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# GLEASON PARTS AND COUNTABLY GENERATED CLOSED IDEALS IN $H^\infty$

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ABSTRACT. It is proved that a countably generated closed ideal in $H^\infty$ whose common zero set is contained in the union set of nontrivial Gleason parts of $H^\infty$ is generated by two Carleson-Newman Blaschke products as a closed ideal.

## 1. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ with the supremum norm $\|\cdot\|_\infty$. We denote by $M(H^\infty)$ the maximal ideal space of $H^\infty$, that is, $M(H^\infty)$ is the family of nonzero multiplicative linear functionals on $H^\infty$ with the weak*-topology. For a subset $E$ of $M(H^\infty)$, we denote by $\overline{E}$ the closure of $E$ in $M(H^\infty)$. We identify a function $f$ in $H^\infty$ with its Gelfand transform $\widehat{f}(m) = m(f), m \in M(H^\infty)$, so we think of $f$ as a continuous function on $M(H^\infty)$. For a sequence $\{a_n\}_n$ in $\mathbb{D}$ satisfying $\sum_{n=1}^\infty (1-|a_n|) < \infty$, we have the Blaschke product

$$b(z) = \prod_{n=1}^\infty \frac{-\overline{a}_n}{|a_n|}\frac{z-a_n}{1-\overline{a}_n z}, \quad z \in \mathbb{D},$$

where if $a_n = 0$, we consider that $-\overline{a}_n/|a_n| = 1$. We call $\{a_n\}_n$ and $b(z)$ interpolating if for any bounded sequence of complex numbers $\{c_n\}_n$ there exists $f$ in $H^\infty$ such that $f(a_n) = c_n$ for every $n \ge 1$. In [2], Carleson gave a characterization of interpolating sequences. A Blaschke product $B$ is said to be Carleson-Newman if $B = \prod_{j=1}^m b_j$ for finitely many interpolating Blaschke products $b_1, b_2, \cdots, b_m$. In this case, there are many ways to give such a factorization. If $m$ is the minimal number of interpolating Blaschke products, $B$ is said to be a Carleson-Newman Blaschke product of order $m$. In the study of the structure of $H^\infty$, Carleson-Newman Blaschke products have played an important role (see [3, 5, 8, 11]). For Blaschke products $b_1$ and $b_2$, we write $b_1 \prec b_2$ if $b_1$ is a subproduct of $b_2$.

For $x, y \in M(H^\infty)$, the pseudo-hyperbolic distance is defined by

$$\rho(x,y) = \sup\left\{|f(x)| : f(y) = 0, f \in H^\infty, \|f\|_\infty \le 1\right\}.$$

A subset $E$ of $M(H^\infty)$ is said to be $\rho$-separated if there is $\varepsilon > 0$ such that $\rho(x,y) \ge \varepsilon$ for every $x, y \in E$ with $x \ne y$. The set

$$P(x) = \{y \in M(H^\infty) : \rho(y,x) < 1\}$$

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is called the Gleason part of $H^\infty$ containing $x \in M(H^\infty)$. If $P(x) \neq \{x\}$, $P(x)$ is said to be nontrivial. We denote by $G$ the union set of all nontrivial Gleason parts in $M(H^\infty)$. In [7] (see also [3]), Hoffman studied the structure of Gleason parts of $H^\infty$ extensively. For $x \in M(H^\infty)$, he proved that $x \in G$ if and only if there is an interpolating Blaschke product $b$ satisfying $b(x) = 0$. He also proved that for an interpolating Blaschke product $b$, there exists $\varepsilon > 0$ such that $\{|b| < \varepsilon\} \subset G$, where

$$\{|b| < \varepsilon\} = \{x \in M(H^\infty) : |b(x)| < \varepsilon\}.$$

This fact shows that $G$ is an open subset of $M(H^\infty)$, and for a Carleson-Newman Blaschke product $B$ there is $\varepsilon > 0$ such that $\{|B| < \varepsilon\} \subset G$. Hoffman also showed that for a nontrivial Gleason part $P(x)$ of $H^\infty$, there is a one-to-one, onto and continuous map $L_x : \mathbb{D} \to P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. For $f \in H^\infty$, we write

$$Z(f) = \{x \in M(H^\infty) : f(x) = 0\}.$$

It is known that if $b$ is an interpolating Blaschke product with zeros $\{z_n\}_n$ in $\mathbb{D}$, then $Z(b) = \overline{\{z_n\}_n}$, $Z(b)$ is $\rho$-separated and homeomorphic to the Stone-Čech compactification of the set of natural numbers, so $Z(b)$ is a totally disconnected set (see [6, 7]). Hence if $B$ is a Carleson-Newman Blaschke product, then $Z(B)$ is also totally disconnected. Let $f \in H^\infty$. For $z \in \mathbb{D}$, we denote by $ord(f, z)$ the order of zero of $f$ at $z$. For $x \in G \setminus \mathbb{D}$, we define $ord(f, x) = ord(f \circ L_x, 0)$. For $x \in M(H^\infty) \setminus G$, we put as usual $ord(f, x) = \infty$ if $f(x) = 0$ and $ord(f, x) = 0$ if $f(x) \neq 0$. Clearly, if $b$ is an interpolating Blaschke product, then $ord(b, x) \leq 1$. If $b$ is a Carleson-Newman Blaschke product of order $m$, then $ord(b, x) \leq m$ for every $x$.

Let $I$ be a closed ideal in $H^\infty$. We write

$$Z(I) = \bigcap_{f \in I} Z(f)$$

and

$$ord(I, x) = \inf_{f \in I} ord(f, x), \quad x \in M(H^\infty).$$

For each $1 \leq j \leq \infty$ and $f \in H^\infty$, we put

$$Z_j(f) = \{x \in M(H^\infty) : ord(f, x) \geq j\}$$

and

$$Z_j(I) = \{x \in M(H^\infty) : ord(I, x) \geq j\}.$$

It seems very difficult to study ideal theory in $H^\infty$ generally (see [1]). In [4], Gorkin, Mortini and the first author proved the following two theorems for a closed ideal $I$ satisfying $Z(I) \subset G$. In this case, by Theorem 2.3 in [5], $I$ contains a Carleson-Newman Blaschke product, so $\sup_{x \in Z(I)} ord(I, x) < \infty$ and $Z(I)$ is totally disconnected (see also [14]).

