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PROOF OF A CONJECTURE BY AHLGREN AND ONO ON THE NON-EXISTENCE OF CERTAIN PARTITION CONGRUENCES

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ABSTRACT. Let $p(n)$ denote the number of partitions of n. Let $A, B \in \mathbb{N}$ with $A > B$ and $\ell \geq 5$ a prime, such that

$$p(An + B) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}.$$

Then we will prove that $\ell | A$ and $\left(\frac{24B-1}{\ell}\right) \neq \left(\frac{-1}{\ell}\right)$. This settles an open problem by Scott Ahlgren and Ken Ono. Our proof is based on results by Deligne and Rapoport.

1. INTRODUCTION

Let $p(n)$ denote the number of partitions of the positive integer n. Ramanujan [25] proved the following congruences,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \end{aligned}$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

We define $p(0) := 1$ and $p(n) := 0$ if $n < 0$.

Following [4], let δ_ℓ be defined by

$$\delta_\ell := \frac{\ell^2 - 1}{24}.$$

Then the Ramanujan congruences above may be written in the form

$$p(\ell n - \delta_\ell) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{Z}. \tag{1}$$

Ahlgren and Boylan [2] proved that (1) holds only for $\ell = 5, 7, 11$. This question was asked by Ramanujan in [25] and is one of the few results on non-existence of congruences for the partition function within infinitely many arithmetic progressions. The proof is partially based on a result by Kiming and Olsson [15] which, to the best of our knowledge, is the first paper that proves non-existence of congruences within infinitely many arithmetic progressions. In both papers [2, 15] the tools involved are modular forms modulo p as developed by Serre and Swinnerton-Dyer [28, 29]. This paper provides another non-existence result, but our methods are entirely different from the ones used in [2, 15].

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Many papers have been written on these three congruences and their extensions (already conjectured, and in some cases proved, by Ramanujan) to arbitrary powers of $5, 7$, and 11; see the fundamental works of Andrews, Atkin, Berndt, Dyson, Garvan, Kim, Ono, Ramanujan, Stanton and Swinnerton-Dyer [5, 6, 7, 9, 11, 13, 14, 26, 23, 24, 25]. According to [3] each of these extensions lies within the class $-\delta_\ell \pmod{\ell}$. We say that a congruence

$$p(An+B) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{Z}$$

lies within the class $\beta \pmod{\ell}$ iff $\{An+B\} \subseteq \{\ell n+\beta\}$. Here for integers x, y we define $\{xn+y\} := \{xn+y : n \in \mathbb{Z}\}$.

The important role that the class $-\delta_\ell \pmod{\ell}$ plays in the theory is illustrated by the work of Kiming and Olsson [15], who proved that if $\ell \geq 5$ is prime and $p(\ell n+\beta) \equiv 0 \pmod{\ell}$ for all n, then $\beta \equiv -\delta_\ell \pmod{\ell}$.

Atkin, Newman, O'Brien, and Swinnerton-Dyer [7, 8, 10, 19] found further congruences modulo ℓ^m for primes $\ell \leq 31$ and small m. Examples by Atkin and Newman in [7] and [19] show that not every congruence lies within the class $-\delta_\ell \pmod{\ell}$. For example when considering $\ell = 13$, we have

$$p(17303n+237) \equiv 0 \pmod{13}. \tag{2}$$

Ahlgren and Ono [1, 20] proved that if $\ell \geq 5$ is prime and m is any positive integer, then there are infinitely many congruences of the form

$$p(An+B) \equiv 0 \pmod{\ell^m}.$$

As in the case of Ramanujan's congruences, all of these congruences lie within the class $-\delta_\ell \pmod{\ell}$. According to [3]: "the research described up to now consists of a systematic theory of congruences that lies within the class $-\delta_\ell \pmod{\ell}$, as well as some sporadic examples (as (2)) of congruences that fall outside of this class. In view of this, it is natural to wonder what role the class $-\delta_\ell \pmod{\ell}$ truly plays". In their paper [3], Ahlgren and Ono show that in general this class is not as distinguished as one might have expected. Namely, they prove that this class is one of $(\ell+1)/2$ classes modulo ℓ in which the partition function enjoys similar congruence properties. The authors naturally ask if, for the remaining $(\ell-1)/2$ classes, there are congruences or not. In [3, Sect. 1] the authors write: "In Section 4, we consider those progressions $\ell n+\beta$ for $\beta \notin S_\ell$. We give heuristics that cast doubt on the existence of congruences within these progressions". In particular, Theorem 1.2, the main result of this paper, implies that indeed there are no such congruences. Before stating Theorem 1.2, we define for $\ell \geq 5$ a prime,

$$\epsilon_\ell := \left(\frac{-6}{\ell}\right)$$

and

$$S_\ell := \{\beta \in \{0, \ldots, \ell-1\} : \left(\frac{\beta+\delta_\ell}{\ell}\right) = 0 \text{ or } -\epsilon_\ell\}.$$

Remark 1.1. For $\ell \geq 5$ a prime, we have

$$S_\ell = \left\{\beta \in \{0, \ldots, \ell-1\} : \left(\frac{24\beta-1}{\ell}\right) \neq \left(\frac{-1}{\ell}\right)\right\}.$$

Motivated by the results in [3], our main result Theorem 1.2 was conjectured by Scott Ahlgren and Ken Ono in another joint paper [4, Conj. 5.4].

Theorem 1.2. *Suppose that $\ell \geq 5$ is prime, $A, B \in \mathbb{N}$ such that $A > B$ and*

$$p(An + B) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}.$$

Then $\ell | A$ and there exists $\beta \in S_\ell$ such that $B \equiv \beta \pmod{\ell}$.

