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ON ALGEBRAS WHICH ARE LOCALLY $\mathbb{A}^1$ IN CODIMENSION-ONE

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Abstract. Let R be a Noetherian normal domain. Call an R-algebra A "locally $\mathbb{A}^1$ in codimension-one" if $R_P \otimes_R A$ is a polynomial ring in one variable over R_P for every height-one prime ideal P in R. We shall describe a general structure for any faithfully flat R-algebra A which is locally $\mathbb{A}^1$ in codimension-one and deduce results giving sufficient conditions for such an R-algebra to be a locally polynomial algebra. We also give a recipe for constructing R-algebras which are locally $\mathbb{A}^1$ in codimension-one. When R is a normal affine spot (i.e., a normal local domain obtained by a localisation of an affine domain), we give criteria for a faithfully flat R-algebra A, which is locally $\mathbb{A}^1$ in codimension-one, to be Krull and a further condition for A to be Noetherian. The results are used to construct intricate examples of faithfully flat R-algebras locally $\mathbb{A}^1$ in codimension-one which are Noetherian normal but not finitely generated.

1. Introduction

Let R be a commutative ring. A polynomial ring in n variables over R is denoted by $R^{[n]}$. An R-algebra A is said to be $\mathbb{A}^1$ if $A = R^{[1]}$ and *locally* $\mathbb{A}^1$ if $A_m(= R_m \otimes_R A) = {R_m}^{[1]}$ for every maximal ideal m of R. A result of Eakin-Heinzer states that any finitely generated locally $\mathbb{A}^1$-algebra over an integral domain R is isomorphic to the symmetric algebra of an invertible ideal of R [7, (3.4)]. However, Eakin-Silver had given an example [8, (3.15)] of a locally $\mathbb{A}^1$-algebra over $R = \mathbb{Z}$ which is *Noetherian* and factorial but still not finitely generated.

In this paper we consider the more general concept of an R-algebra A for which $A_P(= R_P \otimes_R A) = {R_P}^{[1]}$ for every height-one prime ideal P in R. For convenience we shall call such an R-algebra *locally* $\mathbb{A}^1$ *in codimension-one.* Attention to this concept arose out of the surprising discovery that any *finitely generated* faithfully flat algebra over a Noetherian normal domain R, which is locally $\mathbb{A}^1$ in codimension-one, is isomorphic to the symmetric algebra of an invertible ideal of R [4, Theorem 3.4] and related results in [1] and [4]. In view of the Eakin-Silver example, we pay special attention to investigating criteria for finite generation of Noetherian algebras which are locally $\mathbb{A}^1$ in codimension-one.

The investigation of the general structure of algebras which are locally $\mathbb{A}^1$ in codimension-one, but not necessarily finitely generated, was begun in [5, Section 4], to which our present paper is a sequel. The case considered in [5] was that of a Noetherian factorial domain R and a faithfully flat R-algebra A which is locally

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$\mathbb{A}^1$ in codimension-one. It was shown [5, Theorem 4.6] that A is a direct limit of polynomial algebras (in one variable) over R.

The present paper takes up the study of the much harder case of a Noetherian normal domain R and an R-algebra A which is locally $\mathbb{A}^1$ in codimension-one. In Section 2, the general structure of A is described when A is faithfully flat (Theorem 2.3); a further generalisation is given in Section 7 (Theorem 7.2). Unlike the factorial case, the structure theorem (Theorem 2.3) now is quite technical: A is a direct limit of a family of symbolic Rees algebras of divisorial ideals of R. But, from this description, one can deduce pleasant conditions for A to be finitely generated and hence locally polynomial (see Theorems A, B, C below). More significantly, even when A is *not* finitely generated, the structure theorem brings out useful information about the ring A. For instance, it gives neat criteria for A to be Krull and conditions for A to be Noetherian, as will be seen in Section 5. These conditions guide one to examples of A over a two-dimensional normal affine spot R which are Noetherian normal but not finitely generated and which cannot be expressed as a direct limit of polynomial algebras (in one variable) over R (Section 6). The examples, apart from their intrinsic interest, show that the technicality in Theorem 2.3 is unavoidable. The generalised structure theorem (Theorem 7.2) provides a common framework for understanding various results and examples on locally nilpotent derivations of $R[X, Y]$ and $\mathbb{A}^1$-fibrations that were obtained earlier in [1], [2], [4], [5] (cf. Corollaries 7.9, 7.10, 7.11, Theorem 7.12, and Remark 7.13).

We now describe some of the applications of the structure theorem, and related results, proved in our paper. When A is Noetherian, the structure theorem reveals the criterion for A to be finitely generated: A should have a retraction to R, i.e., there should exist an R-algebra homomorphism from A to R (a condition not satisfied by the Eakin-Silver example). More precisely, we get the following result (Theorem 2.11):

Theorem A. *Let R be a Noetherian normal domain and A a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every prime ideal P in R of height one. Then the following conditions are equivalent:*

(i) *A is Noetherian and A has a retraction to R.*
(ii) *$A \cong Sym_R\, J$ for an invertible ideal J of R.*

Apart from the Eakin-Silver example, this result may also be contrasted with the examples in [6] (see Remark 2.13).

In Section 3, we focus on the local situation. When R is factorial, it was shown [5, Corollary 4.11] that at each point P of Spec R, the fibre ring $k(P) \otimes_R A$ is either $k(P)$ or $k(P)^{[1]}$ (where $k(P)$ denotes the field R_P/PR_P). As a consequence, the following result was deduced [5, Theorem 4.12]:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a factorial local domain and A a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every prime ideal P in R of height one. If $\operatorname{tr.deg}_{R/\mathfrak{m}} A/\mathfrak{m}A > 0$, then $A = R^{[1]}$.*

Examples of the Eakin-Silver type show that Theorem 1.1 cannot be extended, by any local-global principle, to regular factorial domains with infinitely many maximal ideals. In the local situation, two questions arise out of Theorem 1.1:

Q.1. Can the condition on the closed fibre be realised by an appropriate hypothesis (like Noetherian) on the algebra A?

Q.2. Can the result be generalised to a Noetherian *normal* local domain R?

Now, a significant consequence of the structure theorem is that over a *complete local* Noetherian normal domain R, any faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one necessarily has a retraction to R (Proposition 3.4). It then follows from Theorem A that Q.1 has an affirmative answer when R is complete and normal (Theorem 3.7):

Theorem B. *Let R be a Noetherian complete normal local domain and A a Noetherian faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every prime ideal P in R of height one. Then $A = R^{[1]}$.*

Example 6.4 shows that Theorem B does not extend to non-complete rings, not even in the factorial situation. Further discussion on the significance of Example 6.4, in light of the Eakin-Silver example, Theorem 1.1 and Theorem B, is made at the beginning of Section 6.

We shall also deduce (Theorem 3.10) that Q.2 has an affirmative answer when R is analytically irreducible. More precisely, we have (Theorem 3.10):

Theorem C. *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian normal local domain and A a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every prime ideal P in R of height one. Then $QA \in \operatorname{Spec} A$ for every prime ideal Q of R. In fact, for each prime ideal Q of R, either $A_Q = {R_Q}^{[1]}$, or the fibre ring $k(Q) \otimes_R A = k(Q)$. In particular, if $R/\mathfrak{m} \subsetneqq A/\mathfrak{m}A$, then $A = R^{[1]}$.*

In Section 4, we shall discuss the converse problem of constructing algebras which are locally $\mathbb{A}^1$ in codimension-one (Proposition 4.3). Further, when R is a normal affine spot with field of fractions K, we show that to any flat R-module N such that $N_P \cong R_P$ for each height-one prime ideal P, an element $x \in N$ which generates $K \otimes_R N$, and any element $\xi \in Ext^1_R(N, R)$, one can associate an appropriate faithfully flat R-algebra B which is locally $\mathbb{A}^1$ in codimension-one (Proposition 4.7).

In Section 4 we shall also see why an exact elegant analogue of the structure theorem for the factorial case [5, Theorem 4.6] does not hold in the normal situation. In fact, our search for a refinement of the structure theorem (Theorem 2.3) had led us to the following:

Q.3. Let $(R, \mathfrak{m})$ be a normal affine spot of dimension two. Suppose that N is a flat R-module such that $N_P \cong R_P$ at every prime ideal P in the punctured spectrum $(\operatorname{Spec} R) \setminus \mathfrak{m}$. Is N necessarily a direct limit of cyclic modules?

One could expect an affirmative answer in view of the theorem of Govorov and Lazard [11, p. 134] where any flat module is a direct limit of finitely generated free modules. However, while the answer is affirmative if R is factorial, Proposition 4.11 shows that Q.3 has a negative answer for the (non-factorial) normal case. Using the module N of Proposition 4.11, one immediately gets a faithfully flat R-algebra A which is locally $\mathbb{A}^1$ in codimension-one but which cannot be expressed as a direct limit of polynomial algebras in one variable over R (Remark 4.12).

The R-algebra A of Remark 4.12 is non-Krull. The next phase of our paper is devoted to establishing principles for a faithfully flat algebra A over a normal affine spot R (which is locally $\mathbb{A}^1$ in codimension-one) to be Krull. Section 5 shows that if $A \neq R^{[1]}$, then A is a Krull domain if and only if A can be embedded in the completion of R (Proposition 5.3). Moreover, we give a codimension-one criterion for A to be Noetherian (Theorem 5.2) and apply it to prove some technical results (Proposition 5.8, Corollary 5.9, Lemma 5.10) which are useful for constructing examples of A which are Noetherian normal but not finitely generated.

In Section 6, the hardest part of the paper, we present our main examples (Proposition 6.1, Examples 6.2, 6.4, 6.5). In Section 7, we give the generalised version of the structure theorem together with a converse (Theorem 7.2) and deduce a few applications (Theorems 7.8, 7.12, 7.14, 7.16 and Corollaries 7.9, 7.10, 7.11).

For the convenience of the reader, we now recall a few preliminary facts about divisorial ideals.

Preliminaries: Divisorial Ideals

Let R be an integral domain with field of fractions K. Recall that for any non-zero fractional ideal I of R, I^{-1} (or $R:_K I$) denotes the R-module $I^{-1} = \{\alpha \in K \mid \alpha I \subseteq R\} \cong \operatorname{Hom}_R(I, R)$. I is said to be *divisorial* if $I = (I^{-1})^{-1}$, that is, if I is reflexive as an R-module. Note that I^{-1} is divisorial for any non-zero fractional ideal I.

For the rest of the section, let R denote a **Noetherian normal domain** (or more generally, a Krull domain). The following notation will be used throughout the paper.

$\Delta :=$ The set of prime ideals in R of height one.

$\Sigma :=$ The set of all finite subsets of Δ.

For a fractional ideal J of R, we state a few equivalent conditions for J to be divisorial. The results are either standard (see, for instance, [9, Corollaries 5.5, 5.6]) or can be easily deduced.

Lemma 1.2. *For a non-zero fractional ideal J of R, the following statements are equivalent:*

(i) *J is divisorial.*
(ii) *$J = \bigcap_{P\in\Delta} J_P$.*

Moreover if, in addition, J is an integral ideal of R, then the above statements are also equivalent to the following:

(iii) *J is an unmixed ideal of height one, namely, J is of the form*

$$J = {P_1}^{(e_1)} \cap \cdots \cap {P_m}^{(e_m)},$$

where $\{P_1, \dots, P_m\} \subseteq \Delta$ and $P_i^{(e_i)} = R \cap {P_i}^{e_i} R_{P_i}$ for each i.

(iv) *$J = R \cap dR_a$ for some $d \in J$ and $a \in R$, where $R_a = R[1/a]$.*

For a divisorial integral ideal J with associated prime ideals $P_1, \dots, P_m$, we define the n-th *symbolic power* of J to be

$$J^{(n)} := R \cap J^n(T^{-1}R), \text{ where } T = R \setminus (P_1 \cup \cdots \cup P_m).$$

Note that $J^{(n)}$ is again divisorial and that $(J^{(n)})^{-1} = (J^n)^{-1}$.

One can also deduce the following equivalence (see the argument in [5, Lemma 2.8]) which will be frequently used in the paper.

Lemma 1.3. *Let M be a torsion-free R-module. Then the following conditions are equivalent:*

(i) *$M = \bigcap_{P\in\Delta} M_P$, where M and $M_P = R_P \otimes_R M$ are identified with their images in $K \otimes_R M$.*

(ii) *If a and b are elements of R such that b is (R/aR)-regular, then b is (M/aM)-regular.*

The following result is an easy application of the above lemma.

Lemma 1.4. *Let L be a torsion-free R-module and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-submodules of L such that for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ satisfying $M_{\lambda_1} \cup M_{\lambda_2} \subseteq M_\lambda$. Set $M = \bigcup_{\lambda\in\Lambda} M_\lambda$. If $M_\lambda = \bigcap_{P\in\Delta}(M_\lambda)_P$ for each $\lambda \in \Lambda$, then $M = \bigcap_{P\in\Delta} M_P$.*

We also deduce the following.

Lemma 1.5. *To each $P \in \Delta$, assign a non-negative integer e_P and let $\{I_\Gamma\}_{\Gamma\in\Sigma}$ be a family of ideals of R such that $I_\Gamma = \bigcap_{P\in\Gamma} P^{(e_P)}$ for each $\Gamma \in \Sigma$. Set $M_n = \bigcup_{\Gamma\in\Sigma}({I_\Gamma}^n)^{-1}$ for each $n \geq 0$ and set $M = M_1$. Then the following assertions hold for the R-modules M_n:*

1. *$(M_n)_P = {M_P}^n \cong R_P$ for every $n \geq 0$ and $P \in \Delta$.*
2. *$M_n = \bigcap_{P\in\Delta}(M_n)_P$ for every $n \geq 0$.*
3. *If M is R-flat, then $M_n = M^n$, and M_n is R-flat for every $n \geq 0$.*

Proof. First of all, note that $I_{\Gamma_1} \supseteq I_{\Gamma_2}$ whenever $\Gamma_1 \subseteq \Gamma_2$, so that $({I_{\Gamma_1}}^n)^{-1} \subseteq ({I_{\Gamma_2}}^n)^{-1}$ for each n.

(1) Let p be a uniformizing parameter of the DVR R_P. Since $(I_\Gamma)_P = P^{e_P}R_P = p^{e_P}R_P$ for Γ with $P \in \Gamma$, by using [9, Corollary 5.4], we can easily check that $(M_n)_P = p^{-ne_P}R_P$. Thus the assertion follows.

(2) This is an immediate consequence of Lemmas 1.2 and 1.4.

(3) Since $M \subseteq K$, if M is R-flat, then $M^n \cong M \otimes \cdots \otimes M$ (an n-fold tensor product), and hence M^n is R-flat. Thus

$$M^n = \bigcap_{P\in\Delta}(M^n)_P = \bigcap_{P\in\Delta}(M_P)^n = \bigcap_{P\in\Delta}(M_n)_P = M_n$$

by (1), (2) and Lemma 1.3. □

We shall call a local domain D an *affine spot* if D is obtained as a localisation of an affine domain (i.e., a finitely generated algebra over a field).

2. The structure theorem

Throughout this section, R will denote a **Noetherian normal domain** with field of fractions K, Δ the set of prime ideals in R of height one, and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one, namely, $A_P\ (= R_P \otimes_R A) = {R_P}^{[1]}$ for every $P \in \Delta$. We shall describe the structure of A (Theorem 2.3), give conditions for A to have a retraction to R (Proposition 2.9) and deduce Theorem A (Theorem 2.11).

We first define a few quantities associated to A. By Lemma 1.3, we have

$$A = \bigcap_{P\in\Delta} A_P. \tag{2.1}$$

Since $A_P = {R_P}^{[1]}$ for $P \in \Delta$, it follows that $K \otimes_R A = K^{[1]}$, and hence there exists an element x in A such that

$$R[x] \subseteq A \subseteq K[x].$$

We choose and fix such an x, and call it a *generic variable* of A.

For a finite subset $\Gamma = \{P_1, \dots, P_n\}$ of Δ, we set

$$R_\Gamma := \bigcap_{Q \in \Delta \setminus \Gamma} R_Q \text{ and}$$

$$A_\Gamma := {S_\Gamma}^{-1} A \cap R_\Gamma[x], \text{ where } S_\Gamma := R \setminus (\bigcup_{P \in \Gamma} P).$$

Thus $R = R_\Gamma \cap {S_\Gamma}^{-1} R$ and ${S_\Gamma}^{-1} A = A_{P_1} \cap \cdots \cap A_{P_n}$. Therefore

$$A_\Gamma = \left(\bigcap_{P \in \Gamma} A_P \right) \cap \left(\bigcap_{P \in \Delta \setminus \Gamma} R_P[x] \right). \tag{2.2}$$

Lemma 2.1. *The following assertions hold for the rings A_Γ defined above:*

(1) *For $P \in \Gamma$, $(R_\Gamma)_P = K$, and hence $(A_\Gamma)_P = A_P$.*
(2) *For $P \in \Delta \setminus \Gamma$, $(R_\Gamma)_P = R_P$, and hence $(A_\Gamma)_P = R_P[x]$.*
(3) *$A_\Gamma = \bigcap_{P \in \Delta} (A_\Gamma)_P$.*
(4) *$A_\Gamma \subseteq A$.*
(5) *If Γ_1 and Γ_2 are finite subsets of Δ with $\Gamma_1 \subseteq \Gamma_2$, then $A_{\Gamma_1} \subseteq A_{\Gamma_2}$ and, moreover, $A_{\Gamma_1} \subsetneq A_{\Gamma_2}$ if and only if there exists $P \in \Gamma_2 \setminus \Gamma_1$ such that $A_P \neq R_P[x]$.*

Proof. The proofs of (1) and (2) are easy; (3) follows from (1) and (2). Now as $R_P[x] \subseteq A_P$ for every $P \in \Delta$, (4) follows from the relations (2.1) and (2.2). Assertion (5) also follows easily, as in [5, Lemmas 4.5, 4.8]. □

Lemma 2.2. *Let Γ be a finite subset of Δ, let $I = R \cap d({S_\Gamma}^{-1} R)$ and let $J = R \cap dR_\Gamma$, where d is a non-zero element of R. Then I and J are divisorial ideals of R and, for every positive integer n,*

$$(I^n)^{-1} = d^{-n}(R \cap d^n R_\Gamma) = d^{-n} J^{(n)}.$$

In particular, if J is invertible, then $J^{(n)} = J^n$ for every $n \geq 0$.

Proof. Clearly I and J are divisorial ideals of R. Let $J_n = R \cap d^n R_\Gamma$. We show that $d^n (I^n)^{-1} = J_n$. It is easy to check that $J_n \subseteq d^n (I^n)^{-1}$. For the converse inclusion, note that $J_n = \bigcap_{P \in \Delta} (J_n)_P$ because J_n is divisorial. Hence it is enough to show that $d^n ((I^n)^{-1})_P \subseteq (J_n)_P$ for every P in Δ. This holds since $I_P = dR_P$ and $(J_n)_P = R_P$ for each $P \in \Gamma$, while $I_P = R_P$ and $(J_n)_P = d^n R_P$ for each $P \in \Delta \setminus \Gamma$. In particular, we have $(J_n)_P = (J^n)_P$ for every $P \in \Delta$. Therefore

$$J^{(n)} = \bigcap_{P \in \Delta} (J^n)_P = \bigcap_{P \in \Delta} (J_n)_P = J_n$$

for every $n \geq 0$. If J is invertible, then $I (= dJ^{-1})$ is also invertible, so $J^{(n)} = d^n (I^n)^{-1} = (dI^{-1})^n = J^n$ for every $n \geq 0$. This completes the proof. □

Now consider the set Σ of all finite subsets Γ of Δ. Lemma 2.1 shows that the rings A_Γ, together with inclusion maps, form a direct system $\{A_\Gamma \mid \Gamma \in \Sigma\}$ indexed by Σ. With this notation, we now state the structure theorem.

