

LAMINATIONS IN THE LANGUAGE OF LEAVES

ALEXANDER M. BLOKH, DEBRA MIMBS, LEX G. OVERSTEEGEN,
AND KIRSTEN I. S. VALKENBURG

This paper is dedicated to the memory of Bill Thurston

ABSTRACT. Thurston defined invariant laminations, i.e. collections of chords of the unit circle \mathbb{S} (called *leaves*) that are pairwise disjoint inside the open unit disk and satisfy a few dynamical properties. To be directly associated to a polynomial, a lamination has to be generated by an equivalence relation with specific properties on \mathbb{S} ; then it is called a *q-lamination*. Since not all laminations are q-laminations, then from the point of view of studying polynomials the most interesting are those which are limits of q-laminations. In this paper we introduce an alternative definition of an invariant lamination, which involves only conditions on the leaves (and avoids gap invariance). The new class of laminations is slightly smaller than that defined by Thurston and is closed. We use this notion to elucidate the connection between invariant laminations and invariant equivalence relations on \mathbb{S} .

1. INTRODUCTION

Invariant laminations, introduced by Thurston in the early 1980s, are used to study the dynamics of individual polynomials and the parameter space of all polynomials, the latter in the quadratic case (an expanded version of Thurston's preprint recently appeared in [Thu09]). Investigating the space of all quadratic invariant laminations played a crucial role in [Thu09]. An important idea of Thurston's was, as we see it, similar to one of the main ideas of dynamics as a whole - to suggest a tool (*laminations*) allowing one to model the dynamics under investigation on a topologically/combinatorially nice object (in the case of [Thu09] one models polynomial dynamics on the Julia set by so-called *topological polynomials*, generated by laminations).

According to Thurston, a *lamination* \mathcal{L} is a closed family of chords inside the open unit disk \mathbb{D} . These chords meet at most in a common endpoint and satisfy some dynamical conditions. These chords are called *leaves (of the lamination)*, and the union of all leaves from \mathcal{L} united with \mathbb{S} is denoted by \mathcal{L}^* . A natural *direct* way to associate a lamination to a polynomial P of degree d with a locally connected Julia set is as follows: (1) define an equivalence relation \sim_P on \mathbb{S} by identifying

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angles if their external rays land at the same point (observe that \sim_P on \mathbb{S} is σ_d -invariant); (2) consider the edges of convex hulls of equivalence classes and declare them to be the *leaves* of the corresponding *lamination* \mathcal{L}_P .

By [Kiw04, BCO08] more advanced methods allow one to associate a lamination to some polynomials with non-locally connected Julia sets (by declaring two angles equivalent if impressions of their external rays are non-disjoint and extending this relation by transitivity). We call laminations, generated by equivalence relations similar to \sim_P above, *q-laminations*. They form an important class of laminations, many of which correspond to complex polynomials with connected Julia sets. In all of these cases the lamination is found through the study of the topology of the Julia set of the polynomial.

The drawback of this approach is that it fails if the topology of the Julia set is complicated (e.g., if a quadratic polynomial has a fixed *Cremer point* [BO06]). Thus, even though ultimately laminations are a tool which allows one to study both individual polynomials and their parameter space, in some cases it is not obvious as to what laminations (or what equivalence relations on the circle) can be directly connected in a meaningful way to certain polynomials. Hence one needs a non-direct way of associating a lamination (or, more generally, some combinatorial structure) to a polynomial with a complicated Julia set.

A possibility here is as follows. For a polynomial $P_c(z) = z^2 + c$, consider sequences of parameters $c_i \rightarrow c$ with $P_{c_i} = P_i$ having locally connected Julia sets and associated lamination \mathcal{L}_{P_i} . These laminations \mathcal{L}_{P_i} (systems of chords of \mathbb{S}) may converge to another lamination (system of chords of \mathbb{S}) in the sense that the continua $\mathcal{L}_{P_i}^*$ may converge to a subcontinuum of \mathbb{D} in the Hausdorff sense, and the limit continuum \mathcal{L}^* then comes from an appropriate lamination \mathcal{L} . In this case the lamination \mathcal{L} is called the *Hausdorff limit* of laminations \mathcal{L}_{P_i} ; one may associate all such Hausdorff limit laminations to c .

Using this notion of convergence one can define the Hausdorff closures of sets of laminations. Hence the space of laminations useful for studying polynomials could be a closed set of laminations which contains the Hausdorff closure of the set of all q-laminations, but is not much larger.

To describe a candidate set of laminations we introduce a new notion of a *sibling invariant lamination* which is slightly more restrictive than the one given by Thurston. The new definition is given intrinsically (i.e., by only listing properties on the leaves of the lamination). We show that the family of all sibling invariant laminations is closed and contains all q-laminations. The new definition significantly simplifies the verification of the fact that a system of chords of \mathbb{S} is an invariant lamination. Thurston [Thu09] introduced the class of *clean* laminations. We use our tools to show that clean laminations are (up to a finite modification) q-laminations. In Section 6 we apply these ideas to the degree 2 case and show that in this case all clean Thurston invariant laminations are q-laminations.

2. LAMINATIONS: CLASSICAL DEFINITIONS

2.1. Preliminaries. Let \mathbb{C} be the complex plane, $\mathbb{S} \subset \mathbb{C}$ the unit circle identified with \mathbb{R}/\mathbb{Z} and let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Define a map $\sigma_d : \mathbb{S} \rightarrow \mathbb{S}$ by $\sigma_d(z) = dz \bmod 1$, $d \geq 2$. By a *chord* in the unit disk we mean a segment of a straight line connecting two points of the unit circle. A *prelamination* \mathcal{L} is a collection of chords in \mathbb{D} , called *leaves*, such that any two leaves of \mathcal{L} meet at most

in a point of \mathbb{S} . If all points of the circle are elements of \mathcal{L} (seen as degenerate leaves) and $\bigcup \mathcal{L} = \mathcal{L}^*$ is closed in \mathbb{C} , then we call \mathcal{L} a *lamination*. Hence, one obtains a lamination by closing a prelamination and adding all points of \mathbb{S} viewed as degenerate leaves. If $\ell \in \mathcal{L}$ and $\ell \cap \mathbb{S} = \{a, b\}$, then we write $\ell = \overline{ab}$. We use the term “leaf” to refer to a non-degenerate leaf, and specify when a leaf may be degenerate, i.e. a point in \mathbb{S} .

Given a leaf $\ell = \overline{ab} \in \mathcal{L}$, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$. If $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf* and $\sigma_d(a)$ a *critical value*. Let $\sigma_d^* : \mathcal{L}^* \rightarrow \overline{\mathbb{D}}$ be the linear extension of σ_d over all the leaves in \mathcal{L} . It is not hard to check that σ_d^* is continuous. Also, σ_d is locally one-to-one on \mathbb{S} , and σ_d^* is one-to-one on any given non-critical leaf. Note that if \mathcal{L} is a lamination, then \mathcal{L}^* is a continuum.

Definition 2.1 (Gap). A *gap* G of a lamination \mathcal{L} is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^*$; its boundary leaves are called *edges (of a gap)*. We also say that a leaf ℓ is an *edge* of ℓ .

For each set $A \subset \overline{\mathbb{D}}$ we denote $A \cap \mathbb{S}$ by $\partial(A)$. If G is a leaf or a gap of \mathcal{L} , then G coincides with the convex hull of $\partial(G)$. If G is a leaf or a gap of \mathcal{L} , we let $\sigma_d(G)$ be the convex hull of $\sigma_d(\partial(G))$. Also, by $\text{Bd}(G)$ we denote the topological boundary of G . The topological boundary of G is a Jordan curve which consists of leaves and points on \mathbb{S} so that $\text{Bd}(G) \cap \mathbb{S} = G \cap \mathbb{S} = \partial(G)$. A gap G is called *infinite* if and only if $\partial(G)$ is infinite. A gap G is called *critical* if $\sigma_d|_{\partial G}$ is not one-to-one. There are two types of degenerate leaves of \mathcal{L}^* which are not endpoints of non-degenerate leaves: (1) certain vertices of gaps, (2) points of \mathbb{S} , separated from other points of \mathbb{S} by a sequence of leaves of \mathcal{L} .

2.2. q-laminations. Let P be a complex polynomial with locally connected Julia set J . Then J is connected and there exists a conformal map $\varphi : \mathbb{C}^* \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C}^* \setminus K$, where K is the *filled-in Julia set* (i.e., the complement of the unbounded component of J in \mathbb{C}). One can choose φ so that $\varphi'(0) > 0$ and $P \circ \varphi = \varphi \circ \sigma_d$, where $\sigma_d(z) = z^d$ and d is the degree of P . Since J is locally connected, φ extends over the boundary \mathbb{S} of \mathbb{D} . We also denote the extended map by φ . Define an equivalence relation \approx_P on \mathbb{S} by $x \approx_P y$ if and only if $\varphi(x) = \varphi(y)$. Since J is the boundary of $\mathbb{C}^* \setminus K$, then J is homeomorphic to \mathbb{S}/\approx_P . Clearly, the map σ_d induces a map $f_d : \mathbb{S}/\approx_P \rightarrow \mathbb{S}/\approx_P$ and the maps $P|_J$ and f_d are conjugate. It is known that all equivalence classes of \approx_P are finite. The collection of boundary edges of convex hulls of all equivalence classes of \approx_P is a lamination denoted by \mathcal{L}_P .

Equivalence relations analogous to \approx_P can be introduced with no reference to polynomials [BL02]. Let \sim be an equivalence relation on \mathbb{S} . Equivalence classes of \sim will be called *(\sim -)classes* and will be denoted by Gothic letters. Also, given a closed set $A \subset \mathbb{C}$, let $\text{CH}(A)$ denote the convex hull of the set A in \mathbb{C} .

Definition 2.2. An equivalence relation \sim is a *(d-)invariant laminational* equivalence relation if:

- (E1) \sim is *closed*: the graph of \sim is a closed set in $\mathbb{S} \times \mathbb{S}$;
 - (E2) \sim is *unlinked*: if \mathfrak{g}_1 and \mathfrak{g}_2 are distinct \sim -classes, then their convex hulls $\text{CH}(\mathfrak{g}_1), \text{CH}(\mathfrak{g}_2)$ in the unit disk \mathbb{D} are disjoint,
 - (D1) \sim is *forward invariant*: for a class \mathfrak{g} , the set $\sigma_d(\mathfrak{g})$ is also a class
- which implies that

- (D2) \sim is *backward invariant*: for a class \mathfrak{g} , its preimage $\sigma_d^{-1}(\mathfrak{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathfrak{g}\}$ is a union of classes;

(D3) for any \sim -class \mathfrak{g} with more than two points, the map $\sigma_d|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \sigma_d(\mathfrak{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S} \setminus \mathfrak{g}$ the arc in the circle $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathfrak{g})$;

(D4) all \sim -classes are finite.

There is an important connection between laminations and (invariant laminational) equivalence relations.

Definition 2.3. Let \mathcal{L} be a lamination. Define the equivalence relation $\approx_{\mathcal{L}}$ by declaring that $x \approx_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of \mathcal{L} joining x to y .

Now we are ready to define *q-laminations*.

Definition 2.4. A lamination \mathcal{L} is called a *q-lamination* if the equivalence relation $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation and \mathcal{L} consists exactly of boundary edges of the convex hulls of all $\approx_{\mathcal{L}}$ -classes together with all points of \mathbb{S} . If an invariant laminational equivalence relation \sim is given and \mathcal{L} is formed by all edges from the convex hulls of all \sim -classes together with all points of \mathbb{S} , then \mathcal{L} is called the *q-lamination (generated by \sim)* and is denoted by \mathcal{L}_{\sim} . Clearly, if \mathcal{L} is a q-lamination, then it is a q-lamination generated by $\approx_{\mathcal{L}}$.

Let \sim be a laminational equivalence relation and $p : \mathbb{S} \rightarrow J_{\sim} = \mathbb{S}/\sim$ be the quotient map of \mathbb{S} onto its quotient space J_{\sim} , and let $f_{\sim} : J_{\sim} \rightarrow J_{\sim}$ be the map induced by σ_d . We call J_{\sim} a *topological Julia set* and the induced map f_{\sim} a *topological polynomial*. It is easy to see from Definition 2.2 that leaves of \mathcal{L}_{\sim} map to leaves of \mathcal{L}_{\sim} under σ_d ; moreover, the map σ_d acting on leaves and gaps of \mathcal{L}_{\sim} also has other more specific properties analogous to (D1) - (D4) above. This leads to the abstract notion of an invariant lamination [Thu09] that allows for laminations which are not directly associated to a laminational equivalence relation and, hence, do not correspond (directly) to a polynomial.

