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THE RESIDUAL EISENSTEIN COHOMOLOGY OF Sp_4 OVER A TOTALLY REAL NUMBER FIELD

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Abstract. Let $G = Sp_4/k$ be the k-split symplectic group of k-rank 2, where k is a totally real number field. In this paper we compute the Eisenstein cohomology of G with respect to any finite–dimensional, irreducible, k-rational representation E of $G_\infty = R_{k/\mathbb{Q}}G(\mathbb{R})$, where $R_{k/\mathbb{Q}}$ denotes the restriction of scalars from k to $\mathbb{Q}$. This approach is based on the work of Schwermer regarding the Eisenstein cohomology for $Sp_4/\mathbb{Q}$, Kim's description of the residual spectrum of Sp_4, and the Franke filtration of the space of automorphic forms. In fact, taking the representation theoretic point of view, we write, for the group G, the Franke filtration with respect to the cuspidal support, and give a precise description of the filtration quotients in terms of induced representations. This is then used as a prerequisite for the explicit computation of the Eisenstein cohomology. The special focus is on the residual Eisenstein cohomology. Under a certain compatibility condition for the coefficient system E and the cuspidal support, we prove the existence of non–trivial residual Eisenstein cohomology classes, which are not square–integrable, that is, represented by a non–square–integrable residue of an Eisenstein series.

Introduction

General background. The cohomology of an arithmetic congruence subgroup Γ of a connected, reductive algebraic k-group G, where k is a number field, is isomorphic to a subspace of the cohomology of the space of automorphic forms. This identification was conjectured by Borel and Harder and first established in a conceptual way by Harder in the case of groups of rank one in [Har73], [Har75] and [Har87]. In all these works he relates the cohomology of Γ and the cohomology of the space of automorphic forms using the fact that the cohomology of Γ is isomorphic to the cohomology of a certain compact space $\Gamma\backslash\overline{X}$, which is an orbifold with orbifold boundary $\partial(\Gamma\backslash\overline{X})$.

More precisely, let $G_\infty = R_{k/\mathbb{Q}}G(\mathbb{R})$ be the Lie group of real points of the algebraic $\mathbb{Q}$-group $R_{k/\mathbb{Q}}G$ obtained from G by the restriction of scalars from k to $\mathbb{Q}$. Let K_∞ be a maximal compact subgroup of G_∞, and $A_{G,\infty} = R_{k/\mathbb{Q}}A_G(\mathbb{R})$ be the real points of the restriction of scalars from k to $\mathbb{Q}$ of a maximal k-split central torus A_G of G. Then $X = G_\infty/K_\infty A^\circ_{G,\infty}$ is the Riemannian symmetric space associated to the Lie group $G_\infty = R_{k/\mathbb{Q}}G(\mathbb{R})$ and $K_\infty A^\circ_{G,\infty}$. The aforementioned space $\Gamma\backslash\overline{X}$ is then the Borel–Serre compactification of the quotient $\Gamma\backslash X$ (locally symmetric if

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Γ is torsionfree). Let E be a finite–dimensional, complex, k-rational representation of G_∞. For simplicity, assume that A_G acts trivially on E. It naturally defines a sheaf $\widetilde{E}$ on $\Gamma\backslash\overline{X}$ and let $H^*(\Gamma\backslash\overline{X},\widetilde{E})$ (respectively $H(\partial(\Gamma\backslash\overline{X}),\widetilde{E})$) denote the corresponding sheaf cohomology spaces.

With this framework in place, Harder showed in the case of groups of rank one (cf. [Har73]) that one can construct the "cohomology at infinity", i.e., a subspace of $H^*(\Gamma\backslash\overline{X},\widetilde{E})$ isomorphic to the image of the natural restriction map

$$H^*(\Gamma\backslash\overline{X},\widetilde{E}) \to H^*(\partial(\Gamma\backslash\overline{X}),\widetilde{E}),$$

by means of Eisenstein series, hence by a special type of automorphic forms. The "cohomology at infinity" forms a natural complement within $H^*(\Gamma\backslash\overline{X},\widetilde{E})$ to the kernel of the above restriction map, which is itself the cohomology of a space of square–integrable automorphic forms. Therefore, all cohomology classes in $H^*(\Gamma\backslash\overline{X},\widetilde{E})$ are representable by automorphic forms.

In the early 90s, J. Franke finally proved in [Fra98] that such an identification of $H^*(\Gamma\backslash\overline{X},\widetilde{E})$ with a subspace of the cohomology of the space of automorphic forms can also be given for an arbitrary connected, reductive algebraic group G. In order to use automorphic forms most effectively, it turns out that it is useful to translate the above picture into the setting of representation theory over groups of adèlic points of G. To this end, let $\mathbb{A}$ be the ring of adèles of k, $\mathbb{A}_f$ the finite adèles, and $\mathfrak{g}_\infty$ the Lie algebra of G_∞. Let $\mathcal{A}$ be the space of automorphic forms on $G(\mathbb{A})$; that is, the space of smooth functions of moderate growth on $G(\mathbb{A})$ that are left invariant for $G(k)$ and $A_G(\mathbb{R})^\circ$, finite for the action of a fixed maximal compact subgroup of $G(\mathbb{A})$, and annihilated by an ideal of finite codimension in the center of the universal enveloping algebra of the complexification of $\mathfrak{g}_\infty$ (cf. [BJ]). It is a $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-module, and its relative Lie algebra cohomology with respect to E is a $G(\mathbb{A}_f)$-module

$$H^q(G,E) := H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}\otimes E)$$

called the *automorphic cohomology* of G/k with respect to E.

As shown in [Fra98], every automorphic form on G can be obtained as the sum of principal values of derivatives of the Eisenstein series attached to a cuspidal or residual representation of a Levi factor of a parabolic k-subgroup of G. Since every residual automorphic representation of a Levi factor is obtained as a residue of a cuspidal Eisenstein series attached to a cuspidal automorphic representation π of a Levi factor L of another parabolic k-subgroup P of G, we may consider the cuspidal support of an automorphic form. Here we allow the case $P = G$ which gives the cuspidal automorphic forms. Having fixed an ideal $\mathcal{J}$ of finite codimension inside the center of the universal enveloping algebra of $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$, let $\mathcal{A}_{\mathcal{J}}$ be the space of those automorphic forms annihilated by some power of $\mathcal{J}$. The discussion above gives rise to a direct sum decomposition of $\mathcal{A}_{\mathcal{J}}$ into

$$\mathcal{A}_{\mathcal{J}} = \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J}}(P) = \bigoplus_{\{P\}} \bigoplus_{\varphi} \mathcal{A}_{\mathcal{J}}(P,\varphi)$$

along the associate classes of parabolic k-subgroups $\{P\}$ and the various cuspidal supports φ. For a precise definition of the spaces $\mathcal{A}_{\mathcal{J}}(P,\varphi)$ see [FS], Section 1. The main tool used to establish this important result is a certain kind of filtration of $\mathcal{A}_{\mathcal{J}}$, introduced by Franke in [Fra98]. If $\mathcal{A}^m_{\mathcal{J}}(P)$ denotes the m-th filtration step of the summand $\mathcal{A}_{\mathcal{J}}(P)$, he showed that each consecutive quotient $\mathcal{A}^m_{\mathcal{J}}(P)/\mathcal{A}^{m+1}_{\mathcal{J}}(P)$

can be described in terms of induced representations from the discrete spectrum of the Levi subgroups containing the one of the given P. More precisely, Franke in fact proved in [Fra98] that each consecutive quotient as above is spanned by main values of the derivatives of cuspidal and residual Eisenstein series.

If we choose $\mathcal{J}$ to be the ideal annihilating the dual representation of E, this moreover induces a decomposition of automorphic cohomology

$$H^q(G,E) = \bigoplus_{\{P\}} \bigoplus_{\varphi} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P,\varphi) \otimes E).$$

As $\mathcal{A}_{\mathcal{J}}(G)$ is the space of cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}$, one calls $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(G) \otimes E)$ the space of *cuspidal cohomology.* Its natural complement in the above decomposition,

$$H^q_{Eis}(G,E) := \bigoplus_{\{P\}\neq\{G\}} \bigoplus_{\varphi} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P,\varphi) \otimes E),$$

is called *Eisenstein cohomology.* Finally, it is a consequence of Franke's aforementioned theorem that taking an appropriate open compact subgroup C_f of $G(\mathbb{A}_f)$, the cohomology of $\Gamma\backslash\overline{X}$ appears as a direct summand in the C_f-invariant points of $H^q(G,E)$. This phenomenon can be rephrased by saying that regarding $H^q(G,E)$, one considers the cohomology of all congruence subgroups at the same time. Moreover, this proves that the cohomology of an arithmetic congruence subgroup Γ of a connected, reductive algebraic k-group G is isomorphic to a subspace of the cohomology of the space of automorphic forms.

The contents of this article. In this paper we study the Eisenstein cohomology of the k-split symplectic group $G = Sp_4/k$ of k-rank 2, where k is a totally real number field. We rely on:

(a) the treatment of the case Sp_4 over $\mathbb{Q}$ done by Schwermer in [Sch86] and [Sch95]; in particular, the points of evaluation of the Eisenstein series that may possibly give non–trivial cohomology classes are given in that work,
(b) the description of the residual spectrum of Sp_4 over an arbitrary number field given by Kim in [Kim],
(c) the filtration of the spaces $\mathcal{A}_{\mathcal{J}}(P)$ used by Franke in the proof of his result in [Fra98].

In the first part of this article we summarize the notation and conventions used in the paper and we give the necessary theoretical background concerning automorphic forms, Eisenstein series and the above-mentioned decomposition along the cuspidal support for the case Sp_4/k. Following Harder's idea for GL_2/k (see [Har87], Sect. 2.8), we also prove that there is no Eisenstein cohomology supported in the Borel subgroup unless the highest weight of the algebraic E has repeating coordinates in the various field embeddings $\sigma : k \hookrightarrow \mathbb{C}$ (cf. Proposition 2.1), whence we take this as a standing assumption.

We then recall the Franke filtration and make it concrete for the case of Sp_4/k. As already mentioned, the evaluation points we must consider are the same as those in [Sch86] and [Sch95], where the case Sp_4 over $\mathbb{Q}$ is treated. The residual spectrum of Sp_4 over k, described in [Kim], is the starting point of the filtration. This finally leads to an explicit description from the representation theoretic point of view of the consecutive quotients $\mathcal{A}^m_{\mathcal{J}}(P,\varphi)/\mathcal{A}^{m+1}_{\mathcal{J}}(P,\varphi)$ and the length of the filtration in dependence of the parabolic P and the cuspidal support φ in question, which is the content of our Theorems 3.3 and 3.6. As a next step, we calculate the cohomology of all the consecutive quotients of the filtration $\mathcal{A}^m_{\mathcal{J}}(P,\varphi)/\mathcal{A}^{m+1}_{\mathcal{J}}(P,\varphi)$ with respect to an arbitrary coefficient system E (cf. Propositions 4.2– 4.6). In particular, we explicitly describe the $G(\mathbb{A}_f)$-module structure of these cohomology spaces. This completes the preparatory work we need.

The second part of this article contains the main results of this paper. By analyzing the long exact sequences in cohomology defined by the short exact sequences coming from forming the filtration quotients $\mathcal{A}^m_{\mathcal{J}}(P,\varphi)/\mathcal{A}^{m+1}_{\mathcal{J}}(P,\varphi)$, we can almost fully determine the summands $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P,\varphi)\otimes E)$ in the Eisenstein cohomology of G indexed by a proper standard parabolic k-subgroup P and a cuspidal support φ. The main theorems are Theorem 5.1 (dealing with the maximal parabolic case) and Theorem 5.4 (describing the minimal parabolic case). Necessary and sufficient conditions for the existence of Eisenstein cohomology classes representable by residues of Eisenstein series are given in our Corollaries 5.2 and 5.6 for the case of a maximal and minimal parabolic subgroup, respectively. In particular, we would like to draw the reader's attention to Corollary 5.6, which says that, under a compatibility condition on the highest weight of the coefficient module E and the cuspidal support, there exist non–trivial Eisenstein cohomology classes which can be represented by *non–square integrable residues of Eisenstein series* attached to the minimal parabolic subgroup. The compatibility condition says that a certain filtration step in the Franke filtration is non-trivial. These residues are themselves obtained from poles of order one, i.e., of non–maximal order, of some Eisenstein series whose cuspidal support is a character of the minimal parabolic subgroup of a certain special form depending on E. As Harder pointed out to the second-named author, he constructed classes of this internal nature for GL_n. For symplectic groups, however, according to our knowledge, classes of this type have not yet been found whence we think of this result as one of the interesting new features compared to existing literature on this subject (cf. [Sch95] for Sp_4 over $k=\mathbb{Q}$ or [Har93]).

Finally, we analyze the case of the trivial representation more closely. As we do so, we obtain an improvement of Borel's result on the injectivity and bijectivity of the Borel map J^q in the case Sp_4/k (cf. Section 6), where k is a totally real number field of degree n over $\mathbb{Q}$. His general theorem implies for our case that J^q is injective for all degrees $q \leq n-1$ and an isomorphism for $q=0,1$. Our Corollary 6.1 improves these bounds. Namely, J^q is injective (at least) up to degree $3n$, and it is an isomorphism up to degree $2n-1$. However, as the referee pointed out, this result also follows from the results regarding the Borel map obtained in the diploma thesis [KR] of Kewenig and Rieband. In their thesis they study the Borel map following the approach of Franke in [Fra08] and describe explicitly the kernel of J^* in the case of the symplectic group of arbitrary rank over any number field. Their result in our case implies that the image of the Borel map is non–trivial in higher degrees than in our Corollary 6.1. Since we were not aware of this thesis while writing this paper, and as it is still unpublished, we follow a suggestion of the referee to include a summary of their result made explicit in our case.

1. NOTATION

1.1. **Number field.** Let k be a totally real number field with n archimedean places, k_v its completion at the place v, and $\mathbb{A}=\mathbb{A}_k$ its ring of adèles. Let S_∞ be the set of archimedean (i.e., real) places and S_f the set of non–archimedean places of k. Let $\mathbb{A}_f$ be the finite adèles.

1.2. **Symplectic group of rank two and parabolic data.** Let $G = Sp_4/k$ be the simple k-split algebraic k-group of k-rank two and Cartan type C$_2$. Let P_0 be a fixed Borel subgroup of G/k. It is a minimal parabolic k-subgroup of G with Levi subgroup L_0 and unipotent radical N_0. We assume that L_0 is realized as the group of diagonal matrices $diag(a_1, a_2, a_1^{-1}, a_2^{-1})$.

Now, define for $t = diag(a_1, a_2, a_1^{-1}, a_2^{-1})$ as usual $e_i(t) = a_i$. We may assume that $\Delta_k = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\}$ is the set of simple k-roots of G with respect to L_0 corresponding to our choice of P_0, and $\Psi_k^+ = \{\alpha_1, \alpha_2, \alpha_3 = e_1 + e_2, \alpha_4 = 2e_1\}$ is the set of positive k-roots.

Let $P_i = L_iN_i$, $i = 1, 2$, be the (maximal) parabolic k-subgroup corresponding to the root α_i, meaning that α_i is the only simple k-root of G vanishing identically on the maximal central k-split torus A_i of L_i, $i = 1, 2$. Hence, $L_1 \cong GL_2$ and $L_2 \cong GL_1 \times SL_2$ and A_i, $i = 1, 2$, is isomorphic to GL_1/k, realized in the following way: A_1 consists of diagonal matrices $diag(a, a, a^{-1}, a^{-1})$, while A_2 consists of diagonal matrices $diag(a, 1, a^{-1}, 1)$. For sake of uniformness of notation, we will also write A_0 for a maximal k-split central torus in L_0.

For a k-algebraic group, let $X^*(H)$ (resp. $X_*(H)$) denote the group of k-rational characters (resp. co-characters) of H. We set $\check{\mathfrak{a}}_{P_i} = X^*(L_i) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_{P_i} = X_*(L_i) \otimes_{\mathbb{Z}} \mathbb{R}$. For $i = 1, 2$, the inclusion $A_i \hookrightarrow A_0$ defines inclusions $\mathfrak{a}_{P_i} \hookrightarrow \mathfrak{a}_{P_0}$ and $\check{\mathfrak{a}}_{P_i} \hookrightarrow \check{\mathfrak{a}}_{P_0}$ and therefore decompositions $\mathfrak{a}_{P_0} = \mathfrak{a}_{P_i} \oplus \mathfrak{a}_0^{P_i}$ and $\check{\mathfrak{a}}_{P_0} = \check{\mathfrak{a}}_{P_i} \oplus \check{\mathfrak{a}}_0^{P_i}$. We will also use $\mathfrak{a}_{P_i}^{P_j}$ to denote the intersection of $\mathfrak{a}_{P_i}$ and $\mathfrak{a}_0^{P_j}$ in $\mathfrak{a}_{P_0}$ and use the analogous notation $\check{\mathfrak{a}}_{P_i}^{P_j}$.

Having fixed positivity on the set of roots defines open positive chambers $\check{\mathfrak{a}}_{P_i}^+$ with closures denoted by $\overline{\check{\mathfrak{a}}_{P_i}^+}$. The cone dual to the positive Weyl chamber $\check{\mathfrak{a}}_{P_i}^+$ is denoted by ${}^+\check{\mathfrak{a}}_{P_i}$ and its closure $\overline{{}^+\check{\mathfrak{a}}_{P_i}}$.

We write $\Delta(P_i, A_i)$ for the set of weights with respect to A_i of the adjoint action of P_i on N_i. As usual, we denote ρ_{P_i} as the half sum of these weights. In particular, the half sum of positive roots ρ is then $\rho = \rho_0 = \rho_{P_0}$.

1.3. **Weyl group.** Let w_1 be the simple reflection with respect to α_1 and w_2 with respect to α_2. Then the k-Weyl group of G with respect to T is

$$W = W_k = \{id, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1, w_2w_1w_2, w_1w_2w_1w_2\}.$$

The absolute Weyl group $W_{\mathbb{C}}$ of $G(k \otimes \mathbb{C})$ is then the direct product of n copies of W. We will also need the set of Kostant representatives for P_i: If $i = 1, 2$ it is defined as $W^{P_i} = \{w \in W | w^{-1}(\alpha_i) > 0\}$, and for $i = 0$ we simply have $W^{P_0} = W$. Note that $W^{P_1} = \{id, w_2, w_2w_1, w_2w_1w_2\}$ and $W^{P_2} = \{id, w_1, w_1w_2, w_1w_2w_1\}$.

