}$ satisfying

$$\lim_{m(I)\to 0} \frac{1}{m(I)} \int_I |u(e^{i\theta}) - u(e^{it})|\, dm(e^{i\theta}) = 0,$$

where I are subarcs of $\partial\mathbb{D}$ centered at e^{it}. By [13, p. 138], $m(L_u) = 1$, and by [13, Theorem 11.23], $u(e^{i\theta}) = \lim_{r\to 1} \widehat{u}(re^{i\theta})$ for $e^{i\theta} \in L_u$, that is, $L_u \subset R_{\widehat{u}}$. By the definition of L_u, we have the following.

Lemma 2.1. *For each $e^{it} \in L_u$ there are a sequence of open subarcs $\{I_n\}_n$ of $\partial\mathbb{D}$ centered at e^{it} and a sequence of measurable subsets $\{E_n\}_n$ such that $E_n \subset I_n \cap L_u$, $m(I_n) \to 0$, $m(E_n)/m(I_n) \to 1$, and*

$$\lim_{n\to\infty} \sup_{e^{i\theta}\in E_n} |u(e^{i\theta}) - u(e^{it})| = 0.$$

We note that the value of the function u is well defined on L_u and

$$\|u\|_{\partial\mathbb{D}} = \sup_{e^{i\theta}\in L_u} |u(e^{i\theta})|.$$

Lemma 2.2. *For $u, v \in L^\infty(\partial\mathbb{D})$, we have the following.*

(i) $L_u \cap L_v \subset L_{uv}$ *and* $\|uv\|_{\partial\mathbb{D}} = \sup_{e^{i\theta}\in L_u\cap L_v} |(uv)(e^{i\theta})|$.

(ii) $L_u \cap L_v \subset L_{u-v}$ *and* $\|u - v\|_{\partial\mathbb{D}} = \sup_{e^{i\theta}\in L_u\cap L_v} |u(e^{i\theta}) - v(e^{i\theta})|$.

(iii) $L_u \cap L_v \subset L_{|u|+|v|}$ *and* $\||u| + |v|\|_{\partial\mathbb{D}} = \sup_{e^{i\theta}\in L_u\cap L_v} (|u| + |v|)(e^{i\theta})$.

For $f \in h^\infty$, we define a function $\widetilde{f}$ on $\mathbb{D} \cup L_{f^*}$ by $\widetilde{f}(z) = f(z)$ for $z \in \mathbb{D}$ and $\widetilde{f}(e^{i\theta}) = f^*(e^{i\theta})$ for $e^{i\theta} \in L_{f^*}$. For $\varphi \in \mathcal{S}(\mathbb{D})$, we have $f \circ \varphi \in h^\infty$ and $(f \circ \varphi)^* = \widetilde{f} \circ \varphi^*$ a.e. on $\partial\mathbb{D}$ (see [9]).

For each number $0 < r < 1$, let

$$\{r < |\varphi^*| < 1\} = \{e^{i\theta} \in L_{\varphi^*} : r < |\varphi^*(e^{i\theta})| < 1\},$$
$$\{|\varphi^*| > r\} = \{e^{i\theta} \in L_{\varphi^*} : |\varphi^*(e^{i\theta})| > r\}$$

and

$$\{|\varphi^*| = 1\} = \{e^{i\theta} \in L_{\varphi^*} : |\varphi^*(e^{i\theta})| = 1\}.$$

Similarly we use $\{|\varphi^*| \le r\}$ and $\{|\varphi^*| < 1\}$, etc.

We denote by $ball(h^\infty)$ and $ball(H^\infty)$ the closed unit balls of h^∞ and H^∞, respectively. For $z, w \in \mathbb{D}$, let

$$\rho(z, w) = \sup\big\{|f(w)| : f \in ball(H^\infty), f(z) = 0\big\}.$$

It is well known that $\rho(z, w) = |(z - w)/(1 - \overline{w}z)|$. For a function F in $L^1(\partial\mathbb{D})$, we denote by $\|F\|_1$ the norm of F in $L^1(\partial\mathbb{D})$. We have

$$\|P_z - P_w\|_1 = \sup\big\{|f(z) - f(w)| : f \in ball(h^\infty)\big\}.$$

By [4, p. 42], we have the following.

Lemma 2.3. *For $z, w \in \mathbb{D}$, we have*

$$\|P_z - P_w\|_1 = 2 - \frac{4}{\pi} \arccos \rho(z, w).$$

Hence $\rho(z, w) \le \|P_z - P_w\|_1 \le 2\rho(z, w)$.

3. Weighted composition operators on h^∞

In this section, we shall study path connected components in $\mathcal{C}_w(h^\infty)$. For $M_uC_\varphi, M_vC_\psi \in \mathcal{C}_w(h^\infty)$, we write $M_uC_\varphi \sim M_vC_\psi$ if there is a continuous map $[0,1] \ni t \to M_{u_t}C_{\varphi_t} \in \mathcal{C}_w(h^\infty)$ such that $M_{u_0}C_{\varphi_0} = M_uC_\varphi$ and $M_{u_1}C_{\varphi_1} = M_vC_\psi$, that is, M_vC_ψ is contained in the path connected component in $\mathcal{C}_w(h^\infty)$ containing M_uC_φ. We note that $M_uC_\varphi \in \mathcal{C}_w(h^\infty)$ if and only if $u \neq 0$.

The following is proved in [9, Theorem 3.2].

Lemma 3.1. *Let $u \in L^\infty(\partial\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then M_uC_φ is compact on h^∞ if and only if $\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} \to 0$ as $r \to 1$.*

Lemma 3.2. *Let $M_uC_\varphi, M_vC_\varphi \in \mathcal{C}_w(h^\infty)$. Then $M_uC_\varphi \sim M_vC_\varphi$ in $\mathcal{C}_w(h^\infty)$.*

Proof. We may assume that $u \neq v$. It is not difficult to find a continuous function $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = 0$, $\gamma(1) = 1$, and $\|(1-\gamma(t))u + \gamma(t)v\|_{\partial\mathbb{D}} \neq 0$ for every $0 \le t \le 1$. Let $u_t = (1-\gamma(t))u + \gamma(t)v$. Then $\|M_{u_t}C_\varphi\|_{h^\infty} \neq 0$ and

$$\begin{aligned}\|M_{u_t}C_\varphi - M_{u_{t_0}}C_\varphi\|_{h^\infty} &= \|u_t - u_{t_0}\|_{\partial\mathbb{D}} \\ &= |\gamma(t) - \gamma(t_0)|\|u - v\|_{\partial\mathbb{D}} \to 0\end{aligned}$$

as $t \to t_0$. Hence $M_uC_\varphi \sim M_vC_\varphi$. □

Lemma 3.3. *Let $M_uC_\varphi, M_vC_\psi \in \mathcal{C}_w(h^\infty)$. If*

$$m\big(\{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\}\big) < 1,$$

then $M_uC_\varphi \sim M_vC_\psi$ in $\mathcal{C}_w(h^\infty)$.

Proof. By the assumption, there is a constant $r, 0 < r < 1$, such that

$$m\big(\{|\varphi^*| > r\} \cup \{|\psi^*| > r\}\big) < 1.$$

Take $p \in L^\infty(\partial\mathbb{D})$ satisfying that $p = 0$ a.e. on $\{|\varphi^*| > r\} \cup \{|\psi^*| > r\}$ and $p \neq 0$. By Lemma 3.2, $M_uC_\varphi \sim M_pC_\varphi$ and $M_vC_\psi \sim M_pC_\psi$. We note that by Lemma 3.1, M_pC_φ and M_pC_ψ are compact. Let $\varphi_t = (1-t)\varphi + t\psi$ for $0 \le t \le 1$. Then $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_t \in \mathcal{S}(\mathbb{D})$. Let $t, t_0 \in [0,1]$ and $t \neq t_0$. By Lemma 2.2, we have $L_{\varphi^*} \cap L_{\psi^*} \subset L_{\varphi_t^*}$ for $0 \le t \le 1$. We have

$$\begin{aligned}&\|M_pC_{\varphi_t} - M_pC_{\varphi_{t_0}}\|_{h^\infty} \\ &= \sup_{f \in ball(h^\infty)} \big\|p\big((f\circ\varphi_t)^* - (f\circ\varphi_{t_0})^*\big)\big\|_{\partial\mathbb{D}} \\ &= \sup_{f \in ball(h^\infty)} \big\|p\chi_{\left(\{|\varphi^*|\le r\}\cap\{|\psi^*|\le r\}\right)}\big((f\circ\varphi_t)^* - (f\circ\varphi_{t_0})^*\big)\big\|_{\partial\mathbb{D}}.\end{aligned}$$

Let $e^{i\theta} \in \{|\varphi^*| \le r\} \cap \{|\psi^*| \le r\}$. Then $|\varphi^*(e^{i\theta})| \le r$, $|\psi^*(e^{i\theta})| \le r$, $|\varphi_t^*(e^{i\theta})| \le r$ and $|\varphi_{t_0}^*(e^{i\theta})| \le r$. Hence

$$\begin{aligned}&\|M_pC_{\varphi_t} - M_pC_{\varphi_{t_0}}\|_{h^\infty} \\ &= \sup_{f \in ball(h^\infty)} \sup_{e^{i\theta} \in \{|\varphi^*|\le r\}\cap\{|\psi^*|\le r\}} \|p\|_{\partial\mathbb{D}}\big|f(\varphi_t^*(e^{i\theta})) - f(\varphi_{t_0}^*(e^{i\theta}))\big| \\ &\le \|p\|_{\partial\mathbb{D}} \sup_{e^{i\theta} \in \{|\varphi^*|\le r\}\cap\{|\psi^*|\le r\}} \|P_{\varphi_t^*(e^{i\theta})} - P_{\varphi_{t_0}^*(e^{i\theta})}\|_1.\end{aligned}$$

By Lemma 2.3,

$$\|P_{\varphi_t^*(e^{i\theta})} - P_{\varphi_{t_0}^*(e^{i\theta})}\|_1 = 2 - \frac{4}{\pi}\arccos\rho(\varphi_t^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta})).$$

We have

$$|\varphi_t^*(e^{i\theta}) - \varphi_{t_0}^*(e^{i\theta})| = |t - t_0||\varphi^*(e^{i\theta}) - \psi^*(e^{i\theta})| \le 2r|t - t_0|.$$

Hence

$$\rho(\varphi_t^*(e^{i\theta}), \varphi_{t_0}^*(e^{i\theta})) \le \frac{2r|t - t_0|}{1 - r^2}.$$

Therefore

$$\begin{aligned} &\|M_p C_{\varphi_t} - M_p C_{\varphi_{t_0}}\|_{h^\infty} \\ &\le \|p\|_{\partial\mathbb{D}} \Big(2 - \frac{4}{\pi} \arccos \frac{2r|t - t_0|}{1 - r^2}\Big) \\ &\to 0 \qquad \text{as } t \to t_0. \end{aligned}$$

Thus we get $M_p C_\varphi \sim M_p C_\psi$. Consequently we have $M_u C_\varphi \sim M_v C_\psi$. □

Let $\mathcal{C}_{w,c}(h^\infty)$ be the set of compact weighted composition operators in $\mathcal{C}_w(h^\infty)$.