2.3. Invariant laminations due to Thurston.

Definition 2.5 (Monotone map). Let X, Y be topological spaces and $f : X \rightarrow Y$ be continuous. Then f is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$. It is known that if f is monotone and X is a continuum, then $f^{-1}(Z)$ is connected for every connected $Z \subset f(X)$.

Definition 2.6 is due to Thurston [Thu09].

Definition 2.6 (Thurston invariant lamination [Thu09]). A lamination \mathcal{L} is *Thurston d-invariant* if it satisfies the following conditions.

- (1) Forward d-invariance: for any leaf $\ell = \overline{pq} \in \mathcal{L}$, either $\sigma_d(p) = \sigma_d(q)$ or $\overline{\sigma_d(p)\sigma_d(q)} = \sigma_d(\ell) \in \mathcal{L}$.
- (2) Backward invariance: for any leaf $\overline{pq} \in \mathcal{L}$, there exists a collection of **d disjoint** leaves in \mathcal{L} (this collection of leaves may not be unique), each joining a preimage of p to a preimage of q .
- (3) Gap invariance: For any gap G , the convex hull H of $\sigma_d(G \cap \mathbb{S})$ is a gap, a leaf, or a single point (of \mathbb{S}).

If H is a gap, $\sigma_d^*|_{\text{Bd}(G)} : \text{Bd}(G) \rightarrow \text{Bd}(H)$ maps as the composition of a monotone map and a covering map to the boundary of the image gap, with positive orientation (the image of a point moving clockwise around $\text{Bd}(G)$ must move clockwise around the image $\text{Bd}(H)$ of G).

By Lemma 2.1 of [BL02], any q -lamination is Thurston invariant (the only part which is not immediate deals with the gap-invariance of infinite gaps of any q -lamination).

3. SIBLING INVARIANT LAMINATIONS

3.1. An alternative definition. Note that in Definition 3.1 we do not require the invariance of gaps.

Definition 3.1 (Sibling d -invariant lamination [Mim10]). A (pre)lamination \mathcal{L} is *sibling d -invariant* if:

- (1) for each $\ell \in \mathcal{L}$ either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(\ell)$ is a point in \mathbb{S} ,
- (2) for each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ such that $\sigma_d(\ell') = \ell$,
- (3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist \mathbf{d} **disjoint** leaves ℓ_1, \dots, ℓ_d in \mathcal{L} such that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i .

Let us make a few remarks. For a continuum/point $K \subset \mathcal{L}^*$ which maps one-to-one onto $\sigma_d^*(K)$, we call a continuum/point $T \subset \mathcal{L}^*$ a *sibling (of K)* if $K \cap T = \emptyset$ and T maps onto $\sigma_d^*(K)$ in a one-to-one fashion (thus, siblings are homeomorphic and disjoint). E.g., the leaves ℓ_2, \dots, ℓ_d from Definition 3.1 are siblings of ℓ . The collection $\{\ell, \ell_2, \dots, \ell_d\}$ of leaves from Definition 3.1 is called a *full sibling collection (of ℓ)*. For a continuum/point $K \subset \mathcal{L}^*$ which maps one-to-one onto $\sigma_d^*(K)$, a collection of d sets made up of K and its pairwise disjoint siblings is called a *full sibling collection (of K)*. Often we talk about siblings without assuming the existence of a full sibling collection (e.g., in the context of Thurston d -invariant laminations).

Let \mathcal{L} be a sibling d -invariant lamination. By Definition 3.1 (1) we see that Definition 2.6 (1) is satisfied. Let $\ell \in \mathcal{L}$ be a leaf. By Definition 3.1 (2) and (3) there are d pairwise disjoint leaves of \mathcal{L} mapped onto ℓ ; hence, Definition 2.6 (2) is satisfied. Thus both sibling d -invariant laminations and Thurston d -invariant laminations satisfy conditions (1) and (2) of Definition 2.6, i.e. are *forward d -invariant* and *backward d -invariant*. These conditions deal with *leaves* and in that respect are intrinsic to \mathcal{L} , defined as a collection of leaves. However, having these conditions is not enough to define a meaningful dynamic collection of leaves; there are examples of laminations satisfying conditions (1)-(2) of Definition 2.6 which are not gap invariant. Hence one needs to add an extra condition to forward and backward invariance.

In Definition 2.6 one deals with gaps, i.e. closures of components of the complement $\mathbb{D} \setminus \mathcal{L}^*$. This is a direct way to ensure that σ_d^* has a nice extension over the plane. A drawback of this approach is that while \mathcal{L} otherwise is defined as a family of chords of \mathbb{D} (leaves), in gap invariance we talk about other objects (gaps). One can argue that gap invariance of \mathcal{L} under σ_d is not sufficiently intrinsic since \mathcal{L} is defined as a collection of leaves. As a consequence it is often more cumbersome to verify gap invariance. This justifies the search for a similar definition of an invariant lamination which deals only with leaves. Above we propose the notion of a *sibling (d)-invariant lamination*.

3.2. Sibling invariant laminations are gap invariant. We show that any sibling d -invariant lamination is Thurston d -invariant. Some complications below are caused by the fact that we do not yet know that the lamination is gap invariant.

E.g., extending the map σ^* over \mathbb{D} to a suitably nice map (i.e., the composition of a monotone and open map) is impossible if the lamination is not gap invariant.

Theorem 3.2. *Suppose that G is a gap of a sibling d -invariant lamination \mathcal{L} . Then either*

- (1) $\sigma_d(G)$ is a point in \mathbb{S} or a leaf of \mathcal{L} or
- (2) there is a gap H of \mathcal{L} such that $\sigma_d(G) = H$, and the map $\sigma_d^*|_{\text{Bd}(G)} : \text{Bd}(G) \rightarrow \text{Bd}(H)$ is the positively oriented composition of a monotone map $m : \text{Bd}(G) \rightarrow S$, where S is a simple closed curve, and a covering map $g : S \rightarrow \text{Bd}(H)$.

Thus, any sibling d -invariant lamination is Thurston d -invariant.

To prove Theorem 3.2 we prove a few lemmas. Given a point x , call any point $\hat{x} \in \mathbb{S}$ with $\sigma_d(\hat{x}) = x$ an x -point. If a lamination is given, by an $\hat{a}\hat{b}$ -leaf, we mean a leaf that maps to \overline{ab} . The word “chord” is used in lieu of “leaf” in reference to a chord of \mathbb{D} which may not be a leaf of \mathcal{L} . By $[a, b]_{\mathbb{S}}$, $a, b \in \mathbb{S}$ we mean the closed arc of \mathbb{S} from a to b , and by $(a, b)_{\mathbb{S}}$ we mean the open arc of \mathbb{S} from a to b . The direction of the arc, clockwise (*negative*) or counterclockwise (*positive*), will be clear from the context; also, sometimes we simply write $[a, b]$, (a, b) , etc. By $<$ we denote the positive (circular) order on \mathbb{S} . If we say that points are *ordered* on \mathbb{S} we mean that they are either positively or negatively ordered. Proposition 3.3 is left to the reader. Observe that in Proposition 3.3 we do not assume the existence of a lamination.

Proposition 3.3. *Suppose that $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ are $2n$ points in the circle. Then for any point a_i and b_j either component of $\mathbb{S} \setminus \{a_i, b_j\}$ contains the same number of a -points and b -points. Thus, if $\hat{a}, \hat{b} \in \mathbb{S}$ are such that $a = \sigma_d(\hat{a}) \neq \sigma_d(\hat{b}) = b$, then either component of $\mathbb{S} \setminus \{\hat{a}, \hat{b}\}$ contains the same number of a -points and b -points.*

Since 2-invariant laminations are invariant under the rotation by $\frac{1}{2}$, then, given a 2-invariant lamination we see that its siblings are rotations of each other. Even though this is not typically true for laminations of higher degree (see Figure 1), Lemma 3.6 states that sibling leaves must connect in the same order. To state it we need Definition 3.4.

Definition 3.4. Consider two sets $A, C \subset \mathbb{S}$ such that $\sigma_d(A) = C$ and $\sigma_d|_A$ is one-to-one. Then A and C are said to have the *same (circular) orientation* if for any three points $x, y, z \in A$, their images $\sigma_d(x), \sigma_d(y), \sigma_d(z)$ have the same orientation as x, y, z . In this case we say that σ_d *preserves orientation* on A . If there is another set B satisfying the same properties as A above (so that C is a one-to-one σ_d -image of B , etc.), we say that A and B have the *same orientation*. Now, if $\sigma_d|_A$ is not one-to-one and for any three points $x, y, z \in A$ their images $\sigma_d(x), \sigma_d(y), \sigma_d(z)$ either have the same orientation as x, y, z or some of them coincide, then we say that $\sigma_d|_A$ *weakly preserves orientation*.

If $\sigma_d(A) = \sigma_d(B) = C$ and σ_d preserves orientation of A and B , then, as we walk along the circle in the positive direction from a point $u \in A$, its sibling $u' \in B$, and its image $\sigma_d(u)$, we will meet points of A , their siblings in B , and their images in C in the same order.

Any two two-point sets which are not critical and map to the same set have the same orientation; this is not necessarily true for sets with more points. The fact

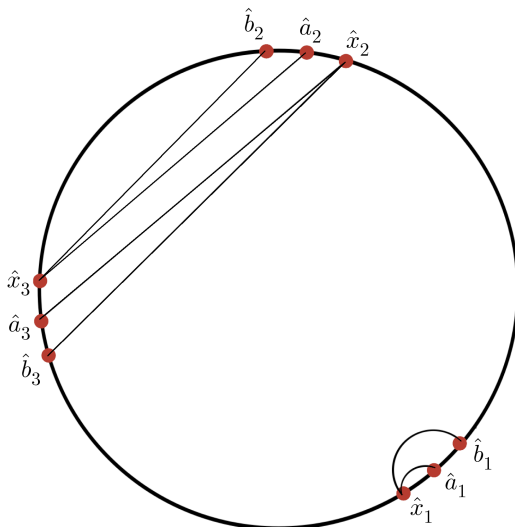


FIGURE 1. Sibling “arcs”

This is an example of sibling “arcs” under σ_3 . Notice that while the arcs connect points in different “patches” and are not found by rigid rotation, the manner in which the endpoints connect preserves order.

that sets have the same orientation sometimes implies “structural” conclusions. For a set $A \subset \mathcal{L}^*$ we write $A \sqsubset \mathcal{L}^*$ if $A \cap \mathbb{S}$ is zero dimensional.

Definition 3.5. A *trioid* is a homeomorphic image of the *simple trioid* τ (the union of three arcs which share a common endpoint). Denote by $B(T)$ the union of the endpoints and the vertex of a trioid T . In what follows we *always* consider trioids $T \subset \mathbb{D}$ with $B(T) \subset \mathbb{S}$. The edge of T , which separates (inside \mathbb{D}) the endpoints of T not belonging to it, is called the *central edge* of T while the other edges of T are said to be *sides* of T . Similarly, if A is the union (the concatenation) of two leaves $\overline{av} \cup \overline{vb}$, we set $B(A) = \{a, v, b\}$.

To avoid ambiguity we call a subarc of \mathbb{S} a *circle arc*. By an *arc in \mathcal{L}^** we mean a *topological arc* (a one-to-one image of an interval). Given a set $A \subset \mathcal{L}^*$ we sometimes need to consider a topological arc in A with endpoints x, y (it will always be clear which arc we actually consider); such an arc will be denoted by $[x, y]_A$. Thus, $[a, b]_{\mathbb{S}}$ is always a circle arc. By default arcs $[a, b], (a, b)$, etc. are circle arcs.

Lemma 3.6. (1) *Let $x_1 < a_1 < b_1 < x_2 < \dots < x_n < a_n < b_n < x_1$ be $3n$ points in \mathbb{S} . If for each i there exists $r(i), m(i) \in \{1, \dots, n\}$ such that arcs $A_i = \overline{x_i a_{r(i)}} \cup \overline{x_i b_{m(i)}}$ are pairwise disjoint ($1 \leq i \leq n$), then $x_i < a_{r(i)} < b_{m(i)}$ for each i .*

(2) *Let $x_1 < a_1 < b_1 < c_1 < x_2 < \dots < x_n < a_n < b_n < c_n < x_1$ be $4n$ points in \mathbb{S} . If for each i there exist $r(i), m(i), l(i) \in \{1, \dots, n\}$ such that trioids $T_i = \overline{x_i a_{r(i)}} \cup \overline{x_i b_{m(i)}} \cup \overline{x_i c_{l(i)}}$ are pairwise disjoint ($1 \leq i \leq d$), then $x_i < a_{r(i)} < b_{m(i)} < c_{l(i)}$ for each i .*

Proof. (1) Let $x_1 < b_{m(1)} < a_{r(1)}$. Then the arc $(x_1, b_{m(1)})$ contains $m(1) - 1$ points $x_2, \dots, x_{m(1)}$, $m(1) - 1$ points $b_1, \dots, b_{m(1)-1}$, but $m(1)$ points $a_1, \dots, a_{m(1)}$. This contradicts the existence of sets A_j .