1.4. **Lie subgroups and Lie algebras.** Fix a maximal compact subgroup $K = \prod_v K_v = K_\infty K_f$ of $G(\mathbb{A})$ in good position with respect to P_0. Denote by $R_{k/\mathbb{Q}}(.)$ the restriction of scalars from k to $\mathbb{Q}$. As usual, we write $H_\infty = R_{k/\mathbb{Q}}(H)(\mathbb{R})$ for the product $\prod_{v \in S_\infty} H(\mathbb{R})$ of the groups of real points of an algebraic k-group H. Then $G_\infty \cong Sp_4(\mathbb{R})^n$ and K_∞ is a maximal compact subgroup of the semi–simple Lie group G_∞. It is isomorphic to the product of n copies of $U(2)$. If Q is any Lie subgroup of G_∞, we write the same but fractional letter (i.e., $\mathfrak{q}$) for its real Lie algebra and $\mathfrak{q}_{\mathbb{C}} = \mathfrak{q} \otimes_{\mathbb{R}} \mathbb{C}$ for its complexification. In particular, in this notation, $\mathfrak{a}_{P_i}$,

$i = 0, 1, 2$, is isomorphic to the Lie algebra of $A_i(\mathbb{R}) = A_i(k_v)$ for every archimedean place $v \in S_\infty$ and $\mathfrak{a}_{P_i,\mathbb{C}}$ is its complexification. We will sometimes also write $\mathfrak{a}_{P_i,\sigma}$ to stress at which place $v \in S_\infty$, identified with the corresponding field embedding $\sigma : k \hookrightarrow \mathbb{C}$, we look at. Furthermore, $\check{\mathfrak{a}}_{P_i}$ is in a natural way isomorphic to the dual space of $\mathfrak{a}_{P_i}$. As $A_i(\mathbb{R})^\circ$ can be diagonally embedded into $L_{i,\infty}$ and G_∞, we can also view $\mathfrak{a}_{P_i}$ (resp. $\check{\mathfrak{a}}_{P_i}$) as being diagonally embedded into the Lie algebras $\mathfrak{l}_{i,\infty}$ and $\mathfrak{g}_\infty$ (resp. their dual spaces). In this setup, if we write $M_{i,\infty} = \bigcap_{\chi \in X^*(L_i)} \ker(|\chi|)$, then we can decompose the Levi factors $L_{i,\infty} = M_{i,\infty} A_i(\mathbb{R})^\circ$, $i = 0, 1, 2$. Back to the case of a general Lie subgroup Q of G_∞, we write $\mathcal{Z}(\mathfrak{q})$ for the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{q}_\mathbb{C})$ and K_Q for the intersection $K_\infty \cap Q$.

1.5. **Coefficient system.** Throughout the paper $E = E_\Lambda$ denotes an irreducible, finite–dimensional representation of G_∞ on a complex vector space determined by its highest weight Λ. We can write $\Lambda = ((\Lambda_1)_\sigma, (\Lambda_2)_\sigma)_\sigma$, where σ runs through the set of field embeddings $k \hookrightarrow \mathbb{R}$ and $(\Lambda_j)_\sigma$ denotes the coordinate with respect to the functional e_j viewed on the copy of $\mathfrak{a}_{P_0,\mathbb{C}}$ corresponding to σ. We abbreviate $\Lambda_\sigma = ((\Lambda_1)_\sigma, (\Lambda_2)_\sigma)$ (so that $\Lambda = (\Lambda_\sigma)_\sigma$). The highest weight, being algebraically integral and dominant, implies that $(\Lambda_1)_\sigma, (\Lambda_2)_\sigma \in \mathbb{Z}$ and $(\Lambda_1)_\sigma \geq (\Lambda_2)_\sigma \geq 0$. We will always assume that E is the complexification of an algebraic representation of G/k. Furthermore, we will assume that the coordinates of Λ are repeating in the field embeddings, i.e., $\Lambda_\sigma = \Lambda_\tau$ for all field embeddings σ, τ. This will turn out to be no restriction (cf. Proposition 2.1), since for all coefficient systems E with a highest weight having non-repeating coordinates, the space of Eisenstein cohomology supported in the Borel subgroup necessarily vanishes.

2. Automorphic forms and Eisenstein cohomology

This section recalls the decomposition of the space of automorphic forms along the cuspidal support, and the corresponding decomposition in cohomology. Although this is well known, it is included here in order to fix the notation. We will also prove that Eisenstein cohomology supported in the Borel subgroup is trivial, unless the coordinates of Λ are repeating in the field embeddings $\sigma : k \hookrightarrow \mathbb{C}$.

2.1. **Automorphic forms.** Let $\mathcal{A}$ be the space of automorphic forms on $G(\mathbb{A})$. Recall that a smooth complex function on $G(\mathbb{A})$ is an automorphic form if it is left $G(k)$-invariant, K-finite, annihilated by an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\infty)$, and of moderate growth; cf. [BJ]. Thus, automorphic forms in $\mathcal{A}$ may be viewed as functions on $G(k)\backslash G(\mathbb{A})$.

As we are only interested in automorphic forms which have non–trivial $(\mathfrak{g}_\infty, K_\infty)$-cohomology with respect to the coefficient system E, we take $\mathcal{J}$ to be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\infty)$ annihilating the dual representation $\check{E}$. Then, we define $\mathcal{A}_\mathcal{J}$ to be the subspace of $\mathcal{A}$ consisting of automorphic forms annihilated by some power of $\mathcal{J}$. It is a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. Only such automorphic forms may represent a non–trivial cohomology class with respect to E; cf. [FS, Rem. 3.4].

2.2. **Induced representations.** Let Π be an automorphic representation of the Levi factor $L_i(\mathbb{A})$ of a standard proper parabolic k-subgroup P_i, where $i = 0, 1, 2$, such that the vector space of Π is the space of smooth K-finite functions in an irreducible constituent of the discrete spectrum of $L_i(\mathbb{A})$. Observe that here we use a standard convention: we say that Π is an automorphic representation of $L_i(\mathbb{A})$, although it is not a representation of $L_i(\mathbb{A})$ at all, but only an $(\mathfrak{l}_{i,\infty}, K_{L_{i,\infty}}; L_i(\mathbb{A}_f))$-module.

Let $\lambda \in \check{\mathfrak{a}}_{P_i,\mathbb{C}}$. Then λ gives rise to a character of $L_i(\mathbb{A})$ by

$$l \mapsto \exp\langle \lambda, H_{P_i}(l)\rangle,$$

where $H_{P_i} : G(\mathbb{A}) \to \mathfrak{a}_{P_i}$ is the standard height function on $G(\mathbb{A})$ (cf., e.g., [Fra98, p. 185]). Then we define

$$I_i(\lambda, \Pi) = \mathrm{Ind}_{P_i(\mathbb{A})}^{G(\mathbb{A})} \left(\Pi \otimes \exp\langle \lambda, H_{P_i}(\cdot)\rangle\right),$$

where the induction is normalized in such a way that it preserves unitarizability.

Let W_Π denote the space of smooth K-finite functions on $L_i(k)N_i(\mathbb{A})\backslash G(\mathbb{A})$ such that for any $g \in G(\mathbb{A})$ the function $f_g(l) = f(lg)$ of $l \in L_i(\mathbb{A})$ belongs to the space of Π. Note that every irreducible constituent of the discrete spectrum of $L_i(\mathbb{A})$ appears with multiplicity one (see [JL] for $i = 1$ and [Ram] for $i = 2$). Then, the space of the induced representation $I_i(\lambda, \Pi)$ may be identified with the space of functions of the form

$$g \mapsto f_\lambda(g) = f(g) \exp\langle \lambda + \rho_{P_i}, H_{P_i}(g)\rangle,$$

where f ranges over all functions in W_Π.

The tensor product $W_\Pi \otimes S(\mathfrak{a}_{P_i,\mathbb{C}})$ of W_Π with the symmetric algebra of $\check{\mathfrak{a}}_{P_i,\mathbb{C}}$ can be endowed with the structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module as in [Fra98, p. 218 and p. 234] and [LS, p. 155]. Since we are just working with the normalized parabolic induction instead of W_Π, this gives rise to a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module structure on

$$I_i(\lambda, \Pi) \otimes S(\mathfrak{a}_{P_i,\mathbb{C}})$$

for a given λ.

Finally, since $I_i(\lambda, \Pi)$ decomposes into a restricted tensor product of local induced representations, we have

$$I_i(\lambda, \Pi) \cong I_i(\lambda, \Pi_\infty) \otimes I_i(\lambda, \Pi_f),$$

where Π_∞ and Π_f are the infinite and finite part of Π, respectively,

$$I_i(\lambda, \Pi_\infty) = \mathrm{Ind}_{(\mathfrak{l}_{i,\infty}, K_{L_{i,\infty}})}^{(\mathfrak{g}_\infty, K_\infty)} \left(\Pi_\infty \otimes \exp_\infty\langle \lambda, H_{P_i}(\cdot)\rangle\right),$$

$$I_i(\lambda, \Pi_f) = \mathrm{Ind}_{P_i(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \left(\Pi_f \otimes \exp_f\langle \lambda, H_{P_i}(\cdot)\rangle\right),$$

and the induction is normalized.

2.3. **Eisenstein series.** Let Π be a discrete spectrum representation of $L_i(\mathbb{A})$ as above. Let f be a function in W_Π and, for any $\lambda \in \check{\mathfrak{a}}_{P_i,\mathbb{C}}$ let f_λ be the function in the space of $I_i(\lambda, \Pi)$ attached to f as above. Then we define the Eisenstein series, at least formally, as

$$E(g, f_\lambda) = \sum_{\gamma \in P_i(k)\backslash G(k)} f_\lambda(\gamma g) = \sum_{\gamma \in P_i(k)\backslash G(k)} f(\gamma g) \exp\langle \lambda + \rho_{P_i}, H_{P_i}(\gamma g)\rangle.$$

The series converges absolutely and locally uniformly in g for λ sufficiently regular (i.e. deep enough in the positive Weyl chamber defined by P). It can be analytically continued to a meromorphic function on all of $\check{\mathfrak{a}}_{P_i,\mathbb{C}}$. Away from its poles it defines an automorphic form on $G(\mathbb{A})$. For a proof of these facts, see Lemma 4.1 and Lemma 6.1 in [Lan] or Section II.1.5, Section IV.1.8, Section IV.3 and Section IV.4 in [MW].

2.4. **Decomposition along the cuspidal support.** There is a decomposition of the space of automorphic forms along their cuspidal support which induces a decomposition of $\mathcal{A}_{\mathcal{J}}$; cf. [FS, Sect. 1], [MW, Thm. III.2.6]. We denote by $\{P\}$ the associate class of parabolic k-subgroups of G represented by a parabolic k-subgroup P of G. In our case, there are four such classes represented by P_0, P_1, P_2, G. As a first step, one has a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module decomposition

$$\mathcal{A}_{\mathcal{J}} = \mathcal{A}_{\mathcal{J}}(P_0) \oplus \mathcal{A}_{\mathcal{J}}(P_1) \oplus \mathcal{A}_{\mathcal{J}}(P_2) \oplus \mathcal{A}_{\mathcal{J}}(G),$$

where for an associate class of parabolic k-subgroups represented by P the space $\mathcal{A}_{\mathcal{J}}(P)$ consists of automorphic forms in $\mathcal{A}_{\mathcal{J}}$ which are negligible along all parabolic k-subgroups not belonging to $\{P\}$. Here negligible along a parabolic k-subgroup Q means that the constant term along Q is orthogonal to the space of cuspidal automorphic forms on the Levi factor of Q. Observe that $\mathcal{A}_{\mathcal{J}}(G)$ is the space of cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}$, and since we are interested in the Eisenstein cohomology (see Section 5), we concentrate on the remaining three subspaces corresponding to classes of proper parabolic k-subgroups.

For the second step in the decomposition, let $\varphi = (\varphi_P)_{P\in\{P_i\}}$ be the associate class of unitary cuspidal automorphic representations of the Levi factors $L_P(\mathbb{A})$ of parabolic k-subgroups $P \in \{P_i\}$, trivial on the diagonally embedded group $A_P(\mathbb{R})^\circ$, and satisfying conditions listed in [FS, Sect. 1.2]. The set of all such φ for a class $\{P_i\}$ is denoted by Φ_i. Then there is a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module decomposition

$$\mathcal{A}_{\mathcal{J}}(P_i) = \bigoplus_{\varphi\in\Phi_i} \mathcal{A}_{\mathcal{J}}(P_i, \varphi),$$

where $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$ is defined as follows. The conditions listed in [FS, Sect. 1.2] ensure that the associate class $\varphi \in \Phi_i$ is obtained by conjugating a single unitary cuspidal automorphic representation π of $L_i(\mathbb{A})$ and that the infinitesimal character of its archimedean component is related in a certain way to the infinitesimal character of $\check{E}$. Then the space $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$ may be defined in two equivalent ways; cf. [FS, Sect. 1]. Roughly speaking, it is spanned by all residues and main values of the derivatives of the Eisenstein series attached to π at certain values of its complex parameter. The condition on the infinitesimal character of the archimedean component of π ensures that the automorphic forms so obtained are indeed annihilated by a power of $\mathcal{J}$.

2.5. **Eisenstein cohomology.** The cohomology of a congruence subgroup of G_∞, with respect to a finite–dimensional representation E, may be interpreted in terms of its automorphic spectrum. Passing to the inductive limit over all congruence subgroups, its study is reduced to the study of automorphic cohomology $H^*(G, E)$ of G with respect to E. It is defined as the relative Lie algebra cohomology of the space of smooth left $G(k)$–invariant functions on $G(\mathbb{A})$ with values in E. However, Borel proved in [Bor83] that it suffices to consider the subspace consisting of K_∞-finite functions of uniform moderate growth. Finally, using his filtration, Franke proved that in fact even

$$H^*(G, E) \cong H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}} \otimes E).$$

The decomposition of the space $\mathcal{A}_{\mathcal{J}}$ of automorphic forms along their cuspidal support gives rise to the decomposition

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P)\otimes E)$$

in the cohomology, where the sum ranges over the associate classes $\{P\}$ of parabolic k-subgroups of G. The cohomology space corresponding to the associate class $\{G\}$ is called the cuspidal cohomology, since $\mathcal{A}_{\mathcal{J}}(G)$ is the subspace of cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}$. The remaining part in the decomposition is called the Eisenstein cohomology. Thus,

$$H^*_{Eis}(G,E) = \bigoplus_{i=0}^{2} \bigoplus_{\varphi\in\Phi_i} H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E).$$

In this paper we describe $H^*_{Eis}(G,E)$ by determining the summands in this decomposition.

2.6. **Repeating coordinates.** We will now justify why we assume that the highest weight Λ of E has repeating coordinates in the field embeddings $\sigma : k \hookrightarrow \mathbb{C}$. Otherwise, $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J}}(P_0)\otimes E)$ vanishes. With this assumption, the infinitesimal character of a $\pi\in\varphi_P$ has repeating coordinates, too. Hence, slightly abusing notation we will consider this infinitesimal character as an element in $\check{\mathfrak{a}}_0^{P_i}$ which is diagonally embedded in $\mathfrak{g}_\infty$, although strictly speaking it is a sum of n copies of such an element.

Proposition 2.1. *Let E be an irreducible, finite-dimensional complex representation of G_∞ of highest weight $\Lambda = (\Lambda_\sigma)_\sigma = ((\Lambda_1)_\sigma, (\Lambda_2)_\sigma)_\sigma$, where σ ranges over all field embeddings $k\hookrightarrow\mathbb{C}$. Assume that E is the complexification of a k-rational representation of G/k. If Λ does not have repeating coordinates, i.e. $\Lambda_\sigma = \Lambda_\tau$ for all field embeddings $\sigma,\tau : k\hookrightarrow\mathbb{C}$, then $H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0)\otimes E) = 0$.*

Proof. We start off more generally. Assume only $H^*_{Eis}(G,E)\neq 0$. By the last section there is hence a proper standard parabolic k-subgroup $P = P_i$, $i\in\{0,1,2\}$, of G and cuspidal support $\varphi\in\Phi_i$ such that

$$H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P,\varphi)\otimes E)\neq 0.$$

Hence, there is a unitary cuspidal automorphic representation $\pi\in\varphi_P$ of $L_P(\mathbb{A})$ and a point $\lambda\in\check{\mathfrak{a}}_{P,\mathbb{C}}$ such that $H^*(\mathfrak{g}_\infty, K_\infty, I_P(\lambda,\pi)\otimes S(\mathfrak{a}_{P,\mathbb{C}})\otimes E)\neq 0$. Applying Frobenius reciprocity and [BW, III Thm. 3.3] shows that for all $\sigma : k\hookrightarrow\mathbb{C}$ there exists a $w_\sigma\in W^P$ such that $\pi_\infty\otimes\mathbb{C}_{\lambda+\rho_P}$ has non-trivial $(\mathfrak{l}_{P,\infty}, K_{L_{P,\infty}})$-cohomology with respect to $S(\mathfrak{a}_{P,\mathbb{C}})\otimes\bigotimes_\sigma F_{w_\sigma}$. Here, $\mathbb{C}_{\lambda+\rho_P}$ denotes the one-dimensional complex representation of $\mathfrak{a}_P\hookrightarrow\mathfrak{l}_{P,\infty}$ on which $a\in\mathfrak{a}_P$ acts by multiplication by $(\lambda+\rho_P)(a)$ and F_{w_σ} is the irreducible finite–dimensional representation of $L_P(\mathbb{R})$ of highest weight $w_\sigma(\Lambda_\sigma+\rho)-\rho$. Recall that this makes sense since ρ has repeating coordinates. Hence the Künneth rule implies that necessarily

$$H^*(\mathfrak{a}_{P,\infty}, \pi|_{A^\circ_{P,\infty}} \otimes \bigotimes_\sigma \mathbb{C}_{w_\sigma(\Lambda_\sigma+\rho)-\rho|_{\mathfrak{a}_{P,\sigma}}} \otimes \mathbb{C}_{\lambda+\rho_P}\otimes S(\mathfrak{a}_{P,\mathbb{C}}))\neq 0. \tag{2.6.1}$$

Observe that, A_P being abelian and π a cuspidal representation, $\pi|_{A^\circ_{P,\infty}} = \widetilde{\chi}|_{A^\circ_{P,\infty}}$ for a unitary character $\widetilde{\chi} : A_P(k)A_P(\mathbb{R})^\circ\backslash A_P(\mathbb{A}) \to \mathbb{C}$. Hence, the non-vanishing of (2.6.1) implies that

$$\widetilde{\chi}^{-1}|_{A^\circ_{P,\infty}} = \bigotimes_\sigma \mathbb{C}_{w_\sigma(\Lambda_\sigma+\rho)-\rho|_{\mathfrak{a}_{P,\sigma}} - \frac{1}{n}(\sum_\sigma w_\sigma(\Lambda_\sigma+\rho)-\rho|_{\mathfrak{a}_{P,\sigma}})}$$

and

$$\lambda = -\frac{1}{n}\sum_\sigma w_\sigma(\Lambda_\sigma+\rho)|_{\mathfrak{a}_{P,\sigma}} = -pr_{\mathfrak{h}_\infty\to\mathfrak{a}_P}((w_\sigma(\Lambda_\sigma+\rho))_\sigma).$$