**Theorem A.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$. Then $I$ coincides with the set of all $f$ in $H^\infty$ satisfying $ord(f, x) \geq ord(I, x)$ for every $x \in Z(I)$.*

This shows that if $I_1, I_2$ are closed ideals in $H^\infty$ such that $Z(I_i) \subset G$ for $i = 1, 2$, $Z(I_1) = Z(I_2)$ and $ord(I_1, x) = ord(I_2, x)$ for every $x \in Z(I_1)$, then we have $I_1 = I_2$.

**Theorem B.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x\in Z(I)} ord(I,x)$. For each $1 \le j \le m$, let $U_j$ be an open subset of $M(H^\infty)$ satisfying $Z_j(I) \subset U_j$. Then $I$ is algebraically generated by Carleson-Newman Blaschke products $B$ of order $m$ in $I$ such that $Z_j(B) \subset U_j$ for $1 \le j \le m$.*

The above two theorems give us a great deal of information about closed ideals $I$ satisfying $Z(I) \subset G$. In [12, 13], the authors studied closed ideals $I$ satisfying $Z(I) \subset G$ extensively.

For a sequence $\{f_n\}_n$ in $H^\infty$, we denote by $I[f_n : n \ge 1]$ the closed ideal in $H^\infty$ generated by functions $f_n, n = 1, 2, \cdots$; that is,

$$I[f_n : n \ge 1] = \overline{\bigcup_{n=1}^{\infty} \sum_{j=1}^{n} f_j H^\infty},$$

where the bar indicates the closure in $H^\infty$. The closed ideal $I[f_n : n \ge 1]$ is called a countably generated closed ideal in $H^\infty$. In this paper, we study the structure of countably generated closed ideals $I$ satisfying $Z(I) \subset G$. For a closed subset $E$ of $M(H^\infty)$, let $I(E) = \{f \in H^\infty : f(x) = 0, x \in E\}$. Then $I(E)$ is a closed ideal in $H^\infty$ and $E \subset Z(I(E))$. For closed ideals $I_1, I_2, \cdots, I_m$ in $H^\infty$, let $\bigotimes_{i=1}^m I_i$ and $\overline{\bigotimes}_{i=1}^m I_i$ be the tensor product and the closed tensor product of $I_1, I_2, \cdots, I_m$, respectively. That is, $\bigotimes_{i=1}^m I_i$ is an ideal generated by functions $\prod_{i=1}^m f_i$, where $f_i \in I_i, 1 \le i \le m$, and $\overline{\bigotimes}_{i=1}^m I_i = \overline{\bigotimes_{i=1}^m I_i}$. In Section 2, we shall prove the following theorem.

**Theorem 1.1.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x\in Z(I)} ord(I,x)$. Then the following conditions are equivalent.*

(i) *$I$ is a countably generated closed ideal.*

(ii) *There are compact $\rho$-separated $G_\delta$-subsets $E_1, E_2, \cdots, E_m$ of $G$ such that $I = \overline{\bigotimes}_{j=1}^m I(E_j)$.*

(iii) *There is a Carleson-Newman Blaschke product $B$ of order $m$ in $I$ such that $ord(B,x) = ord(I,x)$ for every $x \in Z(I)$, and $Z(I)$ is a $G_\delta$-set.*

(iv) *There are two Carleson-Newman Blaschke products $B_1, B_2$ in $I$ such that $I = I[B_1, B_2]$.*

For a compact $\rho$-separated $G_\delta$-subset $E$ of $G$, there is an interpolating Blaschke product $b$ satisfying $E \subset Z(b)$, and $I(E)$ is a countably generated closed ideal. We shall show in Example 2.14 that there exist compact $\rho$-separated $G_\delta$-subsets $E_1$ and $E_2$ of $G$ such that $I(E_1) \cap I(E_2)$ is not countably generated. If $I$ is a countably generated closed ideal in $H^\infty$, then by Theorem 1.1, $Z_j(I)$ is a $G_\delta$-set for every $1 \le j \le \infty$. But if $I$ is the closed ideal given in Example 2.14, then $Z_2(I)$ is not a $G_\delta$-set.

## 2. Countably generated closed ideals

To prove Theorem 1.1, we need some lemmas. For a sequence $\{f_n\}_n$ in $H^\infty$ and $1 \le j \le \infty$, it is not difficult to show that

$$Z_j(I[f_n : n \ge 1]) = \bigcap_{n=1}^{\infty} Z_j(f_n)$$

and

$$ord(I[f_n : n \ge 1], x) = \inf_{n\ge 1} ord(f_n, x), \quad x \in Z(I[f_n : n \ge 1]).$$

**Lemma 2.1.** *Let $B$ be a Carleson-Newman Blaschke product. Then $Z_j(B)$ is a closed $G_\delta$-set for every $1 \le j < \infty$.*

*Proof.* Let $B = \prod_{i=1}^k b_i$, where $b_i$ is an interpolating Blaschke product for every $1 \le i \le k$. Since $ord(b_i, x) \le 1$ for $x \in M(H^\infty)$, we have that $Z_j(B) = \emptyset$ for $j > k$. Suppose that $1 \le j \le k$. Put $E_i = Z(b_i)$. Then $E_i$ is a closed $G_\delta$-set. We have

$$Z_j(B) = \bigcup \Big\{ \bigcap_{\ell=1}^{j} E_{i_\ell} : 1 \le i_1 < i_2 < \cdots < i_j \le k \Big\}.$$

Therefore $Z_j(B)$ is a closed $G_\delta$-set. □

**Lemma 2.2.** *If $f \in H^\infty$ and $f \ne 0$, then $Z_j(f)$ is a closed $G_\delta$-set for every $1 \le j \le \infty$.*

*Proof.* Let $f = Bh$, where $B$ is a Blaschke product and $h \in H^\infty$ satisfying $|h| > 0$ on $\mathbb{D}$. Then $Z_\infty(h) = Z(h)$ and $Z_\infty(h)$ is a closed $G_\delta$-set. By Corollary 3.1 in [9], $Z_\infty(B)$ is a closed $G_\delta$-set. Then $Z_\infty(f) = Z_\infty(B) \cup Z_\infty(h)$ is a closed $G_\delta$-set. We have

$$\begin{aligned} Z(f) \setminus Z_\infty(f) &= (Z(B) \cup Z(h)) \setminus Z_\infty(f) \\ &= (Z(B) \cup Z_\infty(h)) \setminus Z_\infty(f) = Z(B) \setminus Z_\infty(f). \end{aligned}$$

