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MULTIPLICITY ON A RICHARDSON VARIETY IN A COMINUSCULE G/P

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Abstract. We show that in a cominuscule partial flag variety G/P, the multiplicity of an arbitrary point on a Richardson variety $X_w^v = X_w \cap X^v \subset G/P$ is the product of its multiplicities on the Schubert varieties X_w and X^v.

Introduction

Richardson varieties, named after [33], are intersections of a Schubert variety and an opposite Schubert variety inside a partial flag variety G/P (G a connected complex semi-simple group, P a parabolic subgroup). They previously appeared in [9, Ch. XIV, §4] and [36], as well as the corresponding open cells in [6]. They have since played a role in different contexts, such as equivariant K-theory [24], positivity in Grothendieck groups [3], standard monomial theory [4], Poisson geometry [8], positroid varieties [13], and their generalizations [14, 1].

On the other hand, singularities of Schubert varieties have been extensively studied in the last decades. The singular locus of Schubert varieties in Grassmannians has been determined independently in [37] and [27], and more generally in a minuscule G/P in [26]. In the full flag variety of type A_n, it has been determined independently in [2], [5], [12], and [29].

Moreover, the multiplicity of a singular point on a Schubert variety is known in several cases: when G/P is minuscule of arbitrary type, or cominuscule of type C_n, a recursive formula was given in [26]. A direct determinantal formula was given in [34] for G/P a Grassmannian; it has been subsequently interpreted in terms of nonintersecting lattice paths [17]. The multiplicity problem has also been studied in relationship with Hilbert functions and Gröbner degenerations [7, 16, 18, 23, 31, 32], as well as with T-equivariant cohomology [10, 11, 15, 20, 21, 25]. The problem of determining the multiplicity of a point in a Schubert variety in the full flag variety is more complicated; see [39, 28, 40, 41].

For Richardson varieties in a minuscule G/P, the multiplicity of a T-fixed point ($T \subset P$ a maximal torus in G) has been determined by Kreiman and Lakshmibai [22] (for the Gröbner point of view, see also [19] in type A_n and [38] in orthogonal types).

In this paper, we determine the multiplicity of an arbitrary point[1] on a Richardson variety in a cominuscule G/P.

Before stating the main result, let us fix some notation. Let G, P, T be as above, with G adjoint. Let $X(T)$ be the character group of T, $R \subset X(T)$ the root system,

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[1]Note that unlike in the case of a Schubert variety, this no longer follows from information about T-fixed points, as pointed out in the introductions of [22] and [19].

and $W = N_G(T)/T$ its Weyl group. Let $B \subset G$ be a Borel subgroup such that $T \subset B \subset P$: it determines a system of positive roots R^+ and a system of simple roots S. Denote by B^- the opposite Borel subgroup (*i.e.* such that $B \cap B^- = T$).

Let $W_P \subset W$ be the subgroup associated to P (so that $W_G = W$ and W_B is the trivial subgroup). In the quotient $W^P = W/W_P$, every coset wP contains a unique minimal element for the Bruhat order $\leq$ on W, so we shall identify W^P with the set of minimal representatives. The B-orbit (resp. the B^--orbit) of a T-fixed point $e_\tau = \tau P$ is called a Schubert cell (resp. an opposite Schubert cell) in G/P, and denoted by C_τ (resp. C^τ). Its closure is the Schubert variety X_τ (resp. the opposite Schubert variety X^τ).

If $v, w \in W^P$, then the intersection $X_w^v = X_w \cap X^v$ is called a *Richardson variety*; it is non-empty if and only if $v \leq w$ (note that Schubert varieties are the particular cases $X_w = X_w^e$ and $X^v = X_{w_0}^v$, where $e, w_0 \in W$ are the identity and the longest element, respectively).

Now assume P to be maximal, and let α be the associated simple root (so that W_P is generated by the reflections s_δ with $\delta \in S \setminus \{\alpha\}$). Then P (or α) is said to be

- cominuscule if α occurs with a coefficient 1 in the decomposition of the highest root of R^+;
- minuscule if $\alpha^\vee$ is cominuscule in the dual root system $R^\vee$.

The main result of this paper is the following:

Theorem 0.1. *Assume P is cominuscule. Let $m \in X_w^v$ be arbitrary, and denote by μ_w (resp. μ^v, μ_w^v) the multiplicity of m on X_w (resp. X^v, X_w^v). Then*

$$\mu_w^v = \mu_w \ \mu^v. \tag{1}$$

This result indeed determines the multiplicities on X_w^v, since those on X_w and X^v are known: types A_n, D_n, E_6, E_7 are covered by [26], Section 3 (since cominuscule is equivalent to minuscule in those types); type C_n is covered by [26], Section 4. The only remaining case, in type B_n (*cf.* the table below), is elementary, and covered in the Appendix of the present paper for the sake of completeness.

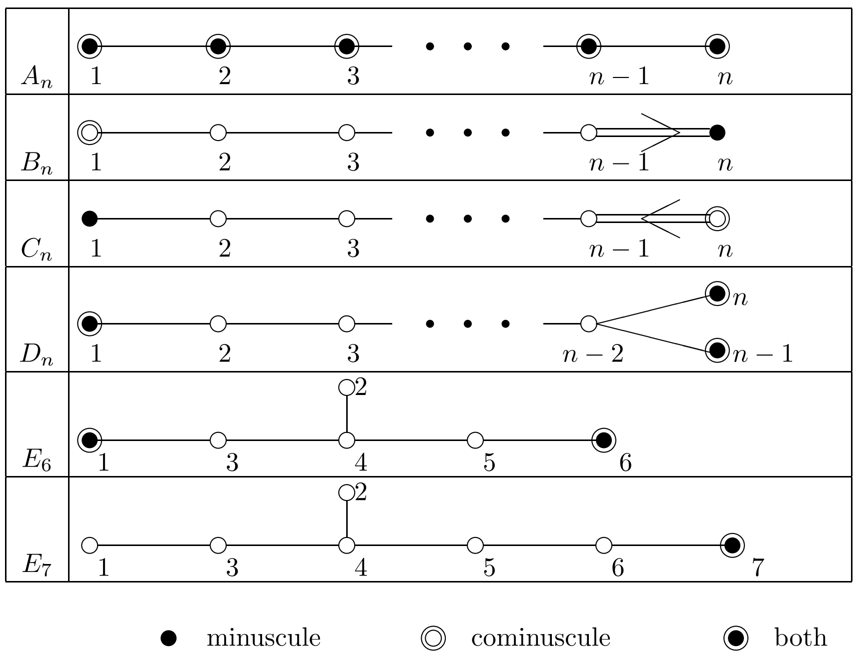
Note that (1) is exactly the result obtained in [22] for a T-fixed point in a minuscule G/P.