Theorem 2.3. *Let R be a Noetherian normal domain, Δ the set of height-one prime ideals of R, and A a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every*

$P \in \Delta$. Then for every finite subset Γ of Δ, there exist elements c_Γ, d_Γ in R such that

$$A_\Gamma = \bigoplus_{n\geq 0} ({I_\Gamma}^n)^{-1}\ (x - c_\Gamma)^n = \bigoplus_{n\geq 0} {J_\Gamma}^{(n)} \left(\frac{x - c_\Gamma}{d_\Gamma}\right)^n, \tag{2.3}$$

where I_Γ and J_Γ are the divisorial ideals defined as $I_\Gamma = R \cap d_\Gamma({S_\Gamma}^{-1}R)$ and $J_\Gamma = R \cap d_\Gamma R_\Gamma$ with $S_\Gamma = R \setminus (\bigcup_{P\in\Gamma} P)$. Moreover,

$$A = \varinjlim{}_{\Gamma\in\Sigma} A_\Gamma\ (= \bigcup\nolimits_{\Gamma\in\Sigma} A_\Gamma).$$

Proof. For simplicity we set $S = S_\Gamma$. Note that $S^{-1}R$ is a semi-local PID and $(S^{-1}A)_Q = {(S^{-1}R)_Q}^{[1]}$ for every Q in Spec $S^{-1}R$. Thus $S^{-1}A = (S^{-1}R)^{[1]}$. Choose $y \in A$ such that $S^{-1}A = (S^{-1}R)[y]$. Then $K[x] = K[y]$ and $x \in A \subseteq S^{-1}A = (S^{-1}R)[y]$. From this we can write

$$x = \frac{dy + d'}{s}$$

for some $d(\neq 0), d' \in R$ and $s \in S$. Then

$$d' = sx - dy \in (s, d)A \cap R = (s, d)R \tag{2.4}$$

by faithful flatness of A over R, so $d' = sc - dc'$ for some $c, c' \in R$. It then follows that

$$x = \frac{dy + sc - dc'}{s} = \frac{d(y - c')}{s} + c.$$

Hence, setting $z = (x - c)/d = (y - c')/s$, we have $S^{-1}R[y] = S^{-1}R[z]$ and

$$A_\Gamma = (S^{-1}R)[z] \cap R_\Gamma[dz] = \bigoplus_{n\geq 0}(S^{-1}R \cap d^n R_\Gamma)z^n = \bigoplus_{n\geq 0}(R \cap d^n R_\Gamma)z^n \tag{2.5}$$

using $R = S^{-1}R \cap R_\Gamma$. Now setting $c_\Gamma := c$ and $d_\Gamma := d$, we see that relation (2.3) holds by Lemma 2.2.

Now let $f \in A$. Since $R[x] \subseteq A \subseteq K[x]$, there exists a non-zero element $a \in R$ such that $af \in R[x]$. Let Γ_a denote the set of prime ideals associated to aR. Then it is easy to see that $f \in A_{\Gamma_a}$. Thus A is the direct limit of the system $\{A_\Gamma \mid \Gamma \in \Sigma\}$. □

Remark 2.4. Let the notation and hypotheses be as in Theorem 2.3.

(1) For $\Gamma \in \Sigma$, we have ${S_\Gamma}^{-1}A = {S_\Gamma}^{-1}A_\Gamma$ by Lemma 2.1 (1), so

$${S_\Gamma}^{-1}A = {S_\Gamma}^{-1}R\left[\frac{x - c_\Gamma}{d_\Gamma}\right] \tag{2.6}$$

by (2.5). In particular,

$$x - c_\Gamma \in d_\Gamma({S_\Gamma}^{-1}A) \cap A = (d_\Gamma({S_\Gamma}^{-1}R) \cap R)A = I_\Gamma A \tag{2.7}$$

by flatness of A over R.

(2) For $P \in \Delta$, let v_P denote the valuation of K whose valuation ring is R_P. Let Γ be a finite subset of Δ such that $P \in \Gamma$. Then

$$A_P = ({S_\Gamma}^{-1}A)_P = R_P\left[\frac{x - c_\Gamma}{d_\Gamma}\right] \tag{2.8}$$

by (2.6). From this we know that $e_P := v_P(d_\Gamma)$ is a constant independent of Γ containing P. Since $I_\Gamma = R \cap d_\Gamma({S_\Gamma}^{-1}R)$, it then follows that

$$I_\Gamma = \bigcap\nolimits_{P \in \Gamma} P^{(e_P)}. \tag{2.9}$$

(3) Note that A is normal by (2.1). Let $P \in \Delta$. Since $A_P = {R_P}^{[1]}$, we have

$$k(P) \otimes_R A = A_P/PA_P = k(P)^{[1]},$$

where $k(P) = R_P/PR_P$. From this it follows that PA is a prime ideal in A of height one, because $A/PA \hookrightarrow A_P/PA_P$ by flatness of A over R. In particular, every prime element of R remains a prime element in A. Now suppose that R is factorial, and let T be the multiplicatively closed set generated by all prime elements of R. Then $T^{-1}R = K$, and hence $T^{-1}A = K^{[1]}$, a factorial domain. Thus, if A is Noetherian, or more generally A is Krull, then, by Nagata's criterion [9, Corollary 7.3], A is also factorial.

(4) Note that when R is factorial, I_Γ becomes a principal ideal, and hence $A_\Gamma = R^{[1]}$. This explains the earlier result [5, Theorem 4.6] from a more general perspective.

(5) Although each A_Γ has a retraction to R, in general, A itself may not have a retraction to R (see the R-algebra A of Proposition 6.1 and Example 6.2, or the D-algebras B and C of Examples 6.4 and 6.5).

(6) We shall later observe (Theorem 7.2) that Theorem 2.3 actually holds over any Krull domain R, and for any algebra satisfying conditions milder than being "faithfully flat" (which we term "semi-faithfully flat") and, conversely, one can give a recipe for constructing all such "semi-faithfully flat" algebras.

For $P \in \Delta$, as in part (2) of Remark 2.4, let v_P denote the valuation of K whose valuation ring is R_P and let $e_P = v_P(d_\Gamma)$ where $\Gamma \in \Sigma$ with $P \in \Gamma$. Now, we define a subset Δ_0 of Δ by

$$\Delta_0 := \{P \in \Delta \mid A_P \neq R_P[x]\} = \{P \in \Delta \mid e_P > 0\}. \tag{2.10}$$

Note that the second equality surely holds in (2.10) because of (2.8).

Lemma 2.5. *With notation as in Theorem* 2.3*, the following assertions hold:*

1. *For $\Gamma \in \Sigma$, setting $\Gamma' = \mathrm{Ass}_R(R/I_\Gamma)$, we have $\Gamma' = \Gamma \cap \Delta_0$, $I_\Gamma = I_{\Gamma'}$ and $A_\Gamma = A_{\Gamma'}$. In particular, $\Gamma' \subseteq \Delta_0$, and $\Gamma = \Gamma'$ if $\Gamma \subseteq \Delta_0$.*
2. *For $\Gamma_1, \Gamma_2 \in \Sigma$, if $\Gamma_1 \subseteq \Gamma_2$, then $I_{\Gamma_1} \supseteq I_{\Gamma_2}$. Moreover, $c_{\Gamma_2} - c_{\Gamma_1} \in I_{\Gamma_1}$.*

Proof. (1) This is an immediate consequence of (2.9) and Lemma 2.1 (5).

(2) The first assertion follows from (2.9). For the second assertion,

$$c_{\Gamma_2} - c_{\Gamma_1} = (x - c_{\Gamma_1}) - (x - c_{\Gamma_2}) \in I_{\Gamma_1}A \cap R = I_{\Gamma_1}$$

by (2.7) and faithful flatness of A over R. □

Corollary 2.6. *The following conditions are equivalent for $\Gamma_1, \Gamma_2 \in \Sigma$:*

(i) $I_{\Gamma_1} = I_{\Gamma_2}$.
(ii) $\Gamma_1 \cap \Delta_0 = \Gamma_2 \cap \Delta_0$.
(iii) $A_{\Gamma_1} = A_{\Gamma_2}$.

In particular, if $\Gamma_1 \subseteq \Delta_0$ and $\Gamma_2 \subseteq \Delta_0$, then $I_{\Gamma_1} = I_{\Gamma_2}$ if and only if $\Gamma_1 = \Gamma_2$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) follow from Lemma 2.5. If $A_{\Gamma_1} = A_{\Gamma_2}$, then $(I_{\Gamma_1})^{-1} = (I_{\Gamma_2})^{-1}$ by (2.3) so that $I_{\Gamma_1} = I_{\Gamma_2}$, since both ideals are divisorial. Thus (iii) ⇒ (i) holds. □

We now define an equivalence relation on Σ as follows: For $\Gamma_1, \Gamma_2 \in \Sigma$, we say that $\Gamma_1 \sim \Gamma_2$ if $I_{\Gamma_1} = I_{\Gamma_2}$. Let Σ_0 be the set of all finite subsets of Δ_0. Then, by Lemma 2.5 and Corollary 2.6, Σ_0 is a subset of Σ comprising exactly one element from each equivalence class.

Note that $\{A_\Gamma \mid \Gamma \in \Sigma\} = \{A_\Gamma \mid \Gamma \in \Sigma_0\}$ by Lemma 2.5 (1), and therefore

$$A = \varinjlim_{\Gamma \in \Sigma_0} A_\Gamma \tag{2.11}$$

by Theorem 2.3. Also note that Σ_0 is finite if and only if Δ_0 is finite.

Corollary 2.7. *Δ_0 is finite if and only if A is isomorphic to the symmetric algebra of an invertible ideal of R.*

Proof. Suppose that Δ_0 is finite. Then Δ_0 is the unique maximal element of Σ_0, and hence, setting $I = I_{\Delta_0}$, $J = J_{\Delta_0}$, $c = c_{\Delta_0}$ and $d = d_{\Delta_0}$, we have

$$A = A_{\Delta_0} = \bigoplus_n (I^n)^{-1}\,(x-c)^n = \bigoplus_n d^{-n} J^{(n)} (x-c)^n \tag{2.12}$$

by (2.11). Since A is R-flat, so is $A_1 := d^{-1}J(x-c)$, because A_1 is a direct summand of A. It then follows that J is also R-flat, and hence an invertible ideal of R. Thus, $J^{(n)} = J^n$ for every $n \geq 0$ by Lemma 2.2, and therefore $A \cong Sym_R\, J$ by (2.12).

Conversely, suppose that $A \cong Sym_R\, J$ for an invertible ideal J of R. Then A is finitely generated over R so that, by (2.11), $A = A_\Gamma$ for some $\Gamma \in \Sigma_0$. If Δ_0 is infinite, then there exists $P \in \Delta_0 \setminus \Gamma$. Then, setting $\Gamma' = \Gamma \cup \{P\}$, we have $A_\Gamma \subsetneqq A_{\Gamma'}$ by Corollary 2.6, a contradiction. Thus Δ_0 is finite. □

We now investigate when the faithfully flat R-algebra A will have a retraction to R.

Lemma 2.8. *Let the notation be as in Theorem* 2.3*, and let $\Gamma \in \Sigma$. Then, for any $c \in R$ satisfying $c - c_\Gamma \in I_\Gamma$, we have*

$$\bigoplus_{n \geq 0} ({I_\Gamma}^n)^{-1}\,(x - c_\Gamma)^n (= A_\Gamma) = \bigoplus_{n \geq 0} ({I_\Gamma}^n)^{-1}\,(x-c)^n.$$

Proof. Since $c - c_\Gamma \in I_\Gamma$, it follows that $({I_\Gamma}^n)^{-1}(c - c_\Gamma)^{n-r} \subseteq ({I_\Gamma}^r)^{-1}$ for each r with $0 \leq r \leq n$. Now using $x - c_\Gamma = (x-c) + (c - c_\Gamma)$, one can easily see that the desired equality holds. □

It follows from Lemma 2.5 (2) that, for $\Gamma_1, \Gamma_2 \in \Sigma$ with $\Gamma_1 \subseteq \Gamma_2$, we have ${I_{\Gamma_1}}^n \supseteq {I_{\Gamma_2}}^n$ for each $n \geq 0$. Thus $({I_{\Gamma_1}}^n)^{-1} \subseteq ({I_{\Gamma_2}}^n)^{-1}$, which implies that the set $\{({I_\Gamma}^n)^{-1}\}_{\Gamma \in \Sigma}$ forms a directed system indexed by Σ. Now, for $n \geq 0$, we define an R-submodule M_n of K by

$$M_n = \varinjlim_{\Gamma \in \Sigma} ({I_\Gamma}^n)^{-1}\, (= \bigcup_{\Gamma \in \Sigma} ({I_\Gamma}^n)^{-1}). \tag{2.13}$$

We set $M := M_1$. (See Lemma 1.5 for some properties of M_n.)

Proposition 2.9. *With notation as in Theorem* 2.3*, the following conditions are equivalent:*

(i) *A has a retraction to R.*

(ii) *There exists* $c \in R$ *such that* $c - c_\Gamma \in I_\Gamma$ *for every* $\Gamma \in \Sigma$.
(iii) *There exists* $c \in R$ *such that* $x - c \in I_\Gamma A$ *for every* $\Gamma \in \Sigma$.
(iv) *There exists* $c \in R$ *such that* $A = \bigoplus_{n\geq 0} M^n(x-c)^n$, *where* $M(= M_1)$ *is the* R*-submodule of* K *defined by* (2.13).
(v) A *is a graded* R*-algebra* $\bigoplus_{n\geq 0} A_n$ *with* $A_0 = R$.
(vi) $A = Sym_R\, N$ *for some flat* R*-module* N *such that* $N_P \cong R_P$ *for every* $P \in \Delta$.

Proof. The implications (v) ⇒ (i), (iv) ⇒ (v) and (vi) ⇒ (v) are clear, and (ii) ⇔ (iii) follows easily from (2.7). It thus suffices to prove that (i) ⇒ (ii), (ii) ⇒ (iv) and (ii) ⇒ (vi).

(i) ⇒ (ii). Let $\varphi\colon A \to R$ denote the retraction and let $c = \varphi(x)$. Then, for $\Gamma \in \Sigma$, we have $x - c_\Gamma \in I_\Gamma A$ by (2.7), and hence $c - c_\Gamma = \varphi(x - c_\Gamma) \in I_\Gamma$.

(ii) ⇒ (iv) and (vi). For $n \geq 0$, let $A_n = M_n(x-c)^n$, where M_n is the R-submodule of K defined by (2.13). Then it follows from Theorem 2.3 and Lemma 2.8 that $A = \bigoplus_{n\geq 0} A_n$. Set $N := A_1(= M(x-c))$. Then $N_P \cong R_P$ for $P \in \Delta$ by Lemma 1.5 (1). Moreover N (a direct summand of A) is R-flat, and hence so is M. Therefore $M_n = M^n$ for each n by Lemma 1.5 (3), so that $A = \bigoplus_{n\geq 0} M^n(x-c)^n = Sym_R\, N$. □

Remark 2.10. (1) By Lemma 2.5 (2), $\{R/I_\Gamma\}_{\Gamma\in\Sigma}$ is an inverse system under the canonical maps, and $\{c_\Gamma\}_{\Gamma\in\Sigma}$ defines an element $\mathbf{c}$ of $\varprojlim R/I_\Gamma$. Let $\phi\colon R \to \varprojlim R/I_\Gamma$ be the canonical R-algebra map. Then condition (ii) of Proposition 2.9 is equivalent to the existence of c in R such that $\phi(c) = \mathbf{c}$.

(2) Let the notation be as in Proposition 2.9. Since $A \subseteq K[x]$, for $f \in A$, there exists $r(\neq 0) \in R$ such that $rf \in R[x]$. From this it follows that if $\varphi\colon A \to R$ is a retraction, then φ is determined by $\varphi(x)$.

Now suppose that A is not the symmetric algebra of an invertible ideal of R. Then the element c satisfying condition (ii) of Proposition 2.9 is unique (if it exists). Indeed, if $c' \in R$ satisfies $c' - c_\Gamma \in I_\Gamma$ for every $\Gamma \in \Sigma$, then $c' - c \in I_\Gamma$, so

$$c' - c \in \bigcap\nolimits_{\Gamma\in\Sigma_0} I_\Gamma = (0)$$

since $\bigcap_{\Gamma\in\Sigma_0} I_\Gamma$ is the intersection of an infinite family of height-one prime ideals by Corollary 2.7. Note that if φ is a retraction, then $\varphi(x)(\in R)$ satisfies the condition that $\varphi(x) - c_\Gamma \in I_\Gamma$ for every $\Gamma \in \Sigma$. Thus the retraction is also unique.

Theorem A mentioned in the Introduction now follows from the above results.

Theorem 2.11. *Let* R *be a Noetherian normal domain and* A *a faithfully flat* R*-algebra such that* $A_P = {R_P}^{[1]}$ *for every prime ideal* P *in* R *of height one. Then the following conditions are equivalent:*

(i) A *is Noetherian and* A *has a retraction to* R.
(ii) A *is finitely generated over* R.
(iii) $A \cong Sym_R\, J$ *for an invertible ideal* J *of* R.

Proof. It suffices to prove that (i) ⇒ (ii) ⇒ (iii).

(i) ⇒ (ii). If A has a retraction to R, then A is a graded R-algebra with $A_0 = R$ by Proposition 2.9. Thus, if further A is Noetherian, then A is finitely generated by [12, Theorem 13.1].