By Lemma 4.6 in [9], $Z(B) \setminus Z_\infty(f)$ is a totally disconnected set. Hence there is a sequence of open and closed subsets $\{E_n\}_n$ of $Z(B)$ such that $Z(B) \setminus Z_\infty(f) = \bigcup_{n=1}^\infty E_n$ and $E_n \cap E_k = \emptyset$ for $n \ne k$. Let $b_n$ be the subproduct of $B$ with zeros $Z(B) \cap E_n \cap \mathbb{D}$ counting multiplicities. Since $Z(B) \cap \mathbb{D} \subset Z(B) \setminus Z_\infty(f)$, we have $B = \prod_{n=1}^\infty b_n$ and $Z(b_n) = E_n$ for every $n \ge 1$. We note that $b_n$ is a Carleson-Newman Blaschke product. For each $1 \le j < \infty$, we have

$$Z_j(f) = Z_\infty(f) \cup \bigcup_{n=1}^{\infty} Z_j(b_n).$$

By Lemma 2.1, $Z_j(b_n)$ is a closed $G_\delta$-set; so is $Z_j(f)$. □

**Lemma 2.3.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x \in Z(I)} ord(I, x)$. Then $I$ is a countably generated closed ideal if and only if $Z_j(I)$ is a closed $G_\delta$-set for every $1 \le j \le m$. In this case, $I$ is generated by countably many Carleson-Newman Blaschke products.*

*Proof.* Suppose that $I = I[f_n : n \ge 1]$ for a sequence $\{f_n\}_n$ in $H^\infty$. For each $1 \le j \le m$, we have $Z_j(I) = \bigcap_{n=1}^\infty Z_j(f_n)$. By Lemma 2.2, $Z_j(I)$ is a closed $G_\delta$-set.

Suppose that $Z_j(I)$ is a closed $G_\delta$-set for every $1 \le j \le m$. For each $1 \le j \le m$, let $\{U_{j,n}\}_n$ be a sequence of open subsets of $G$ such that $Z_j(I) = \bigcap_{n=1}^\infty U_{j,n}$. By Theorem B, there is a sequence of Carleson-Newman Blaschke products $\{\varphi_n\}_n$ in $I$ such that $Z_j(\varphi_n) \subset U_{j,n}$ for every $1 \le j \le m$ and $n \ge 1$. Let $J = I[\varphi_n : n \ge 1]$. Then $J \subset I$ and $Z(I) \subset Z(J)$. We have $Z(J) \subset Z(\varphi_n) \subset U_{1,n}$ for every $n \ge 1$. Then $Z(J) \subset \bigcap_{n=1}^\infty U_{1,n} = Z_1(I) = Z(I)$. Hence $Z(J) = Z(I)$.

Let $x \in Z(I)$ and $\ell = ord(I, x)$. Since $\varphi_n \in I$, $\ell \le ord(\varphi_n, x)$ for every $n \ge 1$. Since $x \notin Z_{\ell+1}(I)$, there is a positive integer $k$ such that $x \notin U_{\ell+1,k}$. Hence

$\ell \le ord(\varphi_k, x) \le \ell$. Therefore

$$\ell = ord(I,x) \le ord(J,x) \le ord(\varphi_k, x) = \ell.$$

Thus we get $ord(J,x) = ord(I,x)$ for every $x \in Z(I)$. By Theorem A, we have $J = I$. □

The following lemma follows from Theorem 3.1 in [10].

**Lemma 2.4.** *Let $E$ be a compact $\rho$-separated subset of $G$ and $U$ be an open subset of $M(H^\infty)$ satisfying $E \subset U$. Then there exists an interpolating Blaschke product $b$ such that $E \subset Z(b) \subset U$.*

**Lemma 2.5.** *Let $E$ be a compact $\rho$-separated $G_\delta$-subset of $G$. Then $I(E)$ is a countably generated closed ideal in $H^\infty$, $E$ is a totally disconnected set, $Z(I(E)) = E$ and $ord(I(E), x) = 1$ for every $x \in E$.*

*Proof.* By Lemma 2.4, there is an interpolating Blaschke product $b$ such that $E \subset Z(b) \subset G$. Hence $ord(I(E), x) = 1$ for every $x \in E$. Since $Z(b)$ is a totally disconnected set, so is $E$. Let $\{U_n\}_n$ be a sequence of open subsets of $G$ satisfying $E = \bigcap_{n=1}^\infty U_n$ and $Z(b) \cap U_n$ be an open and closed subset of $Z(b)$ for every $n \ge 1$. Let $b_n$ be the subproduct of $b$ with zeros $Z(b) \cap U_n \cap \mathbb{D}$. Then $E \subset Z(b_n) \subset U_n$. Let $J = I[b_n : n \ge 1]$. Then we have $J \subset I(E)$ and

$$E \subset Z(I(E)) \subset Z(J) \subset \bigcap_{n=1}^\infty U_n = E.$$

Hence $Z(I(E)) = Z(J) = E$. We have $ord(J,x) = 1$ for every $x \in E$. By Theorem A, we get $J = I(E)$. □

The following lemma follows from the definition of a closed tensor product.

**Lemma 2.6.** *Let $I_1, I_2, \cdots, I_m$ be countably generated closed ideals in $H^\infty$. Then $\overline{\bigotimes}_{j=1}^m I_j$ is a countably generated closed ideal, $Z(\overline{\bigotimes}_{j=1}^m I_j) = \bigcup_{j=1}^m Z(I_j)$ and $ord(\overline{\bigotimes}_{j=1}^m I_j, x) = \sum_{j=1}^m ord(I_j, x)$ for every $x \in Z(\overline{\bigotimes}_{j=1}^m I_j)$.*

For closed ideals $I_1, I_2, \cdots, I_m$ in $H^\infty$ satisfying $Z(I_j) \subset G$ for every $1 \le j \le m$, in [13, Corollary 9.15] the authors proved that $\overline{\bigotimes}_{j=1}^m I_j = \bigotimes_{j=1}^m I_j$.