Observe furthermore that since E is the complexification of a k-rational representation of G/k, $H^*(\mathfrak{n}_{P,\infty}, E) = \bigoplus_{w=(w_\sigma)_\sigma\in(W^P)^n} \bigotimes_\sigma F_{w_\sigma}$ is the complexification of a k-rational representation of L_P/k. In particular, $\bigotimes_\sigma \mathbb{C}_{w_\sigma(\Lambda_\sigma+\rho)-\rho|_{\mathfrak{a}_{P,\sigma}}}$ is the complexification of a rational character of A_P/k. This shows that there is a k-rational, (possibly non-unitary) continuous character $\chi : A_P(k)\backslash A_P(\mathbb{A}) \to \mathbb{C}$ which equals $\widetilde{\chi}$ modulo $A_P(\mathbb{R})^\circ$ and which satisfies that the differential of its restriction to the diagonally embedded group $A_P(\mathbb{R})^\circ$ is $\lambda + \rho_P$. Let $E_0(A_P)$ be the group of units in $A_P(k)$, i.e. of those elements which are in the maximal compact subgroup at all places. Then the same arguments as in [Har87, Sect. 2.5.5] show that χ, being k-rational and continuous, must be trivial on the connected component of the Zariski closure of $E_0(A_P)$. Indeed, every such character has to vanish on some suitable open compact subgroup $C_f \subset A_P(\mathbb{A}_f)$, whence it is trivial on

$$E_+(C_f) := A_P(k) \cap A_P(\mathbb{R})^\circ \cap C_f.$$

Here, we think of $A_P(k)$ as being diagonally embedded in all of $A_P(\mathbb{A})$. By its k-rationality, χ also vanishes on the Zariski closure of $E_+(C_f)$. Further, $E_+(C_f)$ is a subgroup of $E_0(A_P)$ of finite index. Since every such subgroup is necessarily a congruence subgroup, see [Che, Thm. 1], χ must even be trivial on the connected component of the Zariski closure $\overline{E_0(A_P)}$ of $E_0(A_P)$, as claimed. However, as k is totally real, $\overline{E_0(A_P)}$ fits into the following exact sequence:

$$1 \to \overline{E_0(A_P)} \to R_{k/\mathbb{Q}}(A_P) \to A_P/\mathbb{Q} \to 1$$

(see [Har87, Sect.2.8] and [Ser89, Chp. II 3.1-3.3]), implying that $\chi_\sigma^{-1} = \chi_\tau^{-1}$ for all field embeddings σ, τ. In particular, $w_\sigma(\Lambda_\sigma+\rho)|_{\mathfrak{a}_{P,\sigma}} = w_\tau(\Lambda_\tau+\rho)_{\mathfrak{a}_{P,\tau}}$ for all σ and τ. Now, if $P = P_0$, this is only possible if $w_\sigma = w_\tau$ and hence only if $\Lambda_\sigma = \Lambda_\tau$, i.e., if the highest weight of E has repeating coordinates. □

3. The Franke filtration

We briefly recall the filtration of the space of adèlic automorphic forms obtained by Franke in [Fra98, Sect. 6], and its refinement along the cuspidal support by Franke and Schwermer [FS, Sect. 1]. The filtration is valid for any reductive group defined over k, but we write it for $G = Sp_4/k$. In that case we give a precise description of the quotients of consecutive filtration steps in terms of induced representations.

3.1. **Filtration along the cuspidal support.** In [Fra98, Sect. 6], Franke defines a finite descending filtration of the spaces $\mathcal{A}_{\mathcal{J}}(P_i)$ such that the consecutive quotients of the filtration are described as certain induced representations from the discrete spectrum on the Levi factors of parabolic k-subgroups containing P_i. His filtration depends on a choice of a function T defined on a finite subset of $\check{\mathfrak{a}}_0$ with values in $\mathbb{Z}$. Thus, the filtration steps are indexed by integers, although there are only finitely many non–trivial quotients of consecutive filtration steps.

Let $\mathcal{A}^m_{\mathcal{J}}(P_i)$ denote the filtration step corresponding to $m \in \mathbb{Z}$. Then, as in [FS, Sect. 5.2], where the case of a maximal proper parabolic subgroup of GL_n was considered, one can define the filtration of each summand $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$ in the decomposition of $\mathcal{A}_{\mathcal{J}}(P_i)$ by

$$\mathcal{A}^m_i(\varphi) := \mathcal{A}^m_{\mathcal{J}}(P_i) \cap \mathcal{A}_{\mathcal{J}}(P_i, \varphi).$$

Then, $\mathcal{A}^m_i(\varphi)$ consists of those automorphic forms in the filtration step $\mathcal{A}^m_{\mathcal{J}}(P_i)$, which are obtained as residues and main values of derivatives of Eisenstein series attached to $\pi \in \varphi_{P_i}$.

In the rest of this section we explain, following [Fra98, Sect. 5.2 and Sect. 6], how to describe the quotients of the filtration of $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$. The description in our case given below does not hold in general. Here we substantially use the fact that $\mathcal{J}$ annihilates a finite–dimensional representation and that we have fixed the cuspidal support φ, and thus obtain a bit simpler description than the general case in [Fra98].

Since the dual representation $\check{E}$ of E has highest weight $-w_{long,G}(\Lambda) = \Lambda$, where $w_{long,G} = w_1 w_2 w_1 w_2$ is the longest Weyl group element, its infinitesimal character is given by $\Lambda + \rho_0$. Hence, the annihilator $\mathcal{J}$ in $\mathcal{Z}(\mathfrak{g}_\infty)$ of $\check{E}$ annihilates precisely the Weyl group orbit of $\Lambda + \rho_0 = (\Lambda_1 + 2, \Lambda_2 + 1)$, where the coordinates are with respect to the basis $\{e_1, e_2\}$ of $\check{\mathfrak{a}}_0$.

3.2. **Case of minimal parabolic subgroup.** We consider first the associate class $\{P_0\}$ of the fixed minimal parabolic k-subgroup P_0. Let $\varphi = (\varphi_P)_{P \in \{P_0\}}$ be an associate class of cuspidal automorphic representations of the Levi factors of the parabolic k-subgroups in $\{P_0\}$. Let $\mu_1 \otimes \mu_2 \in \varphi_{P_0}$ be a unitary character of $L_0(\mathbb{A})$, trivial on $L_0(k)$, where μ_1 and μ_2 are unitary characters of $k^\times \backslash \mathbb{A}^\times$.

We begin with the following lemma which singles out the possible infinitesimal characters of a discrete spectrum representation of the Levi factor and evaluation points for the corresponding Eisenstein series occurring in the description of the filtration of $\mathcal{A}_{\mathcal{J}}(P_0)$. Since Λ has repeating coordinates as well as the evaluation point, it follows that the possible infinitesimal characters have repeating coordinates as well. As mentioned earlier, we consider them as elements of $\check{\mathfrak{a}}_0^{P_i}$.

Lemma 3.1. *Let $\Lambda = (\Lambda_1, \Lambda_2)$ be the highest weight of E and $\mathcal{J}$ be the ideal annihilating the dual of E. All possible infinitesimal characters $\nu \in \check{\mathfrak{a}}_0^R$ of the infinite component of the discrete spectrum automorphic representation of the Levi factor $L_R(\mathbb{A})$ of a standard parabolic k-subgroup R supported in $\mu_1 \otimes \mu_2 \in \varphi_{P_0}$, and the evaluation points $\lambda \in \check{\mathfrak{a}}_R$ for the corresponding Eisenstein series, such that $\nu + \lambda$ is annihilated by $\mathcal{J}$, are given as follows.*

For P_0 we have $\nu = 0$, and λ is any element of the Weyl group orbit of $\Lambda + \rho_0$. For P_1 we have either

$$\lambda = \pm\left(\frac{3 + \Lambda_1 + \Lambda_2}{2}, \frac{3 + \Lambda_1 + \Lambda_2}{2}\right) \quad \textit{and} \quad \nu = \left(\frac{1 + \Lambda_1 - \Lambda_2}{2}, -\frac{1 + \Lambda_1 - \Lambda_2}{2}\right)$$

or

$$\lambda = \pm\left(\frac{1 + \Lambda_1 - \Lambda_2}{2}, \frac{1 + \Lambda_1 - \Lambda_2}{2}\right) \quad \textit{and} \quad \nu = \left(\frac{3 + \Lambda_1 + \Lambda_2}{2}, -\frac{3 + \Lambda_1 + \Lambda_2}{2}\right).$$

For P_2 we have either

$$\lambda = \pm(2+\Lambda_1, 0) \quad \textit{and} \quad \nu = (0, 1+\Lambda_2)$$

or

$$\lambda = \pm(1+\Lambda_2, 0) \quad \textit{and} \quad \nu = (0, 2+\Lambda_1)\,.$$

For G we have $\lambda = 0$, and ν is the Weyl group orbit of $\Lambda + \rho_0$.

Proof. This is a direct calculation already contained in [Sch86]. It exploits the fact that $\mathcal{J}$ annihilates the Weyl group orbit of $\Lambda + \rho_0$, and thus χ and ξ are just projections of an element in that orbit to $\check{\mathfrak{a}}_0^{P_i}$ and $\check{\mathfrak{a}}_{P_i}$, respectively. □

Since the quotients of the filtration are described using (residual) Eisenstein series evaluated at $\lambda \in \overline{\check{\mathfrak{a}}_R^+}$, we need the following result regarding the analytic behavior of the Eisenstein series for $Sp_4(\mathbb{A})$. Kim in [Kim, Sect. 5] studied these Eisenstein series. We state here only the part of his results which we require in the sequel.

Proposition 3.2 (Kim, [Kim]). *The space $\mathcal{A}_{\mathcal{J}}(P_0, \varphi)$ contains no irreducible constituent of the discrete spectrum of $G(\mathbb{A})$ unless $\Lambda = 0$ and the trivial character of $L_0(\mathbb{A})$ belongs to φ_{P_0}. If $\Lambda = 0$ and the trivial character of $L_0(\mathbb{A})$ belongs to φ_{P_0}, then the only constituent of the discrete series of $G(\mathbb{A})$ belonging to $\mathcal{A}_{\mathcal{J}}(P_0, \varphi)$ is one dimensional and isomorphic to the trivial representation of $G(\mathbb{A})$, i.e. consists of constant functions on $G(\mathbb{A})$.*

The following theorem gives the Franke filtration in the case we consider. However, it depends on the choice of an integer–valued function T defined on a finite subset $S_{\mathcal{J}}$ of $\check{\mathfrak{a}}_0$ with the property

$$T(\lambda_1) < T(\lambda_2) \text{ if } \lambda_1 \neq \lambda_2 \text{ and } \lambda_2 \in \lambda_1 - \overline{{}^+\check{\mathfrak{a}}_0}$$

for all $\lambda_1, \lambda_2 \in \check{\mathfrak{a}}_0$. If λ_1 and λ_2 satisfy the above condition either for $T(\lambda_1) < T(\lambda_2)$ or $T(\lambda_2) < T(\lambda_1)$, we say that they are comparable; otherwise we say that they are incomparable. The subset $S_{\mathcal{J}}$ consists of natural embeddings of those λ obtained in Lemma 3.1 which satisfy $\lambda \in \overline{\check{\mathfrak{a}}_{P_i}^+}$. However, if a particular cuspidal support is fixed, not all elements of $S_{\mathcal{J}}$ play a role. Hence, in order to obtain the filtration of $\mathcal{A}_{\mathcal{J}}(P_0, \varphi)$, we fix a choice of T depending on φ in the course of the proof.

Theorem 3.3. *Let $\{P_0\}$ be the associate class of a minimal parabolic k-subgroup, and let $\varphi \in \Phi_0$ be the associate class of the character $\mu_1 \otimes \mu_2$ of $L_0(\mathbb{A})$, where μ_1 and μ_2 are unitary characters of $k^\times \backslash \mathbb{A}^\times$. The filtration of $\mathcal{A}_{\mathcal{J}}(P_0, \varphi)$, with respect to the function T appropriately chosen during the course of the proof, has at most three non–trivial filtration steps*

$$\mathcal{A}_{\mathcal{J}}(P_0, \varphi) = \mathcal{A}_0^0(\varphi) \supset \mathcal{A}_0^1(\varphi) \supset \mathcal{A}_0^2(\varphi),$$

where $\mathcal{A}_0^2(\varphi)$ is non–trivial if and only if $\Lambda_1 = \Lambda_2 = 0$ and $\mu_1 = \mu_2 = \mathbf{1}$, where $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times$, while $\mathcal{A}_0^1(\varphi)$ is non–trivial if and only if

- *$\Lambda_1 = \Lambda_2$ and $\mu_1 = \mu_2$*
- *or $\Lambda_2 = 0$ and $\mu_2 = \mathbf{1}$.*

If $\mathcal{A}_0^2(\varphi)$ is non-trivial, it is one dimensional and isomorphic as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module to

$$\mathcal{A}_0^2(\varphi) \cong \mathbf{1}_{G(\mathbb{A})},$$

where $\mathbf{1}_{G(\mathbb{A})}$ *is the trivial character of* $G(\mathbb{A})$*, i.e* $\mathcal{A}_0^2(\varphi)$ *consists of constant functions on* $G(\mathbb{A})$*. If* $\mathcal{A}_0^1(\varphi)$ *is non–trivial, then the quotient* $\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)$ *is isomorphic to*

$$\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi) \cong \begin{cases} I_1(\frac{3}{2}+\Lambda, \mu\circ\det)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}}), \\ \qquad \textit{if } \Lambda=\Lambda_1=\Lambda_2 \textit{ and } \mu=\mu_1=\mu_2, \textit{ but } \Lambda\neq 0 \textit{ or } \mu\neq\mathbf{1}, \\ I_2(2+\Lambda_1, \mu\otimes\mathbf{1}_{SL_2(\mathbb{A})})\otimes S(\mathfrak{a}_{P_2,\mathbb{C}}), \\ \qquad \textit{if } \Lambda_2=0 \textit{ and } \mu_2=\mathbf{1}, \textit{ but } \Lambda_1\neq 0 \textit{ or } \mu_1\neq\mathbf{1}, \\ I_1(\frac{3}{2}, \mu\circ\det)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}}) \bigoplus I_2(2, \mu\otimes\mathbf{1}_{SL_2(\mathbb{A})})\otimes S(\mathfrak{a}_{P_2,\mathbb{C}}), \\ \qquad \textit{if } \Lambda_1=\Lambda_2=0 \textit{ and } \mu_1=\mu_2=\mathbf{1} \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where in the first case* Λ *denotes the integer* $\Lambda_1 = \Lambda_2$*,* μ *denotes the character* $\mu_1=\mu_2$*, and* $\mathbf{1}_{SL_2(\mathbb{A})}$ *is the trivial character of* $SL_2(\mathbb{A})$*. Finally, the quotient* $\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)$ *is isomorphic to*

$$\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi) \cong I_0(\Lambda+\rho_0, \mu_1\otimes\mu_2)\otimes S(\mathfrak{a}_{P_0,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module for any* Λ *and* $\mu_1\otimes\mu_2$*.*

Proof. We closely follow [Fra98, Sect. 6], adjusted to the considered situation. As in [Fra98, p. 233], taking into account the cuspidal support, consider the set $M(P_0,\varphi)$ of quadruples (R,Π,ν,λ), such that:

- $R = L_R N_R$ is a standard parabolic k-subgroup of G containing an element of the associate class $\{P_0\}$.
- Π is a discrete spectrum representation of $L_R(\mathbb{A})$ with cuspidal support $\mu_1\otimes\mu_2$ obtained as the iterated residue at the value $\nu\in\check{\mathfrak{a}}_0^R$ of the Eisenstein series on $L_R(\mathbb{A})$ attached to $\mu_1\otimes\mu_2$.
- $\lambda\in\overline{\check{\mathfrak{a}}_R^+}$ is such that $\lambda+\nu$ is annihilated by $\mathcal{J}$.

Observe that the possible pairs (λ,ν) are given in Lemma 3.1, where one should take into account only the cases with $\lambda\in\overline{\check{\mathfrak{a}}_R^+}$.

For $m\in\mathbb{Z}$ let $M^m(P_0,\varphi)$ be the subset of $M(P_0,\varphi)$ consisting of those quadruples for which $T(\lambda)=m$, where λ is viewed as an element in $\check{\mathfrak{a}}_0$ via the natural embedding. Then, by [Fra98, Thm. 14], the quotient

$$\mathcal{A}_0^m(\varphi)/\mathcal{A}_0^{m+1}(\varphi) \cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^m(P_0,\varphi)} I(\lambda,\Pi)\otimes S(\mathfrak{a}_{R,\mathbb{C}}). \tag{3.2.1}$$

Observe at this point that the direct sum on the right hand side is obtained due to the fact that $\mathcal{J}$ annihilates a finite–dimensional representation, and thus it annihilates a Weyl group orbit not intersecting the boundary of the Weyl chambers in $\check{\mathfrak{a}}_0$ (see [Fra98, Thm. 19]). We also introduce the notation $M_R(P_0,\varphi)$ and $M_R^m(P_0,\varphi)$ for the set of all quadruples in $M(P_0,\varphi)$ and $M^m(P_0,\varphi)$, respectively, with a certain parabolic subgroup R as the first entry.