(2) Follows from (1) applied to parts of the triods T_i . □

We will mostly apply the following corollary of Lemma 3.6.

Corollary 3.7. *Let \mathcal{L} be a sibling d -invariant lamination and $T \sqsubset \mathcal{L}^*$ be an arc consisting of two leaves with a common endpoint v or a triod consisting of three leaves with a common endpoint v . Suppose that $S \sqsubset \mathcal{L}^*$ is an arc (triod) such that $\sigma_d^*(S) = T$ and $\sigma_d^*|_S$ is one-to-one. Then the orientation of the sets $B(T)$ and $B(S)$ is the same.*

Proof. The set of all preimages of points of $B(T)$ consists of d triples (if T is an arc) or quadruples (if T is a triod) of points denoted by B_1, \dots, B_d and such that (1) each B_i is contained in a circle arc J_i so that these arcs are disjoint and of the same length less than $\frac{1}{d}$, (2) the circular order of points in B_i is the same as the circular order of σ_d -images of these points.

Denote by v the point from $B(T)$ which is either the vertex of T (if T is a triod) or a point from $B(T)$ which is not an endpoint of T (if T is an arc). Take the leaves which comprise T (two leaves if T is an arc and three leaves if T is a triod). Consider the corresponding leaves comprising S . Each leaf of S gives rise to its full sibling collection (here we use the fact that \mathcal{L} is sibling invariant). Then leaves from those collections “grow” out of points v_1, \dots, v_d which are preimages of v . This gives rise to d unlinked sets S_1, \dots, S_d , where S_i is a union of two (three) leaves growing out of v_i (indeed, no leaf of S_i can coincide with a leaf of S_j where $i \neq j$, while distinct leaves must be disjoint by the properties of laminations). Moreover, we may assume that $S_1 = S$. It now follows from Lemma 3.6 and the above paragraph that all the sets $B(S_i)$ have the same orientation coinciding with the orientation of the set of their σ_d -images, i.e. the set $B(T)$. □

Corollary 3.7 shows that all pullbacks of certain sets have the same orientation as the sets themselves. However it also allows us to study images of some sets. Indeed, by Corollary 3.7, if $S \sqsubset \mathcal{L}^*$ is a triod mapped by σ_d^* one-to-one to T , then the central edge of S maps into the central edge of T .

Lemma 3.6 and Corollary 3.7 are useful in comparing the orientation of arcs (triods) and their images *in the absence of critical leaves*. In the case when there are critical leaves in the arcs and triods we need additional lemmas. We will need the following terminology.

By a *preimage collection (of a chord \overline{ab})* we mean a collection A of several *pairwise disjoint* chords with the same *non-degenerate* image-chord \overline{ab} ; here we do not necessarily assume the existence of a lamination. However, if we deal with a lamination \mathcal{L} , then we always assume that preimage collections consist of leaves of \mathcal{L} . If there are d disjoint chords in A we call it *full*. If X is a preimage collection of a chord \overline{ab} , we denote the endpoints of chords of X by the same letters as for the endpoints of their images but with a hat and distinct subscripts, and call them correspondingly (*a*-points, *b*-points, etc.). Given two sibling chords ℓ' and ℓ'' with non-degenerate image, we say that they are *ordered* if no two sibling endpoints of these chords are adjacent on \mathbb{S} . Thus, if $\hat{a}_1\hat{b}_1, \hat{a}_2\hat{b}_2$ are two sibling chords, then they are ordered if either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$ (equivalently, if $\hat{a}_1\hat{a}_2$

separates $\overline{\hat{a}_1\hat{b}_1} \setminus \{\hat{a}_1\}$ from $\overline{\hat{a}_2\hat{b}_2} \setminus \{\hat{a}_2\}$ in \mathbb{D}). Finally, recall that $\partial(X)$ is the union of all endpoints of chords from X .

Lemma 3.8. *Let X be a full preimage collection of a chord \overline{ab} and let $\ell_1 = \overline{\hat{a}_1\hat{b}_1}$, $\ell_2 = \overline{\hat{a}_2\hat{b}_2}$ be two chords from X . Then the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside \mathbb{D} is even if and only if ℓ_1 and ℓ_2 are ordered. In particular, suppose that one of the following holds:*

- (1) *there exists a concatenation Q of chords connecting \hat{a}_1 and \hat{a}_2 and disjoint with chords of X except for the points \hat{a}_1, \hat{a}_2 ;*
- (2) *there exists a concatenation Q of critical chords connecting \hat{a}_1 and \hat{a}_2 and such that all their endpoints distinct from \hat{a}_1 and \hat{a}_2 belong to the component of $\mathbb{S} \setminus \{\hat{a}_1, \hat{a}_2\}$ not containing \hat{b}_1 .*

Then ℓ_1 and ℓ_2 are ordered.

Proof. See Figure 2. First let us show that if, say, $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$, then the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside \mathbb{D} is even. Indeed, by Proposition 3.3 there are, say, k a -points and k b -points in (\hat{b}_1, \hat{a}_2) . Suppose that among chords of X there are m chords with both endpoints in (\hat{b}_1, \hat{a}_2) . Then there are $2k - 2m$ a - and b -points in (\hat{b}_1, \hat{a}_2) which are exactly all the endpoints of chords from X which cross $\overline{\hat{a}_1\hat{a}_2}$ inside \mathbb{D} . This implies that the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside \mathbb{D} is even.

On the other hand, suppose that the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside \mathbb{D} is even. As before, for definiteness assume that $\hat{a}_1 < \hat{b}_1 < \hat{b}_2 < \hat{a}_1$. For any $Z \subset X$ consider a function $\varphi(I, Z)$ of an arc $I \subset \mathbb{S}$, defined as the difference between the number of a -points in $Z \cap I$ and the number of b -points in $Z \cap I$ taken modulo 2. Clearly, for some k the arc (\hat{b}_1, \hat{b}_2) contains k b -points and $k + 1$ a -points; similarly, for some l the arc (\hat{a}_2, \hat{a}_1) contains l a -points and $l + 1$ b -points. Hence, $\varphi((\hat{b}_1, \hat{b}_2), X) = \varphi((\hat{a}_2, \hat{a}_1), X) = 1$.

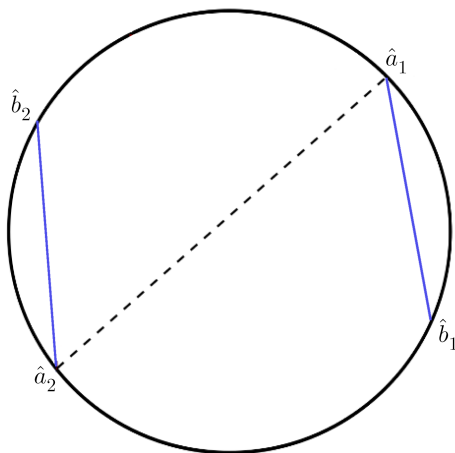


FIGURE 2. Siblings and critical leaves

Remove all the chords from X connecting the arcs (\hat{b}_1, \hat{b}_2) and (\hat{a}_2, \hat{a}_1) to get a new set of chords Y . As we remove one such chord, we change the value of φ

on (\hat{b}_1, \hat{b}_2) by 1, and we change the value of φ on (\hat{a}_2, \hat{a}_1) by 1 as well. By the assumption, to get the set Y we need to remove an even number of chords. Thus, we see that $\varphi((\hat{b}_1, \hat{b}_2), Y) = \varphi((\hat{a}_2, \hat{a}_1), Y) = 1$. However, the remaining points of $\partial(Y) \cap (\hat{b}_1, \hat{b}_2)$ are endpoints of a certain number of chords from X , and hence there must be an equal number of a -points and b -points among them, a contradiction.

Let us prove the remaining claims of the lemma. In case (1) there exists a concatenation Q of chords connecting \hat{a}_1 to \hat{a}_2 so that $Q \cap X = \{\hat{a}_1, \hat{a}_2\}$; then *no chord* of X can cross the chord $\overline{\hat{a}_1 \hat{a}_2}$ inside \mathbb{D} . Hence by the first part ℓ_1 and ℓ_2 are ordered. Now, consider case (2). We may assume that there is a concatenation of critical chords $\overline{\hat{a}_1 \hat{a}^2}, \overline{\hat{a}^2 \hat{a}^3}, \dots, \overline{\hat{a}^i \hat{a}_2}$ such that all points $\hat{a}^2, \dots, \hat{a}^i$ belong to the arc (\hat{a}_1, \hat{a}_2) while $\hat{b}_1 \in (\hat{a}_2, \hat{a}_1)$. Then, by the first part, the chord from X with an endpoint \hat{a}^2 has the other endpoint \hat{b}^2 in (\hat{a}_1, \hat{a}^2) . Repeating this observation, we will finally see that $\hat{b}_2 \in (\hat{a}^i, \hat{a}_2)$, as desired. \square

To prove further lemmas we need more definitions.

Definition 3.9. Let $I \sqsubset \mathcal{L}^*$ be an arc (the image of a homeomorphism $\varphi : [0, 1] \rightarrow I$). We call I a *monotone arc* if its endpoints $\varphi(0), \varphi(1)$ belong to \mathbb{S} and there is a circle arc $T = [\varphi(0), \varphi(1)]_{\mathbb{S}}$ which contains $\partial(I)$ (this implies that the map $\varphi^{-1}|_{\partial(I)}$ is monotone with respect to the circular order on T). Likewise, a triod $T \sqsubset \mathcal{L}^*$ is *monotone* if all its edges are monotone arcs.

As an example of a monotone arc one can consider a single leaf of \mathcal{L} or a subarc of the boundary of a gap of \mathcal{L} . It turns out that in the case of sibling invariant laminations the map preserves local orientation on monotone arcs. More precisely, let $I \sqsubset \mathcal{L}^*$ be a monotone arc. Some leaves (finitely many) in I may be critical.

Definition 3.10. Let $I \sqsubset \mathcal{L}^*$ be a monotone arc with endpoints u and v so that $\partial(I) \subset [u, v]$. Define the natural order $<_I$ on $\partial(I)$ induced by the positive order on the circle arc $[u, v]$. We say that σ_d^* *preserves local orientation* on I if for any point $b \in \partial(I)$ there exists a closed (including the endpoints) neighborhood U of b in the $<_I$ -order topology on $\partial(I)$ such that either $\sigma_d|_U$ weakly preserves orientation or U consists of finitely many points and the corresponding concatenation of leaves maps by σ_d^* to one leaf.

Lemma 3.11. *If \mathcal{L} is a sibling invariant lamination, then σ_d^* preserves local orientation on any monotone arc.*

Proof. Let $I \sqsubset \mathcal{L}^*$ be a monotone arc with endpoints u and v so that $\partial(I) \subset [u, v]$; denote by $<$ the positive order on $[u, v]$. Let \mathcal{I} be the set $\partial(I)$ considered as a topological space endowed with $<_I$ -order topology. Denote by $\overline{[x, y]}_{\mathcal{I}}$ a closed interval in \mathcal{I} with endpoints $x <_I y$. If $b \in \mathcal{I}$ while $\overline{ab} \subset I$ and $\overline{bc} \subset I$ are non-critical leaves, then we can choose $U = [a, b]_{\mathcal{I}}$. Lemma 3.8 implies that either σ_d maps \overline{abc} to one leaf or σ_d *preserves orientation* on $[a, b]_{\mathcal{I}}$. Suppose that there is a concatenation of leaves $\overline{ab_1 \cdots b_k c} \subset I$ such that $\overline{b_i b_{i+1}}$ are critical leaves, $i = 1, \dots, k - 1$. By Lemma 3.8 there exists a leaf $\overline{a' b_k}$ which is a *sibling* of $\overline{ab_1}$ with $b_{k-1} < a' < b_k$. By Lemma 3.8 either σ_d preserves orientation on $\overline{a' b_k c}$ or σ_d maps $\overline{a' b_k c}$ to a leaf. Thus, for each point b_i we can choose $U = [a, c]_{\mathcal{I}}$.

Let $b \in \partial(B)$. Now consider cases when either b is approached on at least one side by points of $\partial(I)$ or b belongs to a maximal concatenation Y of critical leaves in I with at least one endpoint of Y approached by points of $\partial(I)$. In these

cases the arguments are easier because now we can rely upon the fact that $\sigma_d|_{\mathbb{S}}$ preserves local orientation (e.g., this immediately implies our claim in the case when b is non-isolated in $\partial(I)$ on either side) and because we can choose a small semi-neighborhood V of b on the side from which b is non-isolated in $\partial(I)$ so that if there is, say, a non-critical leaf $\overline{ab} \subset I$ coming out of b on the side other than V , the map still preserves orientation on $(V \cap \partial(I)) \cup \{a, b\}$. In the case when there is a concatenation of critical leaves containing b , an obvious modification of these arguments (compare with the first paragraph of the proof) yields the desired result. \square

We are ready to prove the following lemma.