**Lemma 2.7.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $x \in Z(I)$. Let $B$ be a Carleson-Newman Blaschke product in $I$ and $W$ be an open subset of $M(H^\infty)$ satisfying $x \in W$. Then there is an open subset $U$ of $M(H^\infty)$ satisfying that $x \in U \subset G \cap W$ and $Z(I) \cap U$ is an open and closed subset of $Z(I)$, and there is a Carleson-Newman Blaschke product $\varphi$ of order $ord(I,x)$ such that $Z(\varphi) \subset U$, $\varphi \prec B$ and $ord(I,y) \le ord(\varphi, y) \le ord(I,x)$ for every $y \in Z(I) \cap U$.*

*Proof.* Since $Z(I)$ is a totally disconnected set (see [4, Theorem 2.2]), we may take a sufficiently small open subset $U$ of $M(H^\infty)$ such that $x \in U \subset G \cap W$ and $Z(I) \cap U$ is an open and closed subset of $Z(I)$. Since $ord(I,y)$ is upper semicontinuous in $y \in Z(I)$ (see [4, Lemma 1.2]), we may assume that $ord(I,y) \le ord(I,x)$ for every $y \in Z(I) \cap U$. Let

$$I_U = \big\{f \in H^\infty : ord(f,y) \ge ord(I,y), y \in Z(I) \cap U\big\}.$$

Then by Theorem A, $I_U$ is a closed ideal in $H^\infty$, $I \subset I_U$, $Z(I_U) = Z(I) \cap U$ and $ord(I_U, y) = ord(I,y)$ for every $y \in Z(I) \cap U$. By [13, Proposition 8.9], there is a

Carleson-Newman Blaschke product $\varphi$ of order $ord(I,x)$ in $I_U$ such that $Z(\varphi) \subset U$, $\varphi \prec B$ and $ord(\varphi,x) = ord(I_U,x)$. For each $y \in Z(I) \cap U$, we have

$$ord(I,y) = ord(I_U,y) \le ord(\varphi,y) \le ord(I,x).$$

□

**Lemma 2.8.** *Let $I$ be a closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x\in Z(I)} ord(I,x)$. Let $W_1, W_2, \cdots, W_m$ be open subsets of $M(H^\infty)$ such that $Z_j(I) \subset W_j$ for every $1 \le j \le m$ and $W_m \subset W_{m-1} \subset \cdots \subset W_1$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a Carleson-Newman Blaschke product $b$ such that $b \in I$, $b \prec B$ and $ord(b,y) \le j$ for every $y \in Z(I) \cap (W_j \setminus W_{j+1})$ and $1 \le j \le m$, where $W_{m+1} = \emptyset$.*

*Proof.* For each $x \in Z(I)$, since $Z(I) \subset \bigcup_{j=1}^m (W_j \setminus W_{j+1})$ there exists $1 \le j \le m$ such that $x \in W_j \setminus W_{j+1}$. Then $ord(I,x) \le j$. By Lemma 2.7, there is an open subset $U_x$ of $M(H^\infty)$ satisfying that $x \in U_x \subset G \cap W_j$ and $Z(I) \cap U_x$ is an open and closed subset of $Z(I)$, and there is a Carleson-Newman Blaschke product $\varphi_x$ of order $ord(I,x)$ such that $Z(\varphi_x) \subset U_x$, $\varphi_x \prec B$ and $ord(I,y) \le ord(\varphi_x,y) \le ord(I,x)$ for every $y \in Z(I) \cap U_x$.

Since $Z(I)$ is a compact set, there is a finite set $\{x_1, x_2, \cdots, x_s\}$ in $Z(I)$ such that $Z(I) \subset \bigcup_{i=1}^s U_{x_i}$. Let

$$E_1 = Z(I) \cap U_{x_1}, \quad E_2 = (Z(I) \cap U_{x_2}) \setminus (Z(I) \cap U_{x_1}),$$
$$\cdots, \quad E_s = (Z(I) \cap U_{x_s}) \setminus \bigcup_{i=1}^{s-1} (Z(I) \cap U_{x_i}).$$

Then $E_i$ is an open and closed subset of $Z(I)$, $E_i \cap E_j = \emptyset$ for $i \ne j$ and $\bigcup_{i=1}^s E_i = Z(I)$. It may be that $x_i \notin E_i$ for some $1 \le i \le s$. We may take open subsets $V_1, V_2, \cdots, V_s$ of $M(H^\infty)$ satisfying that $E_i \subset V_i \subset U_{x_i}$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ for $i \ne j$. Let $\psi_i$ be the Blaschke subproduct of $\varphi_{x_i}$ with zeros $Z(\varphi_{x_i}) \cap V_i \cap \mathbb{D}$ counting multiplicities. Then $Z(\psi_i) \cap Z(\psi_j) = \emptyset$ for $i \ne j$ and $ord(\psi_i,y) = ord(\varphi_{x_i},y)$ for every $y \in E_i$ and $1 \le i \le s$. Let $b = \prod_{i=1}^s \psi_i$. Then $b \prec B$.

Let $y \in Z(I)$. Then there is the unique $1 \le j \le m$ such that $y \in W_j \setminus W_{j+1}$. Also there is the unique $1 \le i \le s$ such that $y \in E_i$. So we have

$$ord(b,y) = ord(\psi_i,y) = ord(\varphi_{x_i},y) \le ord(I,x_i).$$

Here we have two cases.

*Case* 1. Suppose that $x_i \in W_j \setminus W_{j+1}$. Then we have

$$ord(I,y) \le ord(\varphi_{x_i},y) \le ord(I,x_i) \le j.$$

Hence $ord(I,y) \le ord(b,y) \le j$.

*Case* 2. Suppose that $x_i \in W_k \setminus W_{k+1}$ for some $k \ne j$. If $k < j$, then $ord(I,x_i) \le k < j$. Hence

$$ord(I,y) \le ord(\varphi_{x_i},y) = ord(b,y) < j.$$

If $k > j$, then $y \in U_{x_i} \subset W_k$. Since $y \notin W_{j+1}$ and $W_k \subset W_{j+1}$, we have $y \notin W_k$. This is a contradiction.