To prove the theorem, we shall use a description of the multiplicity using a central projection: namely, given a projective variety $X \subset \mathbf{P}^N$ and a point $m \in X$, we consider the projection p_m, of centre m, onto a hyperplane not containing m. Then the multiplicity of m on X is the difference between the degree of X and the projective degree of p_m. Note that the projective degree of p_m is zero when X is a cone. We apply this description for X, the projective closure of the affine trace $X_w^v \cap \mathcal{O}_\tau$, where $\mathcal{O}_\tau$ is an affine open subset of G/P identified with $\mathbf{A}^N$. One then needs to know whether the affine traces of X_w, X_v, X_w^v are cones or not. In this setting, we can explain why we assume that P is cominuscule:

- it implies that $X_w \cap \mathcal{O}_\tau$ is a cone over *any* point of the cell C_τ (although this may not be the case for $X^v \cap \mathcal{O}_\tau$);
- we relate the central projection p_m to a map which turns out to be a $\mathbf{C}$-action if P is cominuscule. It is this $\mathbf{C}$-action which allows to prove all the necessary properties for p_m.

In Section 1, we give a system of local coordinates in which $X_w \cap \mathcal{O}_\tau$ is a cone over both e_τ and m, and $X^v \cap \mathcal{O}_\tau$ over e_τ. In Section 2, we prove Theorem 0.1 assuming certain formulas for the degrees involved and that $X^v \cap \mathcal{O}_\tau$ is not a cone over m. These assumptions are summarized in Proposition 2.1, and proved in Sections 4 and 5. The proofs are based on a $\mathbf{C}$-action linking the central projections of centres m and e_τ; this action is defined and studied in Section 3.

For the convenience of the reader, we give the minuscule and cominuscule weights in the following table:



There are no minuscule nor cominuscule fundamental weights in types E_8, F_4, G_2.

Assumption. *For the rest of the paper, the parabolic subgroup P is assumed to be cominuscule.*

1. Local coordinates

Use the notation as in the Introduction. Moreover, R_P denotes the root system associated with P,

$$R^+ \setminus R_P^+ = \{\beta \in R^+ \mid U_\beta \subset R_u(P)\},$$

where $R_u(P)$ is the unipotent radical of P, and U_β is the root subgroup associated with β.

Let $m \in X_w^v$. Then m lies in a Schubert cell C_τ for some $\tau \in W^P$. Let

$$U_\tau^- = \prod_{\beta \in \tau(R^+ \setminus R_P^+)} U_{-\beta}$$

and $\mathcal{O}_\tau = U_\tau^-.e_\tau$, where $e_\tau = \tau P$. We identify $U_{-\beta}$ with $\mathbf{C}$ via an isomorphism $\theta_{-\beta} : \mathbf{C} \to U_{-\beta}$ satisfying

$$t\theta_{-\beta}(x)t^{-1} = \theta_{-\beta}\left(\frac{1}{\beta(t)}x\right)$$

for all $t \in T$ and all $x \in \mathbf{C}$. Let N be the cardinality of $R^+ \setminus R_P^+$. We identify $\mathcal{O}_\tau$ with the affine space $\mathbf{A}^N$ via the isomorphism

$$\begin{array}{ccc} \mathbf{A}^N & \longrightarrow & \mathcal{O}_\tau \\ (x_{-\beta})_{\beta\in\tau(R^+\setminus R_P^+)} & \mapsto & \prod_{\beta\in\tau(R^+\setminus R_P^+)} \theta_{-\beta}(x_{-\beta}).e_\tau. \end{array} \tag{2}$$

(In particular, N is the dimension of G/P.)

Lemma 1.1. *Let $\beta \in R$, and $\tau \in W^P$. Then U_β fixes e_τ if and only if $-\beta \notin \tau(R^+ \setminus R_P^+)$.*

Proof. Let $\beta \in R$, and $\tau \in W^P$. Then

$$\begin{aligned} U_\beta.e_\tau = e_\tau &\iff \tau^{-1}U_\beta\ \tau P = P \\ &\iff U_{\tau^{-1}\beta} \subset P \\ &\iff \tau^{-1}\beta \in R^+ \text{ or } -\tau^{-1}\beta \in R_P^+ \\ &\iff -\beta \notin \tau(R^+) \text{ or } -\beta \in \tau(R_P^+) \\ &\iff -\beta \notin \tau(R^+ \setminus R_P^+). \end{aligned}$$

□

Lemma 1.2. *The Schubert cell C_τ is the affine subspace of $\mathcal{O}_\tau$ defined by the vanishing of the coordinates $x_{-\beta}$ with $\beta \in R^+$.*

Proof. Since B is the semi-direct product of T and the unipotent subgroup U, we have $C_\tau = U.e_\tau$. Moreover, for any ordering of positive roots $\{\beta_1, \dots, \beta_p\}$,

$$U = \prod_{i=1}^{p} U_{\beta_i}.$$

We choose an ordering such that the positive roots β with $-\beta \notin \tau(R^+ \setminus R_P^+)$ appear at the end. Then, by the preceding lemma, we have

$$C_\tau = \prod_{\substack{\beta\in\tau(R^+\setminus R_P^+)\\ \beta<0}} U_{-\beta}.e_\tau \subset \mathcal{O}_\tau.$$

□

The following lemma will be useful for the next section.