First consider the case $R=G$. Then always $\lambda=0$, and thus $M_G^m(P_0,\varphi)$ is trivial except possibly for $m=T(0)$. The residual representation Π of $G(\mathbb{A})$ is obtained as a residue of the Eisenstein series attached to $\mu_1\otimes\mu_2$ at $\nu\in\check{\mathfrak{a}}_0$ such that ν is annihilated by $\mathcal{J}$. By Proposition 3.2, the only possibility is that $\mu_1=\mu_2=\mathbf{1}$ and $\Lambda=0$. In that case $\nu=(2,1)$ and $\Pi\cong\mathbf{1}_{G(\mathbb{A})}$. Thus we have determined the quadruples in $M_G(P_0,\varphi)$. Namely,

$$M_G^m(P_0,\varphi) = \begin{cases} \{G, \mathbf{1}_{G(\mathbb{A})}, (2,1), 0\}, & \text{if } m=T(0) \text{ and } \Lambda_1=\Lambda_2=0 \text{ and } \mu_1=\mu_2=\mathbf{1}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $R = P_1$. Since Π is a residual representation of $L_R(\mathbb{A}) \cong GL_2(\mathbb{A})$, it is isomorphic to $\Pi \cong \mu \circ \det$. Hence, necessarily $\mu_1 = \mu_2$ and $\nu = (1/2, -1/2) \in \check{\mathfrak{a}}_0^{P_1}$. By Lemma 3.1, such ν can be obtained only if $\Lambda_1 = \Lambda_2$ and $\lambda = (3/2+\Lambda, 3/2+\Lambda)$, where we denote $\Lambda = \Lambda_1 = \Lambda_2$. Thus, we have

$$M^m_{P_1}(P_0,\varphi) = \begin{cases} \left\{\left(P_1, \mu\circ\det, \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{3}{2}+\Lambda, \frac{3}{2}+\Lambda\right)\right)\right\}, & \text{if } T(\frac{3}{2}+\Lambda, \frac{3}{2}+\Lambda) = m \text{ and} \\ & \Lambda_1 = \Lambda_2 = \Lambda \text{ and } \mu_1 = \mu_2 = \mu, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Similarly, for $R = P_2$, we have that Π is a residual representation of $L_R(\mathbb{A}) \cong GL_1(\mathbb{A}) \times SL_2(\mathbb{A})$. However, the only residual representation of $SL_2(\mathbb{A})$ is the trivial character $\mathbf{1}_{SL_2(\mathbb{A})}$ of $SL_2(\mathbb{A})$. Thus, necessarily μ_2 is the trivial character and $\nu = (0,1)$. By Lemma 3.1, such ν is obtained only if $\Lambda_2 = 0$, and then $\lambda = (2+\Lambda_1, 0)$ is the corresponding λ. So in this case we have

$$M^m_{P_2}(P_0,\varphi) = \begin{cases} \left\{\left(P_2, \mu_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}, (0,1), (2+\Lambda_1, 0)\right)\right\}, & \text{if } T(2+\Lambda_1, 0) = m \text{ and} \\ & \Lambda_2 = 0 \text{ and } \mu_2 = \mathbf{1}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, if $R = P_0$, then $\Pi = \mu_1 \otimes \mu_2$. Hence, $\nu = 0$ and $\lambda = (2+\Lambda_1, 1+\Lambda_2)$. Thus

$$M^m_{P_0}(P_0,\varphi) = \begin{cases} \{(P_0, \mu_1\otimes\mu_2, 0, (2+\Lambda_1, 1+\Lambda_2))\}, & \text{if } T(2+\Lambda_1, 1+\Lambda_2) = m, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The description of $M(P_0,\varphi)$ reveals that for a given Λ the values of a function T are required only at a certain subset of $S_{\mathcal{J}}$. More precisely, $M_{P_1}(P_0,\varphi)$ and $M_{P_2}(P_0,\varphi)$ may possibly be non–empty only for $\Lambda_1 = \Lambda_2 = 0$. Therefore, only in that case does $T(\lambda)$ for λ coming from both cases matter. Note that in this case the two λ are incomparable. We define $T(0) = 2$ and $T(2+\Lambda_1, 1+\Lambda_2) = 0$, and also

$$T\left(\frac{3+\Lambda_1+\Lambda_2}{2}, \frac{3+\Lambda_1+\Lambda_2}{2}\right) = T\left(2+\Lambda_1, 0\right) = 1.$$

Although there exist Λ_1, Λ_2 such that the last two points are comparable, as already explained, both points matter only for $\Lambda_1 = \Lambda_2 = 0$, and in that case they are incomparable. Therefore, we may define T in this way.

Now the theorem follows. Namely, $\mathcal{A}^2_0(\varphi)$ is non–trivial if and only if $M_G(P_0,\varphi)$ is non–trivial, which is if and only if the conditions given in the theorem are satisfied. In that case the only summand in the decomposition (3.2.1) is the trivial representation of $G(\mathbb{A})$.

The space $\mathcal{A}^1_0(\varphi)$ is non–trivial if and only if at least one of $M_{P_1}(P_0,\varphi)$ and $M_{P_2}(P_0,\varphi)$ is non–empty. Note that if $M_G(P_0,\varphi)$ is non–empty, then both $M_{P_i}(P_0,\varphi)$, for $i = 1,2$, are non–empty. Hence, this filtration step is non–trivial exactly if at least one of the two conditions given in the theorem is satisfied. Then the decomposition of the quotient follows directly from (3.2.1).

Finally, $\mathcal{A}^0_0(\varphi)$ is always non–trivial, and the decomposition of the quotient of this filtration step is obtained from (3.2.1). □

3.3. **Case of maximal parabolic subgroups.** Let $P_i = L_i N_i$, for $i = 1,2$, be one of the maximal proper standard parabolic k-subgroups. Let $\varphi = (\varphi_P)_{P\in\{P_i\}} \in \Phi_i$ be an associate class of cuspidal automorphic representations. Let $\pi \in \varphi_{P_i}$, and let

$\chi \in \check{\mathfrak{a}}_0^{P_i}$ be the infinitesimal character of its archimedean component, where $\check{\mathfrak{a}}_0^{P_i}$ is diagonally embedded into $\check{\mathfrak{a}}_{0,\infty}$.

The filtration of $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$, for $i = 1, 2$, depends on the analytic behavior of the Eisenstein series attached to $\pi \in \varphi_{P_i}$. This was studied by Kim in [Kim, Sect. 3 and 4], and we recall the result for convenience of the reader.

Proposition 3.4 (Kim, [Kim]).

1. *In the case of the parabolic subgroup P_1, the Eisenstein series $E(g, f_s)$, attached to a cuspidal automorphic representation π of $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$, has a pole at $s = \nu \in \overline{\check{\mathfrak{a}}_{P_1}^+}$ if and only if $\nu = (1/2, 1/2)$, the central character of π, is trivial and the principal L-function $L(1/2, \pi) \neq 0$. The space spanned by the residues $Res_{s=1/2}E(g, f_s)$ is isomorphic to the unique irreducible quotient $J_1(1/2, \pi)$ of $I_1(1/2, \pi)$.*
2. *In the case of the parabolic subgroup P_2, the Eisenstein series $E(g, f_s)$, attached to a cuspidal automorphic representation $\pi \cong \mu \otimes \sigma$ of $L_2(\mathbb{A}) \cong GL_1(\mathbb{A}) \times SL_2(\mathbb{A})$, has a pole at $s = \nu \in \overline{\check{\mathfrak{a}}_{P_2}^+}$ if and only if $\nu = (1, 0)$, and the Rankin–Selberg L-function $L(s, \mu \times \sigma)$ has a pole at $s = 1$ (see* [Kim, p. 137] *for a more explicit formulation of this condition). The space spanned by the residues $Res_{s=1}E(g, f_s)$ is isomorphic to the unique irreducible quotient $J_2(1, \pi)$ of the induced representation $I_2(1, \pi)$.*

Before proceeding we need the following technical lemma.

Lemma 3.5. *Let $\Lambda = (\Lambda_1, \Lambda_2)$ be the highest weight of E and $\mathcal{J}$ be the ideal annihilating the dual of E. Then the infinitesimal character $\chi \in \check{\mathfrak{a}}_0^{P_i}$ of the archimedean component of $\pi \in \varphi_{P_i}$, where $\varphi = (\varphi_P)_{P \in \{P_i\}} \in \Phi_i$, and the corresponding $\xi \in \check{\mathfrak{a}}_{P_i}$ such that $\xi + \chi$ is annihilated by $\mathcal{J}$ are given as follows. For P_1 we have either*

$$\xi = \pm\left(\frac{3 + \Lambda_1 + \Lambda_2}{2}, \frac{3 + \Lambda_1 + \Lambda_2}{2}\right) \quad \text{and} \quad \chi = \left(\frac{1 + \Lambda_1 - \Lambda_2}{2}, -\frac{1 + \Lambda_1 - \Lambda_2}{2}\right)$$

or

$$\xi = \pm\left(\frac{1 + \Lambda_1 - \Lambda_2}{2}, \frac{1 + \Lambda_1 - \Lambda_2}{2}\right) \quad \text{and} \quad \chi = \left(\frac{3 + \Lambda_1 + \Lambda_2}{2}, -\frac{3 + \Lambda_1 + \Lambda_2}{2}\right).$$

For P_2 we have either

$$\xi = \pm\,(2 + \Lambda_1, 0) \quad \text{and} \quad \chi = (0, 1 + \Lambda_2)$$

or

$$\xi = \pm\,(1 + \Lambda_2, 0) \quad \text{and} \quad \chi = (0, 2 + \Lambda_1)\,.$$

Observe that for each P_i and a fixed cuspidal support φ, at most one of the two possibilities may occur.

Proof. As in Lemma 3.1, this is a direct calculation already contained in [Sch86]. □

Theorem 3.6. *Let the notation be as above. Let $\xi \in \overline{\check{\mathfrak{a}}_{P_i}^+}$ be such that $\xi + \chi \in \check{\mathfrak{a}}_0$ is annihilated by $\mathcal{J}$. The filtration of $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$ has at most two non–trivial steps*

$$\mathcal{A}_{\mathcal{J}}(P_i, \varphi) = \mathcal{A}_i^1(\varphi) \supset \mathcal{A}_i^2(\varphi),$$

where the quotient is isomorphic to

$$\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi) \cong I_i(\xi, \pi) \otimes S(\mathfrak{a}_{P_i,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, and* $\mathcal{A}_i^2(\varphi)$ *is non–trivial if and only if*

- *in the case of* P_1 *we have* $\Lambda_1 = \Lambda_2 = \Lambda$*, the infinitesimal character* $\chi = (\frac{3}{2} + \Lambda, -\frac{3}{2} - \Lambda)$*,* $\xi = (\frac{1}{2}, \frac{1}{2})$*, and there is a section* f_s *of the induced representation* $I_1(s, \pi)$ *such that the Eisenstein series* $E(g, f_s)$ *has a pole at* $s = \xi = (\frac{1}{2}, \frac{1}{2})$*,*
- *in the case of* P_2 *we have* $\Lambda_2 = 0$*, the infinitesimal character* $\chi = (0, 2+\Lambda_1)$*,* $\xi = (1, 0)$*, and there is a section* f_s *of the induced representation* $I_2(s, \pi)$ *such that the Eisenstein series* $E(g, f_s)$ *has a pole at* $s = \xi = (1, 0)$*.*

If non–trivial, it is isomorphic to

$$\mathcal{A}_i^2(\varphi) \cong J_i(\xi, \pi)$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module.*

Proof. This follows from [Fra98, Sect. 6], but we explain for the convenience of the reader in some detail our case, although it is quite similar to the proof of Theorem 3.3. Similarly as in [Fra98, p. 233], but taking into account that we have fixed the cuspidal support, consider the set $M(P_i, \varphi)$ of quadruples (R, Π, ν, λ), such that:

- $R = L_R N_R$ is a standard parabolic k-subgroup of G containing an element of the associate class $\{P_i\}$, i.e. either $R = P_i$ or $R = G$.
- Π is a discrete spectrum representation of $L_R(\mathbb{A})$ with cuspidal support π obtained as the iterated residue at the value $\nu \in \check{\mathfrak{a}}_{P_i}^R$ of the Eisenstein series on $L_R(\mathbb{A})$ attached to π. If $R = P_i$, then $\Pi = \pi$ and $\nu = 0$. If $R = G$, then Π is the residual representation of $G(\mathbb{A})$ with support π and $\nu \in \overline{\check{\mathfrak{a}}_{P_i}^+}$ is the pole of the Eisenstein series attached to π.
- $\lambda \in \overline{\check{\mathfrak{a}}_R^+}$ is such that $\lambda + \nu + \chi$ is annihilated by $\mathcal{J}$. If $R = G$, then $\lambda = 0$, and thus $\nu + \chi$ is annihilated by $\mathcal{J}$. If $R = P_i$, then $\lambda + \chi$ is annihilated by $\mathcal{J}$.

Observe that by the third condition $\xi = \lambda + \nu$ and χ form one of the pairs computed in Lemma 3.5.

For $m \in \mathbb{Z}$ let $M^m(P_i, \varphi)$ be the subset of $M(P_i, \varphi)$ consisting of those quadruples for which $T(\lambda) = m$, where λ is viewed as an element in $\check{\mathfrak{a}}_0$ via the natural embedding. Then, by [Fra98, Thm. 14], the quotient

$$\mathcal{A}_i^m(\varphi)/\mathcal{A}_i^{m+1}(\varphi) \cong \bigoplus_{(R,\Pi,\nu,\lambda) \in M^m(P_i,\varphi)} I(\lambda, \Pi) \otimes S(\mathfrak{a}_{R,\mathbb{C}}). \tag{3.3.1}$$

As in Theorem 3.3, the direct sum on the right hand side is obtained due to the fact that $\mathcal{J}$ annihilates a finite–dimensional representation (see [Fra98, Thm. 19]). We also introduce the notation $M_R(P_i, \varphi)$ and $M_R^m(P_i, \varphi)$ for the set of all quadruples in $M(P_i, \varphi)$ and $M^m(P_i, \varphi)$, respectively, with a parabolic subgroup R as the first entry.

For $R = G$, we always have $\lambda = 0$. Hence, $M_G^m(P_i, \varphi)$ is empty except for $m = T(0)$. Moreover, Π in a quadruple with $R = G$ should be a residual representation of $G(\mathbb{A})$ supported in π. By Proposition 3.4, if π satisfies certain conditions, then the Eisenstein series attached to π has a pole for P_1 only at $\nu = (1/2, 1/2)$ with the residue $\Pi \cong J_1(\nu, \pi)$, and for P_2 at $\nu = (1, 0)$ with the residue $\Pi \cong J_2(\nu, \pi)$. Since $\lambda = 0$, we have $\xi = \nu$, and thus Lemma 3.5 shows that these ξ can be achieved only if $\Lambda_1 = \Lambda_2$ for P_1 and $\Lambda_2 = 0$ for P_2. In both cases, Lemma 3.5 also gives a unique infinitesimal character χ such that $\nu + \chi$ is annihilated by $\mathcal{J}$. More precisely, for

$P = P_1$ it is $\chi = \left(\Lambda + \frac{3}{2}, -\Lambda - \frac{3}{2}\right)$, where $\Lambda = \Lambda_1 = \Lambda_2$, and for $P = P_2$ it is $\chi = (0, 2 + \Lambda_1)$. Thus we have found all quadruples in $M_G(P_i, \varphi)$. Namely,

$$M_G^m(P_1,\varphi) = \begin{cases} \left\{\left(G, J_1(1/2,\pi), \left(\frac{1}{2},\frac{1}{2}\right), 0\right)\right\}, & \begin{array}{l}\text{if } m = T(0) \text{ and } \Lambda_1 = \Lambda_2 = \Lambda \\ \text{and } \chi = \left(\Lambda + \frac{3}{2}, -\Lambda - \frac{3}{2}\right) \\ \text{and } \pi \text{ is as in Prop. 3.4 (1),}\end{array} \\ \emptyset, & \text{otherwise,} \end{cases}$$

while

$$M_G^m(P_2,\varphi) = \begin{cases} \left\{\left(G, J_2(1,\pi), (1,0), 0\right)\right\}, & \begin{array}{l}\text{if } m = T(0) \text{ and } \Lambda_2 = 0 \\ \text{and } \chi = (0, \Lambda_1 + 2) \\ \text{and } \pi \text{ is as in Prop. 3.4 (2),}\end{array} \\ \emptyset, & \text{otherwise.} \end{cases}$$

On the other hand, for $R = P_i$, we have $\Pi = \pi$ and hence $\nu = 0$. Thus, in this case $\xi = \lambda$ and χ form one of the pairs given in Lemma 3.5 with the positive sign taken for ξ. Thus $\lambda \neq 0$, and for a given π and its infinitesimal character χ there is a unique λ forming the quadruple $(P_i, \pi, 0, \lambda) \in M_{P_i}(P_i, \varphi)$. Therefore, having fixed the cuspidal support (and of course the highest weight Λ), we may choose a function T such that $T(\lambda)$ is the same integer satisfying $T(\lambda) < T(0)$ for all $\lambda \neq 0$ appearing among the quadruples. Finally, the sets $M_{P_i}^m(P_i, \varphi)$ are given as

$$M_{P_i}^m(P_i,\varphi) = \begin{cases} \left\{(P_i, \pi, 0, \lambda)\right\}, & \begin{array}{l}\text{if } m = T(\lambda) \text{ and } \lambda \text{ and the infinitesimal character} \\ \chi \text{ form one of the pairs given in Lemma 3.5,}\end{array} \\ \emptyset, & \text{otherwise.} \end{cases}$$

It has no effect on the filtration if we assume that $T(0) = 2$ and $T(\lambda) = 1$ for $\lambda \neq 0$. Then the only non–empty sets $M^m(P_i, \varphi)$ are

$$M^1(P_i, \varphi) = M_{P_i}^1(P_i, \varphi),$$

and possibly

$$M^2(P_i, \varphi) = M_G^2(P_i, \varphi).$$

The second set is non–trivial if and only if the conditions for non–triviality of $\mathcal{A}_i^2(\varphi)$ given in the theorem are satisfied. Therefore, the Franke description of the quotients (3.3.1) shows that

$$\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi) \cong I_i(\xi, \pi) \otimes S(\mathfrak{a}_{P_i,\mathbb{C}}),$$

where $(P_i, \pi, 0, \lambda)$ is the only element of $M^1(P_i, \varphi)$, and $\xi = \lambda$, as claimed, and if $\mathcal{A}_i^2(\varphi)$ is non–trivial,

$$\mathcal{A}_i^2(\varphi) \cong J_i(\xi, \Pi),$$

since the induction is from $G(\mathbb{A})$ to itself, and $\check{\mathfrak{a}}_G$ is trivial. □

4. The cohomology of filtration quotients

4.1. We shall now determine the cohomology of the various quotients

$$\mathcal{A}_i^m(\varphi)/\mathcal{A}_i^{m+1}(\varphi)$$

of the filtration of $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$, with $m \in \mathbb{Z}$ and $i = 0, 1, 2$, using their description given in Theorems 3.3 and 3.6. Therefore, observe that for each archimedean place v of k we may write $L_i(k_v) = L_i(\mathbb{R})$ as a direct product $L_i(\mathbb{R}) = A_i(\mathbb{R})^\circ \times L_i(\mathbb{R})^{ss}$

of the connected component of the group of real points of a maximal central k-split torus $A_i(\mathbb{R})^\circ$ and the semi–simple part $L_i(\mathbb{R})^{ss}$, where

$$L_i(\mathbb{R})^{ss} = \begin{cases} \{\pm 1\} \times \{\pm 1\} = \mathbb{F}_2 \times \mathbb{F}_2, & \text{if } i = 0, \\ \{\pm 1\} \rtimes SL_2(\mathbb{R}) = SL_2^{\pm}(\mathbb{R}), & \text{if } i = 1, \\ \{\pm 1\} \times SL_2(\mathbb{R}) = \mathbb{F}_2 \times SL_2(\mathbb{R}), & \text{if } i = 2. \end{cases}$$

Recall that $SL_2^{\pm}(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) | \det(g) = \pm 1\}$, and $\mathbb{F}_2$ is the multiplicative group of two elements. An irreducible representation of $L_i(\mathbb{R})$ may hence be decomposed into a character of $A_i(\mathbb{R})^\circ$ and an irreducible representation of $L_i(\mathbb{R})^{ss}$. In particular, a finite-dimensional, irreducible representation of $L_i(\mathbb{R})$ is the product of a character of $A_i(\mathbb{R})^\circ$ and a finite-dimensional representation of $L_i(\mathbb{R})^{ss}$. The latter one is either $F^0(a,b) := \mathrm{sgn}^a_{\mathbb{F}_2} \otimes \mathrm{sgn}^b_{\mathbb{F}_2}$ if $i = 0$ or in the case of $i = 1, 2$, the representation $F^i_\ell(a)$, i.e., the unique irreducible representation of $L_i(\mathbb{R})^{ss}$ of dimension ℓ tensored by sgn^a. Recall that the (-1)-element in $L_i(\mathbb{R})^{ss}$ is represented by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if $i = 1$ and by $(-1, id)$ if $i = 2$. In the special case where $\ell = 1$, we will also use the usual notation $F^1_1(a) = \mathrm{sgn}^a_{SL_2^{\pm}(\mathbb{R})}$, resp. $F^2_1(a) = \mathrm{sgn}^a_{\mathbb{F}_2} = \mathrm{sgn}^a_{\mathbb{F}_2} \otimes \mathbf{1}_{SL_2(\mathbb{R})}$.