Corollary 3.4. *The set $C_{w,c}(h^\infty)$ is a path connected set in $\mathcal{C}_w(h^\infty)$.*

Proof. Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,c}(h^\infty)$. By Lemma 3.1, $\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} \to 0$ and $\|v\chi_{\{|\psi^*|>r\}}\|_{\partial\mathbb{D}} \to 0$ as $r \to 1$. In the proof of Lemma 3.3, we have $M_p C_\varphi, M_p C_\psi \in \mathcal{C}_{w,c}(h^\infty)$. By the proof of Lemma 3.2, both paths which connect "$M_u C_\varphi$ and $M_p C_\varphi$" and "$M_v C_\psi$ and $M_p C_\psi$" are contained in $\mathcal{C}_{w,c}(h^\infty)$. By the proof of Lemma 3.3 again, the path which connects $M_p C_\varphi$ and $M_p C_\psi$ is contained in $\mathcal{C}_{w,c}(h^\infty)$. Thus we get the assertion. □

The following is proved in [9, Theorem 4.1].

Lemma 3.5. *Let $u, v \in L^\infty(\partial\mathbb{D})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ with $\varphi \ne \psi$. We put $\Omega = L_u \cap L_v \cap L_{\varphi^*} \cap L_{\psi^*}$. Then*

$$\begin{aligned} \|M_u C_\varphi - M_v C_\psi\|_{h^\infty} = \max\Big\{ &\Big\|(|u| + |v|)\chi_{\left(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\right)}\Big\|_{\partial\mathbb{D}}, \\ &\sup_{e^{i\theta}\in\{|\varphi^*|<1\}\cap\{|\psi^*|<1\}\cap\Omega} \Big\|u(e^{i\theta})P_{\varphi^*(e^{i\theta})} - v(e^{i\theta})P_{\psi^*(e^{i\theta})}\Big\|_1\Big\}. \end{aligned}$$

Theorem 3.6. *Let $\varphi \in \mathcal{S}(\mathbb{D})$. Then we have the following.*

(i) *If $m(\{|\varphi^*| = 1\}) = 1$, then $\{M_u C_\varphi \in \mathcal{C}_w(h^\infty) : u \in L^\infty(\partial\mathbb{D}), u \ne 0\}$ is open and closed, and is a path connected component in $\mathcal{C}_w(h^\infty)$ containing C_φ.*

(ii) *Suppose that $m(\{|\varphi^*| = 1\}) < 1$. Let*

$$X = \big\{M_u C_\psi \in \mathcal{C}_w(h^\infty) : m(\{|\psi^*| = 1\}) < 1, u \in L^\infty(\partial\mathbb{D}), u \ne 0\big\}.$$

Then X is open and closed, and is a path connected component in $\mathcal{C}_w(h^\infty)$ containing C_φ.

Proof. By Lemma 3.2, $C_\varphi \sim M_u C_\varphi$ for $u \in L^\infty(\partial\mathbb{D})$ with $u \ne 0$.

(i) Suppose that $m(\{|\varphi^*| = 1\}) = 1$. Let $u \in L^\infty(\partial\mathbb{D})$, $u \ne 0$, and $M_v C_\psi \in \mathcal{C}_w(h^\infty)$. If $\varphi \ne \psi$, then by Lemma 3.5 we have

$$\|M_u C_\varphi - M_v C_\psi\|_{h^\infty} \ge \||u| + |v|\|_{\partial\mathbb{D}} \ge \|u\|_{\partial\mathbb{D}} > 0.$$

Then $\{M_u C_\varphi \in \mathcal{C}_w(h^\infty) : u \in L^\infty(\partial\mathbb{D}), u \ne 0\}$ is open and is a path connected subset in $\mathcal{C}_w(h^\infty)$.

Suppose that there is a sequence $\{u_n\}_n$ in $L^\infty(\partial\mathbb{D})$ such that $u_n \ne 0$ for every $n \ge 1$ and $M_{u_n} C_\varphi \to M_v C_\psi \in \mathcal{C}_w(h^\infty)$. If $\varphi \ne \psi$, then by Lemma 3.5 we have

$u_n \to 0$ as $n \to \infty$, so $M_{u_n} C_\varphi \to 0$. Hence $M_v C_\psi = 0 \notin \mathcal{C}_w(h^\infty)$. This is a contradiction. Thus we get $\psi = \varphi$. Hence $\{M_u C_\varphi \in \mathcal{C}_w(h^\infty) : u \in L^\infty(\partial\mathbb{D}), u \neq 0\}$ is open and closed in $\mathcal{C}_w(h^\infty)$.

Suppose that there is an element $M_v C_\psi \in \mathcal{C}_w(h^\infty)$ such that $C_\varphi \sim M_v C_\psi$. Let $[0,1] \ni t \to M_{u_t} C_{\varphi_t}$ be a continuous map in $\mathcal{C}_w(h^\infty)$ such that $M_{u_0} C_{\varphi_0} = C_\varphi$ and $M_{u_1} C_{\varphi_1} = M_v C_\psi$. Let $A = \{t \in [0,1] : \varphi_t = \varphi\}$. Then $A \neq \emptyset$. By Lemma 3.5, A is open, and by the last paragraph A is closed. So A is open and closed in $[0,1]$. Hence $A = [0,1]$ and $\psi = \varphi$. Thus $\{M_u C_\varphi \in \mathcal{C}_w(h^\infty) : u \in L^\infty(\partial\mathbb{D}), u \neq 0\}$ is open and closed, and is a path connected component in $\mathcal{C}_w(h^\infty)$ containing C_φ.

(ii) By (i), X is closed in $\mathcal{C}_w(h^\infty)$. Let $\eta \in \mathcal{S}(\mathbb{D})$ such that $\|\eta\|_\infty = 1$ and $m(\{|\eta^*| = 1\}) = 0$. Let $M_v C_\psi \in X$. By Lemma 3.3, $C_\varphi \sim M_v C_\eta \sim M_v C_\psi$. Thus X is a path connected component in $\mathcal{C}_w(h^\infty)$ containing C_φ. Let $\xi \in \mathcal{S}(\mathbb{D})$ and $m(\{|\xi^*| = 1\}) = 1$. By Lemma 3.5, we have

$$\|M_v C_\psi - M_\mu C_\xi\|_{h^\infty} \geq \||v| + |\mu|\|_{\partial\mathbb{D}} \geq \|v\|_{\partial\mathbb{D}} > 0$$

for every $\mu \in L^\infty(\partial\mathbb{D})$ with $\mu \neq 0$. Then X is open in $\mathcal{C}_w(h^\infty)$. Thus we get (ii). $\square$

Next we will consider the connected component in the set $\mathcal{C}_{w,0}(h^\infty)$ of noncompact weighted composition operators on h^∞.

The following lemma follows from the proofs of Lemmas 5 and 6 in [12].

Lemma 3.7. *Let $z_1, z_2 \in \mathbb{D}$ and $z_1 \neq z_2$. For $0 < t < 1$, put $w_t = (1-t)z_1 + tz_2$. For $t_1, t_2 \in [0,1]$ satisfying*

$$0 < |t_1 - t_2| < \frac{1}{\rho(z_1, z_2)} - 1,$$

we have

$$\rho(w_{t_1}, w_{t_2}) \leq \frac{|t_1 - t_2|}{\frac{1}{\rho(z_1,z_2)} - 1 - |t_1 - t_2|}.$$

For $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,0}(h^\infty)$, we write $M_u C_\varphi \approx M_v C_\psi$ if there is a continuous map $[0,1] \ni t \to M_{u_t} C_{\varphi_t} \in \mathcal{C}_{w,0}(h^\infty)$ such that $M_{u_0} C_{\varphi_0} = M_u C_\varphi$ and $M_{u_1} C_{\varphi_1} = M_v C_\psi$.

Let $u \in L^\infty(\partial\mathbb{D})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ satisfy $u \neq 0$ and $\varphi \neq \psi$. Let

$$\Omega_u = L_u \cap L_{\varphi^*} \cap L_{\psi^*}.$$

Then $m(\Omega_u) = 1$. For each $0 < R < 1$, we put

$$\mathcal{T}_R = \left\{e^{i\theta} \in \{|\varphi^*| < 1\} \cap \{|\psi^*| < 1\} : \rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \leq R\right\}.$$

Theorem 3.8. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ and $\varphi \neq \psi$. Suppose that there exists a function $u \in L^\infty(\partial\mathbb{D})$ satisfying the following conditions:*

(i) $M_u C_\varphi \in \mathcal{C}_{w,0}(h^\infty)$,
(ii) $\lim_{R \to 1} \|u\chi_{\mathcal{T}_R^c}\|_{\partial\mathbb{D}} = 0$.

Then $M_u C_\psi \in \mathcal{C}_{w,0}(h^\infty)$ and $M_u C_\varphi \approx M_u C_\psi$ in $\mathcal{C}_{w,0}(h^\infty)$.