The proof of Theorem 1.2 is based on results by Deligne and Rapoport [12] and some results in [18] or [22]. We are also using Theorems 4.2, 4.3 and 4.4 which were proven in a previous paper [21] and are also based on results in [12] and [18] or [22]. Our main contribution to the proof of Theorem 1.2 is Lemma 5.1 which takes the main part of the last section.

The organization of this paper is as follows. In Section 2 we prove Theorem 1.2 by citing several theorems in Section 4. In Section 3 we give some preliminaries to modular forms. In Section 4 we make a classification of congruences in the sense that we show that some congruences $p(An + B) \equiv 0 \pmod{\ell}$ imply that $p(A'n + B') \equiv 0 \mod \ell$, where $\{An + B\} \subseteq \{A'n + B'\}$. In Section 5 we prove our main contribution Lemma 5.1 which is needed for proving Lemma 4.7 in Section 4.

2. The proof

Let $A, B \in \mathbb{N}$ with $A > B$ such that

$$p(An + B) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}.$$

Then by Theorem 4.3 there exists a positive integer Q coprime to 6 dividing A and a $\overline{t} \in \{0, \dots, Q - 1\}$ with $\overline{t} \equiv B \pmod{Q}$ such that

$$p(Qn + \overline{t}) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}. \tag{3}$$

Then $\ell | Q$ because if not, Theorem 4.4 implies that there exists a positive integer n_0 such that $\ell \nmid p(Qn_0 + \overline{t})$ which contradicts (3). Hence we may write $Q = Q_0 \ell^r$ for some positive integers Q_0, r with $\gcd(Q_0, \ell) = 1$. Now if $24\overline{t} - 1 \equiv 0 \pmod{\ell}$, then we are finished. So assume that $24\overline{t} - 1 \not\equiv 0 \pmod{\ell}$. Then by Lemma 4.5,

$$p(Q_0 \ell n + \overline{t}^*) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}, \tag{4}$$

where $\overline{t}^*$ is the minimal non-negative integer such that $\overline{t}^* \equiv \overline{t} \pmod{Q_0 \ell}$. Next we apply Lemma 4.7 to the congruence (4) and we obtain $\left(\frac{24\overline{t}^* - 1}{\ell}\right) \neq \left(\frac{-1}{\ell}\right)$ which together with Remark 1.1 above implies Theorem 1.2.

3. Preliminaries

For f an analytic function on the complex upper half plane $\mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (the set of all 2×2 matrices with integer entries and determinant 1), we define

$$(f|_k \gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau), \quad \tau \in \mathbb{H},$$

where

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

For every positive integer M, we denote by $\Gamma(M)$ the set of all matrices in $\mathrm{SL}_2(\mathbb{Z})$ congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ modulo M. For k an integer and Γ a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ for some minimal N, we denote by $M_k(\Gamma)$ the set of all analytic functions on $\mathbb{H}$ such that:

- for all $\gamma \in \Gamma$ we have $f|_k \gamma = f$;

- for all $\xi \in \mathrm{SL}_2(\mathbb{Z})$ the function $(f|_k\xi)(\tau)$ admits a Laurent series expansion (with finite principal part) in the variable $q_N := e^{2\pi i\tau/N}$. We call this expansion the *q-expansion of $f|_k\gamma$*.

For N a positive integer, let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}.$$

In particular $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N)$. For our purposes, we will need the set

$$\Gamma_0(N)^* := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a, c > 0, \gcd(a, 6) = 1 \right\}$$

and the group

$$\Gamma_1^-(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a^2 \equiv 1 \pmod{N} \right\}.$$

Note in particular that $\Gamma_1^-(N) \supseteq \Gamma_1(N)$.

Lemma 3.1. *$\Gamma_1^-(N)$ is generated by $\Gamma_0(N)^* \cap \Gamma_1^-(N)$.*

Proof. First note that

$$\underbrace{\begin{pmatrix} 6N-1 & -1 \\ 6N & -1 \end{pmatrix}}_{\in \Gamma_0(N)^* \cap \Gamma_1^-(N)} \underbrace{\begin{pmatrix} 1 & -(h+1) \\ 6N & 1-6N(h+1) \end{pmatrix}}_{\in \Gamma_0(N)^* \cap \Gamma_1^-(N)} = \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \tag{5}$$

and

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Note that the two terms on the left hand side of (6) may be written as a product of matrices in $\Gamma_0(N)^* \cap \Gamma_1^-(N)$. Finally let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^-(N)$ with $c > 0$. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(c) & t\,\mathrm{sgn}(c) \\ 0 & \mathrm{sgn}(c) \end{pmatrix} \underbrace{\begin{pmatrix} a\,\mathrm{sgn}(c) - |c|t & b\,\mathrm{sgn}(c) - dt\,\mathrm{sgn}(c) \\ |c| & d\,\mathrm{sgn}(c) \end{pmatrix}}_{\in \Gamma_0(N)^* \cap \Gamma_1^-(N)},$$

where $t \in \mathbb{Z}$ is chosen such that $a\,\mathrm{sgn}(c) - |c|t > 0$ and $\gcd(a \cdot \mathrm{sgn}(c) - |c|t, 6) = 1$. Since we already showed in (5) and (6) that $\begin{pmatrix} \mathrm{sgn}(c) & t\,\mathrm{sgn}(c) \\ 0 & \mathrm{sgn}(c) \end{pmatrix}$ is a product of matrices in $\Gamma_0(N)^* \cap \Gamma_1^-(N)$, the proof is finished. □

4. A classification of congruences

Definition 4.1. For m a positive integer and $t \in \{0, \ldots, m-1\}$, we define $P_m(t)$ to be the set of all $t' \in \{0, \ldots, m-1\}$ such that

$$t' \equiv ta^2 + \frac{1-a^2}{24} \pmod{m},$$

for some $a \in \mathbb{Z}$ with $\gcd(a, 6m) = 1$.