Lemma 1.3. *For all $\beta, \gamma \in \tau(R^+ \setminus R_P^+)$ and for all $x, y \in \mathbf{C}$, the elements $\theta_\beta(x)$ and $\theta_\gamma(y)$ commute.*

Proof. We use the following expansion for the commutator (*cf.* [35], proposition 8.2.3),

$$\theta_\beta(x)\theta_\gamma(y)\theta_\beta(x)^{-1}\theta_\gamma(y)^{-1} = \prod_{\substack{i\beta+j\gamma\in R\\ i,j>0}} \theta_{i\beta+j\gamma}(c_{\beta,\gamma,i,j}\ x^i\ y^j),$$

where $c_{\beta,\gamma,i,j}$ are some constants in $\mathbf{C}$. Since the commutator must lie in U_τ^-, it suffices to prove that the roots of the form $i\beta+j\gamma$ do not lie in $\tau(R^+\setminus R_P^+)$. Now, P is the parabolic subgroup associated with the simple root α. Since α is cominuscule, a positive root δ lies in $R^+\setminus R_P^+$ if and only if α occurs with coefficient 1 in the expression of δ. Clearly, α occurs with a coefficient $i+j$ in $\tau^{-1}(i\beta+j\gamma)$. □

Remark 1.4. Identifying $\mathcal{O}_\tau$ with U_τ^-, it follows from Lemma 1.3 that the isomorphism of algebraic varieties (2) $\mathbf{A}^N\to\mathcal{O}_\tau$ is also an isomorphism of unipotent groups.

Example 1.5. Let $G=SL_n(\mathbf{C})$. It is a group of type A_{n-1}. The torus T is the group of diagonal matrices of determinant 1, and the Borel subgroup B is the group of upper triangular matrices of determinant 1. The roots are denoted $\alpha_{i,j}$, where

$$\alpha_{i,j}:T\to\mathbf{C}^*:\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n\end{pmatrix}\mapsto\frac{t_i}{t_j}.$$

The positive roots are the $\alpha_{i,j}$ with $i<j$, and the simple roots are the $\alpha_i=\alpha_{i,i+1}$ ($i=1,\dots,n-1$). Let $\omega=\omega_d$ be the fundamental weight associated with the simple root α_d. The corresponding parabolic subgroup P is

$$P=\left\{\left(\begin{array}{c|c} * & * \\ \hline 0_{(n-d)\times d} & * \end{array}\right)\right\}.$$

The group G acts transitively on the Grassmannian $G_{d,n}$ of d-spaces in $\mathbf{C}^n$, and P is the isotropy subgroup of the vector space generated by $e_1,\dots,e_d$, where $(e_1,\dots,e_n)$ is the canonical basis of $\mathbf{C}^n$. The Weyl group W of this root system is S_n, and W_P is isomorphic to $S_d\times S_{n-d}$, so

$$W^P=I_{d,n}=\{\mathbf{i}=i_1\dots i_d\mid 1\le i_1<i_2<\cdots<i_d\le n\}.$$

The Lie algebra $\mathfrak{g}$ of G is the space of traceless matrices. Let $\mathfrak{t}$ be the Lie algebra of the torus T. We have the weight decomposition of $\mathfrak{g}$,

$$\mathfrak{g}=\mathfrak{t}\oplus\bigoplus_{i\neq j}\mathbf{C}E_{i,j},$$

where $E_{i,j}$ is the elementary matrix with a 1 on the row i and column j, and zero elsewhere. Thus, the root subgroups are given by

$$U_{\alpha_{i,j}}=\{I_n+xE_{i,j}\mid x\in\mathbf{C}\}$$

and the isomorphism $\theta_{\alpha_{i,j}}$ is just $x\mapsto\exp(xE_{ij})$. Moreover,

$$R^+\setminus R_P^+=\{\alpha_{i,j}\mid i\le d<j\},$$

so in this case, Lemma 1.3 becomes an elementary matrix computation.

Returning to the general case, we denote by $(m_{-\beta}|\beta\in\tau(R^+\setminus R_P^+))$ the coordinates of m, that is,

$$m=\prod_{\beta\in\tau(R^+\setminus R_P^+)}\theta_{-\beta}(m_{-\beta}).e_\tau.$$

Notation 1.6. We set

$$Y_w = X_w \cap \mathcal{O}_\tau, \quad Y^v = X^v \cap \mathcal{O}_\tau, \quad Y_w^v = X_w^v \cap \mathcal{O}_\tau.$$

These sets are affine varieties, *i.e.* Zariski-closed in $\mathcal{O}_\tau = \mathbf{A}^N$.

We now investigate if these affine varieties are cones over m.

Proposition 1.7. *The varieties Y_w, Y^v and Y_w^v are cones over e_τ.*

Proof. Let $\omega^\vee : \mathbf{C}^* \to T$ be the fundamental coweight associated to P. Since $\omega^\vee$ is minuscule, the pairing $\langle \omega^\vee, \gamma \rangle$ is equal to 1 if $\gamma \in R^+ \setminus R_P^+$ (and to 0 if $\gamma \in R_P^+$). Now multiplication in $\mathbf{A}^N$ by a scalar ξ is then given by conjugation in U_τ^- by $\tau(\omega^\vee)(\xi)^{-1} \in T$: indeed, for $\beta = \tau(\gamma)$ with $\gamma \in R^+ \setminus R_P^+$, and for $z \in \mathbf{C}$, we have

$$\tau(\omega^\vee)(\xi)^{-1}\theta_{-\beta}(z)\tau(\omega^\vee)(\xi) = \theta_{-\beta}(\xi^{\langle \tau(\omega^\vee),\beta\rangle} z) = \theta_{-\beta}(\xi^{\langle \omega^\vee,\gamma\rangle} z) = \theta_{-\beta}(\xi z). \tag{3}$$

Let $x \in Y_w$ (resp. $x \in Y^v$), and $(x_{-\beta})$ be its coordinates. Then the point that has coordinates $(\xi x_{-\beta})$ is $t.x$, where $t = \tau(\omega^\vee)(\xi) \in T$. Therefore, this point lies in $X_w \cap \mathcal{O}_\tau$ (resp. in $X^v \cap \mathcal{O}_\tau$), since X_w (resp. X^v) is T-stable. It follows that Y_w, Y^v, and therefore Y_w^v are cones over e_τ. □

Proposition 1.8. *The variety Y_w is a cone over m.*

Proof. Consider the translation that maps e_τ to m. It is given in coordinates by $(x_{-\beta}) \mapsto (x_{-\beta} + m_{-\beta})$. But if x has coordinates $(x_{-\beta})$, then, by Remark 1.4, the point of coordinates $(x_{-\beta} + m_{-\beta})$ corresponds to $b.x$, where $b = \prod_\beta \theta_{-\beta}(m_{-\beta})$. Since $m_{-\beta} = 0$ for all $\beta > 0$, we have $b \in B$ according to Lemma 1.2. Now b leaves Y_w invariant and maps e_τ to m. □

However, the opposite Schubert variety Y^v need not be a cone over m.