Proof. By condition (i) and Lemma 3.1, there exists a positive number ε such that $\|u\chi_{\{|\varphi^*| > r\}}\|_{\partial\mathbb{D}} > \varepsilon$ for every $0 < r < 1$. Since $m(\Omega_u) = 1$, we have $\|u\chi_{(\Omega_u \cap \{|\varphi^*| > r\})}\|_{\partial\mathbb{D}} > \varepsilon$. By condition (ii), there is a number $R_0, 0 < R_0 < 1$, such that

$$\left\|u\chi_{(\Omega_u \setminus \mathcal{T}_{R_0})}\right\|_{\partial\mathbb{D}} < \varepsilon. \tag{3.1}$$

Hence

$$\left\|u\chi_{\left(\Omega_u\cap\{|\varphi^*|>r\}\cap\mathcal{T}_{R_0}\right)}\right\|_{\partial\mathbb{D}}>\varepsilon$$

for every $0<r<1$. Since

$$\Omega_u\cap\big(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\big)\subset\Omega_u\setminus\mathcal{T}_R$$

for every $0<R<1$, by condition (ii) again $u=0$ a.e. on $\Omega_u\cap\big(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\big)$. Thus we get

$$m\big(\Omega_u\cap\{r<|\varphi^*|<1\}\cap\mathcal{T}_{R_0}\big)>0 \tag{3.2}$$

and

$$\left\|u\chi_{\left(\Omega_u\cap\{r<|\varphi^*|<1\}\cap\mathcal{T}_{R_0}\right)}\right\|_{\partial\mathbb{D}}>\varepsilon \tag{3.3}$$

for every $0<r<1$. Take

$$e^{i\theta_r}\in\Omega_u\cap\{r<|\varphi^*|<1\}\cap\mathcal{T}_{R_0}. \tag{3.4}$$

Then

$$r<|\varphi^*(e^{i\theta_r})|<1\quad\text{and}\quad\rho(\varphi^*(e^{i\theta_r}),\psi^*(e^{i\theta_r}))\le R_0.$$

Hence

$$\lim_{r\to1}|\varphi^*(e^{i\theta_r})|=1\quad\text{and}\quad\lim_{r\to1}|\varphi^*(e^{i\theta_r})-\psi^*(e^{i\theta_r})|=0. \tag{3.5}$$

By (3.3)–(3.5) and Lemma 3.1, we have $M_uC_\psi\in\mathcal{C}_{w,0}(h^\infty)$.

For each $0\le t\le1$, put $\varphi_t=(1-t)\varphi+t\psi$. Then $\varphi_t\in\mathcal{S}(\mathbb{D})$, so we put $T_t=M_uC_{\varphi_t}\in\mathcal{C}_w(h^\infty)$. By (3.4) and (3.5), we have

$$\lim_{r\to1}\inf_{e^{i\theta}\in\Omega_u\cap\{r<|\varphi^*|<1\}\cap\mathcal{T}_{R_0}}|\varphi_t^*(e^{i\theta})|=1.$$

By (3.2), $\|\varphi_t\|_\infty=1$. By (3.3) and Lemma 3.1, we have $T_t\in\mathcal{C}_{w,0}(h^\infty)$. We note that $T_0=M_uC_\varphi$ and $T_1=M_uC_\psi$. To show that $M_uC_\varphi\approx M_uC_\psi$, we need to prove that $[0,1]\ni t\to T_t\in\mathcal{C}_{w,0}(h^\infty)$ is a continuous map. Let $\{t_n\}_n$ be a sequence in $[0,1]$ satisfying $t_n\to t_0$. We may assume that $t_n\ne t_0$ for every n. We shall prove that $\|T_{t_n}-T_{t_0}\|_{h^\infty}\to0$ as $n\to\infty$. We have

$$\begin{aligned}\|T_{t_n}-T_{t_0}\|_{h^\infty}&=\sup_{f\in ball(h^\infty)}\|T_{t_n}f-T_{t_0}f\|_\infty\\&=\sup_{f\in ball(h^\infty)}\Big\|u\big(\big(f\circ\big((1-t_n)\varphi+t_n\psi\big)\big)^*\\&\qquad-\big(f\circ\big((1-t_0)\varphi+t_0\psi\big)\big)^*\big)\Big\|_{\partial\mathbb{D}}.\end{aligned}$$

Since $u=0$ a.e. on $\Omega_u\cap(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\})$, we have

$$\begin{aligned}&\Big\|u\big(\big(f\circ\big((1-t_n)\varphi+t_n\psi\big)\big)^*-\big(f\circ\big((1-t_0)\varphi+t_0\psi\big)\big)^*\big)\Big\|_{\partial\mathbb{D}}\\&=\sup_{e^{i\theta}\in\Omega_u\setminus\left(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\right)}\Big|u(e^{i\theta})\big(f\big((1-t_n)\varphi^*(e^{i\theta})+t_n\psi^*(e^{i\theta})\big)\\&\qquad-f\big((1-t_0)\varphi^*(e^{i\theta})+t_0\psi^*(e^{i\theta})\big)\big)\Big|.\end{aligned}$$

We set

$$A=\Omega_u\setminus\big(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\cup\mathcal{T}_{R_0}\big).$$

Then

$$\Omega_u\setminus\big(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\big)\subset A\cup(\Omega_u\cap\mathcal{T}_{R_0}),$$

and by (3.1) we have $\|u\chi_A\|_{\partial\mathbb{D}} < \varepsilon$. Hence

$$\begin{aligned}
&\|T_{t_n} - T_{t_0}\|_{h^\infty} \\
&< 2\varepsilon + \sup_{e^{i\theta}\in\Omega_u\cap\mathcal{T}_{R_0}} |u(e^{i\theta})| \\
&\qquad\qquad \times \Big\|P_{\left((1-t_n)\varphi^*(e^{i\theta})+t_n\psi^*(e^{i\theta})\right)} - P_{\left((1-t_0)\varphi^*(e^{i\theta})+t_0\psi^*(e^{i\theta})\right)}\Big\|_1 \\
&\le 2\varepsilon + 2\|u\|_{\partial\mathbb{D}} \sup_{e^{i\theta}\in\mathcal{T}_{R_0}} \rho\big((1-t_n)\varphi^*(e^{i\theta}) + t_n\psi^*(e^{i\theta}), \\
&\qquad\qquad (1-t_0)\varphi^*(e^{i\theta}) + t_0\psi^*(e^{i\theta})\big) \qquad \text{by Lemma 2.3.}
\end{aligned}$$

Since $m(\mathcal{T}_{R_0}) > 0$, $\varphi^* \neq \psi^*$ a.e. on $\mathcal{T}_{R_0}$. Let $e^{i\theta} \in \mathcal{T}_{R_0}$ and $\varphi^*(e^{i\theta}) \neq \psi^*(e^{i\theta})$. Since $0 < \rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \le R_0$, we have

$$\frac{1-R_0}{R_0} \le \frac{1}{\rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta}))} - 1.$$

For positive integers n satisfying $0 < |t_n - t_0| < (1-R_0)/R_0$, by Lemma 3.7 we have

$$\begin{aligned}
&\rho\big((1-t_n)\varphi^*(e^{i\theta}) + t_n\psi^*(e^{i\theta}), (1-t_0)\varphi^*(e^{i\theta}) + t_0\psi^*(e^{i\theta})\big) \\
&\le \frac{|t_n - t_0|}{\frac{1}{\rho(\varphi^*(e^{i\theta}),\psi^*(e^{i\theta}))} - 1 - |t_n - t_0|} \\
&\le \frac{|t_n - t_0|}{\frac{1-R_0}{R_0} - |t_n - t_0|}.
\end{aligned}$$

Hence

$$\|T_{t_n} - T_{t_0}\|_{h^\infty} < 2\varepsilon + 2\|u\|_{\partial\mathbb{D}} \frac{|t_n - t_0|}{\frac{1-R_0}{R_0} - |t_n - t_0|}.$$

Therefore

$$\limsup_{n\to\infty} \|T_{t_n} - T_{t_0}\|_{h^\infty} \le 2\varepsilon.$$

Since we may take ε sufficiently small, we have $\lim_{n\to\infty} \|T_{t_n} - T_{t_0}\| = 0$. Thus we get $M_uC_\varphi \approx M_vC_\varphi$ in $\mathcal{C}_{w,0}(h^\infty)$. □

Lemma 3.9. *If $\varphi \in \mathcal{S}(\mathbb{D})$ and $\|\varphi\|_\infty = 1$, then $C_\varphi \approx M_uC_\varphi$ in $\mathcal{C}_{w,0}(h^\infty)$ for every $u \in L^\infty(\partial\mathbb{D})$ satisfying $M_uC_\varphi \in \mathcal{C}_{w,0}(h^\infty)$.*

Proof. Since $M_uC_\varphi \in \mathcal{C}_{w,0}(h^\infty)$, by Lemma 3.1 there exists a positive constant δ such that $\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} > \delta$ for every $0 < r < 1$. Let $\{r_n\}_n$ be a sequence of increasing numbers satisfying $0 < r_n < 1$ and $r_n \to 1$. By Lemma 2.2, there is a sequence $\{e^{i\theta_n}\}_n$ in L_u such that $|u(e^{i\theta_n})| > \delta$, $e^{i\theta_n} \in L_{\chi_{\{|\varphi^*|>r_n\}}}$ and $e^{i\theta_n} \in \{|\varphi^*| > r_n\}$. We may assume that $u(e^{i\theta_n}) \to \alpha$ and $|\alpha| \ge \delta$. By Lemma 2.1, there is a sequence of measurable subsets $\{E_n\}_n$ in L_u such that $m(E_n) > 0$, $E_n \subset \{|\varphi^*| > r_n\}$ and

$$\sup_{e^{i\theta}\in E_n} |u(e^{i\theta}) - u(e^{i\theta_n})| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\sup_{e^{i\theta}\in E_n} |u(e^{i\theta}) - \alpha| \to 0 \quad \text{as} \quad n \to \infty.$$

It is easy to see that $M_u C_\varphi \approx M_{\overline{\alpha}u} C_\varphi$. For $0 \le t \le 1$, put $u_t = (1-t)\overline{\alpha}u + t$. Since

$$\sup_{e^{i\theta} \in E_n} |\overline{\alpha}u(e^{i\theta}) - |\alpha|^2| \to 0 \quad \text{as} \quad n \to \infty,$$

we have

$$\sup_{e^{i\theta} \in E_n} \left|u_t(e^{i\theta}) - \left((1-t)|\alpha|^2 + t\right)\right| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\|u_t \chi_{\{|\varphi^*| > r_n\}}\|_{\partial\mathbb{D}} \ge \|u_t \chi_{E_n}\|_{\partial\mathbb{D}} \to (1-t)|\alpha|^2 + t \quad \text{as} \quad n \to \infty.$$

By Lemma 3.1, $M_{u_t} C_\varphi \in \mathcal{C}_{w,0}(h^\infty)$ for every $0 \le t \le 1$. Also, we have

$$\|M_{u_t} C_\varphi - M_{u_{t_0}} C_\varphi\|_{h^\infty} = \|u_t - u_{t_0}\|_{\partial\mathbb{D}} \to 0 \quad \text{as} \quad t \to t_0.$$