(ii) $\Rightarrow$ (iii). This follows from [4, Theorem 3.4]. This implication can also be seen from the proof of Corollary 2.7 by noting that if A is finitely generated, then $A = A_\Gamma$ for some $\Gamma \in \Sigma$. □

Remark 2.12. Let the notation be as in Proposition 2.9. For $f \in A (\subseteq K[x])$, we look at $\deg_x f$, the x-degree of f considered as an element of $K[x]$, and set, for $i \geq 0$,

$$E_i := \{f \in A \mid \deg_x f \leq i\}. \tag{2.14}$$

Then the sequence E_i defines an increasing filtration of R-submodules of A with $E_0 = R$. Set $E := E_1$. Note that, by Theorem 2.3, we have

$$E_n = \varinjlim_{\Gamma \in \Sigma} (R \oplus {I_\Gamma}^{-1}(x - c_\Gamma) \oplus \cdots \oplus ({I_\Gamma}^n)^{-1}(x - c_\Gamma)^n).$$

Thus, for each n, letting M_n be the R-module defined by (2.13), we can define an R-module homomorphism $\rho_n \colon E_n \to M_n$ by

$$\rho_n(f) = \text{the coefficient of } x^n \text{ in } f$$

for $f \in E_n$. Then ρ_n is surjective and $\ker \rho_n = E_{n-1}$. Thus $E_n/E_{n-1} \cong M_n$ for every $n \geq 1$. In particular, we have $E/R \cong M (= M_1)$ so that the sequence

$$0 \to R \to E \xrightarrow{\rho} M \to 0 \tag{2.15}$$

is exact, where $\rho = \rho_1$. Then the splitting of the above short exact sequence is equivalent to the existence of a retraction from A to R. Indeed if there exists a retraction, then $E = R \oplus M(x - c)$ for some $c \in R$ by Proposition 2.9, so that the sequence (2.15) splits. Conversely suppose that there exists an R-module homomorphism $\tau \colon M \to E$ such that $\rho \circ \tau = id_M$. Note that $x \in E$ and $\rho(x) = 1$. Thus, setting $c = x - \tau(1)$, we have $c \in \ker \rho = R$. Let $\Gamma \in \Sigma$ and let β be a non-zero element of I_Γ. Then, for any $\alpha \in {I_\Gamma}^{-1}$, we have $\tau(\beta\alpha) = \beta\tau(\alpha)$ because $\beta \in R$, while $\tau(\beta\alpha) = \beta\alpha\tau(1)$ because $\beta\alpha \in R$. Thus $\beta\tau(\alpha) = \beta\alpha\tau(1)$, which implies that $\alpha(x - c) = \alpha\tau(1) = \tau(\alpha) \in E$. On the other hand $\alpha(x - c_\Gamma) \in E$, so

$$\alpha(c - c_\Gamma) = \alpha(x - c_\Gamma) - \alpha(x - c) \in E \cap K = R.$$

Hence $c - c_\Gamma \in ({I_\Gamma}^{-1})^{-1} = I_\Gamma$ for any $\Gamma \in \Sigma$, from which it follows that A has a retraction to R by Proposition 2.9.

The algebra A is closely related to the modules M and E; the relationship will be discussed in Section 4.

Remark 2.13. It has been shown [6, Examples 5.5, 5.6] that even over a nice discrete valuation ring such as $R = \mathbb{Q}[[t]]$ or $R = k[t]_{(t)}$ (k a field), there exist Noetherian normal birational R-subalgebras of $R[X]$ which are not finitely generated (also see [6, Theorems 4.2, 4.8]). Since R is a DVR, such an algebra A is obviously faithfully flat over R. Further, the condition "A is a normal subalgebra of $R[X]$" ensures that

(1) $A[1/t] = R[1/t]^{[1]}$ and
(2) A has a retraction to R.

However, in contrast to the examples in [8] and [6], Theorem 2.11 shows that if A is a Noetherian faithfully flat algebra over a Noetherian normal domain R such that

(I) A is locally $\mathbb{A}^1$ in codimension-one and
(II) A has a retraction to R,

then A is indeed finitely generated over R.

3. Applications to the local situation

In this section we shall prove a few results over Noetherian normal local domains, including Theorems B and C. We shall use the following theorem due to Chevalley [17, p. 270].

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and $\{I_n\}_{n\geq 1}$ a decreasing sequence of ideals such that $\bigcap_{n\geq 1} I_n = (0)$. Then there exists a positive integer-valued function $s(n)$ which tends to infinity with n such that $I_n \subseteq \mathfrak{m}^{s(n)}$.*

Now let $(R, \mathfrak{m})$ be a Noetherian complete local ring and $\{J_\lambda \mid \lambda \in \Lambda\}$ a family of ideals of R such that given any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ such that $J_\lambda \subseteq J_{\lambda_1} \cap J_{\lambda_2}$. Thus the rings $\{R/J_\lambda\}_{\lambda\in\Lambda}$ form an inverse system under the canonical maps. Assume further that if Λ is infinite, then $\bigcap_{\lambda\in\Omega} J_\lambda = (0)$ for *any* infinite subset Ω of Λ.

In this setup, we deduce the following consequence of Chevalley's theorem:

Proposition 3.2. *The canonical R-algebra map $\phi : R \to \varprojlim_{\lambda\in\Lambda} R/J_\lambda$ is surjective. In particular, ϕ is an isomorphism if Λ is infinite.*

Proof. It is enough to consider the case where Λ is infinite. Note that, in this case, $\bigcap_{\lambda\in\Omega} J_\lambda = (0)$ for any infinite subset Ω of Δ.

For clarity, we first assume that Λ is countably infinite.

Case 1. $|\Lambda| = \aleph_0$.

Let $\Lambda = \{\lambda_1, \dots, \lambda_r, \dots\}$, and for every positive integer n, set

$$J_n := J_{\lambda_1} \cap \cdots \cap J_{\lambda_n}.$$

By Chevalley's result (Theorem 3.1), there exists a positive integer-valued function $s(n)$ tending to infinity such that $J_n \subseteq \mathfrak{m}^{s(n)}$. We first show that the canonical map

$$\psi : R \to \varprojlim_{n\geq 1} R/J_n$$

is surjective.

Let $\{a_1, \dots, a_n, \dots\}$ be a sequence in R such that the images $a_n (\mathrm{mod} J_n)$ form an element $\mathbf{a}$ in $\varprojlim_{n\geq 1} R/J_n$. Thus, for each n,

$$a_{n+1} - a_n \in J_n \subseteq \mathfrak{m}^{s(n)}.$$

Hence $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in the complete local ring $(R, \mathfrak{m})$. Let

$$a = \lim_{n\geq 1} a_n.$$

We show that $a - a_i \in J_i$ for each $i \geq 1$. Fix i. Then, for each $n \geq i$, since

$$a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{i+1} - a_i) + a_i$$

and $a_r - a_{r-1} \in J_i$ whenever $r > i$, we have

$$\bar{a_n} = \bar{a_i} \text{ in } R/J_i,$$

where $\bar{b}$ denotes the image of $b \in R$ in R/J_i. Thus

$$\bar{a} \, (= \lim_{n\geq 1} \bar{a_n}) = \bar{a_i} \text{ in } R/J_i,$$

as claimed. This implies that $\psi(a) = \mathbf{a}$, showing that ψ is surjective.

Now $J_r \subseteq J_{\lambda_r}$ for any $\lambda_r \in \Lambda$, and hence $\{J_n \mid n \geq 1\}$ is a cofinal subset of $\{J_\lambda \mid \lambda \in \Lambda\}$. Therefore the surjective map ψ induces the desired surjection $\phi\colon R \to \varprojlim_{\lambda\in\Lambda} R/J_\lambda$.

We now prove the proposition when Λ is of arbitrary cardinality.

Case 2. $|\Lambda| > \aleph_0$.

Let $\mathbf{b} = (b_\lambda)_{\lambda\in\Lambda}$ be an element of $\varprojlim_{\lambda\in\Lambda} R/J_\lambda$. For each λ in Λ, choose an element a_λ in R such that b_λ is the image of a_λ in R/J_λ. We shall show that there exists a in R such that $a - a_\lambda \in J_\lambda$ for every $\lambda \in \Lambda$, i.e., $\phi(a) = \mathbf{b}$.

For any countably infinite subset $N = \{\lambda_1, \dots, \lambda_r, \dots\}$ of Λ, we shall define an element a_N of R.

For each positive integer r, we first define, inductively, an index $\mu_r \in \Lambda$ and an ideal J_r as follows. Set $\mu_1 := \lambda_1$ and $J_1 := J_{\lambda_1}$. Having defined μ_r and J_r, choose an index $\mu_{r+1} \in \Lambda$ such that

$$J_{\mu_{r+1}} \subseteq J_{\mu_r} \cap J_{\lambda_1} \cap \cdots \cap J_{\lambda_{r+1}}.$$

Set $J_{r+1} := J_{\mu_{r+1}}$. For each $r \geq 1$, set $a_r := a_{\mu_r}$.

Now, for each $r \geq 1$, we have

$$J_{r+1} \subseteq J_r \subseteq m^{s(r)}$$

(where, as before, s is an integer-valued function tending to infinity given by Theorem 3.1). Thus

$$a_{r+1} - a_r \in J_r \subseteq m^{s(r)}$$

for every $r \geq 1$. Hence $\{a_r\}_{r\geq 1}$ is a Cauchy sequence in the complete local ring R. Now set $a_N := \lim_{r\geq 1} a_r$. The argument in the proof of Case 1 shows that for each positive integer r,

$$a_N - a_r \in J_r,$$

that is, the image of a_N in R/J_{μ_r} is b_{μ_r}. Now fix any $\lambda \in N$, say $\lambda = \lambda_r$. By construction,

$$J_r = J_{\mu_r} \subseteq J_{\lambda_r} = J_\lambda.$$

Thus

$$a_N - a_\lambda = (a_N - a_r) + (a_r - a_\lambda) \in J_\lambda;$$

that is, the image of a_N in R/J_λ is b_λ.

We now show that the element a_N constructed above is independent of the choice of the countable set N so that we may set $a = a_N$. To see this, first note that if N and M are two countably infinite subsets of Λ with $N \subseteq M$, then clearly $a_N = a_M$. Now, let N_1, N_2 be any two countably infinite subsets of Λ and let $N = N_1 \cup N_2$. Then $a_{N_1} = a_N = a_{N_2}$.

It is now easy to see that $\phi(a) = \mathbf{b}$. Fix $\lambda \in \Lambda$. We have to show that the image of a in R/J_λ is b_λ. Choose any countably infinite subset T of Λ containing λ and construct a_T. We have already seen that the image of a_T in R/J_λ is b_λ. Since $a = a_T$, the result follows. □

Throughout the rest of this section R will denote a Noetherian normal *local* domain with maximal ideal m and field of fractions K, and A will denote a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. We shall assume R to be *analytically irreducible*, i.e., the completion $\widehat{R}$ is an integral domain. We shall denote the ring $K \otimes_R \widehat{R}$ by R', which is an integral domain, because $R' \cong S^{-1}\widehat{R}$, where $S = R \setminus \{0\}$.

Let the notation be as in Section 2. Then $\{\widehat{R}/I_\Gamma \widehat{R}\}_{\Gamma\in\Sigma}$ forms an inverse system by Lemma 2.5 (2), and hence we can consider $\varprojlim_{\Gamma\in\Sigma} \widehat{R}/I_\Gamma \widehat{R}$. We now deduce a result which we will need for the next proposition. Recall the definition of the set Σ_0 (preceding Corollary 2.7).

Lemma 3.3. *Suppose that Σ_0 is infinite. Then the following assertions hold:*

(1) *For any infinite subset Ω of Σ_0, we have $\bigcap_{\Gamma\in\Omega} I_\Gamma \widehat{R} = (0)$.*
(2) *The canonical map $\phi\colon \widehat{R} \to \varprojlim_{\Gamma\in\Sigma} \widehat{R}/I_\Gamma \widehat{R}$ is an isomorphism.*

Proof. (1) For $P \in \Delta$, let P^* be a minimal prime ideal of $P\widehat{R}$. Then $\mathrm{ht}(P^*) = 1$ and $P^* \cap R = P$ by faithful flatness of $\widehat{R}$ over R. Note that if $\Gamma \in \Omega$ and $P \in \Gamma$, then $P \in \Delta_0$, so $I_\Gamma \subseteq P$ by (2.9). Hence, setting $\mathcal{W} = \bigcup_{\Gamma\in\Omega} \Gamma$, we have

$$\bigcap\nolimits_{\Gamma\in\Omega} I_\Gamma \widehat{R} \subseteq \bigcap\nolimits_{P\in\mathcal{W}} P\widehat{R} \subseteq \bigcap\nolimits_{P\in\mathcal{W}} P^* = (0),$$

because $\widehat{R}$ is a Noetherian domain and $\mathcal{W}$ is infinite.

(2) Since $\{I_\Gamma \mid \Gamma \in \Sigma_0\} = \{I_\Gamma \mid \Gamma \in \Sigma\}$ by Lemma 2.5 (1), we can identify $\varprojlim_{\Gamma\in\Sigma} \widehat{R}/I_\Gamma \widehat{R}$ with $\varprojlim_{\Gamma\in\Sigma_0} \widehat{R}/I_\Gamma \widehat{R}$. Hence the assertion follows from (1) and Proposition 3.2. □

Proposition 3.4. *In the above setup and with notation as in Theorem 2.3, the following assertions hold:*

(1) *There exists $c \in \widehat{R}$ such that $c - c_\Gamma \in I_\Gamma \widehat{R}$ for each $\Gamma \in \Sigma$.*
(2) *With $c \in \widehat{R}$ as in (1), $\widehat{R} \otimes_R A = \bigoplus_{n\geq 0} B_n = Sym_{\widehat{R}} B_1$, where for $n \geq 0$,*

$$B_n = \varinjlim_{\Gamma\in\Sigma} ({I_\Gamma}^n)^{-1} \widehat{R}\ (x - c)^n.$$

(3) *There exists a retraction $\varphi\colon \widehat{R} \otimes_R A \to \widehat{R}$.*

Proof. If Σ_0 is finite, then $A = R^{[1]}$ by Corollary 2.7, and hence the results follow. We assume that Σ_0 is infinite.

(1) This is an immediate consequence of Lemma 3.3 (2).

(2) Since $\widehat{R}$ is faithfully flat over R, for any R-submodule L of K, the natural map from $L \otimes_R \widehat{R}$ to $R'(= K \otimes_R \widehat{R})$ is injective, and hence we may identify $L \otimes_R \widehat{R}$ with $L\widehat{R}$. Likewise, for any R-subalgebra C of $K[x]$, we may identify $\widehat{R} \otimes_R C$ with its image in $R'[x]$.

Note that

$$\widehat{R} \otimes_R A_\Gamma = \bigoplus_{n\geq 0} ({I_\Gamma}^n)^{-1} \widehat{R}\ (x - c_\Gamma)^n$$

by (2.3). Since $c - c_\Gamma \in I_\Gamma \widehat{R}$, arguing as in the proof of Lemma 2.8, we have

$$\widehat{R} \otimes_R A_\Gamma = \bigoplus_{n\geq 0} ({I_\Gamma}^n)^{-1} \widehat{R}\ (x - c)^n. \tag{3.1}$$

On the other hand, since tensor product commutes with direct limit, setting $B = \widehat{R} \otimes_R A$ we have $B = \varinjlim_{\Gamma\in\Sigma} (\widehat{R} \otimes_R A_\Gamma)$ by Theorem 2.3, which implies that $B = \bigoplus_{n\geq 0} B_n$ by (3.1).

We shall show that $B = Sym_{\widehat{R}} B_1$. Note that $B_n = M_n\widehat{R}(x-c)^n$, where M_n is the R-module defined by (2.13). Since A is R-flat, it follows that B is $\widehat{R}$-flat, and hence so is the direct summand B_1. Since $B_1 \cong M_1\widehat{R} = M_1 \otimes_R \widehat{R}$, it follows that $M(= M_1)$ is R-flat by faithfully flat descent. Thus $M_n = M^n$ for each n by Lemma 1.5 (3), so $B_n = {B_1}^n$. Therefore $B = Sym_{\widehat{R}} B_1$, as desired.

(3) follows from (2). □

Remark 3.5. Let the notation and hypotheses be as in Proposition 3.4, and suppose that $A \neq R^{[1]}$.

(1) The element c is unique (cf. Remark 2.10) so that the retraction φ is also unique and given by $\varphi(1 \otimes x) = c$.

(2) Since A is faithfully flat over R, we have $A \hookrightarrow \widehat{R} \otimes_R A$, and hence we can define an R-algebra homomorphism

$$\psi \colon A \to \widehat{R}$$

by $\psi = \varphi|_A$. Note that $\psi(x) = c$. Also note that if $\rho \colon A \to \widehat{R}$ is an R-algebra homomorphism, then ρ can be extended to a retraction from $\widehat{R} \otimes_R A$ to $\widehat{R}$, and hence $\rho = \varphi|_A = \psi$ by uniqueness of φ.

(3) As shown in the proof of (2) of Proposition 3.4, the module M is flat over R, and hence so is $M_n\,(= M^n)$ for each n by Lemma 1.5.

Corollary 3.6. *With notation and hypotheses as in Proposition* 3.4*, suppose that* $A \neq R^{[1]}$*. Then A has a retraction to R if and only if $c \in R$.*

Proof. First suppose that $c \in R$. Then, for $f \in A$, taking $r(\neq 0) \in R$ with $rf \in R[x]$, we have $\varphi(rf) = r\varphi(f) \in R$, so

$$\varphi(A) \subseteq K \cap \widehat{R} = R.$$

Thus $\varphi|_A(= \psi)$ gives a retraction from A to R.

Conversely, suppose that there exists a retraction $\rho \colon A \to R$. Then ρ extends to a retraction from $\widehat{R} \otimes_R A$ to $\widehat{R}$, which implies that $\rho = \varphi|_A$ by uniqueness of φ. Thus $c = \varphi(x) = \rho(x) \in R$, as desired. □

Theorem B mentioned in the Introduction (and stated below) follows from Theorem 2.11 and Proposition 3.4.

Theorem 3.7. *Let R be a Noetherian complete normal local domain and A a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every height-one prime ideal P of R. Then the following conditions are equivalent:*

(i) *A is Noetherian.*
(ii) *A is finitely generated over R.*
(iii) $A = R^{[1]}$.

As a consequence of Proposition 3.4, we also see the following.

Corollary 3.8. *With notation and hypotheses as in Proposition* 3.4*, let* $M(= M_1)$ *and* E_n $(n \geq 0)$ *be the R-modules defined by* (2.13) *and* (2.14)*, respectively. Then the following properties hold:*

(1) *$E_n/E_{n-1} \cong M^n$ for each $n \geq 1$. In particular, E_n/E_{n-1} is R-flat for each n.*

(2) E_n *is* R*-flat for each* $n \geq 0$.
(3) $A = R[E]$, *where* $E = E_1$.

Proof. (1) Since $E_n/E_{n-1} \cong M_n$ by Remark 2.12, the result follows by Remark 3.5 (3).

(2) Since $E_0 = R$, the assertion follows from (1) by induction on n.

(3) By Proposition 3.4, $\widehat{R} \otimes_R A = Sym_{\widehat{R}} B_1$. Thus $\widehat{R} \otimes_R A$ is generated over $\widehat{R}$ by $\widehat{R} \otimes_R E$ $(= B_0 \oplus B_1)$. Since A is faithfully flat over R, from this it follows that A is generated over R by E. □

The following result is a consequence of a theorem of Vasconcelos [16, Theorem 3.1].

Proposition 3.9. *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain and* N *a flat* R*-module of rank one (i.e.,* $K \otimes_R N \cong K$*, where* K *is the field of fractions of* R*). Then either* $N \cong R$ *or* $N = \mathfrak{m}N$.

We now prove Theorem C.

Theorem 3.10. *Let* $(R, \mathfrak{m})$ *be an analytically irreducible Noetherian normal local domain with residue field* k*, and* A *a faithfully flat* R*-algebra such that* $A_P = {R_P}^{[1]}$ *for every prime ideal* P *in* R *of height one. Then, for every prime ideal* Q *of* R*,* $QA \in \operatorname{Spec} A$*. In fact, for each prime ideal* Q *of* R*, either* $A_Q = {R_Q}^{[1]}$ *or the fibre ring* $k(Q) \otimes_R A = k(Q)$*. In particular, if* $k \subsetneqq A/\mathfrak{m}A$*, then* $A = R^{[1]}$.