Example 1.9. We take the same notation as in Example 1.5. In particular, using the identification $W^P = I_{d,n}$, we denote a Schubert variety in $G_{d,n}$ by $X_{i_1 \dots i_d}$, and similarly for opposite Schubert and Richardson varieties. In the Grassmannian $G_{3,7}$, consider the Richardson variety X_{356}^{125}. The coordinates on the open set $\mathcal{O}_{256}$ are parametrized by the set $\{12, 15, 16, 32, 35, 36, 42, 45, 46, 72, 75, 76\}$, where ij stands for the root $\alpha_{i,j}$. More precisely, we have

$$\begin{array}{ccc} \mathbf{A}^{12} & \longrightarrow & \mathcal{O}_{256} \\ (x_{12}, x_{15}, \dots, x_{76}) & \mapsto & \begin{bmatrix} x_{12} & x_{15} & x_{16} \\ 1 & 0 & 0 \\ x_{32} & x_{35} & x_{36} \\ x_{42} & x_{45} & x_{46} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{72} & x_{75} & x_{76} \end{bmatrix}. \end{array}$$

Here, a matrix between brackets actually stands for the 3-space in $\mathbf{C}^7$ generated by its columns. The equations of X_{356} are

$$\begin{cases} x_{72} = x_{75} = x_{76} = 0, \\ x_{42} = 0. \end{cases}$$

The equations of X^{125} are

$$\begin{cases} x_{15}x_{36} - x_{35}x_{16} = 0, \\ x_{15}x_{46} - x_{45}x_{16} = 0, \\ x_{35}x_{46} - x_{45}x_{36} = 0. \end{cases}$$

Let

$$m = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in X_{356}^{125}.$$

We set

$$\begin{cases} y_{16} = x_{16} - 1, \\ y_{36} = x_{36} + 1, \\ y_{ij} = x_{ij}, & \text{if } ij \notin \{16, 36\}. \end{cases}$$

The equations in these new coordinates are

$$\begin{cases} y_{72} = y_{75} = y_{76} = 0, \\ y_{42} = 0, \end{cases}$$

for X_{356} and

$$\begin{cases} y_{15}(y_{36} - 1) - y_{35}(y_{16} + 1) = 0, \\ y_{15}y_{46} - y_{45}(y_{16} + 1) = 0, \\ y_{35}y_{46} - y_{45}(y_{36} - 1) = 0, \end{cases}$$

for X^{125}. While the equations for X_{356} remain homogeneous, those for X^{125} do not.

If Y^v is indeed a cone over m, then we have the following result. The proof is taken from [22], Remark 7.6.6.

Proposition 1.10. *Assume Y^v is a cone over m. Let μ_w (resp. μ^v, μ_w^v) be the multiplicity of m on X_w (resp. X^v, X_w^v). Then*

$$\mu_w^v = \mu_w \, \mu^v. \tag{4}$$

Proof. In this case, $Y_w^v = Y_w \cap Y^v$ is a cone (over m) as well, so we may consider the projective varieties $\mathbf{P}(Y_w)$, $\mathbf{P}(Y^v)$ and $\mathbf{P}(Y_w^v)$, consisting of lines through m. Then μ_w (resp. μ^v, μ_w^v) is just the degree of $\mathbf{P}(Y_w)$ (resp. $\mathbf{P}(Y^v)$, $\mathbf{P}(Y_w^v)$). We conclude with Bézout's theorem since $\mathbf{P}(Y_w)$ and $\mathbf{P}(Y^v)$ intersect transversely (*cf.* [33], Corollary 1.5). □

Assumption 1.11. *For the rest of the paper, we assume that Y^v is not a cone over m.*

It is not clear however whether Y_w^v is a cone or not. This problem will be solved in Section 4.

2. Central projection and proof of Theorem 0.1

We shall compute the multiplicity of a point $m \in Y_w^v$ by relating it to degrees of projections, which requires us to work in a projective setting. More precisely, embed $\mathbf{A}^N$ into $\mathbf{P}^N$ via

$$\begin{array}{rccl} \iota : & \mathbf{A}^N & \hookrightarrow & \mathbf{P}^N = \{[\xi : x_{-\beta}]\} \\ & (x_{-\beta}) & \mapsto & [1 : x_{-\beta}] \end{array}$$

and consider the projective closures

$$Z_w = \overline{\iota(Y_w)}, \qquad Z^v = \overline{\iota(Y^v)}, \qquad Z_w^v = \overline{\iota(Y_w^v)}.$$

We also identify $\mathbf{P}^{N-1}$ with the hyperplane at infinity $\xi = 0$ and consider the central projection $p_m : \mathbf{P}^N \to \mathbf{P}^{N-1}$, sending any point $x \neq m$ to the intersection of the line (mx) with $\mathbf{P}^{N-1}$. If $X \subset \mathbf{P}^N$ is any projective variety and $m \in X$, then we have the following formula (*cf.* [30], Theorem 5.11),

$$\text{(5)} \quad \deg X - \operatorname{mult}_m X = \begin{cases} \deg(p_m)_{|X} \ \deg(p_m X) & \text{if } X \text{ is not a cone over } m, \\ 0 & \text{if } X \text{ is a cone over } m, \end{cases}$$

where $\deg X$ is the degree of X, $\deg(p_m)_{|X}$ is the degree of the rational map p_m restricted to X, and $p_m X$ denotes the Zariski closure of $p_m(X \setminus \{m\})$.

Proposition 2.1.

(a) $\deg Z_w^v = \deg Z_w \deg Z^v$.
(b) Z_w^v *is not a cone over* m.
(c) $\deg(p_m)_{|Z_w^v} = \deg(p_m)_{|Z^v}$.
(d) $\deg(p_m Z_w^v) = \deg Z_w \deg(p_m Z^v)$.

We defer the proof to Section 4.

Proof of Theorem 0.1. Using (5) and Proposition 2.1, we obtain

$$\begin{aligned} \mu_w^v &= \deg Z_w^v - \deg(p_m)_{|Z_w^v} \ \deg(p_m Z_w^v) \\ &= \deg Z_w \deg Z^v - \deg(p_m)_{|Z^v} \ \deg Z_w \deg(p_m Z^v) \\ &= \deg Z_w \big[\deg Z^v - \deg(p_m)_{|Z^v} \ \deg(p_m Z^v)\big] \\ &= \mu_w \ \mu^v. \end{aligned}$$ □

Remark 2.2. In particular, this result enables us to find the singular locus of X_w^v in terms of those of X_w and X^v: the point m is smooth on X_w^v if and only if $\mu_w^v = 1$ if and only if $\mu_w = \mu^v = 1$, that is, if and only if m is smooth on both X_w and X^v. Note that this may also be seen more directly, using the fact that X_w and X^v intersect properly and transversely at any point at which $\mu_w = \mu^v = 1$ (*cf.* [33], Corollary 1.5, or [1], Corollary 2.9).