By the above two cases, we have $ord(I,y) \le ord(b,y) \le j$ for every $y \in Z(I) \cap (W_j \setminus W_{j+1})$. By Theorem A, we have $b \in I$. Thus we get the assertion. □

**Lemma 2.9.** *Let $I$ be a countably generated closed ideal in $H^\infty$ satisfying $Z(I) \subset G$ and $m = \sup_{x\in Z(I)} ord(I,x)$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_1 \prec B$, $b_{n+1} \prec b_n$, $b_n \in I$ for every $n \geq 1$ and for each $x \in Z(I)$ there is a positive integer $n$ satisfying $ord(I,x) = ord(b_n,x)$.*

*Proof.* By Lemma 2.3, $Z_j(I)$ is a closed $G_\delta$-set for every $1 \leq j \leq m$. For each $1 \leq j \leq m$, take a sequence of open subsets $\{W_{j,n}\}_n$ of $M(H^\infty)$ such that $\bigcap_{n=1}^\infty W_{j,n} = Z_j(I)$ and $W_{j,n+1} \subset W_{j,n}$ for every $n \geq 1$. Further we may assume that $W_{j+1,n} \subset W_{j,n}$ for every $1 \leq j \leq m$ and $n \geq 1$, where $W_{m+1,n} = \emptyset$ for every $n \geq 1$. By Lemma 2.8, there is a Carleson-Newman Blaschke product $b_1$ such that $b_1 \in I$, $b_1 \prec B$ and $ord(b_1, y) \leq j$ for every $y \in Z(I) \cap (W_{j,1} \setminus W_{j+1,1})$ and $1 \leq j \leq m$. By Lemma 2.8 again, there is a Carleson-Newman Blaschke product $b_2$ such that $b_2 \in I$, $b_2 \prec b_1$ and $ord(b_2, y) \leq j$ for every $y \in Z(I) \cap (W_{j,2} \setminus W_{j+1,2})$ and $1 \leq j \leq m$. Inductively we may get a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_n \in I$, $b_{n+1} \prec b_n$ and $ord(b_n, y) \leq j$ for every $y \in Z(I) \cap (W_{j,n} \setminus W_{j+1,n})$ and $1 \leq j \leq m$.

Let $x \in Z(I)$ and $t = ord(I,x)$. We consider two cases separately.

*Case* 1. Suppose that $t < m$. Then $x \notin Z_{t+1}(I)$ and there is a positive integer $k$ such that $x \in Z(I) \cap (W_{t,k} \setminus W_{t+1,k})$. Hence $ord(b_k, x) \leq t$. Since $b_k \in I$, we have $t = ord(I,x) \leq ord(b_k,x) \leq t$. Thus we get $ord(I,x) = ord(b_k,x)$.

*Case* 2. Suppose that $t = m$, that is, $ord(I,x) = m$. Then $x \in Z(I) \cap (W_{m,n} \setminus W_{m+1,n})$ for every $n \geq 1$. Hence $ord(b_n,x) \leq m$. Since $b_n \in I$, we have $m \leq ord(b_n,x)$. Thus we get $ord(I,x) = ord(b_n,x)$ for every $n \geq 1$. □

The following is due to Hoffman [7].

**Lemma 2.10.** *For any interpolating Blaschke product $b$ with zeros $\{z_n\}_n$ in $\mathbb{D}$, there exists a positive number $\lambda(b)$ such that a sequence $\{w_n\}_n$ in $\mathbb{D}$ satisfying $\rho(w_n, z_n) < \lambda(b)$ is an interpolating sequence.*

**Lemma 2.11.** *Let $I$ be a closed ideal in $H^\infty$ and $Z(I) \subset G$. Let $B$ be a Carleson-Newman Blaschke product in $I$. Then there is a Carleson-Newman Blaschke product $b$ in $I$ satisfying the following conditions.*

(i) *$ord(b,x) = ord(B,x)$ for every $x \in Z(I) \setminus \mathbb{D}$.*
(ii) *$ord(b,z) = ord(I,z)$ for every $z \in Z(I) \cap \mathbb{D}$.*
(iii) *$ord(b,z) = 1$ for every $z \in (Z(b) \setminus Z(I)) \cap \mathbb{D}$.*

*Proof.* Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be interpolating Blaschke products satisfying $B = \prod_{j=1}^m \varphi_j$. Let $\lambda = \min_{1\leq j\leq m} \lambda(\varphi_j)$. Then $\lambda > 0$. Let $\{z_n\}_n = Z(B) \cap \mathbb{D}$ and $k_n = ord(B, z_n)$. Then $\sup_{n\geq 1} k_n < \infty$. Let $\{\varepsilon_n\}_n$ be a sequence of numbers with $0 < \varepsilon_n < \lambda$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We shall move the zeros of $B$ a little. Let $n$ be a positive integer. If $z_n \notin Z(I)$, then take $\{w_{n,1}, w_{n,2}, \cdots, w_{n,k_n}\}$ in $\mathbb{D}$ such that $\rho(w_{n,i}, z_n) < \varepsilon_n$, $w_{n,i} \neq w_{n,j}$ for $i \neq j$ and

$$\{w_{n,1}, w_{n,2}, \cdots, w_{n,k_n}\} \cap \{z_n\}_n = \emptyset.$$

If $z_n \in Z(I)$, put $\ell_n = ord(I, z_n)$. Then take $\{w_{n,1}, w_{n,2}, \cdots, w_{n,k_n}\}$ in $\mathbb{D}$ as the following: $\rho(w_{n,i}, z_n) < \varepsilon_n$ for every $1 \leq i \leq \ell_n$, $w_{n,1} = w_{n,2} = \cdots = w_{n,\ell_n} = z_n$, $w_{n,i} \neq w_{n,j}$ for every $\ell_n \leq i < j \leq k_n$ and

$$\{w_{n,i} : \ell_n + 1 \leq i \leq k_n\} \cap \{z_n\}_n = \emptyset.$$

Further, we may assume that

$$\{w_{n,1}, w_{n,2}, \cdots, w_{n,k_n}\} \cap \{w_{j,1}, w_{j,2}, \cdots, w_{j,k_j}\} = \emptyset$$

for every $n \neq j$ and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} (1 - |w_{n,i}|) < \infty.$$

Let $b$ be the Blaschke product with zeros $\{w_{n,i}\}_{n,i}$ counting multiplicities. By Lemma 2.10, $b$ is a Carleson-Newman Blaschke product. We have $ord(b, x) = ord(B, x)$ for every $x \in Z(I) \setminus \mathbb{D}$. It is easy to see that $b$ satisfies (ii) and (iii). Since $ord(I, x) \leq ord(b, x)$ for every $x \in Z(I)$, by Theorem A we have $b \in I$. □