In what follows we need to know the cohomological, irreducible unitary representations of $L_i(\mathbb{R})^{ss}$ which is determined in the following lemma. Therefore recall that for every integer $r \geq 2$, $SL_2^{\pm}(\mathbb{R})$ has one discrete series representation D_r indexed by its lowest $O(2)$-type r, while $SL_2(\mathbb{R})$ has two discrete series representations D_r^+ (resp. D_r^-) indexed by the lowest (resp. highest) $SO(2)$-type r (resp. $-r$).

Lemma 4.1. *Let $F^0(a,b)$ and $F^i_\ell(a)$, $i = 1, 2$, be the finite–dimensional irreducible representations of $L_0(\mathbb{R})^{ss}$ and $L_i(\mathbb{R})^{ss}$, respectively, as defined above. Let τ be any irreducible unitary representations of $L_i(\mathbb{R})^{ss}$, $i = 0, 1, 2$.*

$(i = 0)$

$$H^q(\mathfrak{l}_0^{ss}, K_{L_0(\mathbb{R})^{ss}}, \tau \otimes F^0(a,b)) = \begin{cases} \mathbb{C}, & \text{if } q = 0 \text{ and } \tau \cong F^0(a,b), \\ 0, & \text{otherwise.} \end{cases}$$

$(i = 1)$ *If $\ell = 1$, then*

$$H^q(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \tau \otimes \mathrm{sgn}^a_{SL_2^{\pm}(\mathbb{R})}) = \begin{cases} \mathbb{C} & \begin{cases} \text{if } q = 0 \text{ and } \tau \cong \mathrm{sgn}^a_{SL_2^{\pm}(\mathbb{R})}, \\ \text{if } q = 1 \text{ and } \tau \cong D_2, \\ \text{if } q = 2 \text{ and } \tau \cong \mathrm{sgn}^{a+1}_{SL_2^{\pm}(\mathbb{R})} \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

If $\ell > 1$, then

$$H^q(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \tau \otimes F^1_\ell(a)) = \begin{cases} \mathbb{C}, & \text{if } q = 1 \text{ and } \tau \cong D_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

$(i = 2)$ *If $\ell = 1$, then*

$$H^q(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, \tau \otimes \mathrm{sgn}^a_{\mathbb{F}_2}) = \begin{cases} \mathbb{C}, & \begin{cases} \text{if } q = 0, 2 \text{ and } \tau \cong \mathrm{sgn}^a_{\mathbb{F}_2}, \\ \text{if } q = 1 \text{ and } \tau \cong \mathrm{sgn}^a_{\mathbb{F}_2} \otimes D_2^{\pm}, \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

If $\ell > 1$, then

$$H^q(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, \tau \otimes F^2_\ell(a)) = \begin{cases} \mathbb{C}, & \text{if } q = 1 \text{ and } \tau \cong \mathrm{sgn}^a_{\mathbb{F}_2} \otimes D^{\pm}_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows from the Künneth rule and the well–known properties of the cohomological unitary dual of $SL_2^{\pm}(\mathbb{R})$ and $SL_2(\mathbb{R})$; cf. [Sch83, pp. 118–122]. $\square$

4.2. **The first maximal parabolic subgroup.** Let $\varphi = (\varphi_P)_{P\in\{P_1\}} \in \Phi_1$ be an associate class of unitary cuspidal automorphic representations and $\pi \in \varphi_{P_1}$ be a representative. Let $\chi \in \check{\mathfrak{a}}_0^{P_1}$ be the infinitesimal character of its archimedean component, where $\check{\mathfrak{a}}_0^{P_1}$ is diagonally embedded into $\check{\mathfrak{a}}_{0,\infty}$, and take $\xi \in \overline{\check{\mathfrak{a}}_{P_1}^{+}}$ such that $\xi + \chi \in \check{\mathfrak{a}}_0$ is annihilated by $\mathcal{J}$. Which pairs of vectors ξ and χ satisfy this latter condition is listed in Lemma 3.5, but for the reader's convenience we recall that we must have

$$\xi = \frac{3+\Lambda_1+\Lambda_2}{2} \quad \text{and} \quad \chi = \frac{1+\Lambda_1-\Lambda_2}{2}$$

or

$$\xi = \frac{1+\Lambda_1-\Lambda_2}{2} \quad \text{and} \quad \chi = \frac{3+\Lambda_1+\Lambda_2}{2}.$$

In this section we determine the cohomology of the quotients

$$\mathcal{A}_1^1(\varphi)/\mathcal{A}_1^2(\varphi) \quad \text{and} \quad \mathcal{A}_1^2(\varphi),$$

using their explicit description in our Theorem 3.6. We obtain

Proposition 4.2. *Let E be an irreducible representation of G_∞ as in Section* 1.5*, so that its highest weight $\Lambda = (\Lambda_{1,\sigma}, \Lambda_{2,\sigma})_\sigma$ has repeating coordinates in the field embeddings $\sigma : k \hookrightarrow \mathbb{C}$, and may hence be written as $\Lambda = (\Lambda_1, \Lambda_2)$. Then we obtain as a $G(\mathbb{A}_f)$-module*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_1^1(\varphi)/\mathcal{A}_1^2(\varphi)\otimes E) \cong \begin{cases} I_1(\xi,\pi_f)^{m_1(\pi,q)} & \text{if } \pi_v|_{L_1(\mathbb{R})^{ss}} = D_{2\chi+1}\ \forall v \in S_\infty, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$m_1(\pi,q) = \binom{n-1}{q-3n} \quad \text{if} \quad \chi = \frac{3+\Lambda_1+\Lambda_2}{2}$$

and

$$m_1(\pi,q) = \binom{n-1}{q-4n} \quad \text{if} \quad \chi = \frac{1+\Lambda_1-\Lambda_2}{2}.$$

In particular, this space vanishes outside the degrees $3n \leq q \leq 4n-1$ in the first case, and outside the degrees $4n \leq q \leq 5n-1$ in the second case.

If $\mathcal{A}_1^2(\varphi)$ is non–trivial, i.e., if $\Lambda_1 = \Lambda_2 = \Lambda$, $\xi = \frac{1}{2}$, $\chi = \frac{3}{2} + \Lambda$ and π satisfies that its central character is trivial and $L(\frac{1}{2}, \pi) \neq 0$, then

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_1^2(\varphi)\otimes E) \cong \begin{cases} J_1(\xi,\pi_f)^{m_1(q)} & \text{if } \pi_v = D_{2\Lambda+4}\ \forall v \in S_\infty, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$m_1(q) = \#\{(r_1,\dots,r_n)\,|\,r_j \in \{2,4\} \text{ and } \sum_{j=1}^{n} r_j = q\} = \begin{cases} \binom{n}{2n-\frac{q}{2}}, & \text{if } q \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, this cohomology vanishes if q is either odd or not in the range $2n \leq q \leq 4n$.

Proof. We begin by calculating the $(\mathfrak{g}_\infty, K_\infty)$-cohomology of $\mathcal{A}^1_1(\varphi)/\mathcal{A}^2_1(\varphi)$. By Theorem 3.6 we get

$$\begin{aligned} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}^1_1(\varphi)/\mathcal{A}^2_1(\varphi)\otimes E) &\cong H^q(\mathfrak{g}_\infty, K_\infty, I_1(\xi,\pi)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E) \\ &\cong H^q(\mathfrak{g}_\infty, K_\infty, I_1(\xi,\pi_\infty)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E)\otimes I_1(\xi,\pi_f), \end{aligned}$$

where the fist space carries the trivial $G(\mathbb{A}_f)$-module structure. Therefore we only need to show that

$$H^q(\mathfrak{g}_\infty, K_\infty, I_1(\xi,\pi_\infty)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E)$$

is of dimension $m_1(\pi,q)$ if $\pi_v|_{L_1(\mathbb{R})^{ss}} = D_{2\chi+1}$ for all archimedean places v and vanishes otherwise. Now [BW, III Thm. 3.3], together with our Proposition 2.1, shows that there is a unique $w \in W^{P_1}$ for all $\sigma : k \hookrightarrow \mathbb{C}$ such that the representation $\pi_\infty \otimes \mathbb{C}_{\xi+\rho_{P_1}}$ has non-trivial $(\mathfrak{l}_{1,\infty}, K_{L_{1,\infty}})$-cohomology with respect to $S(\mathfrak{a}_{P_1,\mathbb{C}}) \otimes \bigotimes_\sigma F_w$. Here, $\mathbb{C}_{\xi+\rho_{P_1}}$ denotes the one-dimensional complex representation of $\mathfrak{a}_{P_1} \hookrightarrow \mathfrak{l}_{1,\infty}$ on which $a \in \mathfrak{a}_{P_1}$ acts by multiplication by $(\xi+\rho_{P_1})(a)$ and F_w is the irreducible finite-dimensional representation of $L_1(\mathbb{R})$ of highest weight $w(\Lambda+\rho)-\rho$. Again by [BW, III Thm. 3.3] it is clear that either

$$w = w_2 w_1 \quad \text{if} \quad \chi = \frac{3+\Lambda_1+\Lambda_2}{2} \tag{4.2.1}$$

or

$$w = w_2 w_1 w_2 \quad \text{if} \quad \chi = \frac{1+\Lambda_1-\Lambda_2}{2}. \tag{4.2.2}$$

So the length of w is $l(w) = 2$ in case (4.2.1) and $l(w) = 3$ in case (4.2.2), whereas $F_w = \mathbb{C}_{w(\Lambda+\rho)-\rho|_{\mathfrak{a}_{P_1}}} \otimes F^1_{2\chi}(a)$ for some $a \in \{0,1\}$ in both cases. Furthermore, in any case,

$$\begin{aligned} &H^q(\mathfrak{g}_\infty, K_\infty, I_1(\xi,\pi_\infty)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E) \\ &\quad\cong\quad H^{q-l(w)n}(\mathfrak{l}_{1,\infty}, K_{L_1,\infty}, \pi_\infty\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes \mathbb{C}_{\xi+\rho_{P_1}}\otimes \textstyle\bigotimes_\sigma F_w) \\ &\quad\cong\quad H^{q-l(w)n}(\mathfrak{m}_{1,\infty}, K_{M_1,\infty}, \pi_\infty\otimes \textstyle\bigotimes_\sigma F_w). \end{aligned}$$

The first line is [BW, III Thm. 3.3], while the second line follows directly as in [Fra98, p. 256] if we apply the Künneth rule to the decomposition $\mathfrak{l}_{1,\infty} = \mathfrak{m}_{1,\infty} \oplus \mathfrak{a}_{P_1}$.

Now observe that $K_{L_1,\infty} \cap A^\circ_{1,\infty} = \{1\}$. Hence, [BW, II Prop. 3.1] implies together with the Künneth rule that

$$H^{q-l(w)n}(\mathfrak{m}_{1,\infty}, K_{M_1,\infty}, \pi_\infty \otimes \bigotimes_\sigma F_w) \tag{4.2.3}$$

$$\cong \bigoplus_{r+s=q-l(w)n} \left[\bigwedge^r \mathbb{C}^{n-1} \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \pi_v|_{L_1(\mathbb{R})^{ss}} \otimes F^1_{2\chi}(a)) \right].$$

Since a cuspidal automorphic representation $\pi \in \varphi_{P_1}$ cannot have a one–dimensional archimedean component, we conclude by Lemma 4.1 that we must have

$$\pi_v|_{L_1(\mathbb{R})^{ss}} \cong D_{2\chi+1} \quad \forall v \in S_\infty$$

in order to get non–vanishing cohomology. Moreover, Lemma 4.1 says that in this case

$$\bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \pi_v|_{L_1(\mathbb{R})^{ss}} \otimes F^1_{2\chi}(a)) = \begin{cases} \mathbb{C}, & \text{if } s_v = 1 \ \forall v \in S_\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $s = n$, and so the dimension of the vector space (4.2.3) is

$$\dim_{\mathbb{C}}\left(\bigoplus_{r+s=q-l(w)n} \bigwedge^r \mathbb{C}^{n-1} \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \pi_v|_{L_1(\mathbb{R})^{ss}} \otimes F^1_{2\chi}(a)) \right)$$
$$= \binom{n-1}{q-(l(w)+1)n}.$$

But as $l(w) = 2$ in case (4.2.1) and $l(w) = 3$ in case (4.2.2), this shows the claim.

Next we calculate the cohomology of $\mathcal{A}^2_1(\varphi)$ if it is non–trivial. So according to Theorem 3.6 and Proposition 3.4 we assume that $\Lambda_1 = \Lambda_2 = \Lambda$, $\xi = \frac{1}{2}$, $\chi = \frac{3}{2} + \Lambda$, π satisfies that its central character is trivial and $L(\frac{1}{2}, \pi) \neq 0$. By Theorem 3.6 we obtain furthermore that

$$\begin{aligned} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}^2_1(\varphi) \otimes E) &\cong H^q(\mathfrak{g}_\infty, K_\infty, J_1(\xi, \pi) \otimes E) \\ &\cong H^q(\mathfrak{g}_\infty, K_\infty, J_1(\xi, \pi_\infty) \otimes E) \otimes J_1(\xi, \pi_f). \end{aligned}$$

The $(\mathfrak{sp}_4(\mathbb{R}), U(2))$–cohomology of the Langlands quotient $J_1(\xi, \pi_v)$ with respect to $E = E_{(\Lambda,\Lambda)}$ is computed in [BW, VI Thm. 1.7], and together with [BW, VI Lem. 1.5] we obtain

$$H^q(\mathfrak{sp}_4(\mathbb{R}), U(2), J_1(\xi, \pi_v) \otimes E) \cong \begin{cases} \mathbb{C}, & \text{if } q = 2, 4 \text{ and } \pi_v = D_{2\Lambda+4}, \\ 0, & \text{otherwise.} \end{cases}$$

By now applying the Künneth rule, the last assertion of the proposition is given. ☐

4.3. **The second maximal parabolic subgroup.** This section is in complete analogy to the previous one. So, let $\varphi = (\varphi_P)_{P\in\{P_2\}} \in \Phi_2$ be an associate class of unitary cuspidal automorphic representations and $\pi \in \varphi_{P_2}$. Let $\chi \in \check{\mathfrak{a}}_0^{P_2}$ be the infinitesimal character of its archimedean component, where $\check{\mathfrak{a}}_0^{P_2}$ is diagonally embedded into $\check{\mathfrak{a}}_{0,\infty}$, and take $\xi \in \overline{\check{\mathfrak{a}}^+_{P_2}}$ such that $\xi + \chi \in \check{\mathfrak{a}}_0$ is annihilated by $\mathcal{J}$. We recall from Lemma 3.5 that we must have

$$\xi = 2 + \Lambda_1 \quad \text{and} \quad \chi = 1 + \Lambda_2$$

or

$$\xi = 1 + \Lambda_2 \quad \text{and} \quad \chi = 2 + \Lambda_1.$$

The cohomology of the quotients

$$\mathcal{A}^1_2(\varphi)/\mathcal{A}^2_2(\varphi) \quad \text{and} \quad \mathcal{A}^2_2(\varphi)$$

is obtained in the following proposition.

Proposition 4.3. *Let E be an irreducible representation of G_∞ as in Section 1.5, so that its highest weight $\Lambda = (\Lambda_{1,\sigma}, \Lambda_{2,\sigma})_\sigma$ has repeating coordinates in the field* *embeddings* $\sigma : k \hookrightarrow \mathbb{C}$, *and may hence be written as* $\Lambda = (\Lambda_1, \Lambda_2)$. *Then we obtain as a* $G(\mathbb{A}_f)$*-module*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_2^1(\varphi)/\mathcal{A}_2^2(\varphi) \otimes E)$$
$$\cong \begin{cases} I_2(\xi, \pi_f)^{m_2(\pi,q)}, & \textit{if } \pi_v|_{L_2(\mathbb{R})^{ss}} = \operatorname{sgn}_{\mathbb{F}_2}^{\xi} \otimes D_{\chi+1}^{\pm} \ \forall v \in S_\infty, \\ 0, & \textit{otherwise,} \end{cases}$$

where

$$m_2(\pi, q) = \binom{n-1}{q-3n} \quad \textit{if} \quad \chi = \Lambda_1 + 2$$

and

$$m_2(\pi, q) = \binom{n-1}{q-4n} \quad \textit{if} \quad \chi = \Lambda_2 + 1.$$

In particular, this space vanishes outside the degrees $3n \leq q \leq 4n-1$ *in the first case and outside the degrees* $4n \leq q \leq 5n-1$ *in the second case.*

If $\mathcal{A}_2^2(\varphi)$ *is non–trivial, i.e., if* $\Lambda_2 = 0$, $\xi = 1$, $\chi = 2 + \Lambda_1$ *and* $\pi = \mu \otimes \sigma$ *satisfies the fact that* $L(s, \mu \times \sigma)$ *has a pole at* $s = 1$, *then*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_2^2(\varphi) \otimes E) \cong \begin{cases} J_2(\xi, \pi_f)^{m_2(q)}, & \textit{if } \sigma_v = D_{\Lambda_1+3}^{\pm} \ \forall v \in S_\infty, \\ 0, & \textit{otherwise,} \end{cases}$$

where

$$m_2(q) = \#\{(r_1, \dots, r_n) \,|\, r_j \in \{2, 4\} \textit{ and } \sum_{j=1}^{n} r_j = q\} = \begin{cases} \binom{n}{2n-\frac{q}{2}}, & \textit{if } q \textit{ is even,} \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, this cohomology vanishes if q *is either odd or not in the range* $2n \leq q \leq 4n$.