Since $u_0 = \overline{\alpha}u$ and $u_1 = 1$, we have $M_{\overline{\alpha}u} C_\varphi \approx C_\varphi$. Thus we get $C_\varphi \approx M_u C_\varphi$ in $\mathcal{C}_{w,0}(h^\infty)$. □

Let $u \in L^\infty(\partial\mathbb{D})$. For $0 < r < 1$, we also write

$$\{|u| > r\} = \{e^{i\theta} \in L_u : |u(e^{i\theta})| > r\}.$$

Lemma 3.10. *Let $\{M_{u_n} C_{\varphi_n}\}_n$ be a sequence in $\mathcal{C}_{w,0}(h^\infty)$ such that $M_{u_n} C_{\varphi_n} \to M_u C_\varphi \in \mathcal{C}_{w,0}(h^\infty)$. Then $\|u_n - u\|_{\partial\mathbb{D}} \to 0$, $\|M_u(C_{\varphi_n} - C_\varphi)\|_{h^\infty} \to 0$ and $\left\|(\varphi_n^* - \varphi^*)\chi_{\{|u|>r\}}\right\|_{\partial\mathbb{D}} \to 0$ as $n \to \infty$ for every $0 < r < 1$.*

Proof. Since $\left\|(M_{u_n} C_{\varphi_n} - M_u C_\varphi)1\right\|_\infty \to 0$, we have $\|u_n - u\|_{\partial\mathbb{D}} \to 0$. Since $\|M_{u_n - u} C_{\varphi_n}\|_{h^\infty} = \|u_n - u\|_{\partial\mathbb{D}} \to 0$, we have $\|M_u(C_{\varphi_n} - C_\varphi)\|_{h^\infty} \to 0$. Hence $\|u(\varphi_n^* - \varphi^*)\|_{\partial\mathbb{D}} \to 0$. This shows that $\left\|(\varphi_n^* - \varphi^*)\chi_{\{|u|>r\}}\right\|_{\partial\mathbb{D}} \to 0$ for every $0 < r < 1$. □

Let $A(\overline{\mathbb{D}})$ be the disk algebra on the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, that is, $A(\overline{\mathbb{D}})$ is the space of continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$. For even $f \in A(\overline{\mathbb{D}})$, we use its norm by $\|f\|_\infty = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}$. In the following theorem, we determine path connected components in $\mathcal{C}_{w,0}(h^\infty)$. See Theorem 3.6 for path connected components in $\mathcal{C}_w(h^\infty)$.

Theorem 3.11. (i) *Let $\varphi \in \mathcal{S}(\mathbb{D})$. If*

$$0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r\})$$

for $0 < r < 1$ sufficiently close to 1, then $\left\{M_u C_\varphi \in \mathcal{C}_{w,0}(h^\infty) : u \in L^\infty(\partial\mathbb{D})\right\}$ is open and closed, and is a path connected component in $\mathcal{C}_{w,0}(h^\infty)$.

(ii) *The set*

$$X := \left\{M_u C_\varphi \in \mathcal{C}_{w,0}(h^\infty) : u \in L^\infty(\partial\mathbb{D}),\right.$$
$$\left. m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\}) \text{ for any } r, 0 < r < 1\right\}$$

is open and closed, and is a path connected component in $\mathcal{C}_{w,0}(h^\infty)$.

Proof. (i) Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,0}(h^\infty)$ and $\varphi \ne \psi$. By Lemma 3.1, there exists a positive number δ satisfying $\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} > \delta$ for every $0 < r < 1$. Since $0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r\})$ for $0 < r < 1$ sufficiently close to 1, we have $\|u\chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} > \delta$. By Lemma 3.5, we have

$$\|M_u C_\varphi - M_v C_\psi\|_{h^\infty} \ge \|u\chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} > \delta.$$

Then $\left\{M_u C_\varphi \in \mathcal{C}_{w,0}(h^\infty) : u \in L^\infty(\partial\mathbb{D})\right\}$ is an open subset in $\mathcal{C}_{w,0}(h^\infty)$.

Suppose that there is a sequence $\{u_n\}_n$ in $L^\infty(\partial\mathbb{D})$ such that $M_{u_n}C_\varphi \in \mathcal{C}_{w,0}(h^\infty)$ and $\|M_{u_n}C_\varphi - M_vC_\psi\|_{h^\infty} \to 0$. Then $\|u_n\chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} \to 0$. By Lemma 3.10, $\|u_n - v\|_{\partial\mathbb{D}} \to 0$, so we have $M_vC_\varphi = M_vC_\psi$ and $v = 0$ on $\{|\varphi^*| = 1\}$. By Lemma 3.1, M_vC_φ is compact, so $M_vC_\psi \notin \mathcal{C}_{w,0}(h^\infty)$. This is a contradiction, so $\{M_uC_\varphi \in \mathcal{C}_{w,0}(h^\infty) : u \in L^\infty(\partial\mathbb{D})\}$ is closed. By Lemma 3.9, we get (i).

(ii) We divide the proof into three steps.

Step 1. Let $\varphi \in \mathcal{S}(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ satisfying

$$m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\})$$

for every $0 < r < 1$. Then there is a sequence of distinct points $\{e^{i\theta_n}\}_n$ in L_{φ^*} such that $|\varphi^*(e^{i\theta_n})| < 1$ and $|\varphi^*(e^{i\theta_n})| \to 1$. We may assume that $e^{i\theta_n} \to e^{i\theta_0}$, $e^{i\theta_n} \neq e^{i\theta_0}$ for every $n \geq 1$ and $\varphi^*(e^{i\theta_n}) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$. Since $\{e^{i\theta_n}\}_{n\geq 0}$ is a peak interpolation set for $A(\overline{\mathbb{D}})$, there is a function $p \in A(\overline{\mathbb{D}})$ such that $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$ for every $n \geq 1$, $|p(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\}$, $p \neq \varphi$ and $\|p\|_\infty = 1$ (see [3]). Take a sequence of closed subarcs $\{I_n\}_{n\geq 1}$ in $\partial\mathbb{D}$ such that $e^{i\theta_n}$ is the center of I_n and $I_n \cap I_k = \emptyset$ for $n \neq k$. We fix $0 < r_0 < 1$. For $z \in \mathbb{D}$, we write $\Delta(z) = \{w \in \mathbb{D} : \rho(w, z) < r_0\}$. By Lemma 2.1, for each $n \geq 1$ there exists a measurable subset E_n such that $E_n \subset I_n \cap L_{\varphi^*}$, $m(E_n) > 0$ and

$$\sup_{e^{i\theta}\in E_n} |\varphi^*(e^{i\theta}) - \varphi^*(e^{i\theta_n})|$$

is sufficiently small. So we may assume that

$$\{\varphi^*(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

We have $\|\varphi^*\chi_{E_n}\|_{\partial\mathbb{D}} < 1$. Since $p \in A(\overline{\mathbb{D}})$ and $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$, moreover we may assume that

$$\{p(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

By [4, p. 4], we have

$$\rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq \frac{2r_0}{1 + r_0^2} \quad \text{for} \quad e^{i\theta} \in \bigcup_{n=1}^\infty E_n.$$

We define $\eta \in L^\infty(\partial\mathbb{D})$ by $\eta = 1$ on $\bigcup_{n=1}^\infty E_n$ and $\eta = 0$ on $\partial\mathbb{D} \setminus \bigcup_{n=1}^\infty E_n$. We shall prove that $M_\eta C_\varphi \approx M_\eta C_p$ by applying Theorem 3.8. Since $\|\varphi^*\chi_{E_n}\|_{\partial\mathbb{D}} \to 1$, by Lemma 3.1 we have $M_\eta C_\varphi \in \mathcal{C}_{w,0}(h^\infty)$. Let $\Omega_\eta = L_\eta \cap L_{\varphi^*}$, and, as before, for each $0 < R < 1$ we set

$$\mathcal{T}_R = \{e^{i\theta} \in \{|\varphi^*| < 1\} \cap \{|p| < 1\} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq R\}.$$

Take R_0 satisfying

$$\frac{2r_0}{1 + r_0^2} < R_0 < 1.$$

Then we have

$$\begin{aligned}
&\Omega_\eta \setminus \mathcal{T}_{R_0} \\
&\subset \{|\varphi^*| = 1\} \cup \{|p| = 1\} \\
&\quad \cup \{e^{i\theta} \in \Omega_\eta \cap \{|\varphi^*| < 1\} \cap \{|p| < 1\} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) > R_0\} \\
&\subset \partial\mathbb{D} \setminus \bigcup_{n=1}^\infty E_n.
\end{aligned}$$

Hence $\eta = 0$ a.e. on $\Omega_\eta \setminus \mathcal{T}_{R_0}$. Since $\Omega_\eta \setminus \mathcal{T}_R$ is decreasing in R, we have $\|\eta\chi_{(\Omega_\eta\setminus\mathcal{T}_R)}\|_{\partial\mathbb{D}} \to 0$ as $R \to 1$. Thus, by Theorem 3.8, we get that $M_\eta C_p \in \mathcal{C}_{w,0}(h^\infty)$ and $M_\eta C_\varphi \approx M_\eta C_p$. By Lemma 3.9, we have

$$C_\varphi \approx M_\eta C_\varphi \approx M_\eta C_p \approx C_p.$$

Step 2. Suppose that $\psi \in \mathcal{S}(\mathbb{D})$, $\|\psi\|_\infty = 1$ and $m(\{|\psi^*| = 1\}) < m(\{|\psi^*| > r\})$ for every $0 < r < 1$. By Step 1, there exist a function $q \in A(\overline{\mathbb{D}})$ and a point $e^{it_0} \in \partial\mathbb{D}$ satisfying $C_\psi \approx C_q$, $|q(e^{it_0})| = 1$ and $|q(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{it_0}\}$.