The following three theorems were proven in [21].

Theorem 4.2. *Let l, m be positive integers and $t \in \{0, \ldots, m-1\}$ such that*

$$p(mn + t) \equiv 0 \pmod{l}, \quad n \in \mathbb{N}.$$

Then for all $t' \in P_m(t)$ we have

$$p(mn + t') \equiv 0 \pmod{l}, \quad n \in \mathbb{N}.$$

Theorem 4.3. *Let $Q, a, b, \nu \in \mathbb{N}$ and $t \in \{0, \ldots, 2^a 3^b Q - 1\}$ be such that $Q, \nu > 0$ and $\gcd(Q, 6) = 1$. Assume that*

$$p(2^a 3^b Qn + t) \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N}. \tag{7}$$

Then

$$p(Qn + \bar{t}) \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N},$$

where $\bar{t}$ is the minimal non-negative integer such that $t \equiv \bar{t} \pmod{Q}$.

Theorem 4.4. *Let Q, ν be positive integers such that $\gcd(Q, 6\nu) = 1$, $\nu \neq 1$ and $t \in \{0, \ldots, Q-1\}$. Then there exists an integer n such that $\nu \nmid p(Qn + t)$.*

We prove the following lemma.

Lemma 4.5. *Let Q, r, ν be positive integers, $\ell \geq 5$ a prime and $t \in \{0, \ldots, Q\ell^r - 1\}$. Let b be the maximal integer such that $\ell^b | (24t - 1)$. If $r \geq b + 1$ and*

$$p(Q\ell^r n + t) \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N}, \tag{8}$$

then

$$p(Q\ell^{b+1} n + \bar{t}) \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N},$$

where $\bar{t}$ is the minimal non-negative integer such that $t \equiv \bar{t} \pmod{Q\ell^{b+1}}$.

Before we can prove this result we need the following technical lemma.

Lemma 4.6. *Let Q be a positive integer, $v \in \mathbb{Z}$ with $v \neq 0$ and $p \geq 5$ a prime. Let b be maximal such that $p^b | v$. Then for any $l, r \in \mathbb{Z}$ with $r \geq b + 1$ there exists $a_{r,l} \in \mathbb{Z}$ with $\gcd(a_{r,l}, 6Qp) = 1$ such that*

$$a_{r,l}^2 v \equiv v + 24Qlp^{b+1} \pmod{Qp^r}. \tag{9}$$

Proof. Fix $l \in \mathbb{Z}$. By dividing both sides of (9) by p^b, we may assume that $b = 0$ and $p \nmid v$. Then the statement holds for $r = 1$ with $a_{r,l} = 1$. Next we prove that if the statement is true for $r = R \geq 1$, then it is also true for $r = R + 1$. By assumption there exists $a_{R,l}$ such that

$$a_{R,l}^2 v \equiv v + 24Qlp \pmod{Qp^R}. \tag{10}$$

We make the "ansatz" $a_{R+1,l} := a_{R,l} + 24Qp^R x$. Because of (10) there exists $s \in \mathbb{Z}$ satisfying

$$a_{R,l}^2 v - v - 24Qlp = Qp^R s. \tag{11}$$

We need to show that there exists $x \in \mathbb{Z}$ such that

$$(a_{R,l} + 24Qp^R x)^2 v \equiv v + 24Qlp \pmod{Qp^{R+1}}.$$

Together with (11) this reduces to

$$48 a_{R,l} v x + s \equiv 0 \pmod{p},$$

which is solvable for x because of $\gcd(48 a_{R,l} v, p) = 1$. Hence the proof is finished by the induction principle. ∎

Proof of Lemma 4.5. By (8) and Theorem 4.2 we have

$$p(Q\ell^r n + t') \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N}, \quad t' \in P_{Q\ell^r}(t).$$

By Definition 4.1, $t' \in P_{Q\ell^r}(t)$ iff there exists $a \in \mathbb{Z}$ with $\gcd(a, 6Q\ell) = 1$ such that

$$a^2(24t - 1) \equiv 24t' - 1 \pmod{Q\ell^r}, \tag{12}$$

and $t' \in \{0, \ldots, Q\ell^r - 1\}$. By Lemma 4.6 we obtain that for each $u \in \mathbb{Z}$ there exist $a_{r,u}$ with $\gcd(a_{r,u}, 6Q\ell) = 1$ such that

$$a_{r,u}^2(24t - 1) \equiv 24(t + Qu\ell^{b+1}) - 1 \pmod{Q\ell^r},$$

which implies that

$$\overline{t} + Qu\ell^{b+1} \in P_{Q\ell^r}(t).$$

Therefore by Theorem 4.2,

$$p(Q\ell^r n + \overline{t} + Qu\ell^{b+1}) \equiv 0 \pmod{\nu}, \quad n \in \mathbb{N},$$

for every $u \in \{0, \ldots, \ell^{r-b-1} - 1\}$. Since

$$Q\ell^r n + \overline{t} + Qu\ell^{b+1} = Q\ell^{b+1}(\ell^{r-b-1}n + u) + \overline{t}$$

and every non-negative integer m can be written as $m = \ell^{r-b-1}n + u$ for some non-negative integers n, u with $u \in \{0, \ldots, \ell^{r-b-1} - 1\}$, we conclude

$$p(Q\ell^{b+1}m + \overline{t}) \equiv 0 \pmod{\nu}, \quad m \in \mathbb{N}.$$

□

Lemma 4.7. *Let Q be a positive integer and $\ell \geq 5$ a prime such that $\gcd(Q, 6\ell) = 1$. Let $\beta \in \{0, \ldots, Q\ell - 1\}$. Assume that*

$$p(Q\ell n + \beta) \equiv 0 \pmod{\ell}, \quad n \in \mathbb{N}. \tag{13}$$

Then $\left(\frac{24\beta-1}{\ell}\right) \neq \left(\frac{-1}{\ell}\right)$.