Proof. As in the case of P_1, in order to show the assertions on the $(\mathfrak{g}_\infty, K_\infty)$-cohomology of $\mathcal{A}_2^1(\varphi)/\mathcal{A}_2^2(\varphi)$, it is enough to prove that

$$H^q(\mathfrak{g}_\infty, K_\infty, I_2(\xi, \pi_\infty) \otimes S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes E)$$

is of dimension $m_2(\pi, q)$ if $\pi_v|_{L_2(\mathbb{R})^{ss}} = \operatorname{sgn}_{\mathbb{F}_2}^{\xi} \otimes D_{\chi+1}^{\pm}$ for all archimedean places and that it vanishes otherwise. Again, [BW, III Thm. 3.3] together with our Proposition 2.1 shows that there is a unique $w \in W^{P_2}$ for all $\sigma : k \hookrightarrow \mathbb{C}$ such that the representation $\pi_\infty \otimes \mathbb{C}_{\xi+\rho_{P_2}}$ has non-trivial $(\mathfrak{l}_{2,\infty}, K_{L_{2,\infty}})$-cohomology with respect to $S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes \bigotimes_\sigma F_w$. Here, $\mathbb{C}_{\xi+\rho_{P_2}}$ denotes the one-dimensional complex representation of $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{l}_{2,\infty}$ on which $a \in \mathfrak{a}_{P_2}$ acts by multiplication by $(\xi + \rho_{P_2})(a)$ and F_w is the irreducible finite-dimensional representation of $L_2(\mathbb{R})$ of highest weight $w(\Lambda + \rho) - \rho$. Explicitly we get

$$w = w_1 w_2 \quad \text{if} \quad \chi = \Lambda_1 + 2 \tag{4.3.1}$$

and

$$w = w_1 w_2 w_1 \quad \text{if} \quad \chi = \Lambda_2 + 1. \tag{4.3.2}$$

In any of these two cases, $F_w = \mathbb{C}_{w(\Lambda+\rho)-\rho|_{\mathfrak{a}_{P_2}}} \otimes F_\chi^2(\xi)$. Furthermore, as in the case of P_1,

(4.3.3)
$$H^q(\mathfrak{g}_\infty, K_\infty, I_2(\xi, \pi_\infty) \otimes S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes E) \cong H^{q-l(w)n}(\mathfrak{m}_{2,\infty}, K_{M_2,\infty}, \pi_\infty \otimes \bigotimes_\sigma F_w).$$

Again $K_{L_{2,\infty}} \cap A^\circ_{2,\infty} = \{1\}$, whence the latter cohomology space is isomorphic to

$$\bigoplus_{r+s=q-l(w)n} \left[\bigwedge^r \mathbb{C}^{n-1} \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, \pi_v|_{L_2(\mathbb{R})^{ss}} \otimes F^2_\chi(\xi)) \right].$$

Since a cuspidal automorphic representation $\pi \in \varphi_{P_2}$ cannot have a one–dimensional archimedean component, we conclude by Lemma 4.1 that we must have

$$\pi_v|_{L_2(\mathbb{R})^{ss}} \cong \mathrm{sgn}^\xi_{\mathbb{F}_2} \otimes D^\pm_{\chi+1}$$

for all $v \in S_\infty$ in order to get non–vanishing cohomology. Moreover, it follows from Lemma 4.1 that in this case

$$\bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, \pi_v|_{L_2(\mathbb{R})^{ss}} \otimes F^2_\chi(\xi)) = \begin{cases} \mathbb{C}, & \text{if } s_v = 1\ \forall v \in S_\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $s = n$, and so the vector space (4.3.3) has dimension

$$\dim_{\mathbb{C}} \left(\bigoplus_{r+s=q-l(w)n} \bigwedge^r \mathbb{C}^{n-1} \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, \pi_v|_{L_2(\mathbb{R})^{ss}} \otimes F^2_\chi(\xi)) \right)$$
$$= \binom{n-1}{q-(l(w)+1)n}.$$

But as $l(w) = 2$ in case (4.3.1) and $l(w) = 3$ in case (4.3.2), this shows the claim.

It remains to calculate the cohomology of $\mathcal{A}^2_2(\varphi)$ if it is non–trivial. So, according to Theorem 3.6 and Proposition 3.4 we assume that $\Lambda_2 = 0$, $\xi = 1$, $\chi = 2 + \Lambda_1$ and $\pi \cong \mu \otimes \sigma$ satisfies the fact that $L(s, \mu \times \sigma)$ has a pole at $s = 1$. Then, by Theorem 3.6 we obtain

$$\begin{aligned} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}^2_2(\varphi) \otimes E) &\cong H^q(\mathfrak{g}_\infty, K_\infty, J_2(\xi, \pi) \otimes E) \\ &\cong H^q(\mathfrak{g}_\infty, K_\infty, J_2(\xi, \pi_\infty) \otimes E) \otimes J_2(\xi, \pi_f). \end{aligned}$$

The $(\mathfrak{sp}_4(\mathbb{R}), U(2))$–cohomology of the Langlands quotients $J_2(\xi, \pi_v)$ with respect to $E = E_{(\Lambda,\Lambda)}$ is computed in [BW, VI Thm. 1.7] together with [BW, VI Lem. 1.5], which yields

$$H^q(\mathfrak{sp}_4(\mathbb{R}), U(2), J_2(\xi, \pi_v) \otimes E) \cong \begin{cases} \mathbb{C}, & \text{if } q = 2, 4 \text{ and } \sigma_v = D^\pm_{\Lambda_1+3}, \\ 0, & \text{otherwise.} \end{cases}$$

Now the proposition follows. □

4.4. **The minimal parabolic subgroup.** We still have to determine the cohomology of the various filtration quotients coming from the minimal parabolic k-subgroup P_0. As in the notational Section 1.5, a coefficient module E is given represented by its highest weight $\Lambda = (\Lambda_1, \Lambda_2)$. Let $\mu = \mu_1 \otimes \mu_2$ be a unitary character of $L_0(\mathbb{A}) = GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$ representing a cuspidal support $\varphi \in \Phi_0$. We obtain

Proposition 4.4. *There is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$\begin{aligned} &H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi) \otimes E) \\ &\quad \cong \begin{cases} I_0(\Lambda+\rho_0,\mu_f)^{m_0(q)}, & \text{if } \mu_v|_{L_0(\mathbb{R})^{ss}} = \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_1} \otimes \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_2} \quad \forall v \in S_\infty, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$m_0(q) = \binom{2n-2}{q-4n}.$$

In particular, this cohomology space vanishes if q is outside the range $4n \leq q \leq 6n-2$.

Proof. Using Theorem 3.3 we see that

$$\begin{aligned} &H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi) \otimes E) \\ &\quad \cong H^q(\mathfrak{g}_\infty, K_\infty, I_0(\Lambda+\rho_0,\mu) \otimes S(\mathfrak{a}_{P_0,\mathbb{C}}) \otimes E) \\ &\quad \cong H^q(\mathfrak{g}_\infty, K_\infty, I_0(\Lambda+\rho_0,\mu_\infty) \otimes S(\mathfrak{a}_{P_0,\mathbb{C}}) \otimes E) \otimes I_0(\Lambda+\rho_0,\mu_f), \end{aligned}$$

whence it suffices to prove that the space

$$H^q(\mathfrak{g}_\infty, K_\infty, I_0(\Lambda+\rho_0,\mu_\infty) \otimes S(\mathfrak{a}_{P_0,\mathbb{C}}) \otimes E)$$

is of dimension $m_0(q)$ if $\mu_v|_{L_0(\mathbb{R})^{ss}} = \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_1} \otimes \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_2}$ for all $v \in S_\infty$ and vanishes otherwise. Similar to the case of the maximal parabolic subgroups, this can be accomplished harking back to [BW, III Thm. 3.3] and [BW, II Prop. 3.1]. First, we observe that $w = w_2w_1w_2w_1$ is the only element of $W^{P_0} = W$ which may give rise to an $L_0(\mathbb{R})$-module F_w such that $\bigotimes_\sigma F_w$ has non–trivial $(\mathfrak{l}_{0,\infty}, K_{L_{0,\infty}})$-cohomology with respect to $\mu_\infty \otimes S(\mathfrak{a}_{P_0,\mathbb{C}}) \otimes \mathbb{C}_{\Lambda+2\rho_0}$. The module F_w is isomorphic to $F_w = \mathbb{C}_{w(\Lambda+\rho)-\rho} \otimes F^0(\Lambda_1,\Lambda_2)$. Second, we derive as in the proofs of Propositions 4.2 and 4.3 that

(4.4.1)
$$H^q(\mathfrak{g}_\infty, K_\infty, I_0(\Lambda+\rho_0,\mu_\infty) \otimes S(\mathfrak{a}_{P_0,\mathbb{C}}) \otimes E) \cong H^{q-4n}(\mathfrak{m}_{0,\infty}, K_{M_{0,\infty}}, \mu_\infty \otimes \bigotimes_\sigma F_w).$$

Third, applying [BW, II Prop. 3.1] and the Künneth rule to the last cohomology space reveals that it is isomorphic to

$$\bigoplus_{r+s=q-4n} \left[\bigwedge^r \mathbb{C}^{2n-2} \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty}, \\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_0^{ss}, K_{L_0(\mathbb{R})^{ss}}, \mu_v|_{L_0(\mathbb{R})^{ss}} \otimes F^0(\Lambda_1,\Lambda_2)) \right].$$

Here, observe that $K_{M_{0,\infty}}$ has trivial intersection with $A_{0,\infty}^\circ$ and that $\mathfrak{m}_{0,\infty}$ is of dimension $2n-2$. Fourth, checking Lemma 4.1 gives that in order to get non-vanishing cohomology, it is necessary that $\mu_v|_{L_0(\mathbb{R})^{ss}} = F^0(\Lambda_1,\Lambda_2) = \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_1} \otimes \operatorname{sgn}_{\mathbb{F}_2}^{\Lambda_2}$ and $s_v = 0$ for all $v \in S_\infty$, and then

$$\bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_0^{ss}, K_{L_0(\mathbb{R})^{ss}}, \mu_v|_{L_0(\mathbb{R})^{ss}} \otimes F^0(\Lambda_1,\Lambda_2)) = \mathbb{C}.$$

Hence, $s = 0$, too, and we obtain that the dimension of the vector space (4.4.1) is

$$\dim_{\mathbb{C}}\left(\bigoplus_{r+s=q-4n} \bigwedge^{r}\mathbb{C}^{2n-2}\otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v=s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_0^{ss}, K_{L_0(\mathbb{R})^{ss}}, \mu_v|_{L_0(\mathbb{R})^{ss}}\otimes F^0(\Lambda_1,\Lambda_2))\right)$$
$$= \binom{2n-2}{q-4n}.$$

This shows the assertion. ☐

We now deal with the case of the quotient $\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)$ if it is non-trivial, i.e., if μ and Λ satisfy one of the singularity conditions given in Theorem 3.3. That is, if either

$$\Lambda_1 = \Lambda_2 \quad \text{and} \quad \mu_1 = \mu_2 \tag{4.4.2}$$

or

$$\Lambda_2 = 0 \quad \text{and} \quad \mu_2 = \mathbf{1} \tag{4.4.3}$$

or both, i.e.,

$$\Lambda_1 = \Lambda_2 = 0 \quad \text{and} \quad \mu_1 = \mu_2 = \mathbf{1}. \tag{4.4.4}$$

There is the following proposition.

Proposition 4.5. *In each of the three cases* (4.4.2), (4.4.3) *and* (4.4.4), *there is an isomorphism of $G(\mathbb{A}_f)$-modules:*

(1) *In case* (4.4.2),

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E) \cong I_1\left(\Lambda_1+\frac{3}{2}, \mu_f\circ\det\right)^{n_1(\mu,q)},$$

where

$$n_1(\mu,q) = \binom{n-1}{q-3n-2l(\mu)}$$

with

$$l(\mu) = \#\{v\in\ S_\infty|\mu_v|_{L_1(\mathbb{R})^{ss}} = \mathrm{sgn}^{\Lambda_1}_{SL_2^\pm(\mathbb{R})}\}.$$

In particular, this cohomology space vanishes if q is outside the range $3n \le q \le 6n-1$.

(2) *In case* (4.4.3),

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)$$
$$\cong \begin{cases} I_2(\Lambda_1+2, \mu_f\otimes \mathbf{1}_{SL_2(\mathbb{A}_f)})^{n_2(q)} & \textit{if } \mu_v|_{\mathbb{F}_2} = \mathrm{sgn}^{\Lambda_1}_{\mathbb{F}_2}\ \ \forall v\in S_\infty,\\ 0 & \textit{otherwise,}\end{cases}$$

where

$$n_2(q) = \sum_{j=0}^{\lfloor\frac{q-3n}{2}\rfloor}\binom{n-1}{q-3n-2j}\binom{n}{j}.$$

In particular, this cohomology space vanishes if q is outside the range $3n \le q \le 6n-1$.

(3) *Finally, in case* (4.4.4),

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)$$

is isomorphic as a $G(\mathbb{A}_f)$*-module to the direct sum*

$$I_1\left(\frac{3}{2}, \mathbf{1}_{L_1(\mathbb{A}_f)}\right)^{\binom{n-1}{q-5n}} \bigoplus I_2(2, \mathbf{1}_{L_2(\mathbb{A}_f)})^{n_2(q)},$$

where $n_2(q)$ *is as in case (2). So, this space vanishes again if* q *is outside the range* $3n \leq q \leq 6n-1$.

Proof. By the very form of the quotient $\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)$, described in Theorem 3.3, we should determine the $G(\mathbb{A}_f)$-module structure of the cohomology spaces

$$\begin{aligned}
&H^q\left(\mathfrak{g}_\infty, K_\infty, I_1\left(\Lambda_1+\frac{3}{2}, \mu\circ\det\right)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E\right)\\
&\cong H^q\left(\mathfrak{g}_\infty, K_\infty, I_1\left(\Lambda_1+\frac{3}{2}, \mu_\infty\circ\det\right)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E\right)\otimes I_1\left(\Lambda_1+\frac{3}{2}, \mu_f\circ\det\right)
\end{aligned}$$

and

$$\begin{aligned}
&H^q(\mathfrak{g}_\infty, K_\infty, I_2(\Lambda_1+2, \mu\otimes\mathbf{1}_{SL_2(\mathbb{A})})\otimes S(\mathfrak{a}_{P_2,\mathbb{C}})\otimes E)\\
&\cong H^q(\mathfrak{g}_\infty, K_\infty, I_2(\Lambda_1+2, \mu_\infty\otimes\mathbf{1}_{SL_{2,\infty}})\otimes S(\mathfrak{a}_{P_2,\mathbb{C}})\otimes E)\\
&\qquad\otimes I_2(\Lambda_1+2, \mu_f\otimes\mathbf{1}_{SL_2(\mathbb{A}_f)}).
\end{aligned}$$

According to Theorem 3.3, the first one is needed to treat (4.4.2), the second one to treat (4.4.3) and their direct sum to treat (4.4.4).

We will start by determining the first one, i.e., by what we just said, we may assume that $\Lambda_1=\Lambda_2$ and $\mu_1=\mu_2$. A short moment of thought shows that in order to calculate the first cohomology space, one may proceed literally as in the proof of Proposition 4.2 with $w = w_2w_1w_2$ to obtain

$$\begin{aligned}
&H^q\left(\mathfrak{g}_\infty, K_\infty, I_1\left(\Lambda_1+\frac{3}{2}, \mu_\infty\circ\det\right)\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes E\right)\\
&\cong H^{q-3n}(\mathfrak{l}_{1,\infty}, K_{L_1,\infty}, \mu_\infty\otimes S(\mathfrak{a}_{P_1,\mathbb{C}})\otimes\mathbb{C}_{\Lambda_1+2\rho_{P_1}}\otimes\bigotimes_\sigma F_w)\\
&\cong H^{q-3n}(\mathfrak{m}_{1,\infty}, K_{M_1,\infty}, \mu_\infty\otimes\bigotimes_\sigma F_w)\\
&\cong \bigoplus_{r+s=q-3n}\left[\bigwedge^r\mathbb{C}^{n-1}\otimes\bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v=s}}\ \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \mu_v|_{L_1(\mathbb{R})^{ss}}\otimes F_1^1(\Lambda_1+1))\right].
\end{aligned}$$

Recall that $F_1^1(\Lambda_1+1) = \mathrm{sgn}_{SL_2^\pm(\mathbb{R})}^{\Lambda_1+1}$ is one-dimensional. As μ is one-dimensional, too, we must have $\mu_v|_{L_1(\mathbb{R})^{ss}} = \mathrm{sgn}_{SL_2^\pm(\mathbb{R})}^{a_v}$ for some a_v, $v\in S_\infty$. Depending on the parity of a_v, we obtain by our Lemma 4.1 that

$$\begin{aligned}
&H^{s_v}(\mathfrak{l}_1^{ss}, K_{L_1(\mathbb{R})^{ss}}, \mu_v|_{L_1(\mathbb{R})^{ss}}\otimes F_1^1(\Lambda_1+1))\\
&\qquad\cong\begin{cases}\mathbb{C}, & \text{if } a_v\equiv\Lambda_1+1 \mod 2 \text{ and } s_v=0,\\ \mathbb{C}, & \text{if } a_v\equiv\Lambda_1 \mod 2 \text{ and } s_v=2,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Hence, if we let

$$l(\mu) = \#\{v \in \ S_\infty | a_v \equiv \Lambda_1 \mod 2\} = \#\{v \in \ S_\infty | \mu_v|_{L_1(\mathbb{R})^{ss}} = \mathrm{sgn}^{\Lambda_1}_{SL_2^\pm(\mathbb{R})}\},$$

then $s = 2l(\mu)$, and so

$$\dim_\mathbb{C}(H^q(\mathfrak{g}_\infty, K_\infty, I_1(\Lambda_1 + \frac{3}{2}, \mu_\infty \circ \det) \otimes S(\mathfrak{a}_{P_1,\mathbb{C}}) \otimes E)) = \binom{n-1}{q-3n-2l(\mu)}.$$

This proves the assertion in case (4.4.2).