3. C-action on G/P

In this section, we introduce the main tool that will permit us to prove Proposition 2.1 in the next section. Let $e_\tau, m \in \mathcal{O}_\tau$ be as before. We shall construct an action of (the additive group) $\mathbf{C}$ on G/P for which e_τ and m are in the same orbit.

Consider first the map

$$\begin{array}{rccl} \varphi^*: & \mathbf{C}^* & \to & B \\ & \xi & \mapsto & \varphi_\xi = \tau(\omega^\vee)(\xi)^{-1} b \tau(\omega^\vee)(\xi), \end{array}$$

where $b \in B \cap \mathcal{U}_\tau^-$ is the element defined in the proof of Proposition 1.8. The computation (3) shows that this map extends to a group homomorphism $\varphi : \mathbf{C} \to B$. The natural B-action on G/P thus induces a $\mathbf{C}$-action,

$$\begin{array}{rccl} \Phi: & \mathbf{C} \times G/P & \to & G/P \\ & (\xi, x) & \mapsto & \varphi_{-\xi}.x. \end{array}$$

Moreover, $\mathcal{O}_\tau$ is invariant under this action (again by (3)). Actually, $\mathbf{C}$ acts on $\mathcal{O}_\tau = \mathbf{A}^N$ by translations: indeed, we get the following commutative diagram

$$\text{(6)} \qquad \begin{array}{ccc} \mathbf{C} \times \mathbf{A}^N & \xrightarrow{\Phi} & \mathbf{A}^N \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle p_{e_\tau}} \\ \mathbf{P}^N & \underset{p_m}{\dashrightarrow} & \mathbf{P}^{N-1} \end{array} \qquad \begin{array}{ccc} (\xi, x_{-\beta}) & \longmapsto^{\Phi} & (x_{-\beta} - \xi m_{-\beta}) \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle p_{e_\tau}} \\ [\xi : x_{-\beta}] & \underset{p_m}{\longmapsto} & [0 : x_{-\beta} - \xi m_{-\beta}]. \end{array}$$

Let us now restrict to Y_w^v: since it is a cone over e_τ, a point $[\xi : x]$ lies in Z_w^v if and only if $x \in Y_w^v$. It follows that $f(\mathbf{C} \times Y_w^v) = Z_w^v$. Thus, the commutative diagram (6) restricts to

$$\text{(7)} \qquad \begin{array}{ccc} \mathbf{C} \times Y_w^v \setminus \{(\xi, \xi m_{-\beta}) |\ \xi \in \mathbf{C}\} & \xrightarrow{\Phi} & \Phi(\mathbf{C} \times Y_w^v) \setminus \{e_\tau\} \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle p_{e_\tau}} \\ Z_w^v \setminus \{m\} & \xrightarrow[p_m]{} & \mathbf{P}^{N-1}. \end{array}$$

Remark 3.1. Since (6) is a fibre product diagram, any fibre $\Phi^{-1}(\lambda y)$ (for $\lambda \neq 0$ and $[y] \in \mathbf{P}^{N-1}$) is mapped isomorphically via f to the fibre $p_m^{-1}([y])$. Since we have the equalities $f(\mathbf{C} \times Y_w) = Z_w$, $f(\mathbf{C} \times Y^v) = Z^v$, $f(\mathbf{C} \times Y_w^v) = Z_w^v$ and $\mathbf{C} \times Y_w = f^{-1}(Z_w)$, $\mathbf{C} \times Y^v = f^{-1}(Z^v)$, $\mathbf{C} \times Y_w^v = f^{-1}(Z_w)$, the fibres of $\Phi_{|\mathbf{C} \times Y_w}$, $\Phi_{|\mathbf{C} \times Y^v}$, $\Phi_{|\mathbf{C} \times Y_w^v}$ over a point λy are isomorphic to the fibres of $p_{m|Z_w}$, $p_{m|Z^v}$, $p_{m|Z_w^v}$ over the point $[y]$.

In the next section, this remark will allow us to relate the degree of p_m in diagram (7) to that of Φ.

4. Proof of Proposition 2.1

Proof of (a). Since Y_w, Y^v, and Y_w^v are (affine) cones over e_τ, it is clear that $Z_w^v = Z_w \cap Z^v$. In addition, this intersection is proper and generically transverse ([33], Corollary 1.5), hence $\deg Z_w^v = \deg Z_w \deg Z^v$ by Bézout's theorem.

Notation 4.1. We denote by F_w^v the closure in $\mathbf{A}^N$ of $\Phi(\mathbf{C} \times Y_w^v)$, and by d_w^v the degree of $p_m : Z_w^v \setminus \{m\} \to p_m Z_w^v$ whenever it makes sense (*i.e.* when Z_w^v is not a cone). We define F_w, F^v, d^v in a similar way.

Proposition 4.2. *When defined, the degree d_w^v is equal to the degree of $\Phi : \mathbf{C} \times Y_w^v \to F_w^v$.*

Proof. This follows from Remark 3.1. $\square$

Lemma 4.3. *The following properties are equivalent:*

- Z^v_w *is a cone over* m,
- $F^v_w = Y^v_w$,
- *every fibre of* $\Phi : \mathbf{C} \times Y^v_w \to F^v_w$ *has dimension* 1.

In particular, they are true for $v = e$, *hence* $F_w = Y_w = \Phi(\mathbf{C} \times Y_w)$.