In this step, we shall show that if $e^{i\theta_0} \neq e^{it_0}$, then $C_p \approx C_q$. Suppose that $e^{i\theta_0} \neq e^{it_0}$. Then there is a sequence of distinct points $\{e^{it_n}\}_n$ in $\partial\mathbb{D}$ such that $e^{it_n} \to e^{it_0}$ and $e^{it_n} \neq e^{it_0}$ for every n. Moreover, we may assume that $\{e^{i\theta_n}\}_{n\geq 0} \cap \{e^{it_n}\}_{n\geq 0} = \emptyset$. Then $\{e^{i\theta_n}\}_{n\geq 0} \cup \{e^{it_n}\}_{n\geq 0}$ is a peak interpolation set for $A(\overline{\mathbb{D}})$, so there is a function $g \in A(\overline{\mathbb{D}})$ satisfying $g(e^{i\theta_n}) = p(e^{i\theta_n})$, $g(e^{it_n}) = q(e^{it_n})$ for every $n \geq 0$ and $|g(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}, e^{it_0}\}$. By the same argument given in Step 1, we have $C_p \approx C_g$ and $C_q \approx C_g$. Hence $C_p \approx C_q$.

Step 3. Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,0}(h^\infty)$ such that $m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\})$ and $m(\{|\psi^*| = 1\}) < m(\{|\psi^*| > r\})$ for every $0 < r < 1$. If $e^{i\theta_0} \neq e^{it_0}$, then by Step 2 and Lemma 3.9 we have

$$M_u C_\varphi \approx C_\varphi \approx C_p \approx C_q \approx C_\psi \approx M_v C_\psi.$$

Suppose that $e^{i\theta_0} = e^{it_0}$. Take $e^{is_0} \in \partial\mathbb{D} \setminus \{e^{i\theta_0}\}$ and a function $h \in A(\overline{\mathbb{D}})$ such that $|h(e^{i\theta_0})| = |h(e^{is_0})| = 1$ and $|h(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}, e^{is_0}\}$. Then, by Step 2, we have $C_p \approx C_h \approx C_q$. Hence we have $M_u C_\varphi \approx M_v C_\psi$.

By (i), X is a closed set. To prove that X is open in $\mathcal{C}_{w,0}(h^\infty)$, let $M_u C_\varphi \in X$. Suppose that there is a sequence $\{M_{u_n} C_{\varphi_n}\}_n$ in $\mathcal{C}_{w,0}(h^\infty) \setminus X$ such that $M_{u_n} C_{\varphi_n} \to M_u C_\varphi$ as $n \to \infty$. Then $m(\{|\varphi_n^*| = 1\}) > 0$, and for each n there is a number $r_n, 0 < r_n < 1$, such that

$$m(\{r_n < |\varphi_n^*| < 1\}) = 0. \tag{3.6}$$

By Lemma 3.10, $\|M_u(C_{\varphi_n} - C_\varphi)\|_{h^\infty} \to 0$ as $n \to \infty$. By Lemma 3.5,

$$\left\|u\chi_{\left(\{|\varphi_n^*|=1\}\cup\{|\varphi^*|=1\}\right)}\right\|_{\partial\mathbb{D}} \to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $u = 0$ a.e. on $\{|\varphi^*| = 1\}$ and

$$\|u\chi_{\{|\varphi_n^*|=1\}}\|_{\partial\mathbb{D}} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Since $M_u C_\varphi \in X$, $m(\{r < |\varphi^*| < 1\}) > 0$, and by Lemma 3.1 there exists a positive constant δ such that $\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} > \delta$. Hence

$$\|u\chi_{\{r<|\varphi^*|<1\}}\|_{\partial\mathbb{D}} > \delta \tag{3.8}$$

for every r. Let

$$E_r = \{|u| > \delta\} \cap \{r < |\varphi^*| < 1\} \cap \bigcap_{n=1}^{\infty} L_{\varphi_n^*}.$$

By (3.8), $m(E_r) > 0$ for every r, and by (3.7) there is a positive integer n_0 such that $m(E_r \cap \{|\varphi_n^*| = 1\}) = 0$ for $n \geq n_0$. Let

$$G_r = E_r \setminus \bigcup_{n=n_0}^{\infty} \{|\varphi_n^*| = 1\}. \tag{3.9}$$

Then $m(G_r) = m(E_r) > 0$ and $\sup_{e^{i\theta}\in G_r} |\varphi^*(e^{i\theta})| = 1$. By (3.6), $|\varphi_n^*| \le r_n$ a.e. on G_r for $n \ge n_0$. For each $n \ge n_0$, we have

$$\begin{aligned}
&\|M_u(C_{\varphi_n} - C_\varphi)\|_{h^\infty} \\
&= \sup_{f\in ball(h^\infty)} \big\|u\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} \\
&\ge \sup_{f\in ball(h^\infty)} \big\|u\chi_{G_r}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} \\
&\ge \delta \sup_{f\in ball(h^\infty)} \operatorname{ess\,sup}_{e^{i\theta}\in G_r} \big|f(\varphi_n^*(e^{i\theta})) - f(\varphi^*(e^{i\theta}))\big| \qquad \text{by (3.9)} \\
&= \delta \operatorname{ess\,sup}_{e^{i\theta}\in G_r} \big\|P_{\varphi_n^*(e^{i\theta})} - P_{\varphi^*(e^{i\theta})}\big\|_1 \\
&= 2\delta.
\end{aligned}$$

This is a contradiction. Thus X is open in $\mathcal{C}_{w,0}(h^\infty)$, and we get (ii). □

4. Weighted composition operators on H^∞

For $M_uC_\varphi, M_vC_\psi \in \mathcal{C}_w(H^\infty)$, we write $M_uC_\varphi \sim M_vC_\psi$ if there is a continuous map $[0,1] \ni t \to M_{u_t}C_{\varphi_t} \in \mathcal{C}_w(H^\infty)$ such that $M_{u_0}C_{\varphi_0} = M_uC_\varphi$ and $M_{u_1}C_{\varphi_1} = M_vC_\psi$. We note that $M_uC_\varphi \in \mathcal{C}_w(H^\infty)$ if and only if $u \in H^\infty$ and $u \ne 0$.

In [7, Theorem 4.1], Hosokawa and the first and third authors determined path connected components in $\mathcal{C}_w(H^\infty)$ as follows. See Theorem 3.6 for path connected components in $\mathcal{C}_w(h^\infty)$.

Theorem 4.1. (i) *Let $\varphi \in \mathcal{S}(\mathbb{D})$. If $m(\{|\varphi^*| = 1\}) > 0$, then $\{M_uC_\varphi : u \in H^\infty, u \ne 0)\}$ is open and closed, and is a path connected component in $\mathcal{C}_w(H^\infty)$ containing C_φ.*

(ii) *The set*

$$\big\{M_uC_\varphi : u \in H^\infty, u \ne 0, \varphi \in \mathcal{S}(\mathbb{D}), m(\{|\varphi^*| = 1\}) = 0\big\}$$

is closed, and is a path connected component in $\mathcal{C}_w(H^\infty)$, but the set is not open.

Lemma 4.2. *Let $M_uC_\varphi \in \mathcal{C}_w(H^\infty)$. Then M_uC_φ is compact on H^∞ if and only if $\|u^*\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} \to 0$ as $r \to 1$.*

Proof. By [10, Lemma 4.7], M_uC_φ is compact on H^∞ if and only if M_uC_φ is compact on h^∞. By Lemma 3.1, we get the assertion. □

We put

$$d_\infty(z,w) = \sup_{f\in ball(H^\infty)} |f(z) - f(w)| \quad \text{for} \quad z,w \in \mathbb{D}.$$

It is known that

$$d_\infty(z,w) = \frac{2\big(1 - \sqrt{1-\rho(z,w)^2}\big)}{\rho(z,w)}$$

and $\rho(z,w) \le d_\infty(z,w) \le 2\rho(z,w)$ (see [12]).

Let $\mathcal{C}_{w,c}(H^\infty)$ be the set of compact weighted composition operators in $\mathcal{C}_w(H^\infty)$.

Proposition 4.3. *The set $\mathcal{C}_{w,c}(H^\infty)$ is a path connected set in $\mathcal{C}_w(H^\infty)$.*

Proof. Let $M_uC_\varphi, M_vC_\psi \in \mathcal{C}_{w,c}(H^\infty)$, and let $\eta \in \mathcal{S}(\mathbb{D})$ satisfy $\|\eta\|_\infty < 1$. For $0 \le t \le 1$, we write $\varphi_t = (1-t)\varphi + t\eta$. Then $\varphi_0 = \varphi$ and $\varphi_1 = \eta$. Since $\|\varphi_t\|_\infty < 1$ for $0 < t \le 1$, we have $M_uC_{\varphi_t} \in \mathcal{C}_{w,c}(H^\infty)$ for every $0 \le t \le 1$. Let $\{t_n\}_n$ be a sequence in $[0,1]$ such that $t_n \to t_0$. Then $\|\varphi^*_{t_n} - \varphi^*_{t_0}\|_{\partial\mathbb{D}} \to 0$. We have

$$\begin{aligned}
&\|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\| \\
&= \sup_{f\in ball(H^\infty)} \|u(f\circ\varphi_{t_n} - f\circ\varphi_{t_0})\|_\infty \\
&= \sup_{f\in ball(H^\infty)} \max\big\{\big\|u\chi_{\{|\varphi^*|>r\}}\big((f\circ\varphi_{t_n})^* - (f\circ\varphi_{t_0})^*\big)\big\|_{\partial\mathbb{D}}, \\
&\qquad\qquad \big\|u\chi_{\{|\varphi^*|\le r\}}\big((f\circ\varphi_{t_n})^* - (f\circ\varphi_{t_0})^*\big)\big\|_{\partial\mathbb{D}}\big\} \quad (0<r<1) \\
&\le \sup_{f\in ball(H^\infty)} \max\big\{2\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}}, \\
&\qquad\qquad \|u\|_{\partial\mathbb{D}} \sup_{e^{i\theta}\in\{|\varphi^*|\le r\}\cap L_{\eta^*}} \big|f(\varphi^*_{t_n}(e^{i\theta})) - f(\varphi^*_{t_0}(e^{i\theta}))\big|\big\} \\
&= \max\big\{2\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}}, \\
&\qquad\qquad \|u\|_{\partial\mathbb{D}} \sup_{e^{i\theta}\in\{|\varphi^*|\le r\}\cap L_{\eta^*}} d_\infty\big(\varphi^*_{t_n}(e^{i\theta}), \varphi^*_{t_0}(e^{i\theta})\big)\big\}.
\end{aligned}$$