Lemma 3.12. *Let \mathcal{L} be a sibling d -invariant lamination. Suppose that σ_d^* monotonically maps a monotone arc I onto the union of the two sides of a monotone triod T with vertex v whose central edge is a leaf \overline{vm} . Then there exists $\hat{v} \in \partial(I)$ such that $\sigma_d(\hat{v}) = v$ and there exists a leaf $Q = \overline{\hat{v}m}$ such that $I \cup Q$ is a monotone triod with vertex u and central edge Q .*

Proof. Denote the endpoints of T distinct from m by a and b . Then the endpoints of I map to a and b ; denote them \hat{a} and \hat{b} , respectively. For the sake of definiteness assume that $v \in (a, b)$. Consider $\sigma_d|_{(\hat{b}, \hat{a})}$. Clearly, as we move from \hat{b} to \hat{a} we first encounter several semi-open subarcs of (\hat{b}, \hat{a}) which wrap around the circle in a one-to-one fashion. Then the last arc which we encounter connects a b -point with an a -point (namely, with \hat{a}) and maps onto $[b, a]$ in a one-to-one fashion. Hence there is one more m -point in (\hat{b}, \hat{a}) than v -points in (\hat{b}, \hat{a}) . This implies that one m -point belonging to (\hat{b}, \hat{a}) (denote it by \hat{m}) must be connected with a leaf to a v -point belonging to (\hat{a}, \hat{b}) (denote it by \hat{v}). This completes the proof. \square

By a *polygon* we mean a *finite convex polygon*. In what follows, by a *collapsing polygon* we mean a polygon P with edges which are chords of \mathbb{D} such that their images are the same *non-degenerate chord* (thus as we walk along the edges of P , their σ_d -images walk back and forth along the same non-degenerate chord). When we say that Q is a *collapsing polygon of a lamination \mathcal{L}* , we mean that **all** edges of Q are leaves of \mathcal{L} ; we also say that \mathcal{L} *contains a collapsing polygon Q* . However, this does not necessarily imply that Q is a gap of \mathcal{L} , as Q might be further subdivided by leaves of \mathcal{L} *inside* Q .

We often deal with *concatenations of leaves*, i.e. finite collections of pairwise distinct leaves which, when united, form a topological arc in \mathbb{D} . The concatenation of leaves ℓ_1, \dots, ℓ_k is denoted $\ell_1 \cdots \ell_k$. If the leaves are given by their endpoints x_1, \dots, x_k , we denote the concatenation by $\overline{x_1 \cdots x_k}$. We do not assume that points x_1, \dots, x_k are ordered on the circle; however, if they are, we call $\overline{x_1 \cdots x_k}$ an *ordered concatenation*.

Lemma 3.13. *Suppose that \mathcal{L} is a sibling d -invariant lamination which contains two distinct leaves $\ell_0 = \overline{vx}$ and $\ell_1 = \overline{vy}$ such that $\sigma_d(\ell_0) = \sigma_d(\ell_1) = \ell$ is a non-degenerate leaf. Then \mathcal{L} contains a collapsing polygon P with edges ℓ_0 and ℓ_1 such that $\sigma_d(P) = \ell$; also, it contains a critical gap $G \subset P$ with vertex v such that $\sigma_d(G) = \ell$.*

Proof. First assume that $x < v < y$ and that there are no leaves $\ell' = \overline{vz} \in \mathcal{L}$ with $y < z < x$ and $\sigma_d(\ell') = \ell$. Since \mathcal{L} is a *sibling d -invariant lamination*, we can

choose a full sibling collection $A_0 \subset \mathcal{L}$ of ℓ_0 . By Lemma 3.8 there exists $u_1 \in (y, x)$ such that $\overline{yu_1} \in A_0$ and ℓ_0 are siblings (we apply Lemma 3.8 with x and y playing the roles of \hat{a}_1 and \hat{a}_2). Similarly, there exists a full sibling collection $A_1 \subset \mathcal{L}$ of ℓ_1 and a point $u_0 \in (y, x)$ such that $\overline{u_0x} \in A_1$ and ℓ_1 are siblings. Since $\overline{yu_1}$ and $\overline{u_0x}$ are disjoint inside \mathbb{D} , then $y < u_1 \leq u_0 < x$.

Consider all possible choices of points u_0, u_1 so that the above properties hold: $\overline{yu_1}$ and ℓ_0 are siblings, $\overline{u_0x}$ and ℓ_1 are siblings, and $y < u_1 \leq u_0 < x$. Observe that now we do not require that u_0 or u_1 be obtained as endpoints of siblings of ℓ_0 or ℓ_1 coming from full sibling collections, but the existence of such collections shows that the set of the choices is non-empty (see the first paragraph). Choose u_0, u_1 so that the arc $[u_1, u_0]$ is the shortest.

If $u_0 = u_1 = u$, then we obtain a collapsing polygon $P = \text{CH}(vyux)$. Suppose that $u_0 \neq u_1$. Then by the construction and by the choice of u_0 and u_1 no leaf of \mathcal{L} which maps onto ℓ can cross the chords $\overline{vu_0}, \overline{vu_1}$. By Lemma 3.8 applied to A_0 and $\ell_0 \in A_0$, and because of the location of the points found so far, there exists a sibling $\overline{w_0u_0} \in A_0$ of ℓ_0 with $w_0 \in (u_1, u_0)$. Similarly, there exists a sibling $\overline{u_1w_1} \in A_1$ of ℓ_1 with $w_1 \in (u_1, u_0)$. Since leaves $\overline{w_0u_0}$ and $\overline{u_1w_1}$ do not intersect inside \mathbb{D} , we see that $u_1 < w_1 \leq w_0 < u_0$. Similar to what we did before, we can choose w_1 and w_0 so that $\overline{w_0u_0}$ is a sibling of ℓ_0 , $\overline{u_1w_1}$ is a sibling of ℓ_1 , and the arc (w_1w_0) is the shortest possible. We can continue in this manner and obtain a collapsing polygon P with edges ℓ_0 and ℓ_1 .

Now, suppose that there are leaves ℓ' between ℓ_0 and ℓ_1 with $\sigma_d(\ell') = \ell$. Let K be the collection of *all* such leaves ℓ' together with ℓ_0 and ℓ_1 . By the above we can form collapsing polygons for each pair of adjacent leaves from K . If we unite them and erase in that union all leaves of K except for ℓ_0 and ℓ_1 , we will get a collapsing polygon P with edges ℓ_0 and ℓ_1 (leaves of K are diagonals of P). This proves the main claim of the lemma. Let G be any gap of \mathcal{L} contained in P and with edge \overline{vx} . Then $\sigma^*(G) = \sigma_d(\overline{vx})$ and, hence, G is critical. \square

We need the following definition.

Definition 3.14. Given a leaf $\ell = \overline{xy}$, we define the corresponding *open leaf* to be $\ell^\circ = \ell \setminus \{x, y\}$. For a lamination \mathcal{L} , denote its critical leaves by $\bar{c}_i(\mathcal{L}) = \bar{c}_i$. Below we often consider the set $\bigcup_i \bar{c}_i^\circ$ which is the union of *all* open critical leaves of \mathcal{L} .

Lemma 3.15. *Let \mathcal{L} be a sibling d -invariant lamination and $\ell = \overline{ab} \in \mathcal{L}$ be a leaf. If C is a component of $\{(\sigma_d^*)^{-1}(\ell) \setminus \bigcup_i \bar{c}_i^\circ\}$ and G is the convex hull of $\partial(C)$, then G is a leaf or a collapsing polygon of \mathcal{L} .*

Proof. If C is not a leaf, then there exists $x \in \partial(C)$ which is a vertex of at least two leaves from C . Choose leaves ℓ' and ℓ'' in C which form an angle containing all other leaves from C with the endpoint x . By Lemma 3.13 ℓ' and ℓ'' are edges of a collapsing polygon P . By properties of laminations all other leaves of C with an endpoint x (if any) are diagonals of P . Choose a collapsing polygon P , which is maximal by inclusion, with edges ℓ' and ℓ'' .

Let y be a vertex of P . Suppose that a leaf $\overline{yz} \subset C$ comes out of y and is not contained in P . By properties of a lamination then \overline{yz} is in fact disjoint from P (except for y). Choose the edge ℓ''' of P with an endpoint y so that the triangle formed by the convex hull of ℓ''' and \overline{yz} is disjoint from the interior of P . Then by Lemma 3.13 there exists a collapsing polygon P' with edges \overline{yz} and ℓ''' . It

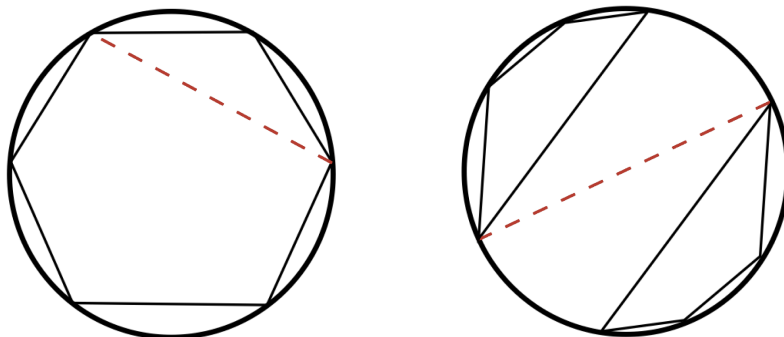


FIGURE 3. Placements of critical leaves

In each picture, the critical leaf is denoted by a dashed line. Notice that in the first example, removing the open critical leaf does not disconnect the polygon, while in the second example, removing the open critical leaf disconnects a previously connected set, resulting in two components.

follows that $P \cup P'$ with ℓ''' removed is a collapsing polygon strictly containing P , a contradiction. Thus, $P = G$, as desired. \square

From now on, by $P_1(\ell), \dots, P_k(\ell)$, with $\ell = \overline{ab}$ non-degenerate, we mean the components of $\{(\sigma_d^*)^{-1}(\ell) \setminus \bigcup_i \overline{c_i^o}\}$; by Lemma 3.15 they are leaves or collapsing polygons. Note that all edges of the sets $P_i(\ell)$ are leaves of \mathcal{L} . We view the set $(\sigma_d^*)^{-1}(\ell)$ as follows. By Definition 3.1, there is a full preimage collection L of ℓ . The endpoints of its leaves are either a -points or b -points (depending on whether they map to a or b). Often these are all the leaves mapped to ℓ . Yet, there might exist other leaves which map into ℓ . Some of these leaves map onto ℓ ; a leaf such as that connects an a -point from one leaf of L to a b -point from another. Some of these leaves are critical and map to a or b ; a leaf such as that connects two a -points (b -points) belonging to distinct leaves from L . Consider the sets $P_1(\ell), \dots, P_k(\ell)$. There might exist leaves from $(\sigma_d^*)^{-1}(\ell)$ inside $P_i(\ell)$, however, we often ignore these leaves (which might or might not be critical). There might also exist other leaves connecting sets $P_i(\ell)$ and $P_j(\ell)$, $i \neq j$. By Lemma 3.15 such leaves must be critical.

Lemma 3.16. *Suppose that $\overline{xv_0 \cdots v_k y} = M$ is an ordered concatenation of leaves of \mathcal{L} such that $\sigma_d(v_i) = w$ for all i and $\sigma_d(x) = \sigma_d(y) \neq w$. Then there exists a collapsing polygon $P_i(\sigma_d^*(\overline{xv_0}))$ which contains M .*

Proof. We see that $\overline{xv_0}$ and $\overline{v_k y}$ have the same non-degenerate image, while all leaves $\overline{v_0 v_1}, \overline{v_1 v_2}, \dots$ are critical. Assume that $x = x_0 < v_0 < \dots < v_k < y = y_0$. Applying Lemma 3.8 to the leaf $\overline{xv_0}$ and its full sibling collection, we obtain a point $x_1, v_0 < x_1 < v_1$ and a leaf $\overline{x_1 v_1}$ which is a sibling of $\overline{xv_0}$. Similarly we obtain points x_2, \dots, x_k located between points v_1, v_2, \dots, v_k and leaves $\overline{x_i v_i}$ which are siblings of $\overline{xv_0}$.