The proof is based on the following theorem by Deligne and Rapoport.

Theorem 4.8 ([12, VII, Cor. 3.12])**.** *Let k, N be positive integers, p a prime number and p^m the highest power of p dividing N, $\gamma = \left(\begin{smallmatrix} a & b \\ p^m c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $f \in M_k(\Gamma(N))$. Let π be a prime ideal in $\mathbb{Z}[e^{2\pi i/N}]$ lying above p. Assume that the coefficients in the q-expansion of f are in $\mathbb{Z}[e^{2\pi i/N}]$. Let ν be a non-negative integer such that $f \equiv 0 \pmod{\pi^\nu}$. Then $f|_k\gamma \equiv 0 \pmod{\pi^\nu}$.*

Remark 4.9. For given positive integers k, N and $f \in M_k(\Gamma(N))$, with the coefficients of the q-expansion of f in $\mathbb{Z}[1/N, e^{2\pi i/N}]$, we obtain by Theorem 4.8 [12, VII, Cor. 3.13] that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the coefficients in the q-expansion of $f|_k\gamma$ have the same property. In this case there also exists a power N^j of N such that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the coefficients in the q-expansion of $N^j f|_k\gamma$ are in $\mathbb{Z}[e^{2\pi i/N}]$ (see for example [12, VII, Cor 3.11]). Consequently for a given prime p and a prime ideal π in $\mathbb{Z}[e^{2\pi i/N}]$ lying above p, it makes sense to write $f|_k\gamma \equiv 0 \pmod{\pi^\nu}$ if there exists $M \in \mathbb{Z}$ with $M \notin \pi$ such that all the coefficients in the q-expansion of $Mf|_k\gamma$ lie in the ideal π^ν.

We also need some additional simple facts.

Definition 4.10. For m a positive integer coprime to 6, $t \in \{0, \ldots, m-1\}$ and $r \in \mathbb{Z}$, we define

$$g(m,t,r,\tau) := \frac{1}{m} \sum_{\lambda=0}^{m-1} e^{\frac{2\pi i \lambda(-24t-r)}{m}} \eta^r \left(\frac{\tau + 24\lambda}{m} \right), \quad \tau \in \mathbb{H}.$$

The proof of the following lemma is simple and can be found in [22, Lem. 1.12].

Lemma 4.11. *Let m be a positive integer coprime to 6, $t \in \{0, \ldots, m-1\}$ and $r \in \mathbb{Z}$. Then*

$$g(m,t,r,\tau) = q^{\frac{24t+r}{24m}} \sum_{n=0}^{\infty} p_r(mn+t) q^n, \quad \tau \in \mathbb{H}, \quad (q = e^{2\pi i \tau}).$$

Proof of Lemma 4.7. Assume that

$$\left(\frac{24\beta - 1}{\ell} \right) = \left(\frac{-1}{\ell} \right).$$

Then there exists $a \in \mathbb{Z}$ such that

$$(24\beta - 1)a^2 \equiv -1 \pmod{\ell}, \tag{14}$$

which implies together with $\gcd(Q, 6\ell) = 1$ that there exists $\overline{a} \in \mathbb{N}$ with $\gcd(\overline{a}, 6Q\ell) = 1$ such that

$$\overline{a} \equiv Qa \pmod{\ell}. \tag{15}$$

Let $\overline{\beta} \in \{0, \ldots, Q\ell - 1\}$ be uniquely defined by the relation

$$\overline{a}^2(24\beta - 1) \equiv (24\overline{\beta} - 1) \pmod{Q\ell}. \tag{16}$$

For $k \in \mathbb{Z}$ we have, by Lemma 4.11,

$$G^{(k)}_{Q\ell,\overline{\beta}} := \eta^{24k} \left(q^{\frac{24\overline{\beta}-1}{24Q\ell}} \sum_{n=0}^{\infty} p(Q\ell n + \overline{\beta}) q^n \right)^{24Q\ell} = \eta^{24k} (g(Q\ell, \overline{\beta}, -1, \tau))^{24Q\ell}, \tag{17}$$

where $\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$, $\tau \in \mathbb{H}$ is the Dedekind eta function and $\eta^{24} \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ (see [27, p. 95, Th. 6]). By [22, Lem. 2.10],

$$(g(Q\ell, \overline{\beta}, -1, \tau))^{24Q\ell}|_{-12Q\ell}\gamma = (g(Q\ell, \overline{\beta}, -1, \tau))^{24Q\ell}$$

for all $\gamma \in \Gamma_0(Q\ell)^*$ such that $a^2 \equiv 1 \pmod{Q\ell}$ or equivalently for all $\gamma \in \Gamma_0(Q\ell)^* \cap \Gamma_1^-(Q\ell)$. Since, by Lemma 3.1, the set $\Gamma_0(Q\ell)^* \cap \Gamma_1^{-1}(Q\ell)$ generates the group $\Gamma_1^-(Q\ell) \supseteq \Gamma_1(Q\ell)$, it follows that for sufficiently large k we have

$$G^{(k)}_{Q\ell,\overline{\beta}} \in M_{12(k-Q\ell)}(\Gamma_1^-(Q\ell)) \supseteq \Gamma_1(Q\ell). \tag{18}$$

From now on we fix k to be any integer such that (18) holds.