**Lemma 2.12.** *Let $B$ be a Carleson-Newman Blaschke product and $\{z_n\}_n$ be an interpolating sequence in $\mathbb{D}$. If $0 < \varepsilon < 1$, then*

$$\inf_n \sup \left\{ |B(z)| : z \in \mathbb{D}, \rho(z, z_n) < \varepsilon \right\} > 0.$$

*Proof.* To prove the assertion, suppose not. Then there exists a subsequence $\{n_j\}_j$ such that

$$\lim_{j \to \infty} \sup \left\{ |B(z)| : z \in \mathbb{D}, \rho(z, z_{n_j}) < \varepsilon \right\} = 0.$$

Let $x$ be a cluster point of $\{z_{n_j}\}_j$ in $M(H^\infty)$. By Hoffman's work [7], it is easy to see that $B \equiv 0$ on $P(x)$, the Gleason part of $x$. By our assumption, $B \not\equiv 0$ on $P(x)$, and this is a contradiction. □

**Lemma 2.13.** *Let $B$ be a Carleson-Newman Blaschke product and $b$ be an interpolating Blaschke product. Let $E$ be a closed $G_\delta$-subset of $Z(b)$. Then there is an interpolating Blaschke product $\varphi$ such that $E \subset Z(\varphi)$ and $Z(B) \cap E = Z(B) \cap Z(\varphi)$.*

*Proof.* If $Z(B) \cap E = Z(B) \cap Z(b)$, then put $\varphi = b$. Then we get the assertion. So we assume that $Z(B) \cap E \subsetneqq Z(B) \cap Z(b)$. By the assumptions, there is a sequence of closed subsets $\{K_n\}_n$ of $Z(b)$ such that

$$(Z(B) \cap Z(b)) \setminus E = \bigcup_{n=1}^{\infty} K_n$$

and $K_n \cap K_k = \emptyset$ for $n \neq k$. We note that

$$\overline{\bigcup_{n=1}^{\infty} K_n} \setminus \bigcup_{n=1}^{\infty} K_n \subset E.$$

Take a sequence of open subsets $\{U_n\}_n$ of $M(H^\infty)$ such that $K_n \subset U_n$, $\overline{U_n} \cap \overline{U_k} = \emptyset$ for $n \neq k$, $E \cap \overline{U}_n = \emptyset$ and $Z(b) \cap U_n$ is an open and closed subset of $Z(b)$ for every $n \geq 1$. Let $b_n$ be the subproduct of $b$ with zeros $\{z_{n,\ell}\}_\ell := Z(b) \cap U_n \cap \mathbb{D}$. Then $K_n \subset Z(b_n)$, $E \cap Z(b_n) = \emptyset$ for every $n \geq 1$ and $b = \prod_{n=0}^{\infty} b_n$ for some interpolating Blaschke product $b_0$. We note that

$$(Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^{\infty} Z(b_n) \subset E.$$

Let $\{\varepsilon_n\}_n$ be a sequence of numbers such that $0 < \varepsilon_n < \lambda(b)$ and $\varepsilon_n \to 0$ as $n \to \infty$. By Lemma 2.12, there is a sequence of positive numbers $\{\delta_n\}_n$ such that

$$\sup \left\{ |B(z)| : z \in \mathbb{D}, \rho(z, z_{n,\ell}) < \varepsilon_n \right\} > \delta_n$$

for every $\ell \geq 1$. For each $\ell \geq 1$, take $w_{n,\ell} \in \mathbb{D}$ satisfying $\rho(w_{n,\ell}, z_{n,\ell}) < \varepsilon_n$ and $|B(w_{n,\ell})| > \delta_n$. By Lemma 2.10, $\{w_{n,\ell}\}_\ell$ is an interpolating sequence for every $n \geq 1$. For each $n \geq 1$, let $\varphi_n$ be the interpolating Blaschke product with zeros $\{w_{n,\ell}\}_\ell$. Then $Z(B) \cap Z(\varphi_n) = \emptyset$ and $E \cap Z(\varphi_n) = \emptyset$ for every $n \geq 1$. Since

$$\sup_{\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) \leq \varepsilon_n \to 0 \quad (n \to \infty),$$

we have

$$Z\Big(\prod_{n=1}^\infty b_n\Big) \setminus \bigcup_{n=1}^\infty Z(b_n) = Z\Big(\prod_{n=1}^\infty \varphi_n\Big) \setminus \bigcup_{n=1}^\infty Z(\varphi_n).$$

Put $\varphi = b_0 \prod_{n=1}^\infty \varphi_n$. Since

$$\sup_{n,\ell \geq 1} \rho(w_{n,\ell}, z_{n,\ell}) < \lambda(b),$$

by Lemma 2.10 $\varphi$ is an interpolating Blaschke product. Since $E \subset Z(b)$ and $E \cap Z(b_n) = \emptyset$ for every $n \geq 1$, we have

$$\begin{aligned} E &\subset Z(b) \setminus \bigcup_{n=1}^\infty Z(b_n) \\ &= \Big(Z(b_0) \cup Z\Big(\prod_{n=1}^\infty b_n\Big)\Big) \setminus \bigcup_{n=1}^\infty Z(b_n) \\ &= \Big(Z(b_0) \setminus \bigcup_{n=1}^\infty Z(b_n)\Big) \cup \Big(Z\Big(\prod_{n=1}^\infty \varphi_n\Big) \setminus \bigcup_{n=1}^\infty Z(\varphi_n)\Big) \\ &= Z(\varphi) \setminus \bigcup_{n=1}^\infty Z(\varphi_n). \end{aligned}$$

Hence $E \subset Z(\varphi)$. Since $Z(B) \cap Z(\varphi_n) = \emptyset$ for every $n \geq 1$, we have

$$\begin{aligned} Z(B) \cap E &\subset Z(B) \cap Z(\varphi) \subset Z(B) \cap \Big(Z(\varphi) \setminus \bigcup_{n=1}^\infty Z(\varphi_n)\Big) \\ &= (Z(B) \cap Z(b)) \setminus \bigcup_{n=1}^\infty Z(b_n) \subset Z(B) \cap E. \end{aligned}$$

Hence we get $Z(B) \cap E = Z(B) \cap Z(\varphi)$. □

*Proof of Theorem* 1.1. (i) ⇒ (ii) By Theorem B, there is a Carleson-Newman Blaschke product $b_1$ of order $m$ in $I$. By Lemma 2.11, we may assume that $ord(b_1, z) = ord(I, z)$ for every $z \in Z(I) \cap \mathbb{D}$ and $ord(b_1, z) = 1$ for every $z \in (Z(b_1) \setminus Z(I)) \cap \mathbb{D}$. By Lemma 2.9, there is a sequence of Carleson-Newman Blaschke products $\{b_n\}_n$ such that $b_n \in I$, $b_{n+1} \prec b_n$ for every $n \geq 1$, and for each $x \in Z(I)$ there is a positive integer $n$ satisfying $ord(b_n, x) = ord(I, x)$.