Now, apply Lemma 3.13 to the leaves $\overline{x_k v_k}$ and $\overline{v_k y_0}$. Then there exists a collapsing polygon P_0 with these leaves as edges. It follows that v_{k-1} is a vertex of P_0 and there is an ordered concatenation of siblings of $\overline{y_0 v_k}$ which begins with $\overline{y_0 v_k}$ and ends with some leaf $\overline{y_1 v_{k-1}}$. We can pair this leaf up with the leaf $\overline{x_{k-1} v_{k-1}}$ and

apply the same arguments. In this manner we will discover a “long” ordered concatenation of siblings which begins with $\overline{y_0 v_k}$ and ends with $\overline{v_0 x_0}$. By Lemma 3.15 there exists a collapsing polygon $P_i(\sigma_d^*(\overline{xv_0}))$ from the collection of collapsing polygons described in Lemma 3.15 which contains this concatenation. It follows that $M \subset P_i(\sigma_d^*(\overline{xv_0}))$, as desired. \square

A lamination is called *gap-free* if it has no gaps. In the next few lemmas we study such laminations. Lemma 3.17 is left to the reader.

Lemma 3.17. *Suppose that \mathcal{L} is a gap-free lamination. Then \mathcal{L} does not contain a collection of leaves with one common endpoint such that their other endpoints fill up an arc $I \subset \mathbb{S}$.*

A continuous interval map $f : I \rightarrow I$ is called a *d-sawtooth map* if it has d intervals of monotonicity of length $\frac{1}{d}$ and the slope on each such interval is $\pm d$.

Lemma 3.18. *If \mathcal{L} is gap-free, then it consists of a family of pairwise disjoint parallel leaves which fill up the entire disk $\overline{\mathbb{D}}$ except for two diametrically opposed points $a, b \in \mathbb{S}$. The factor map p which collapses each leaf to a point, semiconjugates σ_d to a d-sawtooth map.*

Proof. If leaves $\ell_0, \ell_1 \in \mathcal{L}$ have a common endpoint, then by Lemma 3.17 we may assume that ℓ_0, ℓ_1 form a wedge with no leaves of \mathcal{L} in it and that \mathcal{L} is not gap-free, a contradiction. Hence all leaves are pairwise disjoint. Consider an equivalence relation on \mathbb{D} identifying every leaf into one class. The absence of gaps implies that the quotient space is an interval I and there are only two points $a, b \in \mathbb{S}$ which are not endpoints of leaves. Moreover, if $p : \mathbb{S} \rightarrow I$ is the corresponding factor map, then $p(a), p(b)$ are the endpoints of I . Clearly, all leaves of \mathcal{L} must cross the chord \overline{ab} because all points of I separate $p(a)$ and $p(b)$.

Since \mathcal{L} is invariant, then either $\sigma_d(a) = a, \sigma_d(b) = b$, or $\sigma_d(a) = b, \sigma_d(b) = a$, or $\sigma_d(a) = \sigma_d(b) = a$, or $\sigma_d(a) = \sigma_d(b) = b$. Consider the case when $\sigma_d(a) = a, \sigma_d(b) = b$ (others are similar). By continuity it follows that as we travel from a to b along the leaves of \mathcal{L} we meet critical leaves $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{d-1}$ (in this order). Suppose that a critical leaf \bar{c}_i maps to an endpoint of a leaf $\ell \in \mathcal{L}$. Then some preimage-leaf of ℓ will meet \bar{c}_i at one of its endpoints, which is impossible as all leaves are pairwise disjoint. Hence all critical leaves of \mathcal{L} map to a or b . On the other hand, all leaves whose images are non-disjoint from a or b must be critical and have images coinciding with a or b .

It follows from the first and second paragraphs of the proof that the critical leaves $\bar{c}_1 = \overline{x_1 y_1}, \dots, \bar{c}_{d-1} = \overline{x_{d-1} y_{d-1}}$ map alternately to b and a and that their endpoints form the full preimages of a and b . Assume that the positive circular order is given by $b, x_{d-1}, \dots, x_1, a, y_1, \dots, y_{d-1}$. Then the arc $[x_2, a)$ maps one-to-one onto \mathbb{S} and hence has length $\frac{1}{d}$, while the same applies to the arc $(a, y_2]$. Continuing in the same manner we will see that all critical leaves are in fact perpendicular to the chord \overline{ab} which in fact is a diameter of \mathbb{D} . By pulling critical leaves back we complete the proof in the case when $\sigma_d(a) = a, \sigma_d(b) = b$. \square

We are ready to prove the main result of this section.

Proof of Theorem 3.2. Let G be a gap of \mathcal{L} . If there are two adjacent leaves on the boundary of G which have the same image, then by Lemma 3.13 the gap G maps to a leaf. Moreover, if there are two leaves in $\text{Bd}(G)$ which have the same

image and are connected with a finite concatenation of critical leaves in $\text{Bd}(G)$, then by Lemma 3.16 again G maps to a leaf. Hence from now on we may assume that the above two cases do not take place on the boundary of G . This implies that the σ_d^* -image of the boundary of G is not an arc. Indeed, otherwise we can choose an endpoint x of $\sigma_d^*(\text{Bd}(G))$ and a point $\hat{x} \in \text{Bd}(G)$ with $\sigma_d^*(\hat{x}) = x$ so that $x \in \mathbb{S}, \hat{x} \in \mathbb{S}$, too. It is easy to see that under the assumptions made in the first paragraph this is impossible.

Now, move a point t along $\text{Bd}(G)$ in the positive direction (we can start this movement at an edge \overline{ab} of G) and consider the movement of its image σ_d^* . By Lemma 3.11, then the point $\sigma_d^*(t)$ moves along $\text{Bd}(G)$ in the positive direction. The first time the point $\sigma_d^*(t)$ comes back to the already covered part of $\sigma_d^*(\text{Bd}(G))$ must be at the point $\sigma_d^*(a)$ because otherwise by Lemma 3.12 there will have to exist leaves of \mathcal{L} inside G . Similarly, as the point t keeps moving, its image cannot veer off its previous path because then, again by Lemma 3.12, there will have to exist leaves of \mathcal{L} inside G . This proves Theorem 3.2. \square

The basic property defining d -invariant sibling laminations is that every non-critical leaf can be extended to a collection of d pairwise disjoint leaves with the same image (a full sibling collection). We conclude this subsection by showing that this implies the same result for arcs as long as their images are monotone arcs.

Lemma 3.19. *Suppose that σ_d^* homeomorphically maps an arc $A \sqsubset \mathcal{L}^*$ to a monotone arc $B \sqsubset \mathcal{L}^*$. Then there are d pairwise disjoint arcs $A_1 = A, A_2, \dots, A_d$ such that for each i the map σ_d^* homeomorphically maps the arc A_i to B .*

Proof. Suppose that the endpoints of B are p, q and that $B \subset [p, q] = I$. Denote by J_1, \dots, J_d circle arcs which map one-to-one to I . Since B is an arc, there are no more than countably many leaves in B . We can order them so that they form a sequence of leaves $\ell_n \subset B$ with $\text{diam}(\ell_n) \rightarrow 0$. Given a leaf $\ell_n \subset B$ we can choose its unique preimage $\ell_n^1 \subset A$ and then choose a full sibling collection of ℓ_n^1 consisting of leaves $\ell_n^1, \ell_n^2, \dots, \ell_n^d$. In other words, we choose a full preimage collection of leaves for each leaf from B so that this preimage collection includes a leaf from A . Call ℓ_n long if there exists a leaf ℓ_n^j with endpoints coming from distinct sets J_r and J_s .

We claim that there are no more than finitely many long leaves in B . Indeed, suppose otherwise. By continuity we can then choose a subsequence for which we may assume that 1) there is a sequence of leaves $t_n \subset B$ which converges to a point $x \in \mathbb{S}$ from one side, and 2) there is a sequence of their pullback-leaves \hat{t}_n (i.e., $\sigma_d^*(\hat{t}_n) = t_n$) which converges to a critical leaf \hat{x} from one side. Clearly, this is impossible since B is a monotone arc.

Suppose that $\ell_{n_1} = \overline{a_{n_1} b_{n_1}}, \dots, \ell_{n_k} = \overline{a_{n_k} b_{n_k}}$ are all long leaves in B . Without loss of generality we may assume that $a_{n_1} < b_{n_1} \leq a_{n_2} < \dots < b_{n_k}$. Denote closures of components of $B \setminus \bigcup \ell_{n_i}$ by S_1, S_2, \dots, S_{k+1} numbered in the natural order on the circle so that S_1 precedes a_{n_1} , S_2 is located between b_{n_1} and a_{n_2} , etc. (some of these sets may be empty, e.g. if $p = a_{n_1}$, then S_1 is empty).

Then each S_j has d pullbacks, each of which is a monotone arc \widehat{S}_j^r such that $\partial(\widehat{S}_j^r) \subset I_r$. Let us consider the union of all such pullbacks with all leaves from previously chosen full preimage collections of all leaves $\ell_{n_1}, \dots, \ell_{n_k}$ (i.e., of all long leaves in B). Basically, we consider all preimage collections of leaves of B and then take the closure of their union, representing it in a convenient form. We claim

that each component of this union is a monotone arc which maps onto B in a one-to-one fashion. Indeed, by construction each such component X can be extended both clockwise and counterclockwise until it reaches points mapped to p and q respectively. It follows from the construction that these components are pairwise disjoint and that one of them is the originally given arc $A = A_1$. This completes the proof. \square

3.3. The space of all sibling invariant laminations. Our approach is to describe laminations in the “language of leaves”. The main idea is to use sibling invariant laminations for that purpose. By Theorem 3.2 this does not push us outside the class of Thurston invariant laminations. According to the philosophy explained in the Introduction, now we need to verify if the set of all sibling invariant laminations contains all q -laminations and is Hausdorff closed. The first step here is made in Lemma 3.20 in which we relate sibling invariant laminations and q -laminations. Observe that it is well known (and easy to prove) that d -invariant q -laminations are Thurston d -invariant laminations.

Lemma 3.20. *d -invariant q -laminations are sibling d -invariant.*

Proof. Assume that \sim is a d -invariant laminational equivalence relation. Conditions (1) and (2) of Definition 3.1 are immediate. To check condition (3) of Definition 3.1, assume that ℓ is a non-critical leaf of \mathcal{L}_\sim and verify that there are d pairwise disjoint leaves $\ell = \ell_1, \dots, \ell_d$ with the same image. To do so, consider the collection \mathcal{A} of all \sim -classes which map to the \sim -class of $\sigma_d(\ell)$. If a \sim -class $\mathfrak{g} \in \mathcal{A}$ does not contain ℓ and is not critical, then $\text{Bd}(\mathfrak{g})$ contains only a unique sibling of ℓ . If \mathfrak{g} is critical, it maps to the \sim -class of $\sigma_d(\ell)$, say, k -to-1, and we can choose k pairwise disjoint sibling leaves of ℓ on the boundary of $\text{CH}(\mathfrak{g})$. If ℓ is an edge of the set $\text{CH}(\mathfrak{h})$ where \mathfrak{h} is a critical class, we can still find the appropriate number of its pairwise disjoint siblings in the boundary of $\text{CH}(\mathfrak{h})$. The endpoints of leaves from the thus-created list exhaust the list of all points which are preimages of endpoints of $\sigma_d(\ell)$. Thus, we get d siblings, one of which is ℓ , as desired. \square

We will now prove that all limits of q -laminations are sibling d -invariant. Thus, if there is a Thurston d -invariant lamination \mathcal{L} which is not sibling d -invariant, then it is not a Hausdorff limit of q -laminations. This shows that the class of Thurston d -invariant laminations is too wide if we are interested in Hausdorff limits of q -laminations. The above justifies our interest in the next example illustrated in Figure 4.

Suppose that points $\hat{x}_1, \hat{y}_1, \hat{z}_1, \hat{x}_2, \hat{y}_2, \hat{z}_2$ are positively ordered on \mathbb{S} and that $H = \text{CH}(\hat{x}_1\hat{y}_1\hat{z}_1\hat{x}_2\hat{y}_2\hat{z}_2)$ is a critical hexagon of an invariant q -lamination \mathcal{L} such that $\sigma_d^* : H \rightarrow T$ maps H in the 2-to-1 fashion onto the triangle $T = \text{CH}(xyz)$ with $\sigma_d(\hat{x}_i) = x, \sigma_d(\hat{y}_i) = y, \sigma_d(\hat{z}_i) = z$. Now, add to the lamination \mathcal{L} the leaves $\hat{x}_1\hat{z}_1$ and $\hat{x}_1\hat{x}_2$ and all their pullbacks along the backward orbit of H under σ_d^* . Denote the thus-created lamination \mathcal{L}' . It is easy to see that \mathcal{L}' is Thurston d -invariant but not sibling d -invariant because $\overline{\hat{x}_1\hat{z}_1} = \ell$ cannot be completed to a full sibling collection (clearly, H does not contain siblings of ℓ).