Let X, Y be integers such that

$$24^2\ell^2 X + QY = 1. \tag{19}$$

We apply Lemma 5.1 with $t = \overline{\beta}$ and $r = -1$. We then obtain

(20)

$$\begin{aligned} &e^{-\frac{\pi i Q}{12}} e^{-\frac{48\pi i X(24\overline{\beta}-1)}{Q}} g(Q\ell, \overline{\beta}, -1, \gamma\tau)(\ell\tau + QY)^{1/2} \\ =&\frac{1}{Q} \sum_{d|Q} d^{-1/2} e^{-\frac{\pi i(d-1)}{4}} (-1)^{\frac{Q+1}{2}\frac{d-1}{2}} e^{\frac{2\pi i(24t_d-1)d^2\tau}{24Q\ell}} \times \sum_{n=0}^{\infty} e^{\frac{2\pi i n d^2 \tau}{Q}} p(\ell n + t_d) T(n,d), \end{aligned}$$

where t_d is the unique integer satisfying

$$(24\overline{\beta}-1) \equiv d^2(24t_d-1) \pmod{\ell}, \quad 0 \le t_d < \ell. \tag{21}$$

Next we observe that $T(n,Q)=1$ for all $n \in \mathbb{N}$ because of $\left(\frac{a}{1}\right)=1$ for $a \in \mathbb{Z}$. Let $m := Q\ell$ and $q_m := e^{\frac{2\pi i\tau}{m}}$. Because of (14)–(16) and (21), we have $t_Q = 0$ and consequently (20) transforms into

(22)

$$\begin{aligned} & e^{-\frac{\pi i Q}{12}} e^{-\frac{48\pi i X(24\overline{\beta}-1)}{Q}} g(Q\ell, \overline{\beta}, -1, \gamma\tau)(\ell\tau + QY)^{1/2} \\ =& \frac{1}{Q} Q^{-1/2} e^{-\frac{\pi i(Q-1)}{4}} q_m^{-\frac{Q^2}{24}} \sum_{n=0}^{\infty} q_m^{Q^2\ell n} p(\ell n) \\ & + \frac{1}{Q} \sum_{d|Q, d\neq Q} d^{-1/2} e^{-\frac{\pi i(d-1)}{4}} (-1)^{\frac{Q+1}{2}\frac{d-1}{2}} q_m^{\frac{(24t_d-1)d^2}{24}} \sum_{n=0}^{\infty} q_m^{n\ell d^2} p(\ell n + t_d) T(n,d) \\ =& q_m^{-Q^2/24} F(q_m), \end{aligned}$$

where

(23)

$$\begin{aligned} F(q_m) :=& \frac{1}{Q} Q^{-1/2} e^{-\frac{\pi i(Q-1)}{4}} \sum_{n=0}^{\infty} q_m^{Q^2\ell n} p(\ell n) \\ + & \frac{1}{Q} \sum_{d|Q, d\neq Q} d^{-1/2} e^{-\frac{\pi i(d-1)}{4}} (-1)^{\frac{Q+1}{2}\frac{d-1}{2}} q_m^{\frac{(24t_d-1)d^2+Q^2}{24}} \sum_{n=0}^{\infty} q_m^{n\ell d^2} p(\ell n + t_d) T(n,d). \end{aligned}$$

We note from (23) that $F(q_m)$ is a Laurent series[1] in powers of q_m because of $d^2(24t_d-1)+Q^2 \equiv 0 \pmod{24}$ because of $Q^2, d^2 \equiv 1 \pmod{24}$. Again from (23) we observe that the coefficients of $F(q_m)$ belong to $\mathbb{Z}[1/Q, e^{2\pi i/Q}]$ because of

$$T(n,d) \in \mathbb{Z}[1/Q, e^{2\pi i/Q}] \tag{24}$$

and

$$d^{-1/2} e^{-\frac{\pi i(d-1)}{4}} \in \mathbb{Z}[1/Q, e^{2\pi i/Q}], \tag{25}$$

for $d|Q$. The relation (24) is seen directly from the definition of $T(n,d)$. To prove (25) we use the relation $d^{1/2} e^{\frac{\pi i(d-1)}{4}} = \pm\epsilon(d) d^{1/2}$, where

$$\epsilon(d) := \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ i, & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

By [17, p. 87] we have $\epsilon(d) d^{1/2} = \sum_{\lambda=0}^{d-1} e^{2\pi i \lambda^2/d}$. This implies that

$$d^{-1/2} e^{-\frac{\pi i(d-1)}{4}} = d^{-1}(-1)^{\frac{d-1}{2}} \sum_{\lambda=0}^{d-1} e^{2\pi i\lambda^2/d} \in \mathbb{Z}[1/Q, e^{2\pi i/Q}].$$

Furthermore, if π is a prime ideal in $\mathbb{Z}[e^{\frac{2\pi i}{\ell Q}}]$ lying above ℓ, then it makes sense to reduce the coefficients of $F(q_m)$ modulo π because the coefficients of $F(q_m)$ are in $\mathbb{Z}[1/Q, e^{2\pi i/Q}]$ and Q is invertible modulo π. We observe that

$$F(q_m) \not\equiv 0 \pmod{\pi} \tag{26}$$

[1] In fact $F(q_m)$ is a Laurent series in powers of q_m^{ℓ} because of $d^2(24t_d-1)+Q^2 \equiv 0 \pmod{\ell}$ because of (14)–(16) and (21).

because by (22) the order of $F(q_m)$ is 0 and the coefficient of the constant term is equal to $\frac{1}{Q^{3/2}}e^{-\frac{\pi i(Q-1)}{4}} \not\equiv 0 \pmod{\pi}$.