Proof. Let M and $E(= E_1)$ be as in Corollary 3.8. Then M is a flat R-module of rank one by Remark 3.5 (3), and $A = R[E]$ by Corollary 3.8. Let $Q \in \operatorname{Spec} R$. Since $A/QA \hookrightarrow k(Q) \otimes_R A$ by flatness of A, to show that $QA \in \operatorname{Spec} A$ it suffices to show that $k(Q) \otimes_R A$ is a domain.

We have $A_Q = R_Q[E_Q]$. If $QM_Q = M_Q$, then $E_Q = R_Q + QE_Q$, and it follows that the canonical map $k(Q) \to k(Q) \otimes_R A$ is a surjection and hence an isomorphism. On the other hand, if $QM_Q \subsetneqq M_Q$, then $M_Q \cong R_Q$ by Proposition 3.9, and therefore A_Q is finitely generated over R_Q. Hence $A_Q = {R_Q}^{[1]}$ by Theorem 2.11. This completes the proof. □

Corollary 3.11. *Under the hypotheses of Theorem* 3.10*, if* $A \neq R^{[1]}$*, then, for each* $n \geq 1$*, there exists an isomorphism* $R/\mathfrak{m}^n R \to A/\mathfrak{m}^n A$*. Consequently, we have an isomorphism*

$$\varprojlim_{n \geq 1} A/\mathfrak{m}^n A \cong \varprojlim_{n \geq 1} R/\mathfrak{m}^n R = \widehat{R}.$$

In particular, for any $a \in A$*, there exists a Cauchy sequence* $\{a_n\}_{n \geq 1}$ *in* R *such that* $a_{n+1} - a_n \in \mathfrak{m}^n$ *and* $a - a_{n+1} \in \mathfrak{m}^{n+1} A$.

Proof. Since A is faithfully flat over R, the canonical map

$$\phi_n : R/\mathfrak{m}^n R \to A/\mathfrak{m}^n A$$

is injective for each natural number n. On the other hand, since $A \neq R^{[1]}$, we have $R/\mathfrak{m} = A/\mathfrak{m}A$ by Theorem 3.10. Hence $A = R + \mathfrak{m}A$, which implies that $A = R + \mathfrak{m}^n A$ for every $n > 0$. Thus the map $\phi_n : R/\mathfrak{m}^n R \to A/\mathfrak{m}^n A$ is an isomorphism. □

Remark 3.12. With notation as in Theorem 3.10, suppose that $A \neq R^{[1]}$. Then, since there exists a natural homomorphism $A \to \varprojlim_{n\geq 1} A/m^n A$, from Corollary 3.11 we have an R-algebra homomorphism $A \to \widehat{R}$, which coincides with $\varphi|_A(=\psi)$ by Remark 3.5, where φ is the retraction from $\widehat{R} \otimes_R A$ to $\widehat{R}$. In Section 5 we shall study properties of the homomorphism $\psi : A \to \widehat{R}$.

Remark 3.13. With notation as in Corollary 3.8, the increasing filtration $\{E_n\}_{n\geq 0}$ gives rise to a graded R-algebra

$$Gr\,(A) = \bigoplus_{n\geq 0} E_{n+1}/E_n.$$

Then $Gr\,(A) \cong Sym_R\, M$ by Corollary 3.8. Thus, from any faithfully flat R-algebra A which is locally $\mathbb{A}^1$ in codimension-one, one can construct a faithfully flat R-algebra *with retraction* which is locally $\mathbb{A}^1$ in codimension-one.

We end this section with a result on flatness which is a sort-of converse to Proposition 3.9 for two-dimensional normal domains. This result is a consequence of the following elementary criterion for flatness.

Lemma 3.14. *Let N be a torsion-free module over an integral domain R. Then N is flat over R if and only if N/xN is flat over R/xR for every non-zero element x in R.*

Proposition 3.15. *Let (R, m) be a Noetherian normal local domain of dimension two and Δ the set of prime ideals in R of height one. Suppose that N is a torsion-free R-module such that*

(I) $N_P \cong R_P$ *for every* $P \in \Delta$.
(II) $N = \bigcap_{P\in\Delta} N_P$.
(III) $N = mN$.

Then N is a flat R-module.

Proof. By Lemma 3.14, it is enough to show that N/xN is flat over R/xR for every non-zero element x in m.

Fix $x(\neq 0) \in m$. Let $\{P_1, \ldots, P_t\}$ be the set of associated prime ideals of xR. Since R is normal, height of P_i is one for each i $(1 \leq i \leq t)$. Let $S = R\backslash(\bigcup_{1\leq i\leq t} P_i)$.

Note that R/xR is a one-dimensional Cohen-Macaulay ring. Let y be an element of m which is a non-zero divisor of R/xR; that is, x, y form a regular sequence in R. Hence, by hypothesis (II), the R-endomorphism $\mu_y : N/xN \to N/xN$, defined by $\mu_y(f) = yf$, is injective. Since the ideal $J := (x, y)R$ is m-primary (being of height 2), by hypothesis (III), we get $JN = N$. This shows that μ_y is actually an isomorphism. It follows that $S^{-1}(N/xN) = N/xN$. By hypothesis (I), $S^{-1}N \cong S^{-1}R$. Hence $N/xN \cong S^{-1}(R/xR)$, the total quotient ring of R/xR. Thus N/xN is flat over R/xR. □

4. Construction of algebras locally $\mathbb{A}^1$ in codimension-one

In this section we shall consider the converse of Theorem 2.3. Let R be a Noetherian normal domain with field of fractions K, and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Let the notation be as in Section 2. Then, for $P \in \Delta$, by (2.8), we have

$$A_P = R_P\left[\frac{x - c_\Gamma}{p^{e_P}}\right]$$

with $e_P = v_P(d_\Gamma)$, where Γ is any finite subset of Δ containing P, v_P is the discrete valuation of K whose valuation ring is R_P, and p is a uniformizing parameter of the DVR R_P. Thus we have a function

$$e : \Delta \to \mathbb{Z}_{\geq 0}$$

defined by $e(P) := e_P$. Then it follows from (2.9) that

$$I_\Gamma = {P_1}^{(e(P_1))} \cap \cdots \cap {P_n}^{(e(P_n))}, \tag{4.1}$$

where $\Gamma = \{P_1, \ldots, P_n\} \subseteq \Delta$. On the other hand, recall (Remark 2.10) that we can form the inverse limit

$$\widetilde{R}(e) = \varprojlim_{\Gamma \in \Sigma} R/I_\Gamma, \tag{4.2}$$

and $\{c_\Gamma\}_{\Gamma \in \Sigma}$ defines an element $\mathbf{c}$ of $\widetilde{R}(e)$.

Thus, for a faithfully flat R-algebra A which is locally $\mathbb{A}^1$ in codimension-one and an element x in A such that $R[x] \subseteq A \subseteq K[x]$, we have a function e and an element $\mathbf{c}$.

Definition 4.1. We call e the *power function* for A and $\widetilde{R}(e)$ the *filter ring* corresponding to e. The element $\mathbf{c}$ is called the *residue* for A.

Theorem 2.3 asserts that once the element (generic variable) x is chosen so that $R[x] \subseteq A \subseteq K[x]$, A is determined by the power function e and the residue $\mathbf{c}$.

Now, we consider a converse. Let there be given a function $e : \Delta \to \mathbb{Z}_{\geq 0}$. Then, for a finite subset $\Gamma = \{P_1, \ldots, P_n\}$ of Δ, we define the ideal I_Γ by (4.1) and the ring $\widetilde{R}(e)$ by (4.2). Note that I_Γ is a divisorial ideal of R. We denote by d_Γ an element of R such that $I_\Gamma({S_\Gamma}^{-1}R) = d_\Gamma({S_\Gamma}^{-1}R)$, i.e., $I_\Gamma = R \cap d_\Gamma({S_\Gamma}^{-1}R)$, where $S_\Gamma := R \setminus (\bigcup_{P \in \Gamma} P)$.

Let $\mathbf{c}$ be an element of $\widetilde{R}(e)$. Then, for each $\Gamma \in \Sigma$, we can choose $c_\Gamma \in R$ so that $\{c_\Gamma\}_{\Gamma \in \Sigma}$ defines $\mathbf{c}$ in $\widetilde{R}(e)$, namely,

$$\mathbf{c} = (\overline{c}_\Gamma)_{\Gamma \in \Sigma} \in \prod_{\Gamma \in \Sigma} R/I_\Gamma,$$

where $\overline{c}_\Gamma$ denotes the image of c_Γ in R/I_Γ. For $\Gamma = \{P\}$ with $P \in \Delta$, we set $c_{\{P\}} = c_P$. Note that if $\Gamma_1 \subseteq \Gamma_2$, then under the natural map

$$\phi_{\Gamma_1, \Gamma_2} \colon R/I_{\Gamma_2} \to R/I_{\Gamma_1}$$

we have $\phi_{\Gamma_1,\Gamma_2}(\overline{c}_{\Gamma_2}) = \overline{c}_{\Gamma_1}$, and hence $c_{\Gamma_2} - c_{\Gamma_1} \in I_{\Gamma_1}$. In particular, if $P \in \Gamma$, then $c_\Gamma - c_P \in I_{\{P\}} = P^{(e(P))}$.

Let x be an indeterminate over R. Corresponding to e, $\mathbf{c}$ and x, we now define, for each finite subset Γ of Δ, an R-subalgebra B_Γ of $K[x]$ by

$$B_\Gamma = \bigoplus_{n \geq 0} ({I_\Gamma}^n)^{-1}(x - c_\Gamma)^n. \tag{4.3}$$

Lemma 4.2. *The following assertions hold for the rings B_Γ defined above:*

(1) $B_\Gamma = R\left[\dfrac{x - c_\Gamma}{d_\Gamma}\right] \cap R_\Gamma[x]$.

(2) $(B_\Gamma)_P = R_P[x]$ *for* $P \in \Delta \setminus \Gamma$ *and* $(B_\Gamma)_P = R_P\left[\dfrac{x - c_P}{p^{e(P)}}\right]$ *for* $P \in \Gamma$, *where* p *is a uniformizing parameter of* R_P.

(3) $B_\Gamma = \bigcap_{P\in\Delta}(B_\Gamma)_P$.
(4) *If* $\Gamma_1 \subseteq \Gamma_2$, *then* $B_{\Gamma_1} \subseteq B_{\Gamma_2}$.

Proof. Assertion (1) is an immediate consequence of Lemma 2.2. Note that if $P \in \Delta \setminus \Gamma$, then $(R_\Gamma)_P = R_P$, and if $P \in \Gamma$, then $c_\Gamma - c_P \in P^{(e(P))}$ and $e(P) = v_P(d_\Gamma)$. Thus (2) follows from (1), while (3) follows from (1) and (2). (4) follows from (2) and (3). □

It follows from Lemma 4.2 that, together with inclusion maps, $\{B_\Gamma\}_{\Gamma\in\Sigma}$ forms a direct system indexed by Σ. We define an R-algebra $R(x, e, \mathbf{c})$ by

$$R(x, e, \mathbf{c}) = \varinjlim_{\Gamma\in\Sigma} B_\Gamma. \tag{4.4}$$

Thus $B := R(x, e, \mathbf{c})$ is an R-algebra such that $R[x] \subseteq B \subseteq K[x]$.

Moreover, since tensor product commutes with direct limit, we have, for each $P \in \Delta$,

$$B_P = (\varinjlim B_\Gamma) \otimes_R R_P = \varinjlim (B_\Gamma)_P = R_P\left[\frac{x - c_P}{p^{e(P)}}\right] \tag{4.5}$$

by Lemma 4.2. Hence B is an R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Further, $B = \bigcap_{P\in\Delta} B_P$ by Lemma 4.2 (3) and Lemma 1.4.

To summarise, we have obtained the following:

Proposition 4.3. *Let* $e\colon \Delta \to \mathbb{Z}_{\geq 0}$ *be a function, let* $\mathbf{c} = (\overline{c}_\Gamma)_{\Gamma\in\Sigma}$ *be an element of the filter ring* $\widetilde{R}(e)$, *and let* x *be an indeterminate over* R. *Then the* R-*algebra*

$$B := R(x, e, \mathbf{c}) = \varinjlim_{\Gamma\in\Sigma} B_\Gamma, \quad \text{where } B_\Gamma = \bigoplus_{n\geq 0} ({I_\Gamma}^n)^{-1}(x - c_\Gamma)^n,$$

is such that $R[x] \subseteq B \subseteq K[x]$ *and satisfies the following conditions:*
(1) $B_P = {R_P}^{[1]}$ *for every* $P \in \Delta$.
(2) $B = \bigcap_{P\in\Delta} B_P$.

Remark 4.4. (1) With notation as in Proposition 4.3, it follows from (4.5) that

$$B_P = R_P\left[\frac{x - c_\Gamma}{p^{e(P)}}\right] \tag{4.6}$$

for every $P \in \Delta$ and $\Gamma \in \Sigma$ with $P \in \Gamma$, because $c_\Gamma - c_P \in P^{(e(P))}$.

(2) Let A be a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one, and let x be a generic variable, i.e., $R[x] \subseteq A \subseteq K[x]$. Then, regarding x as a variable over R, we may write $A = R(x, e, \mathbf{c})$ by Theorem 2.3, where $e, \mathbf{c}$ are respectively the corresponding power function and the residue defined in Definition 4.1.

Further properties of B will be discussed in Section 7. We define two subsets of Δ which will play a crucial role in our subsequent discussions and examples.

Δ_1: The set of principal prime ideals of R.

Given $e : \Delta \to \mathbb{Z}_{\geq 0}$, set

$\Delta_0 := \{P \in \Delta \mid e(P) > 0\}$.

Throughout the rest of this section $(R, \mathfrak{m})$ will denote an analytically irreducible Noetherian normal local domain and $e, \mathbf{c}, x, B$ will be as above.

It follows from (4.1) that if $\Gamma_1 \subseteq \Gamma_2$, then $I_{\Gamma_1} \supseteq I_{\Gamma_2}$, so $\{(I_\Gamma)^{-1}\}_{\Gamma\in\Sigma}$ forms a directed system indexed by Σ. Thus, corresponding to the function e, as in Section

2, we can associate an R-module $M = M(e)$ defined to be the following submodule of K:

$$M(= M(e)) := \varinjlim_{\Gamma\in\Sigma}(I_\Gamma)^{-1}(= \bigcup_{\Gamma\in\Sigma}(I_\Gamma)^{-1}). \tag{4.7}$$

Recall that M satisfies the properties listed in Lemma 1.5. We also define an R-module $E = E(B)$ by

$$E(= E(B)) := \{f \in B \mid \deg_x f \leq 1\} \tag{4.8}$$

as in Remark 2.12.

The R-algebra B need not be faithfully flat over R in general. (Note that B_Γ may not be faithfully flat over R.) We shall investigate conditions for B to be faithfully flat over R. We first observe:

Proposition 4.5. *B is faithfully flat over R if and only if $M(= M(e))$ is flat over R.*

Proof. If B is faithfully flat over R, then M is R-flat by Remark 3.5 (3).

Conversely, suppose that M is R-flat. Using the arguments in the proof of Proposition 3.4, we can find $c \in \widehat{R}$ such that

$$\widehat{R} \otimes_R B = \bigoplus_{n\geq 0} M_n\widehat{R}(x - c)^n \cong \bigoplus_{n\geq 0} M_n\widehat{R},$$

where $M_n = \varinjlim_{\Gamma\in\Sigma}({I_\Gamma}^n)^{-1}$ as in (2.13). Then, by Lemma 1.5, M_n is R-flat for each n. Thus $M_n\widehat{R} \cong M_n \otimes_R \widehat{R}$ is $\widehat{R}$-flat for each n, which implies that $\widehat{R} \otimes_R B$ is faithfully flat over $\widehat{R}$. Hence B is faithfully flat over R by faithfully flat descent. □

Remark 4.6. The above result shows that faithful flatness of $B = R(x, e, \mathbf{c})$ depends only on the function e; it is independent of the choice of $\mathbf{c}$.

We also observe:

Proposition 4.7. *The following assertions hold for the modules $M(e)$ and $E(B)$ defined above:*

1. *Given a flat R-module N such that $N_P \cong R_P$ for each $P \in \Delta$ and an element $x \in N$ such that $S^{-1}N = Kx$ (where $S = R \setminus \{0\}$), there exists a function $e\colon \Delta \to \mathbb{Z}_{\geq 0}$ such that $M(e) \cong N$.*
2. *Given N and x as in (1) and an element $\xi \in Ext^1_R(N, R)$, there exist $e\colon \Delta \to \mathbb{Z}_{\geq 0}$ and $\mathbf{c} \in \widetilde{R}(e)$ such that ξ corresponds to the short exact sequence $0 \to R \to E(B) \to M(e) \to 0$, where $B := R(x, e, \mathbf{c})$.*

Proof. Since (1) follows from (2) by considering the case $\xi = 0$, i.e., $E(B) = R \oplus N$, it suffices to prove (2). The assertion is in fact the module version of Theorem 2.3, and the proof is essentially the same as in Section 2. We shall give an outline of the proof.

Let

$$0 \to R \xrightarrow{\iota} E \xrightarrow{\rho} N \to 0 \tag{4.9}$$

be the short exact sequence corresponding to ξ. Let w be an element of E such that $\rho(w) = x$, and let $u = \iota(1)$. Note that E is R-flat, because both R and N are R-flat. On the other hand, since $N(\cong E/\iota(R))$ is R-flat, it follows that $\iota(R)(= Ru)$ is a pure submodule of E (cf. [12, p. 54]) so that, for every ideal I of R, the map

$R/I \to E/IE$ induced by ι is injective. Hence E is faithfully flat over R. Moreover, if $a \in R$ is an element such that $au \in IE$, then $a \in I$.

Now let $\Gamma \in \Sigma$ and set $S_\Gamma = R \setminus (\bigcup_{P\in\Gamma} P)$ as in Section 2. Then ${S_\Gamma}^{-1}N$ is a free ${S_\Gamma}^{-1}R$-module of rank one, and hence the induced short exact sequence

$$0 \to {S_\Gamma}^{-1}R \longrightarrow {S_\Gamma}^{-1}E \longrightarrow {S_\Gamma}^{-1}N \to 0$$

splits. Thus if $y_\Gamma \in N$ is such that it is a generator of ${S_\Gamma}^{-1}N$ and $f_\Gamma \in E$ is a pre-image of y_Γ, then ${S_\Gamma}^{-1}E$ is a free ${S_\Gamma}^{-1}R$-module of rank two with basis u, f_Γ. Now write w as a linear combination of u and f_Γ, viz.,

$$w = \frac{d_\Gamma}{s} f_\Gamma + \frac{d'}{s} u,$$

where $d_\Gamma, d' \in R$ and $s \in S_\Gamma$. Then $d'u \in (s, d_\Gamma)E$ and therefore $d' \in (s, d_\Gamma)R$; say $d' = c_\Gamma s - d_\Gamma c'$ for some $c_\Gamma, c' \in R$. It then follows that

$$w = \frac{d_\Gamma}{s}(f_\Gamma - c'u) + c_\Gamma u.$$

Note that $\rho(u) = 0$ so that $\rho(w - c_\Gamma u) = x$ and $\rho(f_\Gamma - c'u) = y_\Gamma$. Let

$$E_\Gamma = {S_\Gamma}^{-1}E \cap (R_\Gamma u + R_\Gamma w),$$

where $R_\Gamma = \bigcap_{Q\in\Delta\setminus\Gamma} R_Q$. Then, setting $I_\Gamma = R \cap d_\Gamma({S_\Gamma}^{-1}R)$, we have

$$E_\Gamma = Ru \oplus {I_\Gamma}^{-1}(w - c_\Gamma u)$$

and $E = \varinjlim_{\Gamma\in\Sigma} E_\Gamma$. Moreover, we have a function $e\colon \Delta \to \mathbb{Z}_{\geq 0}$ such that $I_\Gamma = \bigcap_{P\in\Gamma} P^{(e(P))}$ and such that $(c_\Gamma)_{\Gamma\in\Sigma}$ defines an element $\mathbf{c}$ of $\widetilde{R}(e)$. Let $B := R(x, e, \mathbf{c})$. Then $E \cong E(B)$ and $N \cong M(e)$. We have thus obtained the desired function e and $\mathbf{c} \in \widetilde{R}(e)$. Note that B is faithfully flat over R by Proposition 4.5. □

In view of Proposition 4.5, one would like to know

Question. When is $M = \bigcup_\Gamma (I_\Gamma)^{-1}$ flat over R?