Lemma 3.21. *Take sequences of d sibling leaves $\overline{\hat{a}_i^j\hat{b}_i^j}, 1 \leq j \leq d, i = 1, 2, \dots$, such that $\overline{\hat{a}_i^1\hat{b}_i^1} \rightarrow \overline{\hat{a}^1\hat{b}^1} = \ell^1, \overline{\hat{a}_i^2\hat{b}_i^2} \rightarrow \overline{\hat{a}^2\hat{b}^2} = \ell^2, \dots, \overline{\hat{a}_i^d\hat{b}_i^d} \rightarrow \overline{\hat{a}^d\hat{b}^d} = \ell^d$ and $\sigma_d(\ell^1)$ is not degenerate. Then $\ell^j, 1 \leq j \leq d$ are siblings with non-degenerate image.*

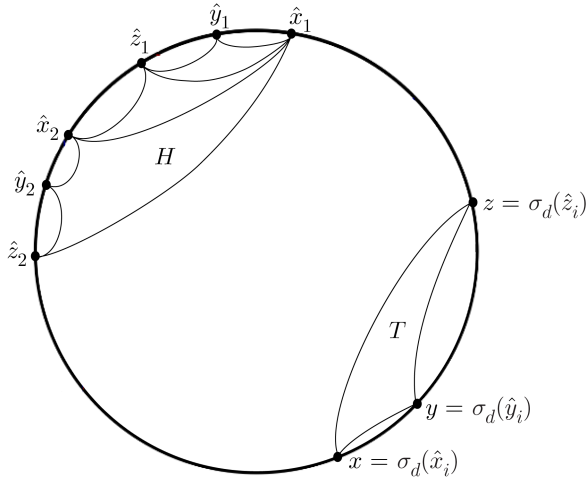


FIGURE 4. An example of a Thurston invariant lamination which is not sibling invariant. The leaf $\overline{x_1 z_1}$ has no siblings in H .

Proof. By continuity, $\sigma_d(\ell^j) = \sigma_d(\ell^1)$ for all j . To show that leaves $\ell^j, 1 \leq j \leq d$ are pairwise disjoint consider i_0 and $\varepsilon > 0$ such that for each $i \geq i_0$ and each $j, 1 \leq j \leq d$ we have that $|\overline{\hat{a}_i^j \hat{b}_i^j}| \geq \varepsilon$. Then it follows that for a fixed i the pairwise distance between points of sets $\{\hat{a}_i^j, \hat{b}_i^j\}$, for $1 \leq j \neq k \leq d$, is bounded away from 0 (\hat{a}_i^j, \hat{a}_i^k cannot be too close because their images coincide and \hat{a}_i^j, \hat{b}_i^k cannot be too close because their images are too far apart). Hence the leaves $\ell^j, 1 \leq j \leq d$ are disjoint, as desired. \square

Suppose that \mathcal{L} is a prelamination. Then by its closure $\overline{\mathcal{L}}$ we mean the set of chords in $\overline{\mathbb{D}}$ which are limits of sequences of leaves of \mathcal{L} . It is easy to see that $\overline{\mathcal{L}}$ is a closed lamination. Corollary 3.22 shows that it is enough to verify the property of being sibling invariant on dense prelaminations. It immediately follows from Lemma 3.21 (so it is given here without proof).

Corollary 3.22. *If \mathcal{L} is a sibling d -invariant prelamination, then its closure $\overline{\mathcal{L}}$ is a sibling d -invariant lamination.*

Theorem 3.23 follows from Lemma 3.20 and Lemma 3.21.

Theorem 3.23. *The Hausdorff limit of a sequence of sibling invariant laminations is a sibling invariant lamination. The space of all sibling invariant laminations is closed in the Hausdorff sense and contains all q -laminations.*

4. PROPER LAMINATIONS

Clearly, not all (sibling, Thurston) invariant laminations are q -laminations (e.g., a lamination with two finite gaps with a common edge is not a q -lamination). In this section we address this issue and describe Thurston d -invariant laminations which almost coincide with appropriate q -laminations. According to the adopted approach, we use the “language of leaves” in our description.

Definition 4.1 (Proper lamination). Two leaves with a common endpoint v and the same image which is a leaf (and not a point) are said to form a *critical wedge*

(the point v is then said to be its vertex). A lamination \mathcal{L} is *proper* if it contains no critical leaf with periodic endpoint and no critical wedge with periodic vertex.

Proper laminations are instrumental for our description of laminations which almost coincide with q -laminations.

Lemma 4.2. *Any q -lamination is proper.*

Proof. Suppose that A is either a critical wedge or a critical leaf which contains a periodic point of period n . Then A is contained in a finite class \mathfrak{g} such that $|\sigma_d(\mathfrak{g})| < |\mathfrak{g}|$, while on the other hand $\sigma_d^n(\mathfrak{g})$ must coincide with \mathfrak{g} , a contradiction. \square

The exact inverse of Lemma 4.2 fails. However, it turns out that proper laminations are very close to q -laminations. To show this we need a few technical definitions.

Definition 4.3. Suppose that A is a polygon with vertices in \mathbb{S} . It is said to be *d-wandering* if for any $m \neq n$ we have $\text{CH}(\sigma_d^m(A \cap \mathbb{S})) \cap \text{CH}(\sigma_d^n(A \cap \mathbb{S})) = \emptyset$.

In Definition 4.3 we do not require that A be a part of some lamination or even that the circular orientation of vertices of A be preserved under σ_d . Still, it follows from Kiwi’s results [Kiw02] (see also a later paper by Childers [Chi07]) that A cannot have too many vertices.

Theorem 4.4 ([Kiw02, Chi07]). *Suppose that A is a polygon with more than d^d vertices. Then it is not d -wandering.*

In Definition 4.3 we assume that images of A have pairwise disjoint convex hulls. As Lemma 4.5 shows, one can slightly weaken this condition and still obtain useful conclusions.

Lemma 4.5. *Suppose that one of the following holds.*

- (1) *A is a polygon, $\partial(A) \subset \mathbb{S}$, and for any $m \neq n$ the interiors of the convex hulls $\text{CH}(\sigma_d^m(A \cap \mathbb{S}))$ and $\text{CH}(\sigma_d^n(A \cap \mathbb{S}))$ are disjoint;*
- (2) *A is a chord of \mathbb{S} , and for any $m \neq n$ two image chords $(\sigma_d^*)^m(A)$ and $(\sigma_d^*)^n(A)$ are disjoint inside \mathbb{D} .*

Then, if $\sigma_d^n|_{\partial(A)}$ is one-to-one for all n , then either A is wandering or A is a chord with a preperiodic endpoint.

Proof. Suppose that $A = \overline{pq}$ is a non-wandering chord and p, q have infinite orbits. We may assume that $\sigma_d^*(A) = \overline{qr}$ and $\sigma_d(p) = q$. Set $A_k = (\sigma_d^*)^k(A)$. Take closures of the two components of $\mathbb{D} \setminus A_k$. By the assumptions, $\bigcup_{i>k} A_i$ is contained in one of them. Then it is easy to see that A_i converges to either a σ_d -fixed point on \mathbb{S} or a σ_d^* -invariant chord. However, this contradicts the fact that σ_d is repelling.

Now suppose that A is a non-wandering polygon. We may assume that $\sigma_d^*(A)$ and A intersect either (1) at a common vertex x or (2) along a common edge $\ell = \overline{xy}$. Choose the vertex u of A with $\sigma_d(u) = x$ and consider the chord \overline{ux} . The chord \overline{ux} satisfies the assumptions of the theorem and is non-wandering. Then by the above u is preperiodic. We may assume that u is fixed. Consider the two edges $\overline{uv_0}$ and $\overline{uw_0}$ of A and the arc $[v_0, w_0]$ in \mathbb{S} not containing u . Similarly, for each n set $v_n = \sigma_d^n(v_0)$ and $w_n = \sigma_d^n(w_0)$. Since the interiors of the convex hulls of $\sigma_d^n(A)$ are pairwise disjoint, the open arcs (v_n, w_n) are also pairwise disjoint, and hence their diameter must converge to zero, a contradiction to the fact that σ_d is expanding. \square

Lemma 4.6. *Suppose that \mathcal{L} is a Thurston invariant lamination. Then there are at most finitely many points x such that there is a critical leaf with an endpoint x or a critical wedge with vertex x .*

Proof. Clearly there are at most finitely many critical leaves. Hence we may suppose that there are infinitely many points x_i such that there are leaves $\overline{a_i x_i}, \overline{b_i x_i}$ with $\sigma_d(a_i) = \sigma_d(b_i) \neq \sigma_d(x_i)$. We may assume that the sets $\overline{\sigma_d(a_i)\sigma_d(x_i)}$ are all distinct and the points $\sigma_d(x_i)$ are all distinct. Clearly, the chords $a_i b_i$ and $a_j b_j$ are disjoint inside \mathbb{D} . Since each such chord is critical, we have a contradiction. \square

Let $E(v)$ be the set of all endpoints of leaves with a common endpoint v (if $E(v)$ accumulates upon v we include v in $E(v)$). Then $E(v)$ is a closed set. Let $C(v)$ be the family of all leaves connecting v and a point of $E(v)$ (it might include $\{v\}$ as a degenerate leaf).

Lemma 4.7. *Suppose that v is a point with infinite orbit and \mathcal{L} is a Thurston d -invariant lamination. Then there are at most finitely many leaves with an endpoint v .*

Proof. Let $E(v)$ be infinite. By Lemma 4.6, we may assume that v and all its images are not endpoints of critical leaves or vertices of critical wedges. If v ever maps to $E(v)$, then by Lemma 4.5 v is preperiodic, a contradiction. Choose $A \subset E(v)$ consisting of d^d points. Consider $\text{CH}(A \cup \{v\})$. By the above for any $n \neq m$, the interiors of $\text{CH}(\sigma_d^n(A \cup \{v\}))$ and $\text{CH}(\sigma_d^m(A \cup \{v\}))$ are disjoint. Then by Lemma 4.5 $\text{CH}(A \cup \{v\})$ is wandering, contradicting Lemma 4.4. \square

Lemma 4.8. *Suppose that \mathcal{L} is a Thurston d -invariant lamination and $A = \overline{x_0 \cdots x_i x_{i+1} \cdots}$, $i = 0, 1, \dots$, is a concatenation of leaves of \mathcal{L} (the set $\partial(A)$ is infinite). Then A has preperiodic vertices.*

Proof. Consider the convex hulls of sets $\overline{\partial(A)} = B_0, \overline{\partial(\sigma_d(A))} = B_1, \dots$. Suppose that all such convex hulls have disjoint interiors. There may exist numbers n such that $\sigma_d|_{B_n}$ is not one-to-one. This means that for each such n there is a critical chord ℓ_n inside the convex hull of B_n . Since we assume that the interiors of sets $\text{CH}(B_n)$ are disjoint, one critical chord can correspond to at most two sets B_n ; otherwise two critical chords ℓ_n and ℓ_m cannot intersect. It follows that there are at most finitely many critical chords ℓ_i constructed as above and that for large enough n the map $\sigma_d|_{B_i}$ is one-to-one. By Lemma 4.5 this is impossible. Hence convex hulls of sets B_n have non-disjoint interiors. Since leaves of \mathcal{L} cannot intersect inside \mathbb{D} , this implies that we can make the following assumption: there exists m, n such that $\sigma_d^m(x_0) = x_n$. We may also assume that $x_n \neq x_0$.

Set $C = \overline{x_0 x_1 \cdots x_{n-1} x_n}$. Then $\sigma_d(C)$ is a concatenation of leaves attached to C , etc. If for some k the set $\sigma_d^k(C)$ is a point we can choose the minimal such k which implies that $\sigma_d^{k-1}(C)$ is a concatenation of leaves whose image is one of its own vertices y . Hence y is periodic as desired. Otherwise we may assume that the number of vertices of C does not drop under iterations of σ_d .

If C has self-intersections, we may refine C to get a concatenation with no self-intersections still connecting x_0 and x_n . We can optimize the situation even more. Indeed, choose the shortest (in terms of the number of its leaves) subchain D of C which ever intersects itself. Then it follows that the image $\sigma_d^k(D)$ of D which intersects D must be concatenated to D at its endpoint (otherwise D can

be shortened even further). Thus, if there are no preperiodic vertices of D , then we get a sequence of σ_d^k -images of D concatenated to each other at their ends. It now follows that $\lim \sigma_d^n(D)$ is either a leaf in \mathcal{L} or a point of \mathbb{S} . In either case this contradicts that σ_d is expanding. \square

Recall that $\approx_{\mathcal{L}}$ was the equivalence relation defined by $x \approx_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of \mathcal{L} connecting x and y . Theorem 4.9 specifies properties of $\approx_{\mathcal{L}}$.