By (22), Definition 4.10 and Lemma 4.11 we have

$$G^{(k)}_{Q\ell,\overline{\beta}}|_\kappa \begin{pmatrix} 1 & -24^2\ell X \\ \ell & QY \end{pmatrix} = \eta^{24k} q^{-Q^2} F^{24Q\ell}(q_m),$$

where $\kappa := 12(k - Q\ell)$. Then (26) implies

$$G^{(k)}_{Q\ell,\overline{\beta}}|_\kappa \begin{pmatrix} 1 & -24^2\ell X \\ \ell & QY \end{pmatrix} \not\equiv 0 \pmod{\pi},$$

and by Theorem 4.8,

$$G^{(k)}_{Q\ell,\overline{\beta}} \not\equiv 0 \pmod{\pi},$$

and consequently $p(Q\ell n + \overline{\beta}) \not\equiv 0 \pmod{\ell}$ for some $n \in \mathbb{N}$ and since $\beta \in P_{Q\ell}(\overline{\beta})$ because of (16) and Definition 4.1, we obtain by Theorem 4.2 that $p(Q\ell n' + \beta) \not\equiv 0 \pmod{\ell}$ for some $n' \in \mathbb{N}$ which is a contradiction to our assumption (13).

5. A modular substitution formula

Lemma 5.1. *Let $\ell \geq 5$ be a prime and Q a positive integer coprime to 6, $t \in \{0, \ldots, m-1\}$, $r \in \mathbb{Z}$ and $X, Y \in \mathbb{Z}$ such that*

$$24^2\ell^2 X + QY = 1. \tag{27}$$

Let $\gamma := \left(\begin{smallmatrix} 1 & -24^2\ell X \\ \ell & QY \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. For any integers s and $d|Q$ such that $\gcd(s, Q/d) = 1$, let $\iota_{s,d}$ be any integer satisfying $s \cdot \iota_{s,d} \equiv 1 \pmod{Q/d}$. Then

(28)
$$\begin{aligned} &e^{\frac{\pi i r Q}{12}} e^{-\frac{48\pi i X(24t+r)}{Q}} g(Q\ell, t, r, \gamma\tau)(\ell\tau + QY)^{-r/2} \\ =&\frac{1}{Q}\sum_{d|Q} e^{\frac{\pi i r(d-1)}{4}} d^{r/2}(-1)^{r\frac{Q+1}{2}\frac{d-1}{2}} e^{\frac{2\pi i(24t_d+r)d^2\tau}{24Q\ell}} \sum_{n=0}^{\infty} e^{\frac{2\pi i n d^2\tau}{Q}} p_r(\ell n + t_d) T(n, d), \end{aligned}$$

where $t_d \in \{0, \ldots, \ell-1\}$ is the unique integer satisfying

$$24t + r \equiv d^2(24t_d + r) \pmod{\ell} \tag{29}$$

and

$$T(n, d) := \sum_{\substack{0 \leq s < Q/d \\ \gcd(s, Q/d)=1}} \left(\frac{24\ell s}{Q/d}\right)^r e^{-\frac{48\pi i X}{Q/d}\{\iota_{s,d}(24(\ell n + t_d)+r) + s(24t+r)\}}.$$

Proof. Note that for any $\lambda \in \{0, \ldots, \ell Q - 1\}$ we have $\lambda \equiv -24\ell X + sd + lQ \pmod{\ell Q}$ for some unique $(d, s, l) \in S$, where

$$S := \{(d, s, l) : d|Q, s \in \{0, \ldots, Q/d\}, \gcd(s, Q/d) = 1, l \in \{0, \ldots, \ell - 1\}\}.$$

This is proven by observing that $|S| := \ell \sum_{d|Q} \varphi(Q/d) = \ell Q$ and by proving that if $\lambda_1 = -24\ell X + s_1 d_1 + l_1 Q$ and $\lambda_2 = -24\ell X + s_2 d_2 + l_2 Q$ satisfy $\lambda_1 \equiv \lambda_2 \pmod{\ell Q}$

for some $(d_1, s_1, l_1), (d_2, s_2, l_2) \in S$, then $(d_1, s_1, l_1) = (d_2, s_2, l_2)$. By Definition 4.10 we obtain

(30)
$$g(Q\ell, t, r, \gamma\tau) = \frac{1}{Q\ell} \sum_{d|Q} \sum_{\substack{0 \leq s < \frac{Q}{d}-1 \\ \gcd(s, mQ/d)=1}} \sum_{l=0}^{m_C-1} e^{\frac{2\pi i(-24\ell X+sd+lQ)(-24t-r)}{Q\ell}} \eta^r\left(\frac{\gamma\tau+24(-24\ell X+sd+lQ)}{Q\ell}\right).$$

We fix $(d, l, s) \in S$ and define

$$\lambda := -24\ell X + sd + lQ. \tag{31}$$

One verifies that

$$\frac{\gamma\tau + 24\lambda}{m} = M_\lambda \frac{d\tau + (-24\ell X + \lambda QY)24^2 x_\lambda}{Q\ell/d}, \tag{32}$$

where $M_\lambda := \begin{pmatrix} \frac{1+24\lambda\ell}{d} & 24\ell y_\lambda(-24\ell X+\lambda QY) \\ \frac{\ell^2 Q}{d} & \ell^2 Q^2 Y y_\lambda + 24 x_\lambda \end{pmatrix}$ and x_λ, y_λ are integers such that

$$24(24\lambda\ell + 1)x_\lambda + \ell^2 Q y_\lambda = d. \tag{33}$$

Knopp [16, p. 51] proved that for each $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with c odd we have

$$\eta(\xi\tau) = (c\tau + d)^{1/2} v_\eta(\xi)\eta(\tau), \quad \tau \in \mathbb{H}, \tag{34}$$

where

$$v_\eta(\xi) := \left(\frac{d}{|c|}\right) e^{\frac{\pi i}{12}(c(a+d)-bd(c^2-1)-3c)}. \tag{35}$$