Since $\varphi^*_{t_n} \to \varphi^*_{t_0}$ uniformly on $\partial\mathbb{D}$ and $\|\varphi^*_{t_0}\chi_{\{|\varphi^*|\le r\}}\|_{\partial\mathbb{D}} < 1$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{e^{i\theta}\in\{|\varphi^*|\le r\}\cap L_{\eta^*}} d_\infty\big(\varphi^*_{t_n}(e^{i\theta}), \varphi^*_{t_0}(e^{i\theta})\big) \\
&\le 2\limsup_{n\to\infty} \sup_{e^{i\theta}\in\{|\varphi^*|\le r\}\cap L_{\eta^*}} \rho\big(\varphi^*_{t_n}(e^{i\theta}), \varphi^*_{t_0}(e^{i\theta})\big) = 0.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\| \le 2\|u\chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}}$$

for every $0 < r < 1$. Letting $r \to 1$, by Lemma 4.2 we get

$$\lim_{n\to\infty} \|M_uC_{\varphi_{t_n}} - M_uC_{\varphi_{t_0}}\| = 0.$$

This shows that $[0,1] \ni t \to M_uC_{\varphi_t}$ is a continuous path in $\mathcal{C}_{w,c}(H^\infty)$. Therefore $M_uC_\varphi \sim M_uC_\eta$ in $\mathcal{C}_{w,c}(H^\infty)$. Similarly we have $M_vC_\psi \sim M_uC_\eta$ in $\mathcal{C}_{w,c}(H^\infty)$. In the same way as Lemma 3.2, we may prove that $M_{u_t}C_\eta \sim M_{u_s}C_\eta$ in $\mathcal{C}_{w,c}(H^\infty)$. As a consequence, we get the assertion. □

In this section, we determine path connected components in $\mathcal{C}_{w,0}(H^\infty)$ which answer questions in [7, Problem 5.3]. We also write $M_uC_\varphi \approx M_vC_\psi$ if there is a continuous path connecting M_uC_φ and M_vC_ψ in $\mathcal{C}_{w,0}(H^\infty)$.

In the same way as in the proof of Lemma 3.9, we may also prove the following.

Lemma 4.4. *If $\varphi \in \mathcal{S}(\mathbb{D})$ and $\|\varphi\|_\infty = 1$, then $C_\varphi \approx M_uC_\varphi$ in $\mathcal{C}_{w,0}(H^\infty)$ for every $u \in H^\infty$ satisfying $M_uC_\varphi \in \mathcal{C}_{w,0}(H^\infty)$.*

In the same way as in the proof of Theorem 3.8, we may prove the following. Recall that for $0 < R < 1$,

$$\mathcal{T}_R = \big\{e^{i\theta} \in \{|\varphi^*| < 1\} \cap \{|\psi^*| < 1\} : \rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \le R\big\}.$$

Theorem 4.5. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $\varphi \neq \psi$. Suppose that there exists a function $u \in H^\infty$ satisfying the following conditions:*

(i) $M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty)$,

(ii) $\lim_{R\to 1} \|u^* \chi_{\mathcal{T}_R^c}\|_{\partial\mathbb{D}} = 0$.

Then $M_u C_\psi \in \mathcal{C}_{w,0}(H^\infty)$ and $M_u C_\varphi \approx M_u C_\psi$ in $\mathcal{C}_{w,0}(H^\infty)$.

In Theorem 4.5, since $M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty)$, we have $u \neq 0$, and by condition (ii) we have $m\big(\{|\varphi^*| = 1\} \cup \{|\psi^*| = 1\}\big) = 0$. In [7, Theorem 5.2], we proved a similar theorem using some work on the maximal ideal space of $L^\infty(\partial\mathbb{D})$. Condition (ii) in Theorem 4.5 is much simpler than the corresponding one in [7].

There is a big difference in properties between $\mathcal{C}_w(h^\infty)$ and $\mathcal{C}_w(H^\infty)$. For a measurable subset E of $\partial\mathbb{D}$ satisfying $0 < m(E) < 1$, there is $u \in h^\infty$ such that $u^* = 0$ a.e. on E and $u \neq 0$, but there are no $u \in H^\infty$ satisfying $u^* = 0$ a.e. on E and $u \neq 0$.

The following follows from [10, Theorem 4.1].

Lemma 4.6. *Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_w(H^\infty)$ and $\varphi \neq \psi$. Then*

$$\|M_u C_\varphi - M_v C_\psi\|_{H^\infty} \geq \big\|(|u^*| + |v^*|)\chi_{\left(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\right)}\big\|_{\partial\mathbb{D}}.$$

The following is the main theorem in this section. Compare Theorem 4.7 with Theorems 3.11 and 4.1.

Theorem 4.7. (i) *Let $\varphi \in \mathcal{S}(\mathbb{D})$. If $m(\{|\varphi^*| = 1\}) > 0$, then $\{M_u C_\varphi : u \in H^\infty, u \neq 0\}$ is open and closed, and is a path connected component in $\mathcal{C}_{w,0}(H^\infty)$ containing C_φ.*

(ii) *The set*

$$X = \big\{M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty) : u \in H^\infty, \varphi \in \mathcal{S}(\mathbb{D}), m(\{|\varphi^*| = 1\}) = 0\big\}$$

is closed, and is a path connected component in $\mathcal{C}_{w,0}(H^\infty)$.

Proof. (i) Let $\varphi \in \mathcal{S}(\mathbb{D})$ and $m(\{|\varphi^*| = 1\}) > 0$. Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,0}(H^\infty)$. We have $\|u^* \chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} > 0$. If $\varphi \neq \psi$, then by Lemma 4.6 we have

$$\begin{aligned} \|M_u C_\varphi - M_v C_\psi\|_{H^\infty} &\geq \big\|(|u^*| + |v^*|)\chi_{\left(\{|\varphi^*|=1\}\cup\{|\psi^*|=1\}\right)}\big\|_{\partial\mathbb{D}} \\ &\geq \|u^* \chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} > 0. \end{aligned}$$

Thus $\{M_u C_\varphi : u \in H^\infty, u \neq 0\}$ is open in $\mathcal{C}_{w,0}(H^\infty)$.

Let $\{M_{u_n} C_\varphi\}_n$ be a sequence in $\mathcal{C}_{w,0}(H^\infty)$ such that $M_{u_n} C_\varphi \to M_v C_\psi \in \mathcal{C}_{w,0}(H^\infty)$. Suppose that $\varphi \neq \psi$. Then we have $\|u_n - v\|_\infty \to 0$ and $\|u_n^* \chi_{\{|\varphi^*|=1\}}\|_{\partial\mathbb{D}} \to 0$ as $n \to \infty$. Hence $v = 0$ a.e. on $\{|\varphi^*| = 1\}$. Since $m(\{|\varphi^*| = 1\}) > 0$, we have $v = 0$, so $M_v C_\psi = 0 \notin \mathcal{C}_{w,0}(H^\infty)$. This is a contradiction. Therefore $\psi = \varphi$ and $\{M_u C_\varphi : u \in H^\infty, u \neq 0\}$ is closed in $\mathcal{C}_{w,0}(H^\infty)$.

Let $M_u C_\varphi, M_v C_\psi \in \mathcal{C}_{w,0}(H^\infty)$. Suppose that $M_u C_\varphi \approx M_v C_\psi$ in $\mathcal{C}_{w,0}(H^\infty)$. Let $[0,1] \ni t \to M_{u_t} C_{\varphi_t} \in \mathcal{C}_{w,0}(H^\infty)$ be a continuous map satisfying $M_{u_0} C_{\varphi_0} = M_u C_\varphi$ and $M_{u_1} C_{\varphi_1} = M_v C_\psi$. Let $A = \{t \in [0,1] : \varphi_t = \varphi\}$. By the first paragraph, A is open in $[0,1]$. By the last paragraph, A is closed, so $A = [0,1]$. Hence $\psi = \varphi$. Therefore we get (i).

(ii) We divide the proof into three steps. Some parts of the proof are similar to those in Theorem 3.11.

Step 1. Let $\varphi \in \mathcal{S}(\mathbb{D})$, $\|\varphi\|_\infty = 1$ and $m(\{|\varphi^*| = 1\}) = 0$. Then $m(\{|\varphi^*| > r\}) > 0$ for every $0 < r < 1$. Take a sequence of distinct points $\{e^{i\theta_n}\}_n$ in L_{φ^*} such that $|\varphi^*(e^{i\theta_n})| < 1$ and $|\varphi^*(e^{i\theta_n})| \to 1$. We may assume that $e^{i\theta_n} \to e^{i\theta_0}$ and $\varphi^*(e^{i\theta_n}) \to \alpha \in \partial\mathbb{D}$. Since $\{e^{i\theta_n}\}_{n\geq 0}$ is a peak interpolation set for $A(\overline{\mathbb{D}})$, there is a function $p \in A(\overline{\mathbb{D}})$ such that $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$ for every $n \geq 1$, $|p(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\}$, $p \neq \varphi$ and $\|p\|_\infty = 1$. We shall show that $C_\varphi \approx C_p$ in $\mathcal{C}_{w,0}(H^\infty)$ by applying Theorem 4.5.

Take a sequence of closed subarcs $\{I_n\}_{n\geq 1}$ in $\partial\mathbb{D}$ such that $e^{i\theta_n}$ is the center of I_n and $I_n \cap I_k = \emptyset$ for $n \neq k$. We fix $0 < R_0 < 1$. For $z \in \mathbb{D}$, we write $\Delta(z) = \{w \in \mathbb{D} : \rho(w, z) < R_0\}$. By Lemma 2.1, for each $n \geq 1$ there exists a measurable subset E_n such that $E_n \subset I_n \cap L_{\varphi^*}$, $m(E_n) > 0$ and

$$\sup_{e^{i\theta} \in E_n} |\varphi^*(e^{i\theta}) - \varphi^*(e^{i\theta_n})|$$

is sufficiently small. So we may assume that

$$\{\varphi^*(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

We have $\|\varphi^*\chi_{E_n}\|_{\partial\mathbb{D}} < 1$ and $\|\varphi^*\chi_{E_n}\|_{\partial\mathbb{D}} \to 1$ as $n \to \infty$. Since $p \in A(\overline{\mathbb{D}})$ and $p(e^{i\theta_n}) = \varphi^*(e^{i\theta_n})$, moreover we may assume that

$$\{p(e^{i\theta}) : e^{i\theta} \in E_n\} \subset \Delta(\varphi^*(e^{i\theta_n})).$$

By [4, p. 4], we have

$$\rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq \frac{2R_0}{1 + R_0^2} \quad \text{for} \quad e^{i\theta} \in \bigcup_{n=1}^{\infty} E_n. \tag{4.1}$$

For each $0 < R < 1$, we put

$$\mathcal{T}_R = \big\{e^{i\theta} \in \{|\varphi^*| < 1\} \cap \{|p| < 1\} : \rho(\varphi^*(e^{i\theta}), p(e^{i\theta})) \leq R\big\}. \tag{4.2}$$

Since $m\big(\{|\varphi^*| = 1\} \cup \{|p| = 1\}\big) = 0$, we have $m(\mathcal{T}_R) \to 1$ as $R \to 1$. When $m(\mathcal{T}_R) = 1$ for some $0 < R < 1$, we set $u = 1$. Then we have $M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty)$ and $\|u^*\chi_{\mathcal{T}_R^c}\|_{\partial\mathbb{D}} = 0$. By Theorem 4.5, we have $C_\varphi \approx C_p$ in $\mathcal{C}_{w,0}(H^\infty)$. So we may assume that $m(\mathcal{T}_R) < 1$ for every $0 < R < 1$.