When R is factorial, and thus each I_Γ is free, M is always a flat module, being the direct limit of free modules. Moreover, if Δ_0 is finite, then M, being finitely generated, is actually free.

We shall now give a partial answer to the above question when R is a non-factorial normal domain. Under our hypotheses (stated before Proposition 4.5), we have the following lemma which will help formulate a sufficient condition for flatness of M.

Lemma 4.8. *If the function $e : \Delta \to \mathbb{Z}_{\geq 0}$ is such that $e(P) > 0$ for infinitely many principal prime ideals P of R (that is, if $\Delta_1 \cap \Delta_0$ is infinite), then $\mathfrak{m}M = M$.*

Proof. Let $y \in M$. Then there exists Γ such that $y \in {I_\Gamma}^{-1}$. Since Γ is finite and $\Delta_1 \cap \Delta_0$ is infinite, there exists a prime element f in Δ_0 such that $f \notin P$ for any $P \in \Gamma$.

Let $r = e(fR)(\geq 1)$ and let $\Gamma_1 = \Gamma \cup fR$. Then $I_{\Gamma_1} = I_\Gamma \cap f^r R = I_\Gamma f^r$. Thus $(I_{\Gamma_1})^{-1} = (I_\Gamma)^{-1} f^{-r}$. Hence $yf^{-r} \in (I_{\Gamma_1})^{-1} \subseteq M$. Thus $y \in f^r M \subseteq \mathfrak{m}M$. □

Consequently, we have the following criterion:

Proposition 4.9. *Let $(R, \mathfrak{m})$ be a two-dimensional analytically irreducible Noetherian normal local domain and $e : \Delta \to \mathbb{Z}_{\geq 0}$ a function such that $e(P) > 0$ for infinitely many principal prime ideals P of R. Then the module $M = \bigcup_{\Gamma \in \Sigma} (I_\Gamma)^{-1}$ is flat over R, and hence the R-algebra $B = R(x, e, \mathbf{c})$ is faithfully flat over R for any $\mathbf{c}$.*

Proof. The assertion follows from Proposition 3.15, Lemma 1.5, Lemma 4.8 and Proposition 4.5. □

The following lemma, which is a simple consequence of "Swan's Bertini theorem", gives a condition for R to have infinitely many principal prime ideals (see [3, Theorem 2.11] for a more detailed version).

Lemma 4.10. *Let k be an algebraically closed field and C an affine normal domain over k of dimension at least 2. Let $\mathfrak{p}$ be a prime ideal of C of height at least 2. For every $a_1, a_2 \in \mathfrak{p}$ with $\operatorname{ht}(a_1, a_2) \geq 2$, there exists $b \in C$ such that $a_1 + ba_2$ is a prime element in C. In particular, $\mathfrak{p}$ contains infinitely many principal prime ideals.*

We shall now use the above results to construct flat modules of rank one which give a negative answer to Q.3 in the Introduction.

Proposition 4.11. *Let k be an algebraically closed field and $(R, \mathfrak{m})$ a normal local domain of dimension two such that R is a localisation of an affine normal domain over k. If R is not factorial, then there exists a flat R-module N such that*

(1) *$N_P \cong R_P$ for every height-one prime ideal P in R.*
(2) *N is not a direct limit of cyclic submodules.*

Proof. Since R is not factorial, there exists a prime ideal Q in R of height one which is not principal. Fix Q. Define $e : \Delta \to \mathbb{Z}_{\geq 0}$ by $e(P) = 1$ if either P is principal or if $P = Q$ and $e(P) = 0$ otherwise. Let $N = M(e)$. By Lemma 4.10, there exist infinitely many principal prime ideals in R so that N is R-flat by Proposition 4.9. Moreover, N satisfies (1) by Lemma 1.5 (1). We shall prove that N satisfies (2).

Let T denote the multiplicatively closed set generated by all prime elements of R. Then $T^{-1}N = (QT^{-1}R)^{-1}$. By construction, $QT^{-1}R$ is a prime ideal of $T^{-1}R$ which is not principal; that is, the $T^{-1}R$-module $(QT^{-1}R)^{-1}$ is not cyclic. But, being an ideal of the Dedekind domain $T^{-1}R$, $T^{-1}N$ is a finitely generated $T^{-1}R$-module. Thus $T^{-1}N$ cannot be a direct limit of cyclic submodules. Therefore, N itself cannot be a direct limit of cyclic submodules. □

Remark 4.12. Take N as in Proposition 4.11, and set $A := Sym_R N$. Then A is locally $\mathbb{A}^1$ in codimension-one since $N_P \cong R_P$ for every height-one prime ideal P in R and A is faithfully flat over R since N is flat over R. Since $T^{-1}A = Sym_{T^{-1}R}\,(T^{-1}N)$ and $T^{-1}N$ is a *finitely generated* non-cyclic rank-one projective module over $T^{-1}R$, it follows that $T^{-1}A \neq (T^{-1}R)^{[1]}$ (cf. [7, Lemma 1.3]). Therefore, $T^{-1}A$, being finitely generated, cannot be a direct limit of polynomial algebras over $T^{-1}R$. Hence A cannot be a direct limit of polynomial algebras over R.

Thus the result [5, Theorem 4.6] does not hold for non-factorial normal domains, not even for normal affine spots. Note that A has a retraction to R and hence, by Theorem 2.11, A is non-Noetherian. In fact, A is non-Krull by a result we shall prove later (Theorem 7.12; see also Remark 5.5).

Over the ring R of Proposition 4.11, one can also construct a Krull domain A which is locally $\mathbb{A}^1$ in codimension-one but not a direct limit of polynomial algebras (Proposition 6.1). For a specific R, we shall also give an example where A could be Noetherian (Example 6.2).

We shall end this section with a few observations on the filter ring $\widetilde{R}(e)$. Suppose that Δ_0 is an infinite set. Then, by Lemma 3.3, the canonical ring homomorphism $\phi : \widehat{R} \to \varprojlim_{\Gamma\in\Sigma} \widehat{R}/I_\Gamma \widehat{R}$ is an isomorphism. Hence we can define a ring homomorphism $\sigma\colon \widetilde{R}(e) \to \widehat{R}$ as the composite of the maps

$$\widetilde{R}(e) = \varprojlim_{\Gamma\in\Sigma} R/I_\Gamma \to \varprojlim_{\Gamma\in\Sigma} \widehat{R}/I_\Gamma\widehat{R} \cong \widehat{R}. \tag{4.10}$$

Then σ is clearly injective.

Lemma 4.13. *Suppose that Δ_0 is infinite. Let $\mathbf{c}$ be an element of $\widetilde{R}(e)$, let x be an indeterminate over R, and let $A = R(x, e, \mathbf{c})$. Then $\psi(x) = \sigma(\mathbf{c})$, where $\psi\colon A \to \widehat{R}$ is the map defined in Remark* 3.5 *and σ is the map defined by* (4.10).

Proof. Let $c = \sigma(\mathbf{c})$. Then it follows from the definition of σ that $c - c_\Gamma \in I_\Gamma\widehat{R}$ for each $\Gamma \in \Sigma$. Thus $\psi(x) = c$ (cf. Remark 3.5). □

Lemma 4.14. *Suppose that $\Delta_0 = \{P_1, P_2, \dots\}$ is a countably infinite set. Set $\Gamma_n = \{P_1, \dots, P_n\}$ and $I_n = I_{\Gamma_n} = {P_1}^{e(P_1)} \cap \cdots \cap {P_n}^{e(P_n)}$. Then*

$$\widetilde{R}(e) = \varprojlim_{n\geq 1} R/I_n.$$

Moreover, if $\{c_n\}_{n\geq 1}$ is a sequence of elements in R such that $\mathbf{c} = (\bar{c_n})_{n\geq 1}$ is an element of $\widetilde{R}(e)$, where $\bar{c_n}$ denotes the image of c_n in R/I_n, then $\{c_n\}_{n\geq 1}$ is a Cauchy sequence with respect to the $\mathfrak{m}$-adic topology.

Proof. For any subset Γ of Δ, there exists a positive integer n such that $I_n \subseteq I_\Gamma$, and hence $\{I_n \mid n \geq 1\}$ is a cofinal subset of $\{I_\Gamma \mid \Gamma \in \Sigma\}$. Therefore the filter ring $\widetilde{R}(e)(= \varprojlim_{\Gamma\in\Sigma} R/I_\Gamma)$ may be identified with $\varprojlim_{n\geq 1} R/I_n$.

By Theorem 3.1, Lemma 3.3 and faithful flatness of $\widehat{R}$ over R, we have $I_n \subseteq \mathfrak{m}^{s(n)}$ for some positive integer-valued function $s(n)$ which tends to infinity with n. If $i < j$, then $c_j - c_i \in I_i \subseteq \mathfrak{m}^{s(i)}$, and hence $\{c_n\}_{n\geq 1}$ is a Cauchy sequence. □

Remark 4.15. With the hypothesis as in Lemma 4.14, let $\mathbf{c} = (\bar{c_n})_{n\geq 1}$ be an element of $\widetilde{R}(e)$, and let c denote the limit of the Cauchy sequence $\{c_n\}_{n\geq 1}$ in $\widehat{R}$. Then $\sigma(\mathbf{c}) = c$, where $\sigma\colon \widetilde{R}(e) \to \widehat{R}$ is the map defined above. Hence, setting $A = R(x, e, \mathbf{c})$, we have $\psi(x) = c$ by Lemma 4.13.

We shall now give an example to show that R need not be algebraically (or even integrally) closed in $\widetilde{R}(e)$.

Lemma 4.16. *Let $(R, \mathfrak{m})$ be a complete local ring containing the field of rationals $\mathbb{Q}$ and let $v \in \mathfrak{m}$. Let $f(T) = T^2 - (1 + v) \in R[T]$. Then $f(T)$ has precisely two roots $\alpha, \beta \in R$ such that $\alpha - \beta \in R^*$, where R^* denotes the set of all units in R.*

Proof. Since R is complete, $\mathbb{Q} \subseteq R$ and $v \in \mathfrak{m}$, there exists $\alpha \in R$ such that $\alpha^2 = 1 + v$. Note that $\alpha \notin \mathfrak{m}$, because $v \in \mathfrak{m}$. Hence, setting $\beta = -\alpha$, we have $\alpha - \beta = 2\alpha \notin \mathfrak{m}$ and $f(T) = (T - \alpha)(T - \beta)$.

If $f(\gamma) = 0$ for $\gamma \in R$, then $(\gamma - \alpha)(\gamma - \beta) = 0$. Since $\alpha - \beta \in R^*$, either $\gamma - \beta \in R^*$ or $\gamma - \alpha \in R^*$. Thus either $\gamma = \alpha$ or $\gamma = \beta$. □

Lemma 4.17. *Let $(R, \mathfrak{m})$ be a complete local ring containing the field of rationals $\mathbb{Q}$ and let $v \in \mathfrak{m}$. Let $I \subseteq J$ be two proper ideals of R. Suppose that t is an element of R such that $t^2 - (1+v) \in J$. Then there exists $w \in R$ such that $w^2 - (1+v) \in I$ and $w - t \in J$.*

Proof. Applying Lemma 4.16 to R/I, we get $w_1, w_2 \in R$ such that ${w_i}^2 - (1+v) \in I$ for $i = 1, 2$ and $w_1 - w_2 \in R^*$. Then either $w_1 + t \in R^*$ or $w_2 + t \in R^*$, so we may assume $w_1 + t \in R^*$. Since ${w_1}^2 - t^2 \in J$, it then follows that $w_1 - t \in J$. □

Example 4.18. Let k be a field of characteristic zero. Let $R = k[[u]][v]_{(u,v)}$, where u and v are indeterminates over k, and let $\mathfrak{m}$ denote the maximal ideal of R. For each $n \geq 0$, set $p_n := v + nu$, $q_n := p_0 p_1 \cdots p_n$ and $I_n := (q_n)R$. Let $R_n = R/I_n$ and $\widetilde{R} = \varprojlim_{n \geq 1} R_n$. Thus, by Lemma 4.14, $\widetilde{R}$ is the filter ring $\widetilde{R}(e)$ corresponding to the function e on Δ defined by $e(p_n R) = 1$ for all n and $e(P) = 0$ otherwise.

Recall that $\widetilde{R}$ is a subring of the completion $\widehat{R}$ of R (cf. (4.10)). Let v_n denote the image of v in R_n. Let $c_0 = 1$. By Lemma 4.17, for each $n \geq 1$, one can define $c_n \in R_n$ such that c_n is a root of $T^2 - (1 + v_n) \in R_n[T]$ and the image of c_n in R_{n-1} is c_{n-1}. Let c be the element of $\widetilde{R}$ defined by the sequence $\{c_n\}_{n \geq 0}$. Then it follows that $c^2 = v + 1$.

Now note that the polynomial $T^2 - (v + 1) \in R[T]$ does not have a root in R (since its image in $(R/uR)[T]$ does not have a root in R/uR). Therefore, $c \notin R$.

Thus c is an element of $\widetilde{R}$ which is integral over R but which does not belong to R.

5. On R-algebras A embedded in the completion $\widehat{R}$

Let R be a Noetherian normal domain with field of fractions K and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. As before, we denote by Δ the set of prime ideals in R of height one, and by x a generic variable for A, namely, x is an element of A such that $K \otimes_R A = K[x]$.

We have seen (Theorem 3.7) that when R is a complete local domain and A is Noetherian, then A must be trivial. In this section, we shall see (Proposition 5.3) that when R is an analytically irreducible local domain and A is a non-trivial Noetherian algebra, then A is necessarily isomorphic to an R-subalgebra of the completion $\widehat{R}$. (This result perhaps sheds additional light as to why Theorem B works.) In fact, Proposition 5.3 will show that if A is non-trivial, then A is Krull if and only if A can be embedded in $\widehat{R}$. Moreover, we shall see that such an algebra A cannot have a retraction to R (Corollary 5.4). We shall also give a codimension-one criterion for such an algebra A to be Noetherian (Theorem 5.2 and Corollary 5.9). Some of the results in this section will form the basis for constructing examples of non-trivial algebras which are locally $\mathbb{A}^1$ in codimension-one and satisfy various additional properties.

We first give the criterion (Theorem 5.2) for A to be Noetherian which, in our setup, provides a sharper version of the Mori-Nishimura Theorem [12, Theorem 12.7]. We shall use the following result [14, Lemma, p. 397].

Lemma 5.1. *Let C be a Krull domain and Q a prime ideal in C of height one. If C/Q is a Noetherian ring, then $C/Q^{(e)}$ is a Noetherian ring for every natural number e.*

Theorem 5.2. *Let R be a Noetherian normal domain with field of fractions K and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Suppose that A is an R-subalgebra of a faithfully flat R-algebra B such that B is a Noetherian domain. Then A is a Krull domain. Moreover, A is Noetherian if and only if A/PA is Noetherian for every $P \in \Delta$.*

Proof. Let $P \in \Delta$. Then PA is a height-one prime ideal of A (Remark 2.4 (3)) and $V_P := A_{PA}$ is a DVR of $K(x)$ such that $A_P = V_P \cap K[x]$, because $A_P = R_P{}^{[1]}$. Thus we have

$$A = \bigcap_{P\in\Delta} A_P = \left(\bigcap_{P\in\Delta} V_P\right) \cap K[x].$$

Hence, for the assertion that A is a Krull domain, it suffices to show that $W := \bigcap_{P\in\Delta} V_P$ is a Krull domain.

Fix $\xi(\neq 0) \in W$, and choose $u(\neq 0) \in A$ such that $u\xi \in A$. For $P \in \Delta$ we denote by v_P the valuation of $K(x)$ whose valuation ring is V_P. Suppose that $v_P(\xi) > 0$. Then $v_P(u\xi) > 0$ so that $u\xi \in PA$. Note that $\mathrm{ht}(PB) = 1$ and $P' \cap R = P$ for any minimal prime ideal P' of PB, because B is Noetherian and faithfully flat over R. Since $u\xi \in PB$, from this it follows that $v_P(\xi) > 0$ for at most finitely many $P \in \Delta$. Thus W is a Krull domain, as desired.

Now assume that A/PA is Noetherian for every $P \in \Delta$. We show that A is Noetherian. By Cohen's Theorem [12, Theorem 3.4], it is enough to show that every prime ideal Q of A is finitely generated.

If $Q \cap R \neq (0)$, then there exists $P \in \Delta$ such that $PA \subseteq Q$. Since A/PA is Noetherian by hypothesis, Q/PA is finitely generated. Therefore, as PA is finitely generated, it follows that Q is finitely generated.

If $Q \cap R = (0)$, then clearly the height of Q is one and there exists $f \in Q$ such that $QK[x] = fK[x]$. Since A is a Krull domain, its proper principal ideal fA has a primary decomposition

$$fA = Q \cap Q_1{}^{(e_1)} \cap \cdots \cap Q_r{}^{(e_r)}$$

for some prime ideals Q_i in A of height one.

Note that, for each i, $1 \leq i \leq r$, $Q_i = P_iA$ for some $P_i \in \Delta$ so that, by hypothesis, $A/Q_i(= A/P_iA)$ is a Noetherian ring. Hence, by Lemma 5.1, $A/Q_i{}^{(e_i)}$ is a Noetherian ring.

Let $J = Q_1{}^{(e_1)} \cap \cdots \cap Q_r{}^{(e_r)}$. Then it is easy to see that A/J is a Noetherian ring. In particular, $(Q+J)/J$ is finitely generated. Since $Q/fA = Q/Q\cap J = (Q+J)/J$, from this it follows that Q is finitely generated. □

In the remaining part of this section, we shall consider the case where R is an analytically irreducible Noetherian normal local domain. We denote by m the maximal ideal of R and by k the residue field of R.

When $A \neq R^{[1]}$, we have the R-algebra homomorphism $\psi : A \to \widehat{R}$ defined in Remark 3.5. By Remark 3.12, ψ is also the composite of the maps

$$A \to \varprojlim_{n\geq 1} A/m^n A \cong \varprojlim_{n\geq 1} R/m^n R = \widehat{R}.$$

Recall that, in the notation of Section 4, we can write $A = R(x, e, \mathbf{c})$ with $\mathbf{c} = (\bar{c_\Gamma})_{\Gamma\in\Sigma} \in \widetilde{R}(e)$, where e is the power function for A and $\widetilde{R}(e)$ is the filter ring corresponding to e (cf. Remark 4.4 (2)). Then $\psi(x) = c$, where c is the unique element in $\widehat{R}$ such that $c - c_\Gamma \in I_\Gamma \widehat{R}$ for every $\Gamma \in \Sigma$ (cf. Proposition 3.4 and Remark 3.5).