Since the order of $b_1$ is equal to $m$, there are interpolating Blaschke products $\varphi_{1,1}, \varphi_{2,1}, \cdots, \varphi_{m,1}$ such that $b_1 = \prod_{j=1}^m \varphi_{j,1}$. Since $b_n \in I$ and $b_{n+1} \prec b_n$ for every $n \geq 1$, we have $ord(b_n, z) = ord(I, z)$ for $z \in Z(I) \cap \mathbb{D}$ and $ord(b_n, z) = 1$ for $z \in (Z(b_n) \setminus Z(I)) \cap \mathbb{D}$. Then there are the unique interpolating Blaschke products $\varphi_{1,n}, \varphi_{2,n}, \cdots, \varphi_{m,n}$ such that $b_n = \prod_{j=1}^m \varphi_{j,n}$ and $\varphi_{j,n+1} \prec \varphi_{j,n}$ for every $1 \leq j \leq m$. We note that if $z \in Z(I) \cap \mathbb{D}$ and $\varphi_{j,1}(z) = 0$, then $\varphi_{j,n}(z) = 0$ for every $n \geq 1$.

For each $1 \le j \le m$, let

$$E_j = Z(I) \cap \bigcap_{n=1}^{\infty} Z(\varphi_{j,n}).$$

By Lemma 2.3, $E_j$ is a compact $G_\delta$-set. Since $\varphi_{j,n}$ is an interpolating Blaschke product, $E_j$ is a $\rho$-separated set. Since $b_n \in I$,

$$Z(I) \subset Z(b_n) = \bigcup_{j=1}^{m} Z(\varphi_{j,n}),$$

so

$$Z(I) = \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})).$$

We have

$$\bigcup_{j=1}^{m} E_j \subset \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I).$$

Suppose that $\bigcup_{j=1}^m E_j \subsetneqq Z(I)$ and $y \in Z(I) \setminus \bigcup_{j=1}^m E_j$. For each $1 \le j \le m$, since $y \notin E_j$ there is a positive integer $n_j$ such that $y \notin Z(I) \cap Z(\varphi_{j,n_j})$. Let $n = \min_{1 \le j \le m} n_j$. Then

$$Z(I) \cap Z(\varphi_{j,n}) \subset Z(I) \cap Z(\varphi_{j,n_j}).$$

Hence

$$y \notin \bigcup_{j=1}^{m} (Z(I) \cap Z(\varphi_{j,n})) = Z(I).$$

But this is a contradiction. Thus we get

$$Z(I) = \bigcup_{j=1}^{m} E_j.$$

Let $x \in Z(I)$. Then there is a positive integer $n_1$ such that $ord(b_{n_1}, x) = ord(I, x)$. We write $\ell = ord(I, x)$. Then there are positive integers $j_1, j_2, \cdots, j_\ell$ such that

$$ord\Big(\prod_{i=1}^{\ell} \varphi_{j_i, n_1}, x\Big) = \ell \quad \text{and} \quad ord\Big(b_{n_1} / \prod_{i=1}^{\ell} \varphi_{j_i, n_1}, x\Big) = 0.$$

Since $b_n \in I$ and $b_n \prec b_{n_1}$ for every $n \ge n_1$, $ord(b_n, x) = \ell$ and $\varphi_{j_i, n}(x) = 0$ for every $1 \le i \le \ell$ and $n \ge n_1$. Thus for any $n \ge n_1$ we have

$$\begin{aligned} ord(I, x) &= ord(b_n, x) = ord\Big(\prod_{j=1}^{m} \varphi_{j,n}, x\Big) \\ &= \sum_{i=1}^{\ell} ord(\varphi_{j_i, n}, x) = \#\{j : x \in E_j, 1 \le j \le m\}, \end{aligned}$$

where $\#A$ denotes the number of elements in a set $A$. Let

$$J = \overline{\bigotimes}_{j=1}^{m} I(E_j).$$

By Lemma 2.5, we have $ord(I(E_j), x) = 1$ for every $x \in E_j$ and $Z(I(E_j)) = E_j$ for every $1 \le j \le m$. Hence by Lemma 2.6, $Z(J) = \bigcup_{j=1}^m E_j = Z(I)$ and

$$ord(J, x) = \sum_{j=1}^{m} ord(I(E_j), x) = \#\{j : x \in E_j, 1 \le j \le m\}$$

for every $x \in Z(I)$. By Theorem A, we have $I = J = \overline{\bigotimes}_{j=1}^m I(E_j)$.

(ii) $\Rightarrow$ (iii) Suppose that condition (ii) holds. By Lemma 2.6,

$$Z(I) = \bigcup_{j=1}^{m} Z(I(E_j)) = \bigcup_{j=1}^{m} E_j,$$

so $Z(I)$ is a $G_\delta$-set. By Lemma 2.4, for each $1 \le j \le m$ there is an interpolating Blaschke product $\varphi_j$ such that $E_j \subset Z(\varphi_j)$. Let $\Phi = \prod_{j=1}^m \varphi_j$. By Lemma 2.13, for each $1 \le j \le m$ there exists an interpolating Blaschke product $b_j$ such that $E_j \subset Z(b_j)$ and $Z(\Phi) \cap Z(b_j) = Z(\Phi) \cap E_j = E_j$. We note that $Z(I) \subset Z(\Phi)$. Let $B = \prod_{j=1}^m b_j$. Then for any $x \in Z(I)$, we have

$$\begin{aligned} ord(B, x) &= ord\Big(\prod_{j=1}^{m} b_j, x\Big) = \sum_{j=1}^{m} ord(b_j, x) \\ &= \#\{j : x \in E_j, 1 \le j \le m\}. \end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$\begin{aligned} ord(I, x) &= ord\Big(\overline{\bigotimes}_{j=1}^{m} I(E_j), x\Big) = \sum_{j=1}^{m} ord(I(E_j), x) \\ &= \#\{j : x \in E_j, 1 \le j \le m\}. \end{aligned}$$

Thus we get $ord(B, x) = ord(I, x)$ for every $x \in Z(I)$. By Theorem A, we have $B \in I$.