We now turn to the case of (4.4.3), i.e., we may assume that $\Lambda_2 = 0$ and $\mu_2 = \mathbf{1}$. As above, we may again proceed precisely as in the corresponding maximal parabolic case, namely as in the proof of Proposition 4.3 with $w = w_1w_2w_1$ in order to analyze

$$H^q(\mathfrak{g}_\infty, K_\infty, I_2(\Lambda_1 + 2, \mu \otimes \mathbf{1}_{SL_2(\mathbb{A})}) \otimes S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes E).$$

We obtain

$$\begin{aligned} &H^q(\mathfrak{g}_\infty, K_\infty, I_2(\Lambda_1 + 2, \mu_\infty \otimes \mathbf{1}_{SL_{2,\infty}}) \otimes S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes E) \\ &\cong \bigoplus_{r+s=q-3n} \bigwedge^r \mathbb{C}^{n-1} \\ &\quad \otimes \bigoplus_{\substack{(s_v)_{v\in S_\infty},\\ \sum_v s_v = s}} \bigotimes_{v\in S_\infty} H^{s_v}(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, (\mu_v \otimes \mathbf{1}_{SL_2(\mathbb{R})})|_{L_2(\mathbb{R})^{ss}} \otimes F_1^2(\Lambda_1)). \end{aligned}$$

As $F_1^2(\Lambda_1) = \mathrm{sgn}^{\Lambda_1}_{\mathbb{F}_2} \otimes \mathbf{1}_{SL_2(\mathbb{R})}$, Lemma 4.1 forces $\mu_v|_{\mathbb{F}_2} = \mathrm{sgn}^{\Lambda_1}_{\mathbb{F}_2}$ and gives

$$H^{s_v}(\mathfrak{l}_2^{ss}, K_{L_2(\mathbb{R})^{ss}}, (\mu_v \otimes \mathbf{1}_{SL_2(\mathbb{R})})|_{L_2(\mathbb{R})^{ss}} \otimes F_1^2(\Lambda_1)) \cong \begin{cases} \mathbb{C} & \text{if } s_v = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the dimension of $H^q(\mathfrak{g}_\infty, K_\infty, I_2(\Lambda_1 + 2, \mu_\infty \otimes \mathbf{1}_{SL_{2,\infty}}) \otimes S(\mathfrak{a}_{P_2,\mathbb{C}}) \otimes E)$ is the number of ways to write $q - 3n$ as a sum $q - 3n = r + \sum_{v\in S_\infty} s_v$ where $0 \leq r \leq n-1$ and for each $v \in S_\infty$, s_v is either 0 or 2. It is an easy exercise in combinatorics to show that this number is actually $n_2(q)$ as predicted by our proposition. Now the proof is complete. ☐

We conclude this section by determining the cohomology of the last remaining filtration step $\mathcal{A}_0^2(\varphi)$. According to Theorem 3.3 it is non–trivial if and only if $\mu = \mathbf{1}_{L_0(\mathbb{A})}$ and $\Lambda = (0, 0)$ and is then isomorphic to

$$\mathcal{A}_0^2(\varphi) = \mathbf{1}_{G(\mathbb{A})}.$$

We show

Proposition 4.6. *Let* $\mu = \mathbf{1}_{L_0(\mathbb{A})}$ *and* $\Lambda = (0, 0)$*. Then the cohomology of* $\mathcal{A}_0^2(\varphi)$ *is isomorphic as a* $G(\mathbb{A}_f)$*-module to*

$$\begin{aligned} H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_0^2(\varphi) \otimes E) &\cong H^q(\mathfrak{g}_\infty, K_\infty, \mathbf{1}_{G(\mathbb{A})}) \\ &\cong \mathbf{1}^{n_0(q)}_{G(\mathbb{A}_f)}, \end{aligned}$$

where

$$n_0(q) = \#\{q = \sum_{j=1}^n r_j | r_j \in \{0, 2, 4, 6\}\}.$$

It therefore vanishes if q *is odd and if* q *is even,*

$$n_0(q) = \sum_{j=0}^{\lfloor \frac{q}{8} \rfloor} (-1)^j \binom{n}{j} \binom{n + \frac{q}{2} - 4j - 1}{n-1}.$$

Proof. It is well known that

$$H^q(\mathfrak{sp}_4(\mathbb{R}), U(2), \mathbf{1}_{G(\mathbb{R})}) \cong \mathbb{C}$$

if $q = 0, 2, 4, 6$ and vanishes otherwise. For instance, see [OS], the table on p. 489. Therefore, it only remains to show that for even degrees q there is the equality

$$n_0(q) := \#\{q = \sum_{j=1}^{n} r_j | r_j \in \{0, 2, 4, 6\}\} = \sum_{j=0}^{\lfloor \frac{q}{8} \rfloor} (-1)^j \binom{n}{j} \binom{n + \frac{q}{2} - 4j - 1}{n-1}.$$

By definition, $n_0(q)$ is the coefficient of x^q in $(x^0 + x^2 + x^4 + x^6)^n$. If we put $y = x^2$, this is the coefficient of $y^{\frac{q}{2}}$ in

$$\begin{aligned} (1 + y + y^2 + y^3)^n &= \frac{(1 - y^4)^n}{(1 - y)^n} \\ &= \sum_j (-1)^j \binom{n}{j} y^{4j} \sum_u \binom{n + u - 1}{n-1} y^u. \end{aligned}$$

Since we want $4j + u = \frac{q}{2}$, it follows that this coefficient is

$$\sum_{j=0}^{\lfloor \frac{q}{8} \rfloor} (-1)^j \binom{n}{j} \binom{n + \frac{q}{2} - 4j - 1}{n-1},$$

which shows the claim. □

5. The main results

5.1. We are now ready to state and prove the main results of this paper on the Eisenstein cohomology of the group $G = Sp_4/k$. Recall that it can be decomposed along the proper parabolic k-subgroups and the various cuspidal supports as a direct sum

$$H^*_{Eis}(G, E) = \bigoplus_{i=0}^{2} \bigoplus_{\varphi \in \Phi_i} H^*(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i, \varphi) \otimes E).$$

We proceed by distinguishing the three standard parabolic k-subgroups P_i and the various cuspidal supports $\varphi \in \Phi_i$, $i = 0, 1, 2$, in question. In order to keep notation at a minimum, we shall abbreviate in this section

$$H^q(\mathcal{A}_i^m(\varphi)/\mathcal{A}_i^{m+1}(\varphi) \otimes E) := H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_i^m(\varphi)/\mathcal{A}_i^{m+1}(\varphi) \otimes E)$$

and similarly

$$H^q(\mathcal{A}_i^m(\varphi) \otimes E) := H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_i^m(\varphi) \otimes E)$$

for the $G(\mathbb{A}_f)$-module of $(\mathfrak{g}_\infty, K_\infty)$-cohomology with respect to E of the quotients of the filtration of $\mathcal{A}_{\mathcal{J}}(P_i, \varphi)$. Furthermore, if M is any $G(\mathbb{A}_f)$-module and S any $G(\mathbb{A}_f)$-submodule of M, we will express this by writing $S = Sb(M)$.

5.2. **Maximal parabolic subgroups.** The case of the maximal parabolic k-subgroups P_i, $i=1,2$, can be treated simultaneously. Let $\varphi = (\varphi_P)_{P\in\{P_i\}} \in \Phi_i$ be an associate class of unitary cuspidal automorphic representations and $\pi \in \varphi_{P_i}$ a representative, i.e., a unitary cuspidal automorphic representation of $L_i(\mathbb{A})$ which is trivial on the diagonally embedded group $A_i(\mathbb{R})^\circ$. Let $\chi \in \check{\mathfrak{a}}_0^{P_i}$ be the infinitesimal character of π_∞, where $\check{\mathfrak{a}}_0^{P_i}$ is diagonally embedded in $\check{\mathfrak{a}}_{0,\infty}$, and $\xi \in \overline{\check{\mathfrak{a}}_{P_i}^{+}}$ such that $\xi+\chi \in \check{\mathfrak{a}}_0$ is annihilated by $\mathcal{J}$, a condition which is explained in Proposition 3.5 and repeated at the beginning of Sections 4.2 and 4.3. Recall from Theorem 3.6 that, if $\mathcal{A}_i^2(\varphi)$ is non–trivial, then it is isomorphic to the residual representation $\mathcal{A}_i^2(\varphi) \cong J_i(\xi,\pi)$. We therefore have a natural morphism of $G(\mathbb{A}_f)$-modules

$$J_i^q(\varphi) : H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_i^2(\varphi)\otimes E) \to H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E)$$

induced by the inclusion $J_i(\xi,\pi) \hookrightarrow \mathcal{A}_{\mathcal{J}}(P_i,\varphi) \hookrightarrow \mathcal{A}_{\mathcal{J}}$. With this notation at hand we obtain the following theorem describing the $G(\mathbb{A}_f)$-module structure of the summand $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E)$ in the Eisenstein cohomology $H^q_{Eis}(G,E)$ of G.

Theorem 5.1. *Let $G = Sp_4/k$ be the split algebraic group of type C_2 over a totally real number field k. Let E be an irreducible, finite-dimensional representation of G_∞ so that its highest weight $\Lambda = (\Lambda_{1,\sigma}, \Lambda_{2,\sigma})_\sigma$ has repeating coordinates in the field embeddings $\sigma : k \hookrightarrow \mathbb{C}$ and may hence be written as $\Lambda = (\Lambda_1, \Lambda_2)$, and assume that E is the complexification of an algebraic representation of G/k. Let $\varphi = (\varphi_P)_{P\in\{P_i\}} \in \Phi_i$, $i=1,2$, and $\pi \in \varphi_{P_i}$ be a unitary cuspidal automorphic representation of $L_i(\mathbb{A})$.*

(1) *If $\mathcal{A}_i^2(\varphi)$ is non–trivial, i.e., if*
  - ($i=1$) *$\Lambda_1 = \Lambda_2 = \Lambda$, $\xi = \frac{1}{2}$, $\chi = \frac{3}{2}+\Lambda$ and π satisfies the fact that its central character is trivial and $L(\frac{1}{2},\pi)\neq 0$,*
  - ($i=2$) *$\Lambda_2 = 0$, $\xi = 1$, $\chi = 2+\Lambda_1$ and $\pi = \mu\otimes\sigma$ satisfies the fact that $L(s,\mu\times\sigma)$ has a pole at $s=1$,*

  then there is the following isomorphism of $G(\mathbb{A}_f)$-modules:

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E)$$
$$\cong \begin{cases} H^q(\mathcal{A}_i^2(\varphi)\otimes E), & 2n \le q \le 3n-1, \\ H^q(\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi)\otimes E) \quad \mathrm{mod}\ J_i^q(\varphi)(H^q(\mathcal{A}_i^2(\varphi)\otimes E)), & 3n \le q \le 4n-1, \\ & q \textit{ even}, \\ Sb(H^q(\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi)\otimes E)), & 3n \le q \le 4n-1, \\ & q \textit{ odd}, \\ J_i^q(\varphi)(H^q(\mathcal{A}_i^2(\varphi)\otimes E)), & q = 4n, \\ 0, & \textit{otherwise}. \end{cases}$$

  Moreover, $J_i^{3n}(\varphi)(H^{3n}(\mathcal{A}_i^2(\varphi)\otimes E)) \cong H^{3n}(\mathcal{A}_i^2(\varphi)\otimes E)$.

(2) *If, however, $\mathcal{A}_i^2(\varphi)$ is trivial, then there is the following isomorphism of $G(\mathbb{A}_f)$-modules in all degrees q:*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E) \cong H^q(\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi)\otimes E) \cong H^q(\mathcal{A}_i^1(\varphi)\otimes E).$$

Before we prove this theorem, we list a couple of remarks and consequences.

Corollary 5.2. (1) *As a consequence of the theorem, if $\mathcal{A}_i^2(\varphi)\neq 0$, then there exist non–trivial Eisenstein cohomology classes in $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi)\otimes E)$ representable by (derivatives of) residues of Eisenstein series at least in all even degrees q, satisfying $2n \le q \le 3n$.*

(2) *Furthermore, again if* $\mathcal{A}_i^2(\varphi) \neq 0$, *there exist non–trivial Eisenstein cohomology classes in* $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi) \otimes E)$ *representable by derivatives of holomorphic main values of Eisenstein series at least in all even degrees* q *in the range* $3n \leq q \leq 4n-1$. *If* $\mathcal{A}_i^2(\varphi) = 0$, *then there exist non–trivial Eisenstein cohomology classes in* $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_i,\varphi) \otimes E)$ *representable by derivatives of holomorphic main values of Eisenstein series in degrees* $3n \leq q \leq 4n-1$, *respectively* $4n \leq q \leq 5n-1$, *depending on the infinitesimal character* χ *of* $\pi \in \varphi_{P_i}$; *cf. Propositions* 4.2 *and* 4.3.

Remark 5.3. (1) We recall that the spaces $H^q(\mathcal{A}_i^2(\varphi) \otimes E)$ and $H^q(\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi) \otimes E)$ used in the statement of the theorem are described explicitly in Propositions 4.2 and 4.3.

(2) We cannot exclude that $J_i^q(\varphi)(H^q(\mathcal{A}_i^2(\varphi) \otimes E)) \neq 0$ in even degrees $3n \leq q \leq 4n$, so holomorphic and residual Eisenstein cohomology classes might not be separated by their degrees.

Proof of Theorem 5.1 *and its Corollary* 5.2. By the very construction of the filtration, we have $\mathcal{A}_{\mathcal{J}}(P_i,\varphi) \cong \mathcal{A}_i^1(\varphi)$. Hence, it suffices to prove the above theorem for $H^q(\mathcal{A}_i^1(\varphi) \otimes E)$. In order to do that, we use the long exact sequence in $(\mathfrak{g}_\infty, K_\infty)$-cohomology obtained from the short exact sequence

$$0 \to \mathcal{A}_i^2(\varphi) \to \mathcal{A}_i^1(\varphi) \to \mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi) \to 0.$$

But having this strategy in mind, the theorem is an easy consequence of the vanishing properties of

$$H^q(\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi) \otimes E) \quad \text{and} \quad H^q(\mathcal{A}_i^2(\varphi) \otimes E)$$

obtained in Propositions 4.2 and 4.3. The corollary now follows from the theorem and the observation that $\mathcal{A}_i^2(\varphi)$ is a residual automorphic representation and that $\mathcal{A}_i^1(\varphi)/\mathcal{A}_i^2(\varphi)$ is spanned by derivatives of holomorphic main values of Eisenstein series. ☐

5.3. **The minimal parabolic subgroup.** We are now considering the case of the minimal parabolic k-subgroup P_0. Therefore, let $\mu = \mu_1 \otimes \mu_2 \in \varphi \in \Phi_0$ be a unitary character of $L_0(\mathbb{A})$ which is trivial on the diagonally embedded group $A_0(\mathbb{R})^\circ$. Recall from Theorem 3.3 that $\mathcal{A}_0^2(\varphi)$ is non–trivial if and only if $\mu = \mathbf{1}_{L_0(\mathbb{A})}$ and $\Lambda = (0,0)$ and is then isomorphic to the residual representation

$$\mathcal{A}_0^2(\varphi) \cong \mathbf{1}_{G(\mathbb{A})}.$$

Hence, we can again consider the morphism

$$J^q : H^q(\mathfrak{g}_\infty, K_\infty, \mathbf{1}_{G(\mathbb{A})}) \to H^q_{Eis}(G,\mathbb{C})$$

induced by the inclusion $\mathbf{1}_{G(\mathbb{A})} \hookrightarrow \mathcal{A}_{\mathcal{J}}(P_0,\varphi) \hookrightarrow \mathcal{A}_{\mathcal{J}}$. We shall now prove a theorem dealing with the summands $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0,\varphi) \otimes E)$ in the Eisenstein cohomology $H^q_{Eis}(G,E)$ of G.

Theorem 5.4. *Let* $G = Sp_4/k$ *be the split algebraic group of type* C_2 *over a totally real number field* k. *Let* E *be an irreducible, finite-dimensional representation of* G_∞ *so that its highest weight* $\Lambda = (\Lambda_{1,\sigma}, \Lambda_{2,\sigma})_\sigma$ *has repeating coordinates in the field embeddings* $\sigma : k \hookrightarrow \mathbb{C}$ *and may hence be written as* $\Lambda = (\Lambda_1, \Lambda_2)$, *and assume that* E *is the complexification of an algebraic representation of* G/k. *Let* $\varphi = (\varphi_P)_{P \in \{P_0\}} \in \Phi_0$, *and* $\mu \in \varphi_{P_0}$ *be a unitary character of* $L_0(\mathbb{A})$.

(1) *If* $\mathcal{A}_0^2(\varphi)$ *is non–trivial, i.e., if* $\mu = \mathbf{1}_{L_0(\mathbb{A})}$ *and* $\Lambda = (0,0)$*, then there is the following isomorphism of* $G(\mathbb{A}_f)$*-modules:*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0,\varphi)) \cong \begin{cases} H^q(\mathbf{1}_{G(\mathbb{A})}), & 0 \le q \le 3n-1, \\ H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)) \mod J^q(H^q(\mathbf{1}_{G(\mathbb{A})})), & 3n \le q \le 4n-1, \\ & q \text{ even}, \\ Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))), & 3n \le q \le 4n-1, \\ & q \text{ odd}. \end{cases}$$

The module $Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)))$ *is non–trivial for all odd* q*,* $3n \le q \le 4n-1$*. Furthermore,* $J^{3n}(H^{3n}(\mathbf{1}_{G(\mathbb{A})})) \cong H^{3n}(\mathbf{1}_{G(\mathbb{A})})$*.*

(2) *If* $\mathcal{A}_0^2(\varphi)$ *is trivial but* $\mathcal{A}_0^1(\varphi)$ *is non–trivial, i.e., if precisely one of the conditions*
- $\Lambda_1 = \Lambda_2$ *and* $\mu_1 = \mu_2$ *or*
- $\Lambda_2 = 0$ *and* $\mu_2 = \mathbf{1}$

is satisfied, then there is the following isomorphism of $G(\mathbb{A}_f)$*-modules:*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0,\varphi)\otimes E) \cong \begin{cases} 0, & 0 \le q \le 3n-1, \\ H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E) \cong H^q(\mathcal{A}_0^1(\varphi)\otimes E), & 3n \le q \le 4n-1. \end{cases}$$

(3) *If, however, even* $\mathcal{A}_0^1(\varphi)$ *is trivial, then there is the following isomorphism of* $G(\mathbb{A}_f)$*-modules in all degrees* q*:*

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0,\varphi)\otimes E) \cong H^q(\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)\otimes E) \cong H^q(\mathcal{A}_0^0(\varphi)\otimes E).$$

Remark 5.5. (1) We recall that the spaces $H^q(\mathcal{A}_0^2(\varphi))$, $H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)$ and $H^q(\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)\otimes E)$ are described explicitly in Propositions 4.6, 4.5 and 4.4, respectively.