Proof. By Remark 3.1, we see that the dimension of a generic fibre of Φ equals the dimension of a generic fibre of p_m. Now Z^v_w is a cone over m if and only if every fibre of p_m has dimension 1, if and only if $\dim F^v_w = \dim Y^v_w$. But $Y^v_w = \Phi(0 \times Y^v_w) \subset F^v_w$ and the varieties Y^v_w and F^v_w are irreducible, so Z^v_w is a cone over m if and only if $F^v_w = Y^v_w$. □

Proof of (b) *and* (c). By Proposition 4.2, it suffices to compare the degree d^v of $\Phi^v : \mathbf{C} \times Y^v \to F^v$ with the degree d^v_w of $\Phi^v_w : \mathbf{C} \times Y^v_w \to F^v_w$. First, the fibre of a point $x \in G/P$ for Φ is

$$\Phi^{-1}(x) = \{(\xi, \Phi(-\xi, x)) \mid \xi \in \mathbf{C}\}.$$

In particular, a point lies in $\mathrm{Im}(\Phi^v)$ (resp. in $\mathrm{Im}(\Phi^v_w)$) if and only if its $\mathbf{C}$-orbit meets Y^v (resp. Y^v_w). There exists an open set Ω^v of F^v in which the fibre of every point y consists of d^v points. Then d^v is just the number of points in the $\mathbf{C}$-orbit of y that belong to Y^v. Now set $y = (y_{-\beta})_{\beta \in \tau(R^+ \setminus R^+_P)}$ and let

$$c = \prod_{\substack{\beta \in \tau(R^+ \setminus R^+_P) \\ \beta < 0}} \theta_{-\beta}(y_{-\beta}), \qquad c^- = \prod_{\substack{\beta \in \tau(R^+ \setminus R^+_P) \\ \beta > 0}} \theta_{-\beta}(-y_{-\beta}),$$

so we have $c.e_\tau = c^-.y =: x$. Since $c \in B$, $x \in C_\tau \subset Y_w$. Now c^- commutes with φ_ξ for all $\xi \in \mathbf{C}$, hence every point in $c^-(\Omega^v)$ has a $\mathbf{C}$-orbit which meets Y^v in exactly d^v points. In particular, $F^v_w \neq Y^v_w$, since otherwise every fibre of Φ^v_w would have dimension 1 (by Lemma 4.3), which is not the case for the fibre of x. This already shows (b), so it makes sense to talk about the degree d^v_w of Φ^v_w. Thus, let Ω^v_w be an open set of F^v_w such that for every point z in Ω^v_w, the fibre of z consists of d^v_w points. Since $x \in c^-(\Omega^v)$, $c^-(\Omega^v) \cap F^v_w$ and Ω^v_w are non-empty open sets of the irreducible variety F^v_w, so they must meet. Taking z in this intersection, we see that $d^v_w = d^v$, which shows (c). □

Proposition 4.4. *The intersection* $F_w \cap F^v$ *is proper and transverse on an open set of* F^v_w.

Proof. The transversality of the intersection $F_w \cap F^v$ on a generic point in F^v_w follows from the transversality of the intersection of a direct Schubert variety and an opposite Schubert variety. More precisely, let $(F_w)_{sm}$ be the open set of smooth points of F_w. Taking a point smooth on Y^v_w shows that $\Omega_w = (F_w)_{sm} \cap F^v_w$ is a non-empty open set of F^v_w. Let $(F^v)_{sm}$ be the open set of smooth points of F^v. Again, $\Omega^v = (F^v)_{sm} \cap F^v_w \neq \emptyset$. Indeed, take a smooth point x of F^v belonging to $\Phi(\mathbf{C} \times Y^v)$. We have seen in the previous proof that from x we can construct an isomorphism c^- of F^v mapping x to a point of F^v_w, which thus remains smooth on F^v. The two non-empty open subsets Ω_w and Ω^v of the irreducible variety F^v_w have a non-empty intersection Ω^v_w. Now $O^v_w = \Phi^{-1}(\Omega^v_w) \cap (\mathbf{P}^1 \times Y^v_w)_{sm} \neq \emptyset$ since $\mathbf{P}^1 \times Y^v_w$ is irreducible. We claim that $\Phi : O^v_w \to \Omega^v_w$ is dominant. Indeed, we must show that every open subset U of Ω^v_w meets $\Phi(O^v_w)$. Since U is open in F^v_w,

$U \cap \Phi(\mathbf{C} \times Y_w^v) \neq \emptyset$. So it makes sense to talk about $\Phi^{-1}(U)$, which is an open set of $\mathbf{C} \times Y_w^v$. Thus, $\Phi^{-1}(U) \cap O_w^v \neq \emptyset$, which implies $U \cap \Phi(O_w^v) \neq \emptyset$. Since $\Phi : O_w^v \to \Omega_w^v$ is dominant, we know that $\Phi(O_w^v)$ contains a non-empty open set Ω of Ω_w^v. Let us summarize the properties of Ω: it is a non-empty open subset of F_w^v, whose every point y is smooth in both F_w and F^v, and $y = \Phi(p)$ with p smooth in $\mathbf{C} \times Y_w^v$, so p is smooth in both $\mathbf{C} \times Y_w$ and $\mathbf{C} \times Y^v$.

Let $y = \Phi(p) \in \Omega$ be such a point. We view the map $\Phi : \mathbf{C} \times \mathbf{A}^N \to \mathbf{A}^N : (\xi, x) \mapsto \varphi_{-\xi}.x$ as a map $\Phi : \mathbf{C}^{N+1} \to \mathbf{C}^N$. It is linear and surjective. Thus,

$$\begin{aligned} \mathbf{C}^N \supset T_y(F_w) + T_y(F^v) &\supset d\Phi_p(T_p(\mathbf{C} \times Y_w)) + d\Phi_p(T_p(\mathbf{C} \times Y^v)) \\ &\supset d\Phi_p(\mathbf{C} \oplus (T_pY_w + T_pY^v)) \\ &\supset d\Phi_p(\mathbf{C} \oplus \mathbf{C}^N) \\ &\supset \mathbf{C}^N. \end{aligned}$$

This transversality result proves that the intersection is proper: indeed, on one hand, $\dim(F_w \cap F^v) \geq \dim(F_w) + \dim(F^v) - N$, but on the other hand,

$$\begin{aligned} \dim(F_w \cap F^v) &\leq \dim(T_y(F_w \cap F^v)) \leq \dim(T_yF_w \cap T_yF^v) \\ &\leq \dim(T_yF_w) + \dim(T_yF^v) - \dim(T_yF_w + T_yF^v) \\ &\leq \dim(F_w) + \dim(F^v) - N. \end{aligned}$$

□

Proposition 4.5. *We have the equality $F_w^v = F_w \cap F^v$. In particular, the intersection $F_w \cap F^v$ is generically transverse.*

This result will be proved in the next section.