We now give conditions for ψ to be injective (that is, conditions for A to be embedded inside $\widehat{R}$).

Proposition 5.3. *Let (R, m) be an analytically irreducible Noetherian normal local domain and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Assume that $A \neq R^{[1]}$, and let $c = \psi(x)$, where $\psi\colon A \to \widehat{R}$ is the R-algebra homomorphism defined in Remark* 3.5*. Then the following conditions are equivalent:*

(i) *ψ is injective.*
(ii) *c is transcendental over R.*
(iii) *A is a Krull domain.*
(iv) *$\bigcap_{n\geq 1} m^n A = (0)$.*

Moreover, if A is Noetherian, then $Q\widehat{R} \cap A = QA$ for every $Q \in$ Spec *R.*

Proof. (i) ⇒ (iii). This implication follows from Theorem 5.2.

(iii) ⇒ (ii). Suppose that c is not transcendental over R. Then there exists $f(X)(\neq 0) \in R[X]$ such that $f(c) = 0$. Recall that $A = R(x, e, \mathbf{c})$, where $\mathbf{c} = (\bar{c_\Gamma})_{\Gamma\in\Sigma} \in \widetilde{R}(e)$, and that $c - c_\Gamma \in I_\Gamma \widehat{R}$. Since $f(c) = 0$, it then follows that

$$f(c_\Gamma) = f(c - (c - c_\Gamma)) \in I_\Gamma \widehat{R} \cap R = I_\Gamma R \subseteq I_\Gamma A$$

for every $\Gamma \in \Sigma$. Therefore, as $x - c_\Gamma \in I_\Gamma A$, we have

$$f(x) = f(x - c_\Gamma + c_\Gamma) \in I_\Gamma A.$$

In particular, $f(x) \in PA$ for every $P \in \Delta_0$. Note that Δ_0 is infinite by Corollary 2.7. As $PA \in \operatorname{Spec} A$ with $\operatorname{ht}(PA) = 1$ and $f(x) \neq 0$, we see that A is not a Krull domain.

(ii) ⇒ (i). If ψ is not injective, then there exists $a(\neq 0) \in A$ such that $\psi(a) = 0$. Since $A \subseteq K[x]$, there exists $r(\neq 0) \in R$ and $f(X) \in R[X] \setminus R$ such that $ra = f(x) \in R[x]$. Now

$$f(c) = \psi(f(x)) = \psi(ra) = r\psi(a) = 0,$$

which shows that c is algebraic over R.

(i) ⇔ (iv). This follows from the fact that $\bigcap_{n\geq 1} m^n A$ is the kernel of the map $A \to \varprojlim_{n\geq 1} A/m^n A$.

We now assume that A is Noetherian. Since we now have an embedding $\psi : A \hookrightarrow \widehat{R}$, we identify A with its image in $\widehat{R}$ and show that $Q\widehat{R} \cap A = QA$ for any $Q \in \operatorname{Spec} R$. By Theorem 3.10, $A/mA = R/m$; in particular, mA is a maximal ideal of A and $m\widehat{R} \cap A = mA$. Thus $B = A_{mA}$ is a Noetherian local ring and $R \subseteq B \subseteq \widehat{R}$. Therefore, the mB-adic completion of B is $\widehat{R}$. In particular, $\widehat{R}$ is faithfully flat over B. Therefore, for $Q \in \operatorname{Spec} R$, we have $Q\widehat{R} \cap B = QB$, and hence $Q\widehat{R} \cap A = QA$. □

The above results imply the following

Corollary 5.4. *Let R and A be as in Proposition* 5.3*, and suppose that there exists an injective R-algebra homomorphism $\rho\colon A \hookrightarrow \widehat{R}$. Then the following assertions hold:*

(1) *$A = R^{[1]}$ if and only if A has a retraction to R.*
(2) *If R is factorial, then A is factorial.*

Proof. (1) Suppose that $A \neq R^{[1]}$. Then $\rho = \psi$ (cf. Remark 3.5), and hence ψ is injective. It then follows from Proposition 5.3 that c is transcendental over R so that $c \notin R$. Thus A cannot have a retraction to R by Corollary 3.6.

(2) If $A = R^{[1]}$, then it is well known that A is factorial. If $A \neq R^{[1]}$, then A is a Krull domain again by Proposition 5.3, and hence the assertion follows from Remark 2.4 (3). □

Remark 5.5. Remark 4.12 gave an example of a non-Krull domain which is locally $\mathbb{A}^1$ in codimension-one over a normal domain and which is also a symmetric algebra of a flat module. Proposition 5.3 shows that there also exist non-Krull domains which are faithfully flat and locally $\mathbb{A}^1$ in codimension-one but which are not obtained as symmetric algebras.

Let $R, e, \mathbf{c}$ be as in Example 4.18 and let $A = R(x, e, \mathbf{c})$. Since R is a factorial domain, A is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. As $c = \psi(x)$ is integral over R (cf. Example 4.18), A is non-Krull by Proposition 5.3. Also, as $c \notin R$ (cf. Example 4.18), A does not have a retraction to R by Corollary 3.6.

In the case R is a two-dimensional normal affine spot, we shall give a criterion (Proposition 5.8) for A/PA to be Noetherian when $P \in \Delta$, which will enable us to apply Theorem 5.2. We first recall an easy application of Cohen's criterion for a ring to be Noetherian [12, Theorem 3.4], which was formulated in [6, Lemma 2.8]:

Lemma 5.6. *Let B be an integral domain. Suppose that there exists a non-zero element π in B such that*

(I) *$B[1/\pi]$ is Noetherian.*
(II) *πB is a maximal ideal of B.*
(III) *$\bigcap_{n>0} \pi^n B = (0)$.*

Then B is a Noetherian ring.

The following version of the above result will be used in our argument.

Corollary 5.7. *Let D be a semi-local PID with field of fractions L. Suppose that C is an integral domain containing D such that*

(I) *$L \otimes_D C$ is a Noetherian ring.*
(II) *πC is a maximal ideal of C for every prime element π of D.*
(III) *$\bigcap_{n>0} \pi^n C = (0)$ for every prime element π in D.*

Then C is a Noetherian ring.

Proof. The assertion follows from Lemma 5.6 by induction on the number of maximal ideals of D. □

We now state the criterion for A/PA to be Noetherian when $P \in \Delta$.

Proposition 5.8. *Let $(R, \mathfrak{m})$ be a local domain of dimension two such that R is a localisation of an affine normal domain over k. Let A be a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Suppose that $A \hookrightarrow \widehat{R}$ and $A \neq R^{[1]}$, and let P be a prime ideal in R of height one for which $J \cap A = PA$ for every associated prime ideal J of $P\widehat{R}$. Then A/PA is a Noetherian domain of dimension one.*

Proof. Note that $\widehat{R}$ is a Noetherian normal local domain [13, Theorem 37.5]. Let $S = R/P$, $L = R_P/PR_P$, the field of fractions of S, and set $B = A/PA$. Let

$\bar{\mathfrak{m}}(=\mathfrak{m}/P)$ denote the image of $\mathfrak{m}$ in S, and let $\widehat{S}=\widehat{R}/P\widehat{R}$, the ($\bar{\mathfrak{m}}$-adic) completion of S.

Since A is faithfully flat over R, $PA\cap R=P$, so $S\hookrightarrow B$ and B is faithfully flat over S. Note that $B\hookrightarrow \widehat{S}$ by hypothesis.

Now let D be the integral closure of S in its field of fractions L. Set $C=D\otimes_S B$ and $C'=D\otimes_S \widehat{S}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} L & \longrightarrow & L\otimes_S B & \longrightarrow & L\otimes_S \widehat{S} \\ \uparrow & & \uparrow & & \uparrow \\ D & \longrightarrow & C=D\otimes_S B & \longrightarrow & C'=D\otimes_S \widehat{S} \\ \uparrow & & \uparrow & & \uparrow \\ S & \longrightarrow & B & \longrightarrow & \widehat{S} \end{array} \tag{5.1}$$

where vertical maps are injective, because both B and $\widehat{S}$ are flat over S. On the other hand, since $S\hookrightarrow B\hookrightarrow \widehat{S}$ and L is S-flat, we have $L\hookrightarrow L\otimes_S B\hookrightarrow L\otimes_S \widehat{S}$ so that $D\hookrightarrow C\hookrightarrow C'$, namely, horizontal maps are also injective. Note that $L\otimes_D C=L\otimes_S B=L^{[1]}$, because $R_P\otimes_R A={R_P}^{[1]}$. Thus $C\hookrightarrow L^{[1]}$, and, in particular, C is an integral domain.

Since S is a one-dimensional affine spot, D is a semi-local PID which is *finite* as an S-module. Hence, to show that B is Noetherian, it suffices, by the Eakin-Nagata theorem [12, Theorem 3.7], to show that the integral domain $C(=D\otimes_S B)$ is a Noetherian ring. We apply Corollary 5.7 to verify this.

We have already seen that $L\otimes_D C(=L^{[1]})$ is Noetherian. Since $A\neq R^{[1]}$, it follows from Theorem 3.10 that $k=A/\mathfrak{m}A$, which implies that $k=B/\bar{\mathfrak{m}}B$. Hence, for every prime element π of D, we have

$$C/\pi C=(D/\pi D)\otimes_S B=(D/\pi D)\otimes_k (B/\bar{\mathfrak{m}}B)=D/\pi D,$$

and hence πC is a maximal ideal of C.

Thus to show that C is Noetherian, it remains to check that $\bigcap_{n>0}\pi^n C=(0)$ for every prime element π in D. Note that C' is the completion of the semi-local ring D because D is a finite S-module, so C' is a direct product of DVRs. Note also that $V:=C'_{\pi C'}$ lies in the DVRs. Let $\sigma\colon C'\to V$ be the natural homomorphism, and let $N=\ker\sigma$. Then N is a minimal prime ideal of C' so that $N\cap\widehat{S}=J/P\widehat{R}$ for some minimal prime ideal J of $P\widehat{R}$. Hence $N\cap B=(0)$ by assumption. Since C is a domain integral over B, from this it follows that $N\cap C=0$, and hence $C\hookrightarrow V$. Therefore $\bigcap_{n>0}\pi^n C\subseteq\bigcap_{n>0}\pi^n V=(0)$, as desired.

Since B is flat over S, it now follows from [12, Theorem 15.1] that $\dim B=\dim S+\dim B/\bar{\mathfrak{m}}B=\dim S=1$. This completes the proof. □

Corollary 5.9. *Under the hypotheses of Proposition* 5.8*, if $J\cap A=PA$ for every $P\in\Delta$ and for every associated prime ideal J of $P\widehat{R}$, then A is a Noetherian domain of dimension two.*

Proof. By Theorem 5.2 and Proposition 5.8, it follows that A is Noetherian. Since $A/\mathfrak{m}A=k$, we have $\operatorname{ht}\mathfrak{m}A=\operatorname{ht}\mathfrak{m}+\dim A/\mathfrak{m}A=2$ by [12, Theorem 15.1]. Since $K\otimes_R A=K^{[1]}$ and $R_P\otimes_R A={R_P}^{[1]}$ for every $P\in\Delta$, it follows that $\dim A=2$. □

We now record a technical result which gives an indication as to how one could construct examples of non-trivial Noetherian algebras which are locally $\mathbb{A}^1$ in codimension-one (see Example 6.2).

Lemma 5.10. *Let R and A be as in Proposition 5.3, and suppose that $A \hookrightarrow \widehat{R}$. Let P be a prime ideal in R of height one, and let J be an associated prime ideal of $P\widehat{R}$. Suppose that there exists an element y in A such that the image of y in $\widehat{R}/J$ is transcendental over R/P. Then $J \cap A = PA$.*

Proof. Let $I = J \cap A$. Then $I \cap R = J \cap R = P$, and hence IA_P is a prime ideal of A_P. Since $A_P/PA_P = k(P)^{[1]}$, if $IA_P \neq PA_P$, then A_P/IA_P is algebraic over $k(P)$ so that A/I is algebraic over R/P, contradicting the hypothesis on y. Thus $IA_P = PA_P$, and hence $I = PA$. □

6. Main examples

In this section, we shall construct our main examples. We shall show (Proposition 6.1) that — in contrast to the case where R is factorial — a faithfully flat algebra A over a Noetherian normal local domain R which is Krull and locally $\mathbb{A}^1$ in codimension-one *need not be* a direct limit of polynomial algebras over R, not even if A is Noetherian (Example 6.2).

Now suppose that D is a Noetherian factorial domain and B a Noetherian factorial faithfully flat D-algebra. Even if $B_P = {D_P}^{[1]}$ for every $P \in \text{Spec } D$, B need not be finitely generated [8, (3.15)] or, equivalently (by Theorem 2.11), B need not have a retraction to D. This leads to

Question. Let $(D, \mathfrak{m})$ be a Noetherian factorial *local* domain and B a Noetherian factorial faithfully flat D-algebra such that $B_P = {D_P}^{[1]}$ for every $P \in (\text{Spec } D) \setminus \mathfrak{m}$. Is $B = D^{[1]}$?

We have seen that Theorem 3.7 gives an affirmative answer to this question when D is complete, while [5, Corollaries 4.9, 4.10] give an affirmative answer when $D/\mathfrak{m} \subsetneq B/\mathfrak{m}B$. We shall show (Example 6.4) that even when $D = \bar{\mathbb{Q}}[u, v]_{(u,v)}$, a regular affine spot over an algebraically closed field, a D-algebra B satisfying the hypotheses in the above question need not have a retraction.

Examples of faithfully flat algebras which are locally $\mathbb{A}^1$ in codimension-one but which do not have retraction appear to be esoteric and intricate; more so if they are to be Noetherian. Over $D = \bar{\mathbb{Q}}[u, v]_{(u,v)}$, we give a simpler but non-Noetherian example (Example 6.5) of a D-algebra C which is factorial, faithfully flat and locally $\mathbb{A}^1$ in codimension-one but which does not have a retraction to D.

Proposition 6.1. *Let k be an algebraically closed field and $(R, \mathfrak{m})$ a normal local domain of dimension two such that R is a localisation of an affine normal domain over k. If R is not factorial, then there exists a Krull domain A which is faithfully flat over R and satisfies the properties:*

1. *$A_P = {R_P}^{[1]}$ for every prime ideal P in R of height one.*
2. *A is not a direct limit of subalgebras of A which are polynomial algebras over R.*

Proof. Fix a non-principal prime ideal Q in R of height one, and let q be an element in Q such that $QR_Q = qR_Q$. By Lemma 4.10, there are infinitely many principal prime ideals of R. Fix a countably infinite set $\{P_1, P_2, \dots, P_n, \dots\}$ of principal prime ideals, and for each n, choose a generator p_n of P_n. Now consider the function $e : \Delta \to \mathbb{Z}_{\geq 0}$ defined by $e(Q) = 1$, $e(P_n) = 1$ for $n \geq 1$ and $e(P) = 0$ otherwise.

Let $I_0 = Q$, and for each $n \geq 1$, let

$$I_n := Q \cap P_1 \cap \cdots \cap P_n (= p_1 \cdots p_n Q).$$

Now set

$$V := \prod_{m \geq 0} k_m, \text{ where } k_m := k \text{ for each } m \geq 0.$$

Let $\widehat{R}$ denote the completion of R, and define a map $\phi \colon V \to \widehat{R}$ by

$$\phi(\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots) = \lambda_0 q + \lambda_1 q p_1 + \cdots + \lambda_n q p_1 p_2 \cdots p_n + \cdots .$$

Clearly ϕ is injective. Thus the image of ϕ is a vector space whose cardinality is larger than that of k. On the other hand, the algebraic closure of R in $\widehat{R}$ has the same cardinality as that of R and hence the same cardinality as that of k. Therefore, there exist $\lambda_0, \lambda_1, \ldots$ in k for which the element

$$c := \lambda_0 q + \lambda_1 q p_1 + \cdots + \lambda_n q p_1 p_2 \cdots p_n + \cdots$$

is transcendental over R. Set $c_0 = 0$, $c_1 = \lambda_0 q$ and, for $n \geq 2$,

$$c_n := \lambda_0 q + \lambda_1 q p_1 + \cdots + \lambda_{n-1} q p_1 p_2 \cdots p_{n-1},$$

and let $\overline{c_n}$ denote the image of c_n in R/I_n. By Lemma 4.14, the elements $\{\overline{c_n}\}_{n \geq 0}$ form an element $\mathbf{c}$ of the filter ring $\widetilde{R}(e)$. Set

$$A := R(x, e, \mathbf{c}) = \bigcup_{n \geq 0} A_n,$$

where x is an indeterminate over R and $A_n = \bigoplus_{j \geq 0} ({I_n}^j)^{-1} (x - c_n)^j$ for $n \geq 0$. Note that, for $n \geq 1$, we have

$$A_n = \bigoplus_{j \geq 0} (Q^j)^{-1} \left(\frac{x - c_n}{p_1 \cdots p_n} \right)^j$$

because $({I_n}^j)^{-1} = (Q^j)^{-1} (p_1 \cdots p_n)^{-j}$. By Propositions 4.3, 4.5 and 4.9, A is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Note that c is the limit of the Cauchy sequence $\{c_n\}_{n \geq 1}$ in $\widehat{R}$, and hence $\psi(x) = c$ by Remark 4.15, where $\psi \colon A \to \widehat{R}$ is the R-algebra homomorphism defined in Remark 3.5. Thus A is a Krull domain by Proposition 5.3.

Let T be the multiplicative set generated by all prime elements of R. Then $T^{-1}A$ is a faithfully flat $T^{-1}R$-algebra which is locally $\mathbb{A}^1$ in codimension-one. Note that the power function for $T^{-1}A$ takes non-zero value only at $T^{-1}Q$, which implies that

$$T^{-1}A = (T^{-1}R)[xQ^{-1}] = Sym_{T^{-1}R}\,(T^{-1}L),$$

where $L = xQ^{-1}$. As $T^{-1}L$ is a *finitely generated* non-cyclic rank-one projective module over the Dedekind domain $T^{-1}R$, it follows (as in Remark 4.12) that A cannot be a direct limit of polynomial algebras. $\square$

Further discussion on Proposition 6.1 will also be made in Section 7 (Remark 7.4). We now show that there exists a Noetherian algebra A with properties mentioned in Proposition 6.1.

Example 6.2. Let $k = \bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, and let

$$R = (k[T_1, T_2, T_3]/(T_1T_2 - T_3^2))_{(\bar{T_1}, \bar{T_2}, \bar{T_3})},$$

where T_i, $1 \leq i \leq 3$, are indeterminates over k. Thus R is a normal affine k-spot of dimension two. Denote the images of T_1, T_2, T_3 in R by t_1, t_2, t_3, respectively. Let Q denote the prime ideal $(t_1, t_3)R$. As before, Δ denotes the set of all height-one prime ideals in R, and Δ_1 the set of all principal prime ideals of R.