(2) Unfortunately, in the case when $\mathcal{A}_0^1(\varphi) \neq 0$ our approach does not give a good description of $H^q(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J}}(P_0,\varphi)\otimes E)$ in the remaining possibly non–trivial degrees $4n \le q \le 6n-2$.

Proof of Theorem 5.4. Observe that by construction of the filtration, we have $\mathcal{A}_{\mathcal{J}}(P_0,\varphi) \cong \mathcal{A}_0^0(\varphi)$. Hence, it is enough to prove the above theorem for $H^q(\mathcal{A}_0^0(\varphi)\otimes E)$. In order to do so, we use as in the case of the maximal parabolic subgroups the long exact sequences in $(\mathfrak{g}_\infty, K_\infty)$-cohomology

$$\text{(5.3.1)} \quad ... \to H^q(\mathcal{A}_0^2(\varphi)\otimes E) \to H^q(\mathcal{A}_0^1(\varphi)\otimes E) \to H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E) \to ...$$

and

$$\text{(5.3.2)} \quad ... \to H^q(\mathcal{A}_0^1(\varphi)\otimes E) \to H^q(\mathcal{A}_0^0(\varphi)\otimes E) \to H^q(\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)\otimes E) \to ...$$

obtained from the short exact sequences

$$0 \to \mathcal{A}_0^2(\varphi) \to \mathcal{A}_0^1(\varphi) \to \mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi) \to 0$$

and

$$0 \to \mathcal{A}_0^1(\varphi) \to \mathcal{A}_0^0(\varphi) \to \mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi) \to 0.$$

By Proposition 4.5, $H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E) = 0$ for $0 \le q \le 3n-1$. Therefore the long exact sequence (5.3.1) yields

$$H^q(\mathcal{A}_0^1(\varphi)\otimes E) \cong H^q(\mathcal{A}_0^2(\varphi)\otimes E) \quad \text{for } 0 \le q \le 3n-1$$

and $J^{3n}(H^{3n}(\mathbf{1}_{G(\mathbb{A})}))\cong H^{3n}(\mathbf{1}_{G(\mathbb{A})})$. Moreover, by Proposition 4.4, $H^q(\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)\otimes E)=0$ for $0\leq q\leq 4n-1$. Hence, the long exact sequence (5.3.2) implies

$$H^q(\mathcal{A}_0^0(\varphi)\otimes E)\cong H^q(\mathcal{A}_0^1(\varphi)\otimes E)\quad\text{for } 0\leq q\leq 4n-1.$$

Keeping this in mind, the vanishing of $H^q(\mathcal{A}_0^2(\varphi)\otimes E)=0$ in odd degrees also implies that

$$\begin{aligned}&H^q(\mathcal{A}_0^1(\varphi)\otimes E)\\ &\quad\cong\begin{cases} H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)\quad\bmod J^q(H^q(\mathcal{A}_0^2(\varphi)\otimes E)), & 3n\leq q\leq 4n-1,\\ & q\text{ even},\\ Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)), & 3n\leq q\leq 4n-1,\\ & q\text{ odd}.\end{cases}\end{aligned}$$

If $\mathcal{A}_0^2(\varphi)$ is trivial, this simplifies to

$$H^q(\mathcal{A}_0^1(\varphi)\otimes E)\cong\begin{cases}0, & 0\leq q\leq 3n-1,\\ H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E), & 3n\leq q\leq 4n-1.\end{cases}$$

Putting the pieces together we obtain

$$\begin{aligned}&H^q(\mathfrak{g}_\infty,K_\infty,\mathcal{A}_{\mathcal{J}}(P_0,\varphi))\\ &\quad\cong\begin{cases} H^q(\mathbf{1}_{G(\mathbb{A})}), & 0\leq q\leq 3n-1,\\ H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))\quad\bmod J^q(H^q(\mathbf{1}_{G(\mathbb{A})})), & 3n\leq q\leq 4n-1,\\ & q\text{ even},\\ Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))), & 3n\leq q\leq 4n-1,\\ & q\text{ odd}.\end{cases}\end{aligned}$$

if $\mathcal{A}_0^2(\varphi)\neq 0$ and

$$\begin{aligned}&H^q(\mathfrak{g}_\infty,K_\infty,\mathcal{A}_{\mathcal{J}}(P_0,\varphi)\otimes E)\\ &\quad\cong\begin{cases}0, & 0\leq q\leq 3n-1,\\ H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)\otimes E)\cong H^q(\mathcal{A}_0^1(\varphi)\otimes E), & 3n\leq q\leq 4n-1\end{cases}\end{aligned}$$

if $\mathcal{A}_0^1(\varphi)\neq 0$ but $\mathcal{A}_0^2(\varphi)=0$. If even $\mathcal{A}_0^1(\varphi)=0$, then $\mathcal{A}_0^0(\varphi)=\mathcal{A}_0^0(\varphi)/\mathcal{A}_0^1(\varphi)$ and the result follows in this case. Hence, it remains to show that $Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)))$ is non–trivial for all odd q, $3n\leq q\leq 4n-1$. This can be seen as follows. If $3n\leq q\leq 4n-1$ is odd, then the integer $n_2(q)$ from Proposition 4.5 is non–zero. Therefore, again by Proposition 4.5,

$$H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))\supseteq I_2(2,\mathbf{1}_{L_2(\mathbb{A})}),$$

and therefore $H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))$ is not finite–dimensional. However, this implies that $Sb(H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)))$ being the kernel of the map $H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi))\to H^{q+1}(\mathbf{1}_{G(\mathbb{A})})$ must be non–trivial, as $H^{q+1}(\mathbf{1}_{G(\mathbb{A})})$ is finite–dimensional. □

Corollary 5.6. *If the highest weight Λ and a unitary character $\mu\in\varphi_{P_0}$ are such that $\mathcal{A}_0^1(\varphi)\neq 0$, then there are non–trivial Eisenstein cohomology classes in all degrees $3n\leq q\leq 4n-1$ which are representable by the main values of derivatives of residual Eisenstein series obtained from a simple pole of a cuspidal Eisenstein series attached to μ. Thus, their main values are residues of Eisenstein series which are not square–integrable (and do not come from a pole of the highest possible order 2).*

Proof. If the highest weight Λ and a unitary character $\mu \in \varphi_{P_0}$ are such that $\mathcal{A}_0^1(\varphi) \neq 0$, then Theorem 5.4 shows that there are non–trivial Eisenstein cohomology classes in all degrees $3n \leq q \leq 4n-1$ which are elements of the cohomology spaces $H^q(\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi) \otimes E)$. As the quotient $\mathcal{A}_0^1(\varphi)/\mathcal{A}_0^2(\varphi)$ is spanned by main values of derivatives of residual Eisenstein series which are obtained from a simple pole of a cuspidal Eisenstein series attached to μ, the assertion follows. □

6. ON THE CONTRIBUTION OF THE TRIVIAL REPRESENTATION TO AUTOMORPHIC COHOMOLOGY

We would like to finish with a more detailed discussion of the actual contribution of the trivial representation $\mathbf{1}_{G(\mathbb{A})}$ to Eisenstein cohomology of $G = Sp_4/k$ over a totally real number field k. More precisely, we consider the $G(\mathbb{A}_f)$-morphism

$$J^q : H^q(\mathbf{1}_{G(\mathbb{A})}) \to H^q_{Eis}(G, \mathbb{C})$$

induced by the inclusion $\mathbf{1}_{G(\mathbb{A})} \hookrightarrow \mathcal{A}_{\mathcal{J}}(P_0, \mathbf{1}_{L_0(\mathbb{A})}) \hookrightarrow \mathcal{A}_{\mathcal{J}}$, usually called the Borel map.

The approach taken in this paper, more precisely the results of Section 5, only provides an incomplete description of the image of the Borel map, which we summarize in Corollary 6.1 below. As pointed out by the referee, the true approach to resolve this problem is the one of Kewenig and Rieband in their Diplomarbeit [KR], following Franke [Fra08]. As we were not aware of their work [KR], which is still unpublished and quite difficult to find (we found a copy in the library of the *Mathematisches Institut der Universität Bonn*), following a suggestion by the referee, we decided to include in Section 6.2 a complete summary of the results obtained by Kewenig and Rieband in [KR], made explicit in the specific case Sp_4 over a totally real number field.

6.1. We begin with a corollary that is a consequence of our computations in Section 5. It describes the Borel map up to degree $q = 3n$, but fails in higher possible degrees. However, this is already an improvement of a general result of Borel (cf. [Bor74, Thm. 7.5]) in the case $G = Sp_4$. For a complete description of the image of the Borel map see Section 6.2, where a summary of [KR] in the case $G = Sp_4$ over a totally real number field is given.

Corollary 6.1. *The full space of Eisenstein cohomology $H^q_{Eis}(G, \mathbb{C})$, with respect to the trivial coefficient system $E = \mathbb{C}$, is entirely spanned by the cohomology of the trivial representation $\mathbf{1}_{G(\mathbb{A})}$ in degrees $0 \leq q \leq 2n-1$, so*

$$H^q_{Eis}(G, \mathbb{C}) \cong H^q(\mathbf{1}_{G(\mathbb{A})}) \cong \mathbf{1}_{G(\mathbb{A}_f)}^{n_0(q)}, \qquad \text{for } 0 \leq q \leq 2n-1$$

in the notation of Proposition 4.6. Moreover, the morphism J^q determining the contribution of $H^q(\mathbf{1}_{G(\mathbb{A})})$ to Eisenstein cohomology is injective (at least) up to degree $q = 3n$.

Proof. This is a direct consequence of Theorems 5.4 and 5.1. □

Remark 6.2. As mentioned above, the corollary – although a partial result – is already an improvement of Borel's result on the contribution of the trivial representation to the cohomology of arithmetic groups (cf. [Bor74, Thm. 7.5]) in the case $G = Sp_4/k$.

Indeed, denote by $c(R_{k/\mathbb{Q}}(G))$ the maximum of all degrees q such that $\rho_0 - \nu > 0$ for all weights ν of A_0 in $\bigwedge^q \mathfrak{n}_{0,\infty}$, and by $m(G_\infty) + 1$ the smallest degree in which a non-trivial irreducible unitary representation of G_∞ may have cohomology. Then Borel proved that J^q is injective for $q \leq c(R_{k/\mathbb{Q}}(G))$ and an isomorphism for $q \leq \min(c(R_{k/\mathbb{Q}}(G)), m(G_\infty))$. It is easy to make these numbers explicit in the considered case $G = Sp_4/k$: we obtain $c(R_{k/\mathbb{Q}}(G)) = n-1$ and $m(G_\infty) = 1$. Hence, the claim follows.

This is in analogy to the case SL_2/k, k being any number field with more than one real place, as it was observed by Harder in [Har75, Prop. 2.3.(iv)]. See also [Bor74, Example 7.7].

6.2. We now give a complete summary of the results of Kewenig and Rieband in their Diplomarbeit [KR], made explicit for the case Sp_4 over a totally real number field. Following the approach of Franke, applied in [Fra08] to the special linear group, they determined the kernel of $J^* := \bigoplus_{q \geq 0} J^q$ very explicitly for the symplectic and odd special orthogonal groups of arbitrary rank over any number field. This makes it possible to determine the image as well.

Their strategy is as follows. The cohomology $H^*(\mathfrak{g}_\infty, K_\infty, \mathbf{1}_{G_\infty})$ of the trivial representation $\mathbf{1}_{G_\infty}$ of G_∞ can be identified with the cohomology $H^*(X_G^{(c)})$ of the connected compact dual $X_G^{(c)}$ of the symmetric space $X_G = G_\infty/K_\infty$, attached to G_∞ and its maximal compact subgroup K_∞. In particular,

$$H^*(\mathbf{1}_{G(\mathbb{A})}) \cong H^*(X_G^{(c)}).$$

We need some notation by Kewenig–Rieband. They denote by $H^*(X_G^{(c)})_{\mathrm{kernel}}$ the kernel of the Borel map J^* viewed as a subspace of $H^*(X_G^{(c)})$,

$$H^*(X_G^{(c)})_{\mathrm{kernel}} := \ker J^* \subseteq H^*(X_G^{(c)}).$$

The crucial fact, due to Franke [Fra08], is that this kernel can be computed as the Poincaré orthogonal complement in $H^*(X_G^{(c)})$ of another subspace, denoted by $H^*(X_G^{(c)})_{\mathrm{image}}$,

$$H^*(X_G^{(c)})_{\mathrm{kernel}} \cong \left(H^*(X_G^{(c)})_{\mathrm{image}}\right)^{\perp}.$$

This latter subspace is the image of the Poincaré dual of the Borel map restricted to a certain subspace of $H_c^*(G, \mathbb{C})$, defined by Franke in [Fra08]. In view of this latter interpretation of $H^*(X_G^{(c)})_{\mathrm{kernel}}$, Franke [Fra08, (7.2)] now provides an effective description of the kernel of the Borel map.

The results for the symplectic group over a totally real number field are obtained, using the above strategy, in Section 12.1 of [KR]. They first determine in Satz 12.1.1 the subspace $H^*(X_G^{(c)})_{\mathrm{image}}$ of

$$H^*(X_G^{(c)}) \cong \bigotimes_{v \in S_\infty} H^*\left(Sp(4)/U(2)\right).$$

(Here we used the Lie group theorists' notation $Sp(4)$ for the real compact from of $Sp_4(\mathbb{C})$, rather than $USp_4(\mathbb{C})$ used in [KR].) It is the ideal spanned by the top Chern classes X_v, attached to the factor $H^*\left(Sp(4)/U(2)\right)$ corresponding to $v \in S_\infty$. Since top Chern classes are self–orthogonal with respect to the Poincaré pairing (see [KR, Korollar 10.1.5]), one obtains an explicit description of the kernel of the Borel map as well. This is done in [KR, Satz 12.1.2] and is summarized in the next theorem.

Theorem 6.3 ([KR] Satz 12.1.2, Kor. 12.1.3)**.** *Let $G = Sp_4/k$ be the split symplectic group of k-rank two over a totally real number field k of degree n over $\mathbb{Q}$. Viewed as a subspace of $H^*(X_G^{(c)}) \cong \bigotimes_{v\in S_\infty} H^*\left(Sp(4)/U(2)\right)$, the kernel of the morphism $J^* = \bigoplus_{q\geq 0} J^q$ is the ideal which is spanned by the product $\prod_{v\in S_\infty} X_v \otimes \mathbf{1}_{G(\mathbb{A}_f)}$, X_v being the top Chern class associated to $H^*\left(Sp(4)/U(2)\right)$ at the place v. In particular, its dimension is $\dim Ker J^* = 2^n$.*

From the description of the kernel of J^* one can determine its image. In this way, as a direct consequence of Theorem 6.3, one obtains the following corollary, which shows that J^q is non–trivial in even higher degrees than what could be determined using our approach in Corollary 6.1.

Corollary 6.4. *The dimension of the image of the Borel map $J^* = \bigoplus_{q\geq 0} J^q$ for $G = Sp_4/k$ is given by*

$$\dim \mathrm{Im} J^* = 2^n(2^n - 1).$$

Moreover, if $n \geq 2$, then the trivial representation $\mathbf{1}_{G(\mathbb{A})}$ contributes non-trivially to Eisenstein cohomology above the middle degree $q = 3n = \frac{1}{2}\dim G_\infty/K_\infty$.

Proof. By induction on the degree $n = [k : \mathbb{Q}]$, one shows using the proof of Proposition 4.6 that $\dim H^*(\mathbf{1}_{G(\mathbb{A})}) = 2^{2n}$. Now, the first part of the corollary is a consequence of Theorem 6.3. For the last assertion recall that the $(\mathfrak{g}_\infty, K_\infty)$-cohomology of $\mathbf{1}_\infty$ satisfies Poincaré duality and the fact that for $n \geq 2$, $\dim \mathrm{Im} J^* = 2^n(2^n - 1) > 2^{2n-1} = \frac{1}{2}\dim H^*(\mathbf{1}_{G(\mathbb{A})})$. This shows the last assertion in the case of odd n. If $n = 2\ell \geq 2$ is even, we need to prove that $\dim \mathrm{Im} J^* > \frac{1}{2}\dim H^*(\mathbf{1}_{G(\mathbb{A})}) + \frac{1}{2}n_0(3n)$. An easy observation, again the proof of Proposition 4.6 and Poincaré duality, shows that this is equivalent to $2^n < \sum_{j=0}^{3\ell-1} a_j$, where a_j is the coefficient of y^j in the polynomial $(1 + y + y^2 + y^3)^n$. But this follows by induction on ℓ. ☐

Remark 6.5. The last assertion of Corollary 6.4 is in accordance with the case SL_2/k considered by Harder; cf. [Har75, Prop. 2.3.(iv)]. He proved that if $k \neq \mathbb{Q}$, then $\mathbf{1}_{SL_2(\mathbb{A})}$ contributes non–trivially to Eisenstein cohomology of SL_2/k in some degrees greater than the middle one, i.e., greater than half the dimension of the symmetric spaces associated to $SL_{2,\infty}$ and a maximal compact subgroup.

Example 6.6. For the convenience of the reader, and as an example, we make in Table 6.1 the contribution of the trivial representation to Eisenstein cohomology, i.e., the behavior of the Borel map $J^q : H^q(\mathbf{1}_{G(\mathbb{A})}) \to H^q_{Eis}(G, \mathbb{C})$, explicit for the group Sp_4 over a real quadratic extension $k/\mathbb{Q}$, i.e., $n = 2$.

TABLE 6.1. The behavior of the Borel map for the group Sp_4 over a real quadratic extension of $\mathbb{Q}$

q	0	2	4	6	8	10	12
$\mathrm{Im}J^q$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^3$	$\mathbb{C}^4$	$\mathbb{C}^2$	0	0
$\mathrm{Ker}J^q$	0	0	0	0	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$
J^q is	bij.	bij.	inj.	inj.	×	$\equiv 0$	$\equiv 0$

We use the notation "bij." for bijective, "inj." for injective but not surjective, the symbol × for neither injective nor surjective, and $\equiv 0$ for the trivial map. The middle degree in this example is $q = 3n = 6$. Hence, as predicted by Corollary 6.4, we see that there is a non–trivial contribution in degree $q = 8$, which is above the middle degree.

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