Proof of (d). Since $y = \Phi(\xi, x)$ implies $zy = \Phi(z\xi, zx)$ for all $z \in \mathbf{C}$, $\Phi(\mathbf{C} \times Y_w^v)$ is a cone over e_τ, and so is its closure F_w^v. But by the commutative diagram (7),

$$p_{e_\tau}(F_w^v \setminus \{e_\tau\}) \subset \overline{p_{e_\tau}(\Phi(\mathbf{C} \times Y_w^v) \setminus \{e_\tau\})} = p_m Z_w^v.$$

Comparing dimensions, we see that $p_{e_\tau} F_w^v = p_m Z_w^v$, *i.e.* $p_m Z_w^v$ is the projective variety at infinity of the cone F_w^v. In particular, $\deg(p_m Z_w^v) = \deg(F_w^v)$, and similarly $\deg(p_m Z_w) = \deg(F_w)$ and $\deg(p_m Z^v) = \deg(F^v)$. Equality (d) now follows from Proposition 4.4 and Bézout's theorem, noting that $\deg(p_m Z_w) = \deg(Z_w)$. □

5. Proof of Proposition 4.5

Since $\Phi(\mathbf{C} \times Y_w^v) \subset \Phi(\mathbf{C} \times Y_w) \cap \Phi(\mathbf{C} \times Y^v)$, we obtain $F_w^v \subset F_w \cap F^v$. Moreover, the first inclusion is an equality: indeed, if $z = \Phi(\xi, x) \in Y_w$ with $\xi \in \mathbf{C}, x \in Y^v$, then $x = \Phi(-\xi, z) \in Y_w$ since $\Phi(\mathbf{C} \times Y_w) = Y_w$, so $z = \Phi(\xi, x) \in \Phi(\mathbf{C} \times Y_w^v)$.

However, the inclusion $F_w \cap F^v \subset F_w^v$ requires some work. Let

$$\mathcal{U} = \{(\xi, x, \Phi(\xi, x)) | \xi \in \mathbf{C}, x \in G/P\}$$

and Γ be its closure in $\mathbf{P}^1 \times G/P \times G/P$ (so Γ is the graph of Φ viewed as a rational map). We have a commutative diagram,

$$\begin{array}{ccc} \Gamma & & \\ {\scriptstyle \pi_1 \times \pi_2} \downarrow & \searrow^{\pi_3} & \\ \mathbf{P}^1 \times G/P & \overset{\Phi}{\dashrightarrow} & G/P \end{array} \qquad \begin{array}{ccc} (\xi, x, y) & & \\ {\scriptstyle \pi_1 \times \pi_2} \downarrow & \searrow^{\pi_3} & \\ (\xi, x) & \overset{\Phi}{\longmapsto} & \Phi(\xi, x). \end{array}$$

The morphism $\pi_1 \times \pi_2 : \Gamma \to \mathbf{P}^1 \times G/P$ is surjective, and restricts to an isomorphism between $\mathcal{U}$ and $\mathbf{C} \times G/P$. In particular, Γ is an irreducible projective variety of dimension $N+1$.

Likewise, let $\mathcal{U}_w = \{(\xi, x, \Phi(\xi, x)) | \xi \in \mathbf{C}, x \in X_w\}$ and Γ_w be its closure, and similarly for $\mathcal{U}^v, \mathcal{U}^v_w, \Gamma^v, \Gamma^v_w$. Then $\pi_3(\Gamma_w) = \pi_3(\overline{\mathcal{U}_w}) = \overline{\pi_3(\mathcal{U}_w)}$ in G/P, so $\pi_3(\Gamma_w) \cap \mathcal{O}_\tau$ is the closure of $\pi_3(\mathcal{U}_w) \cap \mathcal{O}_\tau = \Phi(\mathbf{C} \times Y_w)$ in $\mathcal{O}_\tau$. Proceeding similarly with Γ^v and Γ^v_w, we obtain

$$\pi_3(\Gamma_w) \cap \mathcal{O}_\tau = F_w, \quad \pi_3(\Gamma^v) \cap \mathcal{O}_\tau = F^v, \quad \pi_3(\Gamma^v_w) \cap \mathcal{O}_\tau = F^v_w.$$

We now need to study the π_3-fibre of a point in F_w. Actually, if y is in Y_w, then its fibre lies entirely in Γ_w. Indeed, U^-_τ naturally acts on G/P and on Γ via $g.(\xi, x, y) = (\xi, g.x, g.y)$ (since U^-_τ is Abelian), and the morphism π_3 is U^-_τ-equivariant. It follows that whenever two points in G/P belong to the same U^-_τ-orbit, their fibres are isomorphic. Now since $\pi_3 : \Gamma \to G/P$ is dominant, there is an open set in G/P in which every point has a fibre of pure dimension 1. Since $\mathcal{O}_\tau$ is open in G/P, it meets this open set, and since $\mathcal{O}_\tau$ is a U^-_τ-orbit in G/P, y itself has a fibre of pure dimension 1.

Now fix an irreducible component C of $\pi_3^{-1}(y)$. Then

$$\big(\pi_1 \times \pi_2(C)\big) \cap (\mathbf{C} \times G/P) \subset \Phi^{-1}(y).$$

If $C \cap \mathcal{U} \neq \emptyset$, then the left hand side of this inclusion is non-empty and of dimension 1. Since $\Phi^{-1}(y)$ is isomorphic to the $\mathbf{C}$-orbit of y, it is itself irreducible of dimension (at most) 1, hence the inclusion becomes an equality. Taking closures, we then obtain $C = \overline{\{(\xi, x, y) | (\xi, x) \in \Phi^{-1}(y)\}}$; in particular, C is the unique irreducible component of $\pi_3^{-1}(y)$ that intersects $\mathcal{U}$. Note also that $C \subset \Gamma_w$.