By (32) and (34) we obtain

$$\eta\left(\frac{\gamma\tau + 24\lambda}{Q\ell}\right) = (d(\ell\tau + QY))^{1/2} v_\eta(M_\lambda)\eta\left(\frac{d\tau + 24^2 x_\lambda(-24\ell X + \lambda QY)}{Q\ell/d}\right). \tag{36}$$

We have $v_\eta(M_\lambda) = \left(\frac{24x_\lambda}{Q/d}\right) e^{\frac{\pi i Q}{12}(2-3d)}$ because of $M_\lambda \equiv \begin{pmatrix} d & 0 \\ Qd & d \end{pmatrix} \pmod{24}$ because of $d^2 \equiv 1 \pmod{24}$ and because of $y_\lambda \equiv Qd \pmod{24}$ by (33). By (31) and (33) we obtain

$$24x_\lambda \cdot 24\ell s \equiv 1 \pmod{Q/d} \tag{37}$$

which implies $\left(\frac{24x_\lambda}{Q/d}\right) = \left(\frac{24\ell s}{Q/d}\right)$ by using standard properties of the Jacobi symbol. Consequently, we obtain

$$v_\eta(M_\lambda) = \left(\frac{24\ell s}{Q/d}\right) e^{\frac{\pi i Q}{12}(2-3d)} = \left(\frac{24\ell s}{Q/d}\right) e^{\frac{\pi i(d-1)}{4}} e^{-\frac{\pi i Q}{12}} (-1)^{\frac{Q+1}{2}\frac{d-1}{2}}. \tag{38}$$

By (36) and (38) we obtain

(39)
$$e^{\frac{\pi i Q}{12}}(\ell\tau + QY)^{-1/2}\eta\left(\frac{\gamma\tau + 24\lambda}{Q\ell}\right) = d^{1/2} e^{\frac{\pi i(d-1)}{4}} \left(\frac{24\ell s}{Q/d}\right)(-1)^{\frac{Q+1}{2}\frac{d-1}{2}} \eta\left(\frac{d\tau + 24^2 x_\lambda(-24\ell X + \lambda QY)}{Q\ell/d}\right).$$

Next we obtain an expression for $H := 24x_\lambda(-24\ell X + \lambda QY)$ modulo $\ell Q/d$.

Multiplying both sides of (37) by $\ell X \iota_{s,d}$ we obtain $x_\lambda \equiv \ell X \iota_{s,d} \pmod{Q/d}$ because of $s\iota_{s,d} \equiv 1 \pmod{Q/d}$ and $24^2\ell^2 X \equiv 1 \pmod{Q/d}$ because of (27). This implies that

$$H \equiv -24^2 \ell X \cdot \ell X \iota_{s,d} \equiv -X i_{s,d} \pmod{Q/d}. \tag{40}$$

By (33) we obtain $x_\lambda \equiv d/24 \pmod{\ell}$. This implies together with (31) that

$$H \equiv 24(sd + lQ)QY \cdot d/24 \equiv sd^2 + lQd \pmod{\ell} \tag{41}$$

because of $QY \equiv 1 \pmod{\ell}$ because of (27).

Because of (41) there exists an integer v such that $H = sd^2 + lQd + v\ell$ which implies, together with (40) and $24^2\ell^2 X \equiv 1$ by (27), that

$$v \equiv 24^2 X\ell(-X\iota_{s,d} - sd^2 - lQd).$$

Consequently,

$$\begin{aligned} H \equiv & sd^2 + lQd + 24^2\ell^2 X(-X\iota_{s,d} - sd^2 - lQd) = QY(sd^2 + lQd) - 24^2\ell^2X^2\iota_{s,d} \\ \equiv & sd^2YQ + ldQ - 24^2\ell^2X^2\iota_{s,d} \pmod{\ell Q/d}. \end{aligned}$$

Next note that if v_1 and v_2 are integers such that $v_2 = v_1 + t(24Q\ell/d)$ for some integer t, then $\eta\left(\frac{d\tau+v_2}{\ell Q/d}\right) = \eta\left(\frac{d\tau+v_1}{\ell Q/d} + 24t\right) = \eta\left(\frac{d\tau+v_1}{Q\ell/d}\right)$, because of $\eta(\tau+24) = \eta(\tau)$. Using this fact with $v_1 = 24H$ and $v_2 = 24(sd^2YQ + ldQ - 24^2\ell^2X^2\iota_{s,d})$ on (39), we obtain

$$\begin{aligned} & e^{\frac{\pi i Q}{12}}(\ell\tau + QY)^{-1/2}\eta\left(\frac{\gamma\tau + 24\lambda}{Q\ell}\right) \\ =& d^{1/2} e^{\frac{\pi i(d-1)}{4}}\left(\frac{24\ell s}{Q/d}\right)(-1)^{\frac{Q+1}{2}\frac{d-1}{2}}\eta\left(\frac{\tau'(s,d) + 24d^2 l}{\ell}\right), \end{aligned} \tag{42}$$

where

$$\tau'(s,d) := \frac{d\tau + 24(sd^2YQ - 24^2\ell^2X^2\iota_{s,d})}{Q/d}.$$

By (42) and (30)