Theorem 4.9. *Let \mathcal{L} be a proper Thurston invariant lamination. Then $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation.*

Proof. Let us show that any point $v \in \mathbb{S}$ is an endpoint of at most finitely many leaves of \mathcal{L} . Otherwise by Lemma 4.7 we may assume that v is fixed. Take the infinite invariant set $E' = E(v) \cup \{v\}$. Since σ_d is expanding and because there are no infinite compact spaces admitting one-to-one expanding maps, E' contains points x, x' with $\sigma_d(x) = \sigma_d(x')$ contradicting the fact that \mathcal{L} is proper.

Suppose next that $A = \overline{x_0 \cdots x_i x_{i+1} \cdots}$, $i = 0, 1, \dots$, is an infinite concatenation of leaves of \mathcal{L} . By Lemma 4.8 we may assume that $x_0 = x$ is fixed. Let us show that if $\ell_1 = \overline{x_0 x_1}$, \dots , $\ell_k = \overline{x_{k-1} x_k}$ are all the leaves with the endpoint x , then $\sigma_d(e_i) = e_i$ for all i . Since \mathcal{L} is proper, the leaves ℓ_1, \dots, ℓ_k have distinct non-degenerate σ_d^* -images. Hence all points $e_i, 1 \leq i \leq k$ are periodic. If $k = 1$, then e_1 is fixed and we are done. If $k = 2$, then ℓ_1, ℓ_2 are edges of some gap G , and the fact that the orientation is preserved under σ_d implies that both ℓ_1, ℓ_2 are fixed. Suppose that $k \geq 3$. We may assume that $e_1 < e_2 < \dots < e_k$. Since gaps map to gaps and the orientation is preserved on them, the fact that x is fixed implies that then $\sigma_d(e_1) < \sigma_d(e_2) < \dots < \sigma_d(e_k)$ and hence in fact $\sigma_d(e_i) = e_i, 1 \leq i \leq k$. Thus, the leaf $\overline{x_0 x_1}$ is fixed, the leaf $\overline{x_1 x_2}$ is fixed and, by induction, all the leaves $\overline{x_i x_{i+1}}$ are fixed, a contradiction.

By the above \mathcal{L} contains no infinite cones and no infinite concatenations of leaves. Let us show that all $\approx_{\mathcal{L}}$ -classes are finite (and hence closed). Otherwise let E be an infinite $\approx_{\mathcal{L}}$ -class and let $x_0 \in E$. For each $y \in E$ fix a concatenation of leaves L_y from x_0 to y containing the least number of leaves. Then there are infinitely many sets $L_y, y \in E$. Since there are only finitely many leaves of \mathcal{L} with an endpoint x_0 , we can choose $x_1 \in E$ so that there are infinitely many sets $L_y, y \in E$ whose first leaf is $\overline{x_0 x_1}$. Since there only finitely many leaves of \mathcal{L} with an endpoint x_1 , we can choose $x_2 \in E$ so that there are infinitely many sets $L_y, y \in E$ whose second leaf is $\overline{x_1 x_2}$. Continuing in this manner we will find an infinite concatenation of leaves of \mathcal{L} , a contradiction (by the choice of sets $L_y, y \in E$, the points x_0, x_1, \dots are all distinct).

Take convex hulls of $\approx_{\mathcal{L}}$ -classes. Clearly, these convex hulls are pairwise disjoint. It follows that if a non-constant sequence of $\approx_{\mathcal{L}}$ -classes converges, then it converges to a leaf of \mathcal{L} or a point. Hence $\approx_{\mathcal{L}}$ is a closed equivalence relation. To show that $\approx_{\mathcal{L}}$ is invariant and laminational we have to prove that $\approx_{\mathcal{L}}$ -classes map onto $\approx_{\mathcal{L}}$ -classes in a covering way (i.e., we need to check conditions (D1) and (D3) of Definition 2.2). Let us show that for any $x \in \mathbb{S}$ the $\approx_{\mathcal{L}}$ -class maps onto the $\approx_{\mathcal{L}}$ -class of $\sigma_d(x)$. Indeed, let y belong to the $\approx_{\mathcal{L}}$ -class of $\sigma_d(x)$. Choose a finite concatenation $\ell_1 \ell_2 \cdots \ell_k$ of leaves connecting $\sigma_d(x)$ and y (here x is an endpoint of ℓ_1 and y is an endpoint of ℓ_k). Take a pullback-leaf $\overline{x x_1}$ of ℓ_1 with an endpoint x , then a pullback-leaf $\overline{x_1 x_2}$ of ℓ_2 , etc., until we get a finite concatenation of leaves

connecting x and some point y' with $\sigma_d(y') = y$. This implies that the $\approx_{\mathcal{L}}$ -class maps onto the $\approx_{\mathcal{L}}$ -class of $\sigma_d(x)$, as desired.

It remains to prove that $\approx_{\mathcal{L}}$ satisfies condition (D3) from Definition 2.2 (i.e., that σ is covering on $\approx_{\mathcal{L}}$ -classes). Observe that if \mathcal{L} is sibling invariant, this immediately follows from Corollary 3.7. In the case when \mathcal{L} is Thurston invariant we need an extra argument.

So, suppose that \mathfrak{g} is a $\approx_{\mathcal{L}}$ -class. We may assume that $|\mathfrak{g}| > 2$. Let ℓ be an edge of $\text{CH}(\mathfrak{g})$. Then either (1) ℓ is approached from the outside of $\text{CH}(\mathfrak{g})$ by leaves ℓ_i of \mathcal{L} or (2) there is an infinite gap G on the other side of ℓ , opposite to the side where $\text{CH}(\mathfrak{g})$ is located. Below we refer to these as cases (1) and (2).

Let us show that if one of the edges of $\text{CH}(\mathfrak{g})$ is critical, then all are critical. Indeed, let $\ell = \overline{ab}$ be a critical edge of $\text{CH}(\mathfrak{g})$. In case (1) images of ℓ_i separate $\sigma_d(\ell)$ from the rest of the circle; hence all points of \mathfrak{g} map to the same point, as desired. In case (2) both endpoints of ℓ are limit points of vertices of G because otherwise we could extend the $\approx_{\mathcal{L}}$ -class \mathfrak{g} . Since \mathcal{L} is Thurston invariant, we conclude that $\sigma_d(\ell)$ is a vertex of an infinite gap $\sigma_d(G)$, and $\sigma_d(\ell)$ is approached from either side on \mathbb{S} by vertices of $\sigma_d(G)$. Hence no leaves can come out of $\sigma_d(\ell)$, and again all points of \mathfrak{g} map to the same point. Thus, in this case Definition 2.2(D3) is satisfied for $\text{CH}(\mathfrak{g})$.

Now suppose that all edges of $\text{CH}(\mathfrak{g})$ are non-critical. We claim that if ℓ is an edge of $\text{CH}(\mathfrak{g})$, then $\sigma_d(\ell)$ is an edge of $\text{CH}(\sigma_d(\mathfrak{g}))$. Indeed, in case (1) $\sigma_d(\ell)$ is approached by leaves of \mathcal{L} from one side, and in case (2) $\sigma_d(\ell)$ borders an infinite gap of \mathcal{L} from one side. In either case $\sigma_d(\ell)$ cannot be a diagonal of the gap $\text{CH}(\sigma_d(\mathfrak{g}))$, and the claim is proved.

It remains to show that as we walk along the boundary of $\text{CH}(\mathfrak{g})$, the σ_d -image of the point walks in the positive direction along the boundary of $\text{CH}(\sigma_d(\mathfrak{g}))$. Indeed, suppose first that $\sigma_d(\mathfrak{g})$ consists of two points. Then by the above there are no critical edges of $\text{CH}(\mathfrak{g})$, and the condition we want to check is automatically satisfied. Otherwise let $\text{CH}(\sigma_d(\mathfrak{g}))$ be a gap. Let \overline{ab} be an edge of $\text{CH}(\mathfrak{g})$ such that moving from a to b along \overline{ab} takes place in the *positive* direction on the boundary of $\text{CH}(\mathfrak{g})$. Then in case (1) the set $\sigma_d(\mathfrak{g})$ must be located so that movement from $\sigma_d(a)$ to $\sigma_d(b)$ takes place in the positive direction on the boundary of $\text{CH}(\sigma_d(\mathfrak{g}))$ as on the other side of $\sigma_d^*(\overline{ab})$ images of leaves, approximating \overline{ab} , approximate $\sigma_d^*(\overline{ab})$. In case (2) we make the same conclusion because on the other side of $\sigma_d^*(\overline{ab})$ the image of an infinite gap adjacent to $\text{CH}(\mathfrak{g})$ along \overline{ab} is located (and this image cannot contain $\text{CH}(\sigma_d(\mathfrak{g}))$). Hence the map is positively oriented on $\text{Bd}(\text{CH}(\mathfrak{g}))$, as desired. \square

Theorem 4.9 shows that, up to a “finite” restructuring, a lamination is a q -lamination if and only if it is proper. The appropriate claim is made in Corollary 4.10, whose proof is left to the reader.

Corollary 4.10. *A proper Thurston invariant lamination \mathcal{L} is a q -lamination if and only if for each $\approx_{\mathcal{L}}$ -class \mathfrak{g} the edges of its convex hull $\text{CH}(\mathfrak{g})$ belong to \mathcal{L} , while no leaf of \mathcal{L} is contained in the interior of $\text{CH}(\mathfrak{g})$.*

5. CLEAN LAMINATIONS

Thurston defined clean laminations. In this section we show that every clean Thurston invariant lamination is a proper sibling invariant lamination. Thus, up to

a minor modification every clean Thurston invariant lamination is a q-lamination. We show in the next section that every clean Thurston 2-invariant lamination is a q-lamination.

Definition 5.1. Let \mathcal{L} be a lamination. Then \mathcal{L} is *clean* if no point of \mathbb{S} is the common endpoint of three distinct leaves of \mathcal{L} .

Theorem 5.2. *Let \mathcal{L} be a Thurston d -invariant clean lamination. Then \mathcal{L} is a proper sibling d -invariant lamination.*

Proof. Let \mathcal{L} be a clean Thurston d -invariant lamination. Suppose first that \mathcal{L} contains a critical leaf \overline{xy} with a periodic endpoint. Assume that x is fixed. There must exist d disjoint leaves which map to \overline{xy} . One of them must have x as an endpoint. Label this leaf \overline{xz} (since $\sigma_d^*(\overline{xy}) = x, y \neq z$). Similarly, there must exist d leaves which map to \overline{xz} ; one of them must be \overline{xw} (and, as above, $\overline{xy}, \overline{xz}, \overline{xw}$ are distinct). Hence \mathcal{L} is not clean, a contradiction. The case when \mathcal{L} contains a critical wedge is similar. Thus, \mathcal{L} is proper.

Suppose next that $\ell = \overline{xy} \in \mathcal{L}$ and $\sigma_d(\ell)$ is non-degenerate. To show that \mathcal{L} is sibling d -invariant we need to show that there are $d - 1$ siblings of ℓ . Since \mathcal{L} is a Thurston d -invariant lamination, there exists a collection B of d pairwise disjoint leaves ℓ_1, \dots, ℓ_d so that $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i . If $\ell = \ell_i$ for some i , we are done. Otherwise there exists $i \neq j$ so that $\ell_i \cap \ell \neq \emptyset \neq \ell_j \cap \ell$. Let $\ell_i = \overline{xz}, \ell_j = \overline{yt}$ and consider two cases.

(1) Points z and t are located in distinct components of $\mathbb{S} \setminus \{x, y\}$. Then ℓ_i and ℓ are edges of a certain gap G because \mathcal{L} is clean. Since $\sigma_d^*|_{\text{Bd}(G)}$ is positively oriented in case $\text{CH}(\sigma_d(\partial G))$ is a gap, G must be a finite gap of \mathcal{L} , collapsing to a leaf. Hence there exists an edge of G with an endpoint y , contradicting the assumption that \mathcal{L} is clean.

(2) Points z and t belong to the same component of $\mathbb{S} \setminus \{x, y\}$. Similar to (1), there exists a gap G with edges ℓ_i, ℓ, ℓ_j (and possibly other edges), collapsed onto $\sigma_d(\ell)$ under σ_d . Since \mathcal{L} is clean, every leaf of \mathcal{L} , which intersects G , is contained in $\text{Bd}(G)$. Hence $\text{Bd}(G)$ consists of $2n$ leaves all of which map to $\sigma_d(\ell)$, and, possibly, to some critical leaves.