Now let C' be an irreducible component of $\pi_3^{-1}(y)$ different from C, so that $C' \subset \{\infty\} \times G/P \times \{y\}$. Let $\Gamma_\infty \subset \Gamma$ be the subvariety $\pi_1^{-1}(\infty)$. We have a U^-_τ-equivariant morphism $\pi : \Gamma_\infty \to G/P : (\infty, x, y) \mapsto y$, so C' is an irreducible subvariety of the fibre $\pi^{-1}(y)$. Since $\Gamma_\infty \subsetneq \Gamma$, its dimension is at most N. Because of the equivariance of π, we see that $\mathcal{O}_\tau$ is in the image of π, so π is surjective. Decomposing Γ_∞ into irreducible components $\Gamma_\infty = C_1 \cup \cdots \cup C_r$, we obtain $G/P = \pi(C_1) \cup \cdots \cup \pi(C_r)$, so that for some i, $\pi : C_i \to G/P$ is onto. Renumbering the C_i, we may assume that for some $t \geq 1$, $C_1, \ldots, C_t$ are mapped surjectively to G/P, and $C_{t+1}, \ldots, C_r$ are not. For $i \leq t$, there is an open set U_i of G/P such that each element on U_i has a finite fibre in C_i. For $i > t$, let U_i be the open set $G/P \setminus \pi(C_i)$. Taking the intersection $U = \bigcap_{i=1}^n U_i$, we obtain a non-empty open set of G/P satisfying the following property: for each $z \in U$, the fibre of z in Γ_∞ consists of a finite number of points. Again, U meets the open orbit $\mathcal{O}_\tau$, so this property is true for every point in $\mathcal{O}_\tau$, in particular for y. So C' is included in the finite fibre $\pi^{-1}(y)$; a contradiction. Therefore, C' cannot exist, *i.e.* $\pi_3^{-1}(y) = C \subset \Gamma_w$ is irreducible, and not contained in $\{\infty\} \times G/P \times G/P$.

Assume now that $F^v_w \neq F_w \cap F^v$. By Proposition 4.4, F^v_w and $F_w \cap F^v$ have the same dimension, thus $F_w \cap F^v$ is not irreducible. Let F be an irreducible component of the intersection $F_w \cap F^v$ different from F^v_w. Let $y \in F$, and assume that $y \notin F^v_w$. Then $y \notin \pi_3(\mathcal{U}^v)$, so $\pi_3^{-1}\{y\} \subset \Gamma^v \setminus \mathcal{U}^v \subset \{\infty\} \times G/P \times G/P$. But $y \in F_w$, and we have seen that in this case $\pi_3^{-1}(y)$ is never contained in $\{\infty\} \times G/P \times G/P$. This gives a contradiction. □

Appendix. Singularities of Schubert varieties in $SO(2n+1)/P_1$

In this Appendix, we shall determine the singular locus of Schubert varieties in G/P, where G is of type B_n and P is cominuscule. So let $V = \mathbf{C}^{2n+1}$ together with a non-degenerate symmetric bilinear form $(.,.)$ given in the canonical basis $(e_1, \ldots, e_{2n+1})$ by the anti-diagonal matrix E with 1's all along the anti-diagonal. The expression of the quadratic form Q associated with $(.,.)$ is

$$Q(x_1, \ldots, x_{2n+1}) = x_{n+1}^2 + 2\sum_{i=1}^{n} x_i x_{2n+2-i}.$$

Let $G = SO(V)$, $B \subset G$ the subgroup of upper triangular matrices, and $T \subset G$ the subgroup of diagonal matrices. Then B is a Borel subgroup of G and T is a maximal torus of G. The group G acts naturally on V, and e_1 is a highest weight vector, of weight ω_1 (the unique cominuscule weight of G), so that G/P_1 gets identified with the G-orbit of the line generated by e_1,

$$G/P_1 = \{[x_1 : \cdots : x_{2n+1}] \mid Q(x_1, \ldots, x_{2n+1}) = 0\}.$$

In this setting, the Schubert varieties are given by

$$X_i = \{[x_1 : \cdots : x_i : 0 : \cdots : 0] \mid Q(x_1, \ldots, x_i, 0, \ldots, 0) = 0\},$$

with $1 \le i \le 2n+1$, but $i \ne n+1$. Indeed, let $x = [x_1 : \cdots : x_{i-1} : 1 : 0 : \cdots : 0]$ with $Q(x) = 0$, and let us prove that $x \in C_i$, that is, there exists $b \in B$ such that $x = b.e_i$. A straightforward calculation shows that we may take the columns $b_1, \ldots, b_{2n+1}$ of b as follows:

- Case 1: $i < n+1$.

$$b_j = \begin{cases} e_j, & \text{if } j \ne i \text{ and } j \le 2n+2-i, \\ x, & \text{if } j = i, \\ e_j - x_{2n+2-j} e_{2n+2-i}, & \text{otherwise.} \end{cases}$$

- Case 2: $i > n+1$.

$$b_j = \begin{cases} e_j, & \text{if } j \le 2n+2-i, \\ x, & \text{if } j = i, \\ e_j - x_{2n+2-j} e_{2n+2-i}, & \text{otherwise.} \end{cases}$$

The Jacobian criterion easily shows that Sing X_i is equal to X_{2n+1-i} if $i > n+1$, and empty if $i < n+1$. Moreover, since X_i is defined by a single quadratic equation, the multiplicity of a singular point must be equal to 2. Hence there are two cases for the multiplicity $\mu_i(x)$ of a point $x = [x_1 : \cdots : x_i : 0 : \cdots : 0]$ on X_i:

- Case 1: $i < n+1$. Then $\mu_i(x) = 1$.
- Case 2: $i > n+1$. Then

$$\mu_i(x) = \begin{cases} 2, & \text{if } x_i = \cdots = x_{2n+2-i} = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Of course, we have the same result for the opposite Schubert varieties

$$X^j = \{[0 : \cdots : 0 : x_j : \cdots : x_{2n+1}] \mid Q(0, \ldots, 0, x_j, \ldots, x_{2n+1}) = 0\}.$$

There are again two cases for the multiplicity $\mu^j(x)$ of $x = [0 : \cdots : 0 : x_j : \cdots : x_{2n+1}]$ on X^j:

- Case 1: $j < n+1$. Then

$$\mu^j(x) = \begin{cases} 2, & \text{if } x_j = \cdots = x_{2n+2-j} = 0, \\ 1, & \text{otherwise.} \end{cases}$$

- Case 2: $j > n+1$. Then $\mu^j(x) = 1$.

Note that a Richardson variety X_i^j ($j \leq i$) also is a quadric in a projective space, so the multiplicity of a point $m \in X_i^j$ must be at most 2. But by Theorem 0.1, if m were singular in both X_i and in X^j, then its multiplicity would be 4. This means that $\operatorname{Sing} X_i \cap \operatorname{Sing} X^j = \emptyset$, a fact that can also be verified directly: indeed, if this intersection is non-empty, then $2n+3-j \leq 2n+1-i$, so $j \leq i \leq j-2$; a contradiction.

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