To show $C_\varphi \approx C_p$, take a sequence of increasing numbers $\{R_k\}_k$ such that $2R_0/(1 + R_0^2) < R_k < 1$ and $R_k \to 1$. By (4.1) and (4.2), we have

$$\bigcup_{n=1}^{\infty} E_n \subset \mathcal{T}_{R_1}. \tag{4.3}$$

Let $A_0 = \mathcal{T}_{R_1}$ and $A_k = \mathcal{T}_{R_{k+1}} \setminus \mathcal{T}_{R_k}$ for $k \geq 1$. Since $\mathcal{T}_{R_k} \subset \mathcal{T}_{R_{k+1}}$, $\{A_k\}_{k\geq 0}$ is a set of mutually disjoint sets and

$$m\Big(\bigcup_{k=0}^{\infty} A_k\Big) = 1.$$

Let $\{a_k\}_{k\geq 1}$ be a sequence of positive numbers such that $0 < a_k < 1$, $\lim_{k\to\infty} a_k = 0$ and

$$\sum_{k=0}^{\infty} m(A_k) \log a_k > -\infty.$$

We define a function $\eta \in L^\infty(\partial\mathbb{D})$ by

$$\eta(e^{i\theta}) = \begin{cases} 1, & e^{i\theta} \in A_0, \\ a_k, & e^{i\theta} \in A_k,\ k \geq 1. \end{cases} \tag{4.4}$$

Then we have

$$\int_{\partial\mathbb{D}} \log \eta(e^{i\theta})\, dm(e^{i\theta}) = \sum_{k=1}^{\infty} m(A_k) \log a_k > -\infty.$$

By [5, p. 53], there exists a function $u \in H^\infty$ such that $|u^*| = \eta$ a.e. on $\partial\mathbb{D}$. By (4.3), $|u^*| = 1$ a.e. on $\bigcup_{n=1}^\infty E_n$. Since $\|\varphi^* \chi_{E_n}\|_{\partial\mathbb{D}} \to 1$ as $n \to \infty$, we have $\|u^* \chi_{\{|\varphi^*|>r\}}\|_{\partial\mathbb{D}} \to 1$ as $r \to 1$. By Lemma 4.2, we get

$$M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty). \tag{4.5}$$

We shall prove that

$$\lim_{k\to\infty} \|u^* \chi_{\mathcal{T}_{R_k}^c}\|_{\partial\mathbb{D}} = 0. \tag{4.6}$$

By (4.4) we have

$$\|u^* \chi_{\mathcal{T}_{R_k}^c}\|_{\partial\mathbb{D}} = \left\|u^* \chi_{\left(\bigcup_{j=k}^\infty A_j\right)}\right\|_{\partial\mathbb{D}} = \max_{k\leq j<\infty} a_j \to 0 \quad \text{as} \quad k \to \infty.$$

Thus we get (4.6). By (4.5), (4.6) and Theorem 4.5, we have $M_u C_p \in \mathcal{C}_{w,0}(H^\infty)$ and $M_u C_\varphi \approx M_u C_p$. By Lemma 4.4, we have $C_\varphi \approx C_p$ in $\mathcal{C}_{w,0}(H^\infty)$.

Step 2. Suppose that $\psi \in \mathcal{S}(\mathbb{D})$, $\|\psi\|_\infty = 1$ and $m(\{|\psi^*| = 1\}) = 0$. By Step 1, there are a point $e^{it_0} \in \partial\mathbb{D}$ and a function $q \in A(\overline{\mathbb{D}})$ such that $|q(e^{it_0})| = 1$, $|q(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{it_0}\}$ and $C_\psi \approx C_q$ in $\mathcal{C}_{w,0}(H^\infty)$.

In this step, suppose that $e^{i\theta_0} \neq e^{it_0}$. Then there are sequences $\{e^{i\theta_n}\}_n$ and $\{e^{it_n}\}_n$ such that $e^{i\theta_n} \to e^{i\theta_0}$, $e^{it_n} \to e^{it_0}$ and $\{e^{i\theta_n}\}_{n\geq 0} \cap \{e^{it_n}\}_{n\geq 0} = \emptyset$. Here $\{e^{i\theta_n}\}_{n\geq 0} \cup \{e^{it_n}\}_{n\geq 0}$ is a peak interpolation set for $A(\overline{\mathbb{D}})$. Then there exists a function $g \in A(\overline{\mathbb{D}})$ such that $g(e^{i\theta_n}) = p(e^{i\theta_n})$, $g(e^{it_n}) = q(e^{it_n})$, $|g(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}, e^{it_0}\}$ and $\|g\|_\infty = 1$. By the same argument given in Step 1, we have $C_p \approx C_g$ and $C_q \approx C_g$. Hence $C_p \approx C_q$ in $\mathcal{C}_{w,0}(H^\infty)$.

Step 3. Let $M_u C_\varphi, M_v C_\psi \in X$. Then $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ and $m(\{|\varphi^*| = 1\}) = m(\{|\psi^*| = 1\}) = 0$. We shall show that $M_u C_\varphi \approx M_v C_\psi$ in $\mathcal{C}_{w,0}(H^\infty)$. By Steps 1 and 2, there are functions $p, q \in A(\overline{\mathbb{D}})$ such that $C_\varphi \approx C_p$, $C_\psi \approx C_q$, $\|p\|_\infty = \|q\|_\infty = 1$, $|p(e^{i\theta_0})| = |q(e^{it_0})| = 1$, $|p(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\}$ and $|q(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{it_0}\}$. By Step 2, if $e^{i\theta_0} \neq e^{it_0}$, then $C_p \approx C_q$, so by Lemma 4.4 we have

$$M_u C_\varphi \approx C_\varphi \approx C_p \approx C_q \approx C_\psi \approx M_v C_\psi.$$

Suppose that $e^{i\theta_0} = e^{it_0}$. Take $e^{is_0} \in \partial\mathbb{D} \setminus \{e^{i\theta_0}, e^{it_0}\}$ and $h \in A(\overline{\mathbb{D}})$ such that $|h(e^{is_0})| = 1$ and $|h(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{is_0}\}$. By Step 2, we have $C_p \approx C_h$ and $C_q \approx C_h$, so $C_p \approx C_q$. Hence we have $M_u C_\varphi \approx M_v C_\psi$. Therefore X is a path connected component in $\mathcal{C}_{w,0}(H^\infty)$.

By (i), X is closed in $\mathcal{C}_{w,0}(H^\infty)$. This completes the proof. □

In Theorem 4.7, X is not open. To show this fact, we need the following lemma (for example, see [13, Theorem 15.4]).

Lemma 4.8. *Let $\{M_n\}_n$ be a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} M_n < \infty$. Let $\{g_n\}_n$ be a sequence in $A(\overline{\mathbb{D}})$ such that $\|g_n\|_\infty = 1$ and $\|g_n - 1\|_\infty \le M_n$ for every n. Then $\prod_{n=1}^{\infty} g_n$ converges uniformly on $\overline{\mathbb{D}}$ and $\prod_{n=1}^{\infty} g_n \in A(\overline{\mathbb{D}})$.*

Example 4.9. We shall show that X is not open in $\mathcal{C}_{w,0}(H^\infty)$. Let $\{M_n\}_n$ be a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} M_n < 1$. Let $\{\theta_j\}_j$ be a sequence of positive numbers satisfying

$$\left\{z \in \mathbb{D} : |z - e^{i\theta_j}| < |1 - e^{i\theta_j}|\right\} \subset \{z \in \mathbb{D} : |z - 1| < M_j\}.$$

Then $\theta_j \to 0$ as $j \to \infty$. Put

$$I_j = \left\{e^{i\theta} \in \partial\mathbb{D} : |e^{i\theta} - e^{i\theta_j}| < |1 - e^{i\theta_j}|\right\}.$$

By the theorems of Riemann mapping and Carathéodory, there is a function $g_j \in A(\overline{\mathbb{D}})$ such that g_j is a conformal map from $\mathbb{D}$ onto

$$\left\{z \in \mathbb{D} : |z - e^{i\theta_j}| < |1 - e^{i\theta_j}|\right\},$$

$|g_j| = 1$ on I_j, $g_j(I_j) = I_j$, $g_j(1) = 1$ and $|g_j| < 1$ on $\overline{\mathbb{D}} \setminus \overline{I_j}$. We note that $|g_j(z) - 1| \le M_j$ for every $z \in \overline{\mathbb{D}}$ and $j \ge 1$. Also, there are a function $h_0 \in A(\overline{\mathbb{D}})$ and a sequence of distinct numbers $\{t_k\}_k$ in $(-1, 0)$ satisfying $t_k \to 0$ such that $|h_0| = 1$ on I_1, $h_0(I_1) = I_1$, $h_0(1) = h_0(e^{it_k}) = 1$ for every k, and $|h_0| < 1$ on $\overline{\mathbb{D}} \setminus (I_1 \cup \{e^{it_k}\}_k)$.