Let us show that G has no critical edges. Suppose that \overline{uv} is a critical edge of G such that all vertices of G are contained in the circle arc $I = [v, u]$. Each leaf of \mathcal{L} close to \overline{uv} and with endpoint from (u, v) will have the image which crosses $\sigma_d(\ell)$. Hence there are no such leaves and \overline{uv} is an edge of a gap H whose vertices belong to $[u, v]$. Since \mathcal{L} is clean, there are no edges of H with endpoint u or v except for \overline{uv} . Hence there exist sequences $u_i \in \partial(H)$ converging to u and $v_i \in \partial(H)$ converging to v . Then points $\sigma_d(u_i)$ and $\sigma_d(v_i)$ are on opposite sides of $\sigma_d(u)$. It follows that the leaf $\sigma_d(\ell)$ cuts the image of H , a contradiction with the assumption that \mathcal{L} is a Thurston d -invariant lamination. Thus, $\text{Bd}(G)$ consists of $2n$ leaves, all of which map to $\sigma_d(\ell)$.

This implies that in the collection $\{\ell_1, \dots, \ell_d\} = B$ there are exactly n edges of G ; denote their collection by A . Since \mathcal{L} is clean, for each k either $\ell_k \cap G = \emptyset$ or $\ell_k \subset \text{Bd}(G)$. Hence there is a set A' of $d - n$ leaves in the collection ℓ_1, \dots, ℓ_d which are disjoint from G . Now, starting with ℓ , select n disjoint siblings of ℓ from $\text{Bd}(G)$ and unite them with A' to get a full set of siblings of ℓ . As this can be done for any ℓ , we see that \mathcal{L} is sibling d -invariant. □

Let \mathcal{L} be a clean Thurston d -invariant lamination and let $\approx_{\mathcal{L}}$ be the equivalence relation defined in Definition 2.3; by Theorem 5.2 and Theorem 4.9 $\approx_{\mathcal{L}}$ is a d -invariant laminational equivalence relation. By Corollary 4.10 and since \mathcal{L} is clean, \mathcal{L} is a q -lamination if and only if every chord in the boundary of the convex hull of an equivalence class of $\approx_{\mathcal{L}}$ is a leaf of \mathcal{L} . Below we further compare the laminations \mathcal{L} and $\mathcal{L}_{\approx_{\mathcal{L}}}$ (Definition 2.4). For an equivalence class \mathfrak{g} , let $A_{\mathfrak{g}}$ be the union of all leaves of \mathcal{L} which join points of \mathfrak{g} . Since \mathcal{L} is clean, each $A_{\mathfrak{g}}$ is a point, a simple closed curve, a single leaf, or an arc containing at least two leaves. In all but the last case all leaves of $A_{\mathfrak{g}}$ are leaves of $\mathcal{L}_{\approx_{\mathcal{L}}}$. Thus, $[\mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}] \cup [\mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}]$ is contained in the countable union of the convex hulls of the equivalence classes \mathfrak{g}_i for which $A_{\mathfrak{g}_i}$ is an arc with at least two leaves. We specify this in Corollary 5.3.

Corollary 5.3. *Let \mathcal{L} be a clean Thurston d -invariant lamination and \mathfrak{g} an equivalence class of $\approx_{\mathcal{L}}$ such that $A_{\mathfrak{g}}$ is an arc which contains at least two leaves of \mathcal{L} . Suppose that $\ell = \overline{ab} \subset \text{CH}(\mathfrak{g})$. If $\ell \in \mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}$, then there exists an infinite gap U of \mathcal{L} such that $\ell \setminus \{a, b\}$ is contained in the interior of U and the subarc of $A_{\mathfrak{g}}$, which connects a and b , is a maximal concatenation of leaves in $\text{Bd}(U)$. On the other hand, if $\ell' = \overline{uv} \in \mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}$, then $\ell' \setminus \{u, v\}$ is contained in the interior of $\text{CH}(\mathfrak{g})$ and ℓ' is the intersection of two infinite gaps of \mathcal{L} .*

Proof. Suppose that $\ell = \overline{ab} \subset \text{CH}(\mathfrak{g})$, $\ell \in \mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}$. Clearly, ℓ cannot be approximated by leaves of \mathcal{L} . Moreover, since \mathfrak{g} is finite, no leaf of \mathcal{L} intersects ℓ inside \mathbb{D} . Hence there exists an infinite gap U of \mathcal{L} such that $\ell \setminus \{a, b\}$ is contained in the interior of U . Since \mathcal{L} is clean, the subarc $[a, b]_{A_{\mathfrak{g}}}$ of $A_{\mathfrak{g}}$ is contained in the boundary of U . Moreover, since ℓ is an edge of $\text{CH}(\mathfrak{g})$, $[a, b]_{A_{\mathfrak{g}}}$ is a maximal concatenation of leaves in $\text{Bd}(U)$ (if it were not maximal, we could extend \mathfrak{g}).

Now, let $\ell = \overline{uv} \in \mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}$. Then $\ell \setminus \{u, v\}$ is contained in the interior of $\text{CH}(\mathfrak{g})$. Hence ℓ is isolated and there exist two gaps U, V of \mathcal{L} so that $\ell = U \cap V$. If one of these gaps is finite, then its boundary is a subset of $A_{\mathfrak{g}}$, a contradiction. \square

6. QUADRATIC INVARIANT LAMINATIONS

In this section we study quadratic laminations. First we show that Corollary 4.10 can be made more precise in the quadratic case.

If a 2-invariant q -lamination \mathcal{L} has a finite critical gap L , then one can insert a critical diameter connecting two vertices of L and then pull it back along the backward orbit of L . Also, if L has six vertices or more, one can insert a critical quadrilateral (with two edges which are edges of L) inside L and then pull it back along the backward orbit of L . Such a quadrilateral can itself split into two triangles by a diameter and then can be pulled back along the backward orbit of L . In this way one can create proper sibling invariant laminations which are not q -laminations. If a lamination contains a finite critical polygon L which contains a critical leaf (collapsing quadrilateral) in the interior of its convex hull, then we say that it has a *critical splitting (by a leaf, or, resp., by a quadrilateral)*. Corollary 6.1 shows that these mechanisms are *the only* ways a proper quadratic lamination cannot be a q -lamination.

Corollary 6.1. *A quadratic sibling invariant lamination is a q -lamination if and only if it is proper and does not have a critical splitting.*

Proof. Every q -lamination is proper and has no critical splitting. Now let \mathcal{L} be a proper sibling invariant lamination without critical splitting. Define $\approx_{\mathcal{L}}$ and show that for each $\approx_{\mathcal{L}}$ -class \mathfrak{g} the edges of $\text{CH}(\mathfrak{g})$ belong to \mathcal{L} . Suppose that for a $\approx_{\mathcal{L}}$ -class \mathfrak{g} there is an edge of $\text{CH}(\mathfrak{g})$ not included in \mathcal{L} . By definition, there are finite concatenations of edges of \mathcal{L} connecting all points of \mathfrak{g} . Hence $\text{CH}(\mathfrak{g})$ cannot be a leaf and \mathfrak{g} consists of more than two points. By Thurston's No Wandering Triangles Theorem [Thu09] gaps of quadratic laminations are (pre)critical or (pre)periodic. Hence \mathfrak{g} is either (pre)periodic or (pre)critical (\mathfrak{g} can first map into a critical class of $\approx_{\mathcal{L}}$ and then into a periodic class of $\approx_{\mathcal{L}}$, but not vice versa because \mathcal{L} is proper).

Consider cases. Suppose that \mathfrak{g} is (pre)periodic but not (pre)critical. Then for some n the $\approx_{\mathcal{L}}$ -class $\sigma_2^n(\mathfrak{g})$ is periodic. By [Thu09], the edges of $\text{CH}(\sigma_2^n(\mathfrak{g}))$ form one periodic orbit of edges. Hence no diagonal of $\text{CH}(\sigma_2^n(\mathfrak{g}))$ can be a leaf of \mathcal{L} so that one edge of $\text{CH}(\sigma_2^n(\mathfrak{g}))$ is a leaf of \mathcal{L} , and hence all edges of $\text{CH}(\sigma_2^n(\mathfrak{g}))$ are leaves of \mathcal{L} . Since \mathfrak{g} maps onto $\sigma_2^n(\mathfrak{g})$ one-to-one by our assumptions, then all edges of $\text{CH}(\mathfrak{g})$ are in \mathcal{L} , as desired.

Now, suppose that \mathfrak{g} is precritical and $\sigma_2^n(\mathfrak{g})$ is critical. Again, we may assume that $\text{CH}(\mathfrak{g})$ is not a leaf. Since $\sigma_2^n(\mathfrak{g})$ is a critical $\approx_{\mathcal{L}}$ -class, it must have $2k$ -edges and must map onto its image two-to-one. It follows that the edges of $\sigma_2^n(\mathfrak{g})$ are limits of sequences of $\approx_{\mathcal{L}}$ -classes. Indeed, otherwise there are gaps of $\approx_{\mathcal{L}}$ sharing common edges with $\sigma_2^n(\mathfrak{g})$. By construction this would mean that these gaps are infinite, and hence a forward image of one of these gaps is a critical $\approx_{\mathcal{L}}$ -class. Since we deal with quadratic laminations and \mathfrak{g} is also critical, then this is impossible. Thus, the edges of $\sigma_2^n(\mathfrak{g})$ are limits of sequences of $\approx_{\mathcal{L}}$ -classes which implies that edges of $\sigma_2^n(\mathfrak{g})$ are leaves of \mathcal{L} . As before, since \mathcal{L} is a Thurston invariant lamination, then all edges of \mathfrak{g} are leaves of \mathcal{L} . Thus, in any case, if \mathfrak{g} is a $\approx_{\mathcal{L}}$ -class, then its edges are leaves of \mathcal{L} .

It remains to show that $\text{CH}(\mathfrak{g})$ cannot contain any leaves of \mathcal{L} in its interior. Indeed, suppose otherwise. We may assume that \mathfrak{g} has at least 4 vertices. Suppose that \mathfrak{g} is (pre)critical and $\sigma_2^n(\mathfrak{g})$ is critical. Let us show that any leaf inside $\sigma_2^n(\mathfrak{g})$ must have an image which is an edge or a vertex of $\sigma_2^{n+1}(\mathfrak{g})$. Indeed, it suffices to consider the case when $\sigma_2^n(\mathfrak{g})$ has at least six vertices. By the No Wandering Triangles Theorem [Thu09] for some $m \geq 0$, the gap $\sigma_2^{n+m}(\mathfrak{g})$ is periodic. By the above quoted result of [Thu09], the edges of $\text{CH}(\sigma_2^{n+m}(\mathfrak{g}))$ form one periodic orbit of edges. Hence if there is a leaf of \mathcal{L} inside $\text{CH}(\sigma_2^{n+m}(\mathfrak{g}))$, it will cross itself under the appropriate power of σ_2 , a contradiction. Thus, any leaf inside $\sigma_2^n(\mathfrak{g})$ must have the image which is an edge or a vertex of $\sigma_2^{n+1}(\mathfrak{g})$.

We next show that such a leaf cannot exist. Since \mathcal{L} does not admit a critical splitting, we also need to show that no other splitting of $\text{CH}(\sigma_2^n(\mathfrak{g}))$ by leaves of \mathcal{L} is possible. Indeed, suppose that there are leaves of \mathcal{L} inside $\text{CH}(\sigma_2^n(\mathfrak{g}))$. Since \mathcal{L} does not have critical splitting, it cannot be just one critical leaf, nor can it be a quadrilateral or a quadrilateral with a critical leaf inside. Now, suppose that there is a unique leaf ℓ of \mathcal{L} inside $\sigma_2^n(\mathfrak{g})$ such that $\sigma_2(\ell)$ is an edge of $\sigma_2^{n+1}(\mathfrak{g})$. Then it has to have a sibling leaf which will also be a leaf inside $\sigma_2^n(\mathfrak{g})$. Hence $\sigma_2^n(\mathfrak{g})$ contains a collapsing quadrilateral, a contradiction. As these possibilities exhaust all possibilities for leaves inside $\text{CH}(\sigma_2^n(\mathfrak{g}))$, it follows that there are no leaves inside $\text{CH}(\sigma_2^n(\mathfrak{g}))$ and hence no leaves inside $\text{CH}(\mathfrak{g})$, as desired. \square

Corollary 6.2. *Suppose that \mathcal{L} is a clean Thurston 2-invariant lamination. Then \mathcal{L} is a q -lamination.*

Proof. Suppose that \mathcal{L} is a clean, Thurston 2-invariant lamination. By Theorem 5.2, \mathcal{L} is proper and sibling invariant. Moreover, since \mathcal{L} is clean, it does not have critical splitting. Hence the result follows from Corollary 6.1. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, ALABAMA 35294-1170

E-mail address: `ablokh@math.uab.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, ALABAMA 35294-1170

Current address: Department of Mathematics, Lee University, Cleveland, Tennessee 37320-3450

E-mail address: `dmimbs@leeuniversity.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, ALABAMA 35294-1170

E-mail address: `overstee@math.uab.edu`

FACULTEIT DER EXACTE WETENSCHAPPEN, AFDELING WISKUNDE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: `kirstenvalkenburg@gmail.com`