By Lemma 4.10, Δ_1 is infinite. Since R is countable, it follows that both Δ and Δ_1 are countably infinite. Hence the elements of Δ_1 and $\Delta \setminus \Delta_1$ can be labelled as $\Delta_1 = \{P_1, P_2, \dots\}$ and $\Delta \setminus \Delta_1 = \{Q_0, Q_1, \dots\}$, where $Q_0 = Q$. For convenience we set $P_0 = 0$, the null ideal. For each $P_n \in \Delta_1$, choose a generator p_n of the principal prime ideal P_n.

Now set

$$V := \prod_{m \geq 0} k_m, \text{ where } k_m := k \text{ for each } m \geq 0.$$

For $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n, \dots) \in V$, we define the element $\widehat{\lambda} \in \widehat{R}$ by

$$\widehat{\lambda} := \lambda_0 + \lambda_1 p_1 + \lambda_2 p_1 p_2 + \cdots + \lambda_n p_1 p_2 \cdots p_n + \cdots .$$

Fix n (≥ 0). Then we set

$$\widehat{\lambda}_n := \lambda_n + \lambda_{n+1} p_{n+1} + \lambda_{n+2} p_{n+1} p_{n+2} + \cdots \in \widehat{R}$$

so that

$$\widehat{\lambda} = (\lambda_0 + \lambda_1 p_1 + \cdots + \lambda_{n-1} p_1 p_2 \cdots p_{n-1}) + (p_1 \cdots p_n)\widehat{\lambda}_n.$$

Note that if $P \in \Delta$, then $\widehat{R}/P\widehat{R}$ is a reduced ring [13, Theorem 36.4]. Let Φ_n and Ψ_n be the sets of associated prime ideals of $Q_n\widehat{R}$ and $P_n\widehat{R}$, respectively. Then, for each $J \in \Phi_n$, we define a k-linear map

$$\phi_{nJ} \colon V \to \widehat{R}/J$$

by

$$\phi_{nJ}(\lambda) := \text{ the image of } \widehat{\lambda} \text{ in } \widehat{R}/J.$$

Similarly, for $N \in \Psi_n$, we define

$$\rho_{nN} \colon V \to \widehat{R}/N$$

by

$$\rho_{nN}(\lambda) := \text{ the image of } \widehat{\lambda}_n \text{ in } \widehat{R}/N.$$

Note that if $P \in \Delta$, then $\widehat{R}/P\widehat{R}$ is faithfully flat over R/P, and hence any non-zero divisor in R/P remains a non-zero divisor in $\widehat{R}/P\widehat{R}$. Since $p_m \notin Q_n$ for $m \geq 1$ and $p_m \notin P_n$ for $m > n$, from this one can easily check that the k-linear map ϕ_{nJ} is injective and the kernel of ρ_{nN} is a vector space over k of dimension n. Hence the image of V under either map is a vector space of uncountable dimension.

For $n \geq 0$, we set

$$V_{nJ} := \{\lambda \in V \mid \phi_{nJ}(\lambda) \text{ is algebraic over } R/Q_n\}$$

and

$$V_n = \sum_{J \in \Phi_n} V_{nJ}.$$

We also set

$$W_{nN} := \{\lambda \in V \mid \rho_{nN}(\lambda) \text{ is algebraic over } R/P_n\}$$

and

$$W_n = \sum_{N \in \Psi_n} W_{nN}.$$

Then V_n and W_n are countable k-subspaces of V so that, setting

$$U := \sum_{n \geq 0} V_n + \sum_{n \geq 0} W_n,$$

we get a countable subspace U of V. Let Z be a subspace of V such that $V = U \oplus Z$. Then Z is uncountable.

For any ideal I of $\widehat{R}$, the ring $\widehat{R}/I$ is an R-module, so for any element $t \in R$ and any $f \in \widehat{R}/I$, we denote by tf the product of f with the image of t in $\widehat{R}/I$.

Let $\gamma = (\gamma_n)_{n \geq 0}$ be a non-zero element in Z, and set

$$c := t_3 \widehat{\gamma} = \sum_{i \geq 0} \gamma_i t_3 p_1 \cdots p_i \in \widehat{R}.$$

By construction, for each $n \geq 1$ and $J \in \Phi_n$, the image of c in $\widehat{R}/J$ (that is, $t_3 \phi_{nJ}(\gamma)$) is transcendental over R/Q_n (since t_3 remains a non-zero divisor in $\widehat{R}/Q_n\widehat{R}$ and $\gamma \in Z \setminus \{0\}$); it also follows that c is transcendental over R.

Define a sequence $\{c_n\}_{n \geq 0}$ of elements in R by $c_0 = 0$ and

$$c_n := \sum_{i=0}^{n-1} \gamma_i t_3 p_1 \cdots p_i \in R$$

for $n \geq 1$. Thus $\{c_n\}_{n \geq 0}$ is a Cauchy sequence whose limit is c. Let $s_0 = c$, and for each $n \geq 1$, let

$$s_n := t_3 \widehat{\gamma}_n = \frac{c - c_n}{p_1 \cdots p_n} \in \widehat{R}.$$

Then, for each $n \geq 1$ and $N \in \Psi_n$, the image of s_n in $\widehat{R}/N$ is transcendental over R/P_n because t_3 remains a non-zero divisor in $\widehat{R}/P_n\widehat{R}$ and $\gamma \in Z \setminus \{0\}$.

Now consider the element

$$u = \frac{t_2}{t_3} c = t_2 \widehat{\gamma}.$$

For each $J \in \Phi_0$, the image of u in $\widehat{R}/J$ is $t_2 \phi_{0J}(\gamma)$. Since $\gamma \in Z \setminus \{0\}$, we have $\gamma \notin V_{0J}$, and hence $\phi_{0J}(\gamma)$ is transcendental over R/Q. As $t_2 \notin Q$, it follows that $t_2 \phi_{0J}(\gamma)$ is transcendental over R/Q. Thus the image of u in $\widehat{R}/J$ is transcendental over R/Q.

Now consider the function $e : \Delta \to \mathbb{Z}_{\geq 0}$ defined by $e(P_n) = 1$ for $n \geq 1$, and $e(Q) = 1$ and $e(Q_n) = 0$ for $n \geq 1$. Let $I_0 = Q$ and, for each $n \geq 1$, let

$$I_n := Q \cap P_1 \cap \cdots \cap P_n (= p_1 \cdots p_n Q).$$

By Lemma 4.14, the filter ring $\widetilde{R}(e)$ (as defined in Section 4) may be identified with $\varprojlim_{n \geq 1} (R/I_n)$. In particular, the sequence $\{c_n\}_{n \geq 1}$ forms an element $\mathbf{c}$ of $\widetilde{R}(e)$. Now let x be an indeterminate over R and set

$$A := R(x, e, \mathbf{c}).$$

By Propositions 4.3, 4.5 and 4.9, A is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Note that

$$A = \bigcup_{n \geq 0} A_n$$

where, for each $n \geq 0$, setting $x_n = \dfrac{x - c_n}{p_1 \cdots p_n}$ with $x_0 = x$, we have

$$A_n := \bigoplus_{j \geq 0} (Q^j)^{-1} {x_n}^j.$$

Since c is the limit of $\{c_n\}_{n \geq 1}$ in $\widehat{R}$, we have $\psi(x) = c$ by Remark 4.15, where $\psi\colon A \to \widehat{R}$ is the R-algebra homomorphism defined in Remark 3.5 so that $A \hookrightarrow \widehat{R}$ by Proposition 5.3. We now demonstrate the local generators of A at each $P \in \Delta$. Recall that $\Delta_0 = \{Q, P_1, P_2, \dots\}$.

Case (i). $P = P_n = p_n R$ with $n \geq 1$.

In this case, it follows from Remark 4.4 (1) that $A_{P_n} = R_{P_n}[x_n]$. Note that $x_n \in A_n \subseteq A$.

Case (ii). $P = Q$.

Since $QR_Q = t_3 R_Q$, we have $A_Q = R_Q[x/t_3]$ again by Remark 4.4 (1). Thus $A_Q = R_Q[t_2 x/t_3]$ because t_2 is a unit in R_Q. Note that $t_2/t_3 \in Q^{-1}$, and hence $t_2 x/t_3 \in A_0 \subseteq A$.

Case (iii). $P = Q_n$ with $n \geq 1$.

Since $Q_n \notin \Delta_0$, in this case we have $A_{Q_n} = R_{Q_n}[x]$.

Now, note that $\psi(x_n) = s_n$ for $n \geq 0$ and that $\psi(t_2 x/t_3) = t_2 c/t_3 = u$. Thus, given any prime ideal P in R of height one, we have seen, through case-by-case analysis, that $A_P = R_P[y_P]$, where y_P is an element of A whose image in $\widehat{R}/J$ is transcendental over R/P for every associated prime ideal J of $P\widehat{R}$. Therefore, A is Noetherian by Lemma 5.10 and Corollary 5.9.

As in the proof of Proposition 6.1, A is not a direct limit of polynomial algebras.

Remark 6.3. In the above example, $Q^{-1} = R + (t_3 {t_1}^{-1})R$, $(Q^{2n})^{-1} = {t_1}^{-n}R$, and $(Q^{2n+1})^{-1} = ({t_1}^{-n}, t_3 {t_1}^{-n-1})R$. One can check that

$$A_m = R[x_m,\ t_3 {t_1}^{-1} x_m,\ {t_1}^{-1} {x_m}^2]$$

for any m.

Example 6.4. Let k, R, A be as in Example 6.2. Let $D = k[u,v]_{(u,v)}$ and identify R with the subring $k[u^2, uv, v^2]_{(u^2,uv,v^2)}$, with t_1, t_2, t_3 corresponding to u^2, v^2, uv, respectively. Let $B = D \otimes_R A$. As D is a finite R-module and A is a Noetherian faithfully flat R-algebra, it follows that B is a Noetherian faithfully flat D-algebra.

Let $\mathfrak{p}$ be a prime ideal in D of height one. We show that $B_{\mathfrak{p}} = {D_{\mathfrak{p}}}^{[1]}$. Let $P = \mathfrak{p} \cap R$. Then $\operatorname{ht} P = 1$. Let $S = R \setminus P$. As $A_P = {R_P}^{[1]}$, we have

$$S^{-1}B = S^{-1}(D \otimes_R A) = S^{-1}D \otimes_{R_P} A_P = (S^{-1}D)^{[1]}.$$

Hence, by further localisation, $B_{\mathfrak{p}} = {D_{\mathfrak{p}}}^{[1]}$. Thus B is a D-algebra which is locally $\mathbb{A}^1$ in codimension-one. By Remark 2.4 (3), B is factorial.

Since A is not finitely generated over R and D is integral over R, it follows that B is not finitely generated over D and hence B does not have any retraction to D by Theorem 2.11.

Example 6.5. Example 6.4 is existential. Over $D = k[u,v]_{(u,v)} (k = \bar{\mathbb{Q}})$ as in Example 6.4, below we give a more concrete construction of a faithfully flat factorial D-algebra C (inside the completion $k[[u,v]]$ of D) which is locally $\mathbb{A}^1$ in codimension-one but which does not have a retraction. However, C is a *non-Noetherian* factorial domain.

For each $n \in \mathbb{N}$, set $q_n := (u/n) + v$; $a_0 := 1$ and inductively define a sequence $\{a_n \mid n \geq 1\}$ by the recurrence relation

$$a_n := a_{n-1} + q_1 \cdots q_n.$$

Clearly $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in D. Set

$$z := 1 + q_1 + q_1 q_2 + \cdots + q_1 q_2 \cdots q_n + \cdots ,$$

the limit of the sequence $\{a_n\}_{n\geq 1}$ in $\widehat{D}$. Then z is transcendental over D because the image $\bar{z}$ of z in $\widehat{D}/v\widehat{D} (= k[[u]])$ is given by

$$\bar{z} = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots ,$$

which is transcendental over D/vD. Set $z_0 := z$, and for any $n \geq 1$, set $z_n := (z_{n-1} - 1)/q_n$. Now let $C = D[z_0, z_1, \ldots, z_n, \ldots]$. Then C is a faithfully flat D-algebra which is locally $\mathbb{A}^1$ in codimension-one. In fact, $C = D(z, e, \mathbf{z})$, where e is the function defined by $e(P) = 1$ when $P = q_n R$ and $e(P) = 0$ otherwise, and $\mathbf{z}$ is the element in the inverse limit $\widetilde{D}(e)$ defined by the sequence $\{z_n\}_{n\geq 0}$. It is factorial and does not have retraction to D (Corollary 5.4).

Note that, for each $n \geq 0$, setting $\alpha = (1-v)^{-1} \in D$ we have $z_n = \alpha + u\xi_n$ for some $\xi_n \in \widehat{D}$. It thus follows that $C/(u\widehat{D} \cap C) = D/uD$, and hence the canonical map $C/uC \to \widehat{D}/u\widehat{D}$ is not injective because C/uC is transcendental over D/uD. Hence C is not Noetherian by Proposition 5.3. In fact, since we have a strict chain $(0) \subsetneqq uC \subsetneqq u\widehat{D} \cap C \subsetneqq mC$ of prime ideals in C, we see that the height of the two-generated ideal mC exceeds 2.

The ring C is an example of a non-Noetherian Krull domain which is an overring of the three-dimensional Noetherian domain $D[z]$ (that is, $D[z]$ is a birational subring of C). Note that, by a result of Heinzer [10], any Krull domain which is an overring of a two-dimensional Noetherian domain is necessarily Noetherian.

Remark 6.6. Let R be as in Proposition 6.1. Using the above techniques, one can construct a Noetherian R-algebra A such that $A \neq R^{[1]}$, but A is locally $\mathbb{A}^1$ in codimension-one and A is a direct limit of polynomial algebras. For instance, one can construct an element $d' = \sum_i \lambda_i p_1 \ldots p_i$ in $\widehat{R}$ which is transcendental over R and which is such that, for every n, the image of $d'_n = \lambda_n + \sum_{i\geq 1} \lambda_{n+i} p_{n+1} \ldots p_{n+i}$ in $\widehat{R}/J$ is transcendental over R/P_n for every associated prime ideal J of $P_n\widehat{R}$. Now consider the integer-valued function e' on Δ defined by $e'(P) = 1$ when $P = P_n$ for some n and $e'(P) = 0$ otherwise. Let $\boldsymbol{d'}$ be the element in $\widetilde{R}(e')$ formed, as before, by the partial sums d'_n of d'. Then $R(x, e', \boldsymbol{d'})$ has the required properties.

7. Generalised structure theorem and applications

Recall that the R-algebra $B := R(x, e, \mathbf{c})$ constructed in Section 4 need not be faithfully flat. However it satisfies two properties of faithful flatness listed below.

Lemma 7.1. *The following assertions hold for the R-algebra $B := R(x, e, \mathbf{c})$ defined by* (4.4)*:*

(1) $B = \bigcap_{P\in\Delta} B_P$.
(2) $(a, b)B \cap R = (a, b)R$ *for any two elements a and b of R.*

Proof. (1) This is already proved in Proposition 4.3.

(2) Let I be an arbitrary ideal of R. For $\Gamma \in \Sigma$, we have from Lemma 4.2 (1) that

$$B_\Gamma \subseteq {S_\Gamma}^{-1}R\left[\frac{x - c_\Gamma}{d_\Gamma}\right] \cap R_\Gamma\left[\frac{x - c_\Gamma}{d_\Gamma}\right] = R\left[\frac{x - c_\Gamma}{d_\Gamma}\right] = R^{[1]}$$

using $R = {S_\Gamma}^{-1}R \cap R_\Gamma$. Hence $IB_\Gamma \cap R = I$, which implies that $IB \cap R = I$, because $B = \varinjlim B_\Gamma$. This completes the proof. □

For the sake of convenience we call an R-algebra B *semi-faithfully flat* over R if B satisfies conditions (1) and (2) in Lemma 7.1. Note that condition (2) is satisfied if B is a subalgebra of a faithfully flat R-algebra.

Checking the proof precisely, we realize that Theorem 2.3 holds when A is semi-faithfully flat over R. Indeed, faithful flatness is used only to ensure that $A = \bigcap_{P\in\Delta} A_P$ (by using Lemma 1.3) and that equation (2.4) holds. Moreover, the proof of Theorem 2.3 and the arguments and proofs of Section 4 up to Proposition 4.3 work over any Krull domain R. We have thus obtained the following theorem.

Theorem 7.2. *Let R be a Krull domain with field of fractions K, let $e\colon \Delta \to \mathbb{Z}_{\geq 0}$ be a function, let $\mathbf{c}$ be an element of the filter ring $\widetilde{R}(e)$, and let x be an indeterminate over R. Then the R-algebra $R(x, e, \mathbf{c})$ defined by* (4.4) *is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one.*

Conversely, let A be a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one, and let x be a generic variable of A, i.e., $R[x] \subseteq A \subseteq K[x]$. Then $A = R(x, e, \mathbf{c})$, where e is the power function for A and $\mathbf{c}$ is the residue of A.

Remark 7.3. (1) Let $\mathbf{c}$ and $\mathbf{b}$ be elements of the filter ring $\widetilde{R}(e)$. Then $R(x, e, \mathbf{c}) = R(x, e, \mathbf{b})$ if and only if $\mathbf{c} = \mathbf{b}$.

(2) The R-algebra $B = R(x, e, \mathbf{c})$ has the property that $QB \cap R = Q$ for every ideal Q of R. If R is locally factorial, then B is faithfully flat over R.

(3) Taking $\mathbf{c} = \mathbf{0}$ in $R(x, e, \mathbf{c})$, one obtains an R-algebra with a retraction to R. In fact, all semi-faithfully flat algebras with retractions which are locally $\mathbb{A}^1$ in codimension-one can be obtained in this way.

Remark 7.4. Let R be a Krull domain and A a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Fix an element x in A which is a generic variable of A. Clearly A satisfies the following conditions:

(1) A is normal.
(2) R is an inert subring of A.
(3) $PA \cap R = P$ for every $P \in \Delta$.
(4) Every prime element p in R remains a prime element in A.

Moreover, if A is faithfully flat over R, then we also have

(5) $PA \in \operatorname{Spec} A$ for every $P \in \Delta$.

Some of our investigations in this paper amounted to exploring whether A can be expressed as a direct limit of *finitely generated* subalgebras A_α where each A_α is a semi-faithfully flat R-algebra containing x and satisfying conditions (1) to (4), and moreover, if A is given to be faithfully flat, then whether A_α can be chosen to be a faithfully flat R-algebra satisfying (1) to (5).

Now suppose in addition that $(R, \mathfrak{m})$ is a normal affine spot over a field. In this case it is easy to see that one can always choose finitely generated R-subalgebras A_α of A which satisfy (1), (2) and (3) and the condition $(a,b)A_\alpha \cap R = (a,b)R$ for every a, b in R.

One can also see that when R is factorial, then $A_\alpha = R^{[1]}$ if and only if A_α is semi-faithfully flat over R and satisfies (4). Thus the result ([5, Theorem 4.6]) shows that when R is factorial, one can indeed choose A_α satisfying the five properties of A.

When R is normal, one can again see that $A_\alpha = R^{[1]}$ if and only if A_α is faithfully flat over R and satisfies (5). Thus, by Proposition 6.1 and Example 6.2, it may not be possible in this case to express A as a direct limit of subalgebras A_α satisfying the above properties.