Let

$$\varphi = g_1 \prod_{j=2}^{\infty} g_j \circ h_0.$$

Then $\|g_j \circ h_0 - 1\|_\infty \le M_j$ for every $j \ge 2$, so by Lemma 4.8 we have that

$$\varphi_n := g_1 \prod_{j=2}^{n} g_j \circ h_0 \to \varphi$$

uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$, $\varphi \in A(\overline{\mathbb{D}})$, $\|\varphi\|_\infty = 1$, $\{|\varphi| = 1\} = \{1\}$ and $\varphi(1) = 1$. Also, we have that $|g_1(e^{it_k})| < 1$, $g_1(e^{it_k}) \to 1$ as $k \to \infty$ and $(g_j \circ h_0)(e^{it_k}) = g_j(1) = 1$ for every $j \ge 2$ and $k \ge 1$. Thus we get that $\varphi_n(e^{it_k}) = g_1(e^{it_k}) = \varphi(e^{it_k})$ for every $k \ge 1$ and $n \ge 2$, $|\varphi(e^{it_k})| < 1$ and $\varphi(e^{it_k}) \to 1$ as $k \to \infty$. We have

$$\begin{aligned}
\{|\varphi_n| = 1\} &= \{|g_1| = 1\} \cap \bigcap_{j=2}^{n} \{|g_j \circ h_0| = 1\} \\
&= I_1 \cap \bigcap_{j=2}^{n} \{e^{i\theta} \in I_1 : h_0(e^{i\theta}) \in I_j\} \\
&= \{e^{i\theta} \in I_1 : h_0(e^{i\theta}) \in I_n\}.
\end{aligned}$$

Hence $m(\{|\varphi_n| = 1\}) > 0$, $\{|\varphi_n| = 1\}$ is a closed subarc of $\partial\mathbb{D}$ and $\{|\varphi_n| = 1\}$ converges to the point 1. Since $\varphi_n \to \varphi$ uniformly on $\overline{\mathbb{D}}$ and $\varphi_n(e^{it_k}) = \varphi(e^{it_k})$, there is a closed subarc J_k of $\partial\mathbb{D}$ centered at e^{it_k} such that for each fixed k,

$$\lim_{n\to\infty} \sup_{e^{i\theta} \in J_k} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) = 0 \tag{4.7}$$

and

$$\lim_{k\to\infty} \sup_{e^{i\theta} \in J_k} \sup_{n \ge 2} \rho(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})) = 0. \tag{4.8}$$

Moreover, we may assume that $J_k \cap J_\ell = \emptyset$ for $k \ne \ell$. Then $1 \notin J_k$ for every k.

Let

$$L_1 = \Big\{ e^{i\theta} \in \partial\mathbb{D} \setminus \bigcup_{k=1}^{\infty} J_k : |e^{i\theta} - 1| > 1 \Big\},$$

and for $j \geq 2$ let

$$L_j = \Big\{ e^{i\theta} \in \partial\mathbb{D} \setminus \bigcup_{k=1}^{\infty} J_k : \frac{1}{j+1} < |e^{i\theta} - 1| \leq \frac{1}{j} \Big\}.$$

Take a sequence of numbers $\{a_j\}_j$ in $(0,1)$ satisfying that

$$\sum_{j=1}^{\infty} m(L_j) \log a_j > -\infty.$$

Let

$$\eta(e^{i\theta}) = \begin{cases} 1, & e^{i\theta} \in \bigcup_{k=1}^{\infty} J_k, \\ a_j, & e^{i\theta} \in L_j, \\ 0, & e^{i\theta} = 1. \end{cases}$$

Then $\eta \in L^\infty(\partial\mathbb{D})$ and

$$\int_{\partial\mathbb{D}} \log \eta(e^{i\theta})\, dm(e^{i\theta}) = \sum_{j=1}^{\infty} m(L_j) \log a_j > -\infty.$$

By [5, p. 53], there is a function $u \in H^\infty$ such that $|u^*| = \eta$ a.e. on $\partial\mathbb{D}$. We have $|u^*| = 1$ a.e. on $\bigcup_{k=1}^{\infty} J_k$. We note that $\|u\|_\infty = 1$ and $|u^*| = a_j$ a.e. on L_j. Since $\varphi(1) = 1$, we have $\|\varphi\chi_{J_k}\|_{\partial\mathbb{D}} \to 1$, so $\|u^*\chi_{\{|\varphi|>r\}}\|_{\partial\mathbb{D}} = 1$ for every $0 < r < 1$. Hence, by Lemma 4.2, $M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty)$ and $M_u C_\varphi \in X$. Since $m(\{|\varphi_n| = 1\}) > 0$, we have $M_u C_{\varphi_n} \notin X$.

For each positive integer $n \geq 2$, we have

$$\begin{aligned} &\|M_u C_{\varphi_n} - M_u C_\varphi\| \\ &= \sup_{f \in ball(H^\infty)} \|u(f \circ \varphi_n - f \circ \varphi)\|_\infty \\ &= \sup_{f \in ball(H^\infty)} \max\Big\{ \sup_{k \geq 1} \big\|u^* \chi_{J_k} \big((f \circ \varphi_n)^* - (f \circ \varphi)^*\big)\big\|_{\partial\mathbb{D}}, \\ &\qquad\qquad \sup_{j \geq 1} \big\|u^* \chi_{L_j} \big((f \circ \varphi_n)^* - (f \circ \varphi)^*\big)\big\|_{\partial\mathbb{D}} \Big\}. \end{aligned} \tag{4.9}$$

Since $|u^*| = 1$ a.e. on J_k, we have

$$\begin{aligned} &\sup_{f \in ball(H^\infty)} \big\|u^* \chi_{J_k} \big((f \circ \varphi_n)^* - (f \circ \varphi)^*\big)\big\|_{\partial\mathbb{D}} \\ &= \sup_{f \in ball(H^\infty)} \big\|\chi_{J_k} \big((f \circ \varphi_n)^* - (f \circ \varphi)^*\big)\big\|_{\partial\mathbb{D}} \\ &= \sup_{f \in ball(H^\infty)} \sup_{e^{i\theta} \in J_k} \big|f(\varphi_n(e^{i\theta})) - f(\varphi(e^{i\theta}))\big| \\ &= \sup_{e^{i\theta} \in J_k} d_\infty\big(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})\big) \\ &\leq 2 \sup_{e^{i\theta} \in J_k} \rho\big(\varphi_n(e^{i\theta}), \varphi(e^{i\theta})\big). \end{aligned} \tag{4.10}$$

Take $\varepsilon > 0$ arbitrarily. By (4.8), there exists a positive integer k_0 such that

$$\sup_{e^{i\theta}\in J_k}\sup_{n\geq 2}\rho\big(\varphi_n(e^{i\theta}),\varphi(e^{i\theta})\big) < \varepsilon \quad \text{for} \quad k \geq k_0.$$

Hence, by (4.10), we have

$$\sup_{f\in ball(H^\infty)}\big\|u^*\chi_{J_k}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} < 2\varepsilon \quad \text{for} \quad k \geq k_0.$$

For $1 \leq k < k_0$, by (4.7) there exists a positive integer n_0 such that

$$\sup_{e^{i\theta}\in J_k}\rho(\varphi_n(e^{i\theta}),\varphi(e^{i\theta})) < \varepsilon \quad \text{whenever} \quad n \geq n_0.$$

Hence, for $n \geq n_0$, by (4.10) again we have

$$\sup_{f\in ball(H^\infty)}\big\|u^*\chi_{J_k}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} < 2\varepsilon \quad \text{whenever} \quad 1 \leq k < k_0.$$

Thus

$$\sup_{f\in ball(H^\infty)}\big\|u^*\chi_{J_k}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} < 2\varepsilon$$

for $k \geq 1$ and every $n \geq n_0$. Therefore, by (4.9), for $n \geq n_0$ we have

$$\begin{aligned}(4.11)\qquad &\|M_uC_{\varphi_n} - M_uC_\varphi\| \\ &\leq \max\Big\{2\varepsilon, \sup_{f\in ball(H^\infty)}\sup_{j\geq 1}\big\|u^*\chi_{L_j}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}}\Big\}.\end{aligned}$$

Since $a_j \to 0$, there exists a positive integer j_0 such that $a_j < \varepsilon$ for every $j \geq j_0$. Since $|u^*| = a_j$ a.e. on L_j, we have

$$(4.12)\qquad \sup_{f\in ball(H^\infty)}\big\|u^*\chi_{L_j}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} < 2\varepsilon$$

for $j \geq j_0$ and every $n \geq 1$. Since $\{|\varphi_n| = 1\}$ is a closed subarc of $\partial\mathbb{D}$ and converges to 1 as $n \to \infty$, we may assume that $\overline{L}_j \cap \{|\varphi_n| = 1\} = \emptyset$ for every $1 \leq j < j_0$ and $n \geq n_0$. For $1 \leq j < j_0$ and $n \geq n_0$, we have

$$\begin{aligned}&\sup_{f\in ball(H^\infty)}\big\|u^*\chi_{L_j}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} \\ &\leq \sup_{f\in ball(H^\infty)}\sup_{e^{i\theta}\in L_j}\big|f(\varphi_n(e^{i\theta})) - f(\varphi(e^{i\theta}))\big| \\ &= \sup_{e^{i\theta}\in L_j} d_\infty\big(\varphi_n(e^{i\theta}),\varphi(e^{i\theta})\big) \\ &\leq 2\sup_{e^{i\theta}\in L_j}\rho\big(\varphi_n(e^{i\theta}),\varphi(e^{i\theta})\big).\end{aligned}$$

Since $\varphi_n \to \varphi$ uniformly on $\overline{\mathbb{D}}$ and $\sup_{e^{i\theta}\in L_j}|\varphi(e^{i\theta})| < 1$, we may further assume that

$$\sup_{e^{i\theta}\in L_j}\rho\big(\varphi_n(e^{i\theta}),\varphi(e^{i\theta})\big) < \varepsilon$$

for $1 \leq j < j_0$ and every $n \geq n_0$. Thus, by (4.12), we get

$$\sup_{f\in ball(H^\infty)}\sup_{j\geq 1}\big\|u^*\chi_{L_j}\big((f\circ\varphi_n)^* - (f\circ\varphi)^*\big)\big\|_{\partial\mathbb{D}} \leq 2\varepsilon \quad \text{for} \quad n \geq n_0.$$

Combining this with (4.11), we get

$$\|M_uC_{\varphi_n} - M_uC_\varphi\| \leq 2\varepsilon \quad \text{for} \quad n \geq n_0.$$

Therefore we have $M_u C_{\varphi_n} \to M_u C_\varphi$ as $n \to \infty$. Since $M_u C_\varphi \in X$ and $M_u C_{\varphi_n} \notin X$, X is not open in $\mathcal{C}_{w,0}(H^\infty)$.

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