$$\begin{aligned} & e^{\frac{\pi i r Q}{12}}(\ell\tau + YQ)^{-r/2} g(\ell Q, t, r, \gamma\tau) \\ =& \frac{1}{\ell Q}\sum_{d|Q}\sum_{\substack{0\le s<\frac{Q}{d}-1 \\ \gcd(s,Q/d)=1}}\sum_{l=0}^{\ell-1} e^{\frac{2\pi i(-24\ell X + sd + lQ)(-24t-r)}{\ell Q}} \\ & \times d^{r/2} e^{\frac{\pi i r(d-1)}{4}}(-1)^{r\frac{Q+1}{2}\frac{d-1}{2}}\left(\frac{24\ell s}{Q/d}\right)^r \eta^r\left(\frac{\tau'(s,d) + 24d^2 l}{\ell}\right) \\ =& e^{\frac{48\pi i X(24t+r)}{Q}}\frac{1}{\ell Q}\sum_{d|Q} d^{r/2} e^{\frac{\pi i r(d-1)}{4}}\sum_{\substack{0\le s<\frac{m_0}{d}-1 \\ \gcd(s,Q/d)=1}} e^{\frac{2\pi i s(-24t-r)}{\ell Q/d}} \\ & \times \sum_{l=0}^{\ell-1} e^{\frac{2\pi i l(-24t-r)}{\ell}} e^{\frac{\pi i r(d-1)}{4}}(-1)^{r\frac{Q+1}{2}\frac{d-1}{2}}\left(\frac{24\ell s}{Q/d}\right)^r \eta^r\left(\frac{\tau'(s,d) + 24d^2 l}{\ell}\right). \end{aligned} \tag{43}$$

Summing in the last sum over any set of modulo ℓ representatives does not change the value of the sum. In particular, we make the substitution $l = (YQ/d)^2 l'$ and observe that $d^2(YQ/d)^2 \equiv 1 \pmod{\ell}$ because of (27). Thus we obtain together with Definition 4.10 and (29),

(44)
$$
\begin{aligned}
&e^{\frac{\pi i r Q}{12}} e^{-\frac{48\pi i X(24t+r)}{Q}} g(m,t,r,\gamma\tau)(\ell\tau+YQ)^{-r/2}\\
=&\frac{1}{Q}\sum_{d|Q} d^{r/2} e^{\frac{\pi i r(d-1)}{4}}(-1)^{r\frac{Q+1}{2}\frac{d-1}{2}} \sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{24\ell s}{Q/d}\right)^r e^{\frac{2\pi i(-24t-r)s}{\ell Q/d}} g(\ell,t_d,r,\tau'(s,d)).
\end{aligned}
$$

By Lemma 4.11,

$$
\begin{aligned}
&\sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{s}{Q/d}\right)^r e^{\frac{2\pi i(-24t-r)s}{\ell Q/d}} g(\ell,t_d,r,\tau'(s,d))\\
=&\sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{s}{Q/d}\right)^r e^{\frac{2\pi i(-24t-r)s}{\ell Q/d}} e^{\frac{2\pi i(24t_d+r)\tau'(s,d)}{24\ell}} \sum_{n=0}^{\infty} p_r(\ell n+t_d) e^{2\pi i n\tau'(s,d)}\\
=&\sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{s}{Q/d}\right)^r e^{\frac{2\pi i(-24t-r)s}{\ell Q/d}} e^{\frac{2\pi i(24t_d+r)\left\{\frac{d\tau+24(sd^2YQ-24^2\ell^2X^2\iota_{s,d})}{Q/d}\right\}}{24\ell}}\\
&\times\sum_{n=0}^{\infty} p_r(\ell n+t_d) e^{2\pi i n\left\{\frac{d\tau+24(sd^2YQ-24^2\ell^2X^2\iota_{s,d})}{Q/d}\right\}}\\
=&e^{\frac{2\pi i d^2(24t_d+r)\tau}{24\ell Q}} \sum_{n=0}^{\infty} p_r(\ell n+t_d) e^{\frac{2\pi i n d^2\tau}{Q}}\\
&\times \sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{s}{Q/d}\right)^r e^{-\frac{2\pi i}{\ell Q/d}\{24^2\ell^2X^2\iota_{s,d}((24t_d+r)+24\ell n)+s(24t+r-(24t_d+r)d^2QY)\}}\\
=&e^{\frac{2\pi i d^2(24t_d+r)\tau}{24\ell Q}} \sum_{n=0}^{\infty} p_r(\ell n+t_d) e^{\frac{2\pi i n d^2\tau}{Q}}\\
&\times \underbrace{\sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)}} \left(\frac{s}{Q/d}\right)^r e^{-\frac{48\pi i\ell X}{m/d}\{24\ell X\iota_{s,d}(24(n\ell+t_d)+r)+24\ell s(24t+r)\}}}_{=T(n,d)},
\end{aligned}
$$

by first substituting $d^2QY(24t_d+r) \equiv QY(24t+r) \pmod{\ell Q}$ which follows from (29) and next substituting $1-QY = 24^2\ell^2X$ because of (27). Next we exploit the identity,

$$
\begin{aligned}
T(n,d) &= \sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)=1}} \left(\frac{s}{Q/d}\right)^r e^{-\frac{48\pi i\ell X}{m/d}\{24\ell x\iota_{s,d}(24(n\ell+t_d)+r)+24\ell s(24t+r)\}}\\
&= \sum_{\substack{0\le s<Q/d\\ \gcd(s,Q/d)=1}} \left(\frac{24\ell s}{Q/d}\right)^r e^{-\frac{48\pi i X}{Q/d}\{\iota_{s,d}(24(n\ell+t_d)+r)+s(24t+r)\}},
\end{aligned}
$$

because $s \mapsto 24\ell X s$ is a bijection modulo Q/d together with $24^2\ell^2 x \equiv 1 \pmod{Q/d}$.

Finally substituting in (44) we obtain (28). □

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