We now give a few applications of the general structure theorem (Theorem 7.2). In particular, we shall show that the previous results on $\mathbb{A}^1$-fibrations and locally nilpotent derivations over Noetherian normal domains obtained in [1], [4], [2], [5] emanate from the common generalisation.

We begin with a preliminary result which generalises [15, Proposition 2.11] to (not necessarily Noetherian) integral domains.

Proposition 7.5. *Let $R \subseteq A$ be integral domains with* $\operatorname{tr.deg}_R A = n$*. Then the following conditions are equivalent:*

(i) *A is an R-subalgebra of a finitely generated R-algebra.*
(ii) *For any $D \subseteq A$ such that $D = R^{[n]}$, there exists $f(\neq 0) \in D$ such that A_f is a finite D_f-module; in particular, A_f is a finitely generated R-algebra.*

Moreover, if A is normal and if D is a birational subring of A satisfying (ii)*, then $D_f = A_f$.*

Proof. It suffices to prove that (i) $\Rightarrow$ (ii). Suppose that A is an R-subalgebra of a finitely generated R-algebra B. Let $T = A \setminus \{0\}$, and let $T^{-1}Q$ be a maximal ideal of $T^{-1}B$, where Q is a prime ideal of B. Then $Q \cap A = (0)$ and $T^{-1}B/T^{-1}Q$ is algebraic over $T^{-1}A$. Hence, replacing B by B/Q, we may assume that B is an integral domain algebraic over A. Let $D = R[z_1, \ldots, z_n]$. Since B is finitely generated and algebraic over D, there exists $g(\neq 0) \in D$ such that B_g is a finite D_g-module. Since $A_g \subseteq B_g$, it follows from Lemma 7.6 below that there exists $h(\neq 0) \in D_g$ such that $(A_g)_h$ is a finite $(D_g)_h$-module. We may assume that $h \in D$. Then, setting $f = gh$, we know that A_f is a finite D_f-module, and hence A_f is a finitely generated R-algebra. □

We now prove an analogous result for modules.

Lemma 7.6. *Let R be an integral domain, let M be a torsion-free finite R-module and let N be an R-submodule of M. Then there exists $f(\neq 0) \in R$ such that N_f is a finite R_f-module.*

Proof. Since M is a finite R-module, there exists a free R-module F of finite rank and a surjective R-module homomorphism $\varphi \colon F \to M$. Let $L = \varphi^{-1}(N)$. Suppose that there exists $f(\neq 0) \in R$ such that L_f is a finite R_f-module. Since the surjective R-module homomorphism $\varphi|_L \colon L \to N$ induces a surjective R_f-module homomorphism $L_f \to N_f$, it then follows that N_f is a finite R_f-module. It thus suffices to show the assertion in the case where M is a free R-module. We use induction on $n := \operatorname{rank} M$.

We first consider the case where $n = 1$. Write $M = Ru$ with $u \in M$, and let $I = N :_R u$. Then I is an ideal of R, so $I_f = R_f$ for any $f(\neq 0) \in I$. Hence $N_f = M_f$, and therefore N_f is a finite R_f-module.

We now consider the general case. Let $u_1, \ldots, u_n$ be a free basis of M and let $M_1 = Ru_1$. Then M/M_1 is a free R-module of rank $n-1$ and the image of N in M/M_1 is isomorphic to $N/(N \cap M_1)$. By induction hypothesis, there exists $h(\neq 0) \in R$ such that $(N/(N \cap M_1))_h$ is a finite R_h-module. On the other hand, since $N \cap M_1$ is an R-submodule of $M_1 = Ru_1$, there exists $g(\neq 0) \in R$ such that $(N \cap M_1)_g$ is a finite R_g-module. Let $f = gh$. Then both $(N/(N \cap M_1))_f$ and $(N \cap M_1)_f$ are finite R_f-modules. Since $(N/(N \cap M_1))_f = N_f/(N \cap M_1)_f$, from this it follows that N_f is a finite R_f-module. This completes the proof. □

We also observe:

Lemma 7.7. *Let J be an ideal of a Krull domain R such that J is flat as an R-module. Then J is invertible.*

Proof. Any flat ideal J of a Krull domain R, being a divisorial ideal, is of the form $J = dR_a \cap R$ for some $d \in J$ and $a \in R$ (cf. Lemma 1.2). Hence J is invertible by [5, Lemma 2.5 and Remark 2.6]. □

Note that if A is a faithfully flat R-algebra locally $\mathbb{A}^1$ in codimension-one, then all results of Section 2, except Theorem 2.11, also hold in the case where the base ring R is a Krull domain. (In the proof of Corollary 2.7, one may use Lemma 7.7 to conclude that J is invertible; all other arguments in Section 2 go through over any Krull domain R.) Theorem 2.11 has generalisations in Theorems 7.12 and 7.14.

We shall now prove a few results on semi-faithfully flat algebras over Krull domains. Corollary 2.7 is generalised in the implication (iii) ⇔ (iv) of Theorem 7.8 below.

Theorem 7.8. *Let R be a Krull domain with field of fractions K and A a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Then the following conditions are equivalent:*

(i) *A_f is finitely generated over R for some $f(\neq 0) \in R$.*
(ii) *A is a subalgebra of a finitely generated R-algebra B.*
(iii) *The set Δ_0 corresponding to power function e for A is finite.*
(iv) *$A \cong \bigoplus_{n \geq 0} J^{(n)}$ for some divisorial ideal J of R.*

Proof. By Theorem 7.2, we can write $A = R(x, e, \mathbf{c})$ for some power function e and residue $\mathbf{c}$. The implication (iv) ⇒ (i) is obvious because $J_f = R_f$ for any $0 \neq f \in J$, and (i) ⇔ (ii) is proved in Proposition 7.5. Moreover, under condition (iii), we can apply the same argument as in the proof of Corollary 2.7 to obtain equation (2.12). Hence (iii) ⇒ (iv) holds. It thus suffices to prove that (i) ⇒ (iii).

Set $D = R[x]$. Then $D \subseteq A \subseteq K[x]$, and hence, by Proposition 7.5, there exists $f \in D$ such that $D[1/f] = A[1/f]$. Let b be the coefficient of the highest degree term of f as a polynomial in $R[x]$. We will show that $e(P) = 0$ for $P \in \Delta$ such that $b \notin P$. Indeed, suppose on the contrary that $r := e(P) > 0$ for some P with $b \notin P$. We have

$$A_P[1/f] = R_P[x, 1/f]. \tag{7.1}$$

Let p be a uniformizing parameter of PR_P. Then $A_P = R_P[p^{-r}(x-c)]$ for some $c \in R$ by Remark 4.4 (1) so that, by (7.1), we have $p^{-r}(x-c)f^n \in R_P[x]$ for a sufficiently large integer n. From this it follows that $b^n \in p^r R_P$, and hence $b \in PR_P \cap R = P$, a contradiction. Therefore $e(P) = 0$ for P with $b \notin P$, which implies that $\Delta_0 \subseteq \mathrm{Ass}_R\, R/bR$. Thus Δ_0 is a finite set. □

Corollary 7.9. *Let R be a Krull domain with field of fractions K. For an R-algebra A, the following conditions are equivalent:*

(i) *A is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one and A is a subalgebra of a finitely generated R-algebra.*
(ii) *$A \cong \bigoplus_{n\geq 0} J^{(n)}$ for some divisorial ideal J of R.*

As one application of our results, we show that [2, Proposition 3.3] and [2, Theorem 3.5] on inert subrings and kernels of locally nilpotent R-derivations of $R[X,Y]$ can be generalised to Krull domains of characteristic zero in the following form.

Corollary 7.10. *Let R be a Krull domain and A an R-algebra which is an inert subring of $R[X_1,\dots,X_m]$ $(= R^{[m]})$ of transcendence degree one over R. Then A is isomorphic to $\bigoplus_{n\geq 0} J^{(n)}$ for some divisorial ideal J of R.*

Proof. For every P in Δ, R_P is a DVR, and hence A_P, being an inert subring of the factorial ring $R_P[X_1,\dots,X_m]$, is a factorial ring of transcendence degree one over R_P. Thus $A_P = {R_P}^{[1]}$ for every $P \in \Delta$. If a, b is a regular sequence in R, it remains a regular sequence in $R[X_1,\dots,X_m]$ and hence in the inert subring A. Therefore, as $A \subseteq R[X_1,\dots,X_m]$, it is semi-faithfully flat. The result now follows from Corollary 7.9. □

Corollary 7.11. *Let R be a Krull domain of characteristic zero, D a non-zero locally nilpotent R-derivation of $R[X,Y]$ and $A = \ker\, D$. Then A is isomorphic, as a graded R-algebra, to $\bigoplus_{n\geq 0} J^{(n)}$ for some divisorial ideal J of R. Conversely, if $B = \bigoplus_{n\geq 0} J^{(n)}$ for some divisorial ideal J of R, then there exists a locally nilpotent R-derivation D of $R[X,Y]$ whose kernel is isomorphic to B as a graded R-algebra. In particular, B can be embedded as an inert subring of $R[X,Y]$.*

Proof. The first part follows from Corollary 7.10. For the converse, let $\Gamma = \mathrm{Ass}_R\,(R/J)$ and let $S = R \setminus (\bigcup_{P\in\Gamma} P)$. Since J is divisorial, there exists $y \in S$ and $a \in J$ such that $J = R \cap aR_y$. Then $J^{(n)} = R \cap a^n R_y$. The assertion now follows from the same argument as in the proof of [2, Theorem 3.5]. □

We now give criteria for a flat R-algebra to be the symmetric algebra of an invertible ideal of R. We shall use the notation as in Section 2.

Theorem 7.12. *Let R be a Krull domain with field of fractions K and A a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Then the following conditions are equivalent:*

(i) *A is finitely generated over R.*
(ii) *A is a subalgebra of a finitely generated R-algebra.*
(iii) *Δ_0 is a finite set.*
(iv) *A is a Krull domain and A has a retraction to R.*
(v) *The module $M (= M_1)$ defined by (2.13) is a finite R-module.*
(vi) *$A \cong Sym_R\, J$ for an invertible ideal J of R.*

Proof. The implications (vi) $\Rightarrow$ (i) $\Rightarrow$ (ii) and (vi) $\Rightarrow$ (iv) are obvious, while (ii)$\Leftrightarrow$(iii) is proved in Theorem 7.8. It thus suffices to show (iv) $\Rightarrow$ (iii)$\Rightarrow$ (vi) and (iii) $\Leftrightarrow$ (v).

(iv) $\Rightarrow$ (iii). Suppose that Δ_0 is an infinite set. Since A has a retraction to R, by Proposition 2.9, there exists $c \in R$ such that $x - c \in PA$ for each $P \in \Delta_0$, where x is a generic variable. Note that if $P, P' \in \Delta$ and $P \neq P'$, then $PA \neq P'A$ because of faithful flatness. Note also that the height of the prime ideal PA is one, because $A_P = {R_P}^{[1]}$. Hence $x - c$ is contained in infinitely many prime ideals in A of height one, which contradicts the assumption that A is a Krull ring.

(iii) $\Rightarrow$ (vi). By Theorem 7.8, we have $A \cong \bigoplus_{n\geq 0} J^{(n)}$ for some divisorial ideal J of R. Then the ideal J is R-flat because A is R-flat, and J is isomorphic to a direct summand of A. Hence J is invertible by Lemma 7.7, and $J^{(n)} = J^n$ for every $n \geq 0$.

(iii) $\Rightarrow$ (v). Set $\Gamma = \Delta_0$. Then $M = (I_\Gamma)^{-1}$ by definition (2.13), because Γ is the unique maximal element of Σ_0. On the other hand, we have

$$A = A_\Gamma = \bigoplus_n ({I_\Gamma}^n)^{-1}(x - c_\Gamma)^n$$

by (2.12), and hence M is R-flat, because $M \cong M(x - c_\Gamma)$ and $M(x - c_\Gamma)$ is a direct summand of A. Since $M(= (I_\Gamma)^{-1}) = {d_\Gamma}^{-1} J_\Gamma$ again by (2.12), it thus follows that J_Γ is R-flat. Therefore J_Γ is invertible by Lemma 7.7. Thus J_Γ is a finite R-module, and hence so is M, because $M \cong J_\Gamma$.

(v) $\Rightarrow$ (iii). Write $M = Rf_1 + \cdots + Rf_n$. Since $M = \bigcup_\Gamma {I_\Gamma}^{-1}$, there exists $\Gamma \in \Sigma$ such that $f_1, \ldots, f_n \in {I_\Gamma}^{-1}$. For such Γ, we have $M = {I_\Gamma}^{-1}$, which implies that ${I_{\Gamma'}}^{-1} = {I_\Gamma}^{-1}$, and hence $I_\Gamma = I_{\Gamma'}$, for any $\Gamma' \in \Sigma$ with $\Gamma \subseteq \Gamma'$. Now suppose that Δ_0 is an infinite set. Take $P \in \Delta_0 \setminus \Gamma$ and set $\Gamma' = \Gamma \cup \{P\}$. Then $\Gamma' \cap \Delta_0 \neq \Gamma \cap \Delta_0$, and hence $I_{\Gamma'} \neq I_\Gamma$ by Corollary 2.6, a contradiction. Thus Δ_0 is a finite set. $\square$

Remark 7.13. Let R be a Krull domain with field of fractions K. Then the following conditions are equivalent for any R-algebra A (see the argument in [5, Theorem 3.5]):

(a) $A_P = {R_P}^{[1]}$ for every $P \in \Delta$.
(b) $k(P) \otimes_R A = k(P)^{[1]}$ for every P in Spec R of height ≤ 1.
(c) $K \otimes_R A = K^{[1]}$, and for each P in Δ, $k(P) \otimes_R A$ is an integral domain, $k(P)$ is algebraically closed in $k(P) \otimes_R A$ and $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$.

Hence, the implication (ii) $\Rightarrow$ (vi) of Theorem 7.12 shows that the earlier results on affine fibrations in [1, Theorem 3.10], [4, Theorem 3.4] and [5, Theorem 3.5] hold over an arbitrary Krull domain and are really special cases of Theorem 7.2 (see also [5, Remark 3.8]).

We shall now deduce a result when A has a retraction to R.

Theorem 7.14. *Let R be a Krull domain with field of fractions K and A a flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Suppose that there exists a retraction $\varphi\colon A \to R$. If $\mathfrak{a} := \ker \varphi$ is a finitely generated ideal, then $A \cong Sym_R\, J$ for an invertible ideal J in R.*

Proof. Note that, for every maximal ideal m of R, we have $m = \varphi(mA)$, so $mA \neq A$. Thus A is faithfully flat over R. Hence, by Theorem 7.12, it suffices to show that Δ_0 is a finite set.

Write $A = R(x, e, \mathbf{c})$ by Theorem 7.2. Then $R[x] \subseteq A \subseteq K[x]$, and hence, letting $g_1, \ldots, g_m$ be generators of $\mathfrak{a}$, we can find a non-zero element $d \in R$ such that $dg_i \in R[x]$ for each $i = 1, \ldots, m$. Set $\Gamma_d := \{P \in \Delta \mid d \in P\}$, and let P be an element in $\Delta \setminus \Gamma_d$. We will show that $r := e(P) = 0$; if this is the case, then $\Delta_0 \subseteq \Gamma_d$, and hence Δ_0 is a finite set.

Suppose on the contrary that $r > 0$, and let $c = \varphi(x) \in R$. Then $A_P = R_P[p^{-r}(x-c)]$ by (4.6), because $c - c_\Gamma \in I_\Gamma$ for every $\Gamma \in \Sigma$ by Proposition 2.9. Thus, replacing x by $x - c$, we may assume that $\varphi(x) = 0$ and $A_P = R_P[z]$ where $z = p^{-r}x$. Then, letting $\varphi_P\colon A_P \to R_P$ be the retraction induced by φ, we have $\ker \varphi_P = zR_P[z]$. Note that $\ker \varphi_P = \mathfrak{a}_P$. Note also that $g_i \in R_P[x]$ for each i, because $dg_i \in R[x]$ and $d \notin P$. Thus, for each i, we have $g_i \in R_P[x] \cap zR_P[z] = xR_P[x]$ so that $g_i = xh_i(x)$, where $h_i(x) \in R_P[x]$. Now, since $z \in \mathfrak{a}_P$ and $\mathfrak{a}_P = (g_1, \ldots, g_m)R_P[z]$, we can write

$$p^{-r}x = xh_1(x)u_1(z) + \cdots + xh_m(x)u_m(z)$$

for some $u_1(z), \ldots, u_m(z) \in R_P[z]$. Dividing both sides of the above equation by x and substituting $x = 0$, we have $p^{-r} = h_1(0)u_1(0) + \cdots + h_m(0)u_m(0) \in R_P$. This is a contradiction, as desired. □

Remark 7.15. We make a few observations regarding the necessity of the various hypotheses in Theorem 7.14.

(1) The example of Eakin-Silver ([8, (3.15)]) shows that, in the above result, the hypothesis "A has a retraction to R" is necessary even to conclude that A is finitely generated.

(2) Even if A is finitely generated, the condition "A has a retraction to R" would be necessary to conclude that A is a symmetric algebra. (Consider $R = k[[t_1, t_2]]$, where k is any field and $A = R[X, Y]/(t_1X + t_2X - 1)$.)

(3) The condition "$\mathfrak{a}$ is finitely generated" is also necessary. Consider $R = \mathbb{Z}$ and $A = \mathbb{Z}[\{X/p \mid p \text{ a prime in } \mathbb{Z}\}]$. Over any DVR (R, tR), consider $A = R[X, X/t, X/t^2, \ldots]$.

In the case where R is a Noetherian normal domain, we have the following.

Theorem 7.16. *Let R be a Noetherian normal domain with field of fractions K and A a flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Suppose that there exists a retraction $\varphi : A \to R$. Let $\mathfrak{a} = \ker \varphi$. Then the following conditions are equivalent:*

- (i) *A is a Krull domain.*
- (ii) *$\mathfrak{a}$ is a finitely generated ideal of A.*
- (iii) *$\mathfrak{a}/\mathfrak{a}^2$ is a finitely generated R-module.*
- (iv) *$A \cong Sym_R\, J$ for some invertible ideal J of R.*

Proof. The equivalences (i) ⇔ (iv) ⇔ (ii) are immediate consequences of Theorems 7.12 and 7.14. It thus suffices to show (ii) ⇒ (iii) ⇒ (iv).

(ii) ⇒ (iii). Let $\mathfrak{a} = Af_1 + \cdots + Af_n$. Since $A = R + \mathfrak{a}$, from this it follows that $\mathfrak{a} = Rf_1 + \cdots + Rf_n + \mathfrak{a}^2$. Thus $\mathfrak{a}/\mathfrak{a}^2$ is a finite R-module.

(iii) ⇒ (iv). By Proposition 2.9, we have $A = \bigoplus_{n \geq 0} M^n(x - c)$ for some $c \in R$, where $M = \varinjlim (I_\Gamma)^{-1}$. Set $A_n = M^n(x - c)$ for $n \geq 0$. Then $\mathfrak{a} = \bigoplus_{n \geq 1} A_n$, and we have $A_1 \hookrightarrow \mathfrak{a}/\mathfrak{a}^2$. From this it follows that A_1 is a finite R-module, because so is $\mathfrak{a}/\mathfrak{a}^2$ and R is Noetherian. Thus M is a finite R-module, so (iv) holds by Theorem 7.12. This completes the proof. □

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