(iii) $\Rightarrow$ (iv) Suppose that condition (iii) holds. Let $B_1$ be a Carleson-Newman Blaschke product of order $m$ in $I$ satisfying $ord(B_1, x) = ord(I, x)$ for every $x \in Z(I)$. Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be interpolating Blaschke products satisfying $B_1 = \prod_{j=1}^m \varphi_j$. For each $1 \le j \le m$, let $E_j = Z(I) \cap Z(\varphi_j)$. Since $Z(I)$ is a $G_\delta$-set, $E_j$ is a closed $G_\delta$-set. By Lemma 2.13, there is an interpolating Blaschke product $b_j$ such that $Z(B_1) \cap Z(b_j) = E_j$. Let $B_2 = \prod_{j=1}^m b_j$. For any $x \in Z(I)$, we have

$$\begin{aligned} ord(B_2, x) &= \sum_{j=1}^{m} ord(b_j, x) = \#\{j : x \in E_j, 1 \le j \le m\} \\ &= ord(B_1, x) \ge ord(I, x). \end{aligned}$$

By Theorem A, we have $B_2 \in I$. We also have

$$\begin{aligned} Z(B_1) \cap Z(B_2) &= Z(B_1) \cap \bigcup_{j=1}^{m} Z(b_j) = \bigcup_{j=1}^{m} E_j \\ &= Z(I) \cap Z(B_1) = Z(I). \end{aligned}$$

Let $J = I[B_1, B_2]$. Then $Z(J) = Z(I)$ and $ord(J, x) = ord(I, x)$ for every $x \in Z(I)$. By Theorem A again, we have $J = I$.

(iv) $\Rightarrow$ (i) is trivial. □

In the following example, we shall show that there exist compact $\rho$-separated $G_\delta$-subsets $E_1$ and $E_2$ of $G$ such that the ideal $I(E_1) \cap I(E_2)$ is not countably generated.

**Example 2.14.** Let $\{\theta_k\}_k$ be a sequence of numbers such that $0 < \theta_{k+1} < \theta_k < 1$ and $\theta_k \to 0$ as $k \to \infty$. It is known that there is an interpolating Blaschke product $B_1$ with zeros $\{z_n\}_n$ in $\mathbb{D}$ such that

$$\overline{\{z_n\}_n}^{\mathbb{C}} \setminus \{z_n\}_n = \{e^{i\theta_k} : k \geq 1\} \cup \{1\},$$

where $\overline{\{z_n\}_n}^{\mathbb{C}}$ is the closure of $\{z_n\}_n$ in $\mathbb{C}$. Let $\mathbb{N}$ be the set of positive integers. We may divide $\mathbb{N}$ as $\mathbb{N} = \bigcup_{k=1}^{\infty} N_k$ such that $N_k \cap N_j = \emptyset$ for $k \neq j$ and

$$\overline{\{z_n : n \in N_k\}}^{\mathbb{C}} \setminus \{z_n : n \in N_k\} = \{e^{i\theta_k}\}, \quad k \in \mathbb{N}.$$

Let $b_k$ be the subproduct of $B_1$ with zeros $\{z_n : n \in N_k\}$. Then $B_1 = \prod_{k=1}^{\infty} b_k$. Let $\{\varepsilon_k\}_k$ be a sequence of numbers such that $0 < \varepsilon_k < 1$ and $\varepsilon_k \to 0$ as $k \to \infty$. Let $q_k(z) = (b_k(z) - \varepsilon_k)/(1 - \varepsilon_k b_k(z))$. Taking smaller $\varepsilon_k$, we may assume that $B_2 := \prod_{k=1}^{\infty} q_k$ is an interpolating Blaschke product and

$$\Big(\bigcup_{k=1}^{\infty} Z(b_k)\Big) \cap \Big(\bigcup_{k=1}^{\infty} Z(q_k)\Big) = \emptyset.$$

Let

$$E_1 = Z(B_1) \setminus \mathbb{D} \quad \text{and} \quad E_2 = Z(B_2) \setminus \mathbb{D}.$$

Then $E_1, E_2$ are compact $\rho$-separated $G_\delta$-subsets of $G$,

$$E_1 = \Big(\bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D})\Big) \cup \Big(E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k)\Big)$$

and

$$E_2 = \Big(\bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D})\Big) \cup \Big(E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k)\Big).$$

By Lemma 2.5, $I(E_1)$ and $I(E_2)$ are countably generated closed ideals in $H^\infty$. Let $I = I(E_1) \cap I(E_2)$. Then $I = I(E_1 \cup E_2)$. By the construction, we may check that

$$E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k) = E_2 \setminus \bigcup_{k=1}^{\infty} Z(q_k)$$

and

$$\begin{aligned} \overline{\bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D})} \setminus \bigcup_{k=1}^{\infty} Z(b_k) &= \overline{\bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D})} \setminus \bigcup_{k=1}^{\infty} Z(q_k) \\ &\subsetneq E_1 \setminus \bigcup_{k=1}^{\infty} Z(b_k). \end{aligned}$$

Let $\Omega$ be the set of all subproducts $q$ of $B_2$ satisfying

$$\bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D}) \subset Z(q).$$

Then we have $B_1 q \in I$ for every $q \in \Omega$ and

$$\bigcap_{q\in\Omega} Z(q) = \overline{\bigcup_{k=1}^{\infty} (Z(q_k) \setminus \mathbb{D})}.$$

By this fact, we have

$$Z_2(I) = \overline{\bigcup_{k=1}^{\infty} (Z(b_k) \setminus \mathbb{D})} \setminus \bigcup_{k=1}^{\infty} Z(b_k) = \overline{\bigcup_{k=1}^{\infty} Z(b_k)} \setminus \bigcup_{k=1}^{\infty} Z(b_k),$$

and $Z_2(I)$ is not a $G_\delta$-set (see Example 2.9 in [12]). By Lemma 2.3, $I$ is not countably generated. We note that $I = I(E_1)\overline{\otimes} I(\overline{E_2 \setminus E_1})$. □

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