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THE TRUNCATED MATRIX-VALUED K-MOMENT PROBLEM ON $\mathbb{R}^d$, $\mathbb{C}^d$, AND $\mathbb{T}^d$

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ABSTRACT. The truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$ will be considered. The truncated matrix-valued K-moment problem on $\mathbb{R}^d$ requires necessary and sufficient conditions for a multisequence of Hermitian matrices $\{S_\gamma\}_{\gamma\in\Gamma}$ (where Γ is a finite subset of $\mathbb{N}_0^d$) to be the corresponding moments of a positive Hermitian matrix-valued Borel measure σ, and also the support of σ must be contained in some given non-empty set $K \subseteq \mathbb{R}^d$, i.e.,

$$S_\gamma = \int_{\mathbb{R}^d} \xi^\gamma d\sigma(\xi), \quad \text{for all } \gamma \in \Gamma, \tag{0.1}$$

and

$$\text{supp } \sigma \subseteq K. \tag{0.2}$$

Given a non-empty set $K \subseteq \mathbb{R}^d$ and a finite multisequence, indexed by a certain family of finite subsets of $\mathbb{N}_0^d$, of Hermitian matrices we obtain necessary and sufficient conditions for the existence of a minimal finitely atomic measure which satisfies (0.1) and (0.2). In particular, our result can handle the case when $\Gamma = \{\gamma \in \mathbb{N}_0^d \colon 0 \le |\gamma| \le 2n+1\}$. We will also discuss a similar result in the multivariable complex and polytorus setting.

1. INTRODUCTION

Recall from [16], [4] the odd-case of the *truncated Stieltjes moment problem on* $\mathbb{R}$. Given the real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma = \{0,\dots,2m+1\}$, there exists a positive Borel measure σ on $\mathbb{R}$, with convergent moments (i.e., $\int_{\mathbb{R}} \xi^n d\sigma(\xi)$ exists for all $n = 0,1,\dots$) so that

$$s_\gamma = \int_{\mathbb{R}} \xi^\gamma d\sigma(\xi), \quad \text{for all } \gamma \in \Gamma,$$

and

$$\text{supp } \sigma \subseteq [0,\infty),$$

if and only if

$$\Phi := \begin{pmatrix} s_0 & \cdots & s_m \\ \vdots & \iddots & \vdots \\ s_m & \cdots & s_{2m} \end{pmatrix} \text{ and } \Phi_1 := \begin{pmatrix} s_1 & \cdots & s_{m+1} \\ \vdots & \iddots & \vdots \\ s_{m+1} & \cdots & s_{2m+1} \end{pmatrix}$$

are positive semidefinite (denoted by $\Phi, \Phi_1 \ge 0$) and Ran $\Phi_1 \subseteq$ Ran Φ. If Φ is positive definite (denoted by $\Phi > 0$), then it is easy to see that $\Theta_1 := \Phi^{-1}\Phi_1 = \Phi^{-\frac{1}{2}}(\Phi^{-\frac{1}{2}}\Phi_1\Phi^{\frac{1}{2}})\Phi^{-\frac{1}{2}}$ has non-negative eigenvalues. We will see that supp σ can be chosen to be the set of eigenvalues of Θ_1. Furthermore, this notion will generalize to the matrix-valued multivariable case (see Remark 2.10). The *truncated Hamburger*

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moment problem (see [1], [4]) is as follows. Given a real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma = \{0, \dots, m\}$, we wish to find necesary and sufficient conditions on $\{s_\gamma\}_{\gamma\in\Gamma}$ for the existence of a positive Borel measure σ on $\mathbb{R}$, with convergent moments, so that

$$s_\gamma = \int_{\mathbb{R}} \xi^\gamma d\sigma(\xi), \quad \text{for all } \gamma \in \Gamma.$$

The truncated Stieltjes and truncated Hamburger moment problems are particular cases of the *truncated K-moment problem on* $\mathbb{R}$ (see [4]), when $K = [0, \infty)$ and $K = \mathbb{R}$, respectively. Given a non-empty set $K \subseteq \mathbb{R}$ and a real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma = \{0, \dots, m\}$, we wish to determine necessary and sufficient conditions on $\{s_\gamma\}_{\gamma\in\Gamma}$ so that there is a positive Borel measure σ on $\mathbb{R}$, with convergent moments, so that

$$s_\gamma = \int_{\mathbb{R}} \xi^\gamma d\sigma(\xi), \quad \text{for all } \gamma \in \Gamma,$$

and

$$\text{supp } \sigma \subseteq K.$$

The *full Hamburger moment problem on* $\mathbb{R}$ (see [14], [1]) can be posed similarly. Given an infinite real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma = \{0, 1, \dots\}$, we wish to determine necessary and sufficient conditions on $\{s_\gamma\}_{\gamma\in\Gamma}$ so that there exists a positive Borel measure σ on $\mathbb{R}$ which satisfies

$$s_\gamma = \int_{\mathbb{R}} \xi^\gamma d\sigma(\xi), \quad \text{for all } \gamma \in \Gamma.$$

The *full Stieltjes moment problem* has the additional requirement that supp $\sigma \subseteq [0, \infty)$. Similarly, the full Hamburger and full Stieltjes moment problems are particular cases of the *full K-moment problem on* $\mathbb{R}$ when $K = \mathbb{R}$ or $K = [0, \infty)$, respectively. The full K-moment problem on $\mathbb{R}$ will not be of interest in this paper. Note that a solution can be found in [19].

We will now introduce frequently used definitions and notation. Frequently used sets are $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{C}$, which stand for the sets of positive integers, nonnegative integers, integers, complex numbers of modulus one, real numbers, and complex numbers, respectively. In addition, given a set E, we will let

$$E^d = \{(m_1, \dots, m_d) \colon m_j \in E,\ 1 \le j \le d\}.$$

In particular, when $E = \mathbb{N}_0$, let

$$\mathrm{e}_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^d,$$

where 1 is in the j-th entry, $1 \le j \le d$. We will let $0_d = (0, \dots, 0) \in \mathbb{N}_0^d$. Next, $\mathcal{H}_p \subset \mathbb{C}^{p\times p}$ will denote the real Hilbert space of $p \times p$ Hermitian matrices. Given $m = (m_1, \dots, m_d) \in \mathbb{N}_0^d$, we define the *length* of m by $|m| = m_1 + \cdots + m_d$. If $\xi = (x_1, \dots, \xi_d) \in \mathbb{R}^d$ and $m = (m_1, \dots, m_d) \in \mathbb{N}_0^d$, then let $\xi^m = \xi_1^{m_1} \cdots \xi_d^{m_d}$. We say that a finite set $\Gamma \subset \mathbb{N}_0^d$ is a *lattice set* when, for all $\gamma \in \Gamma$, there exist $\gamma_1 = 0_d, \gamma_2, \dots, \gamma_k \in \Gamma$ and $j_1, \dots, j_k \in \{1, \dots, d\}$ so that

$$\gamma_2 = \gamma_1 + \mathrm{e}_{j_1},$$

$$\vdots$$

$$\gamma = \gamma_k + \mathrm{e}_{j_k},$$

where $k = |\gamma|$. For example, in $\mathbb{N}_0^2$, $\{(0,0),(0,1),(1,1),(1,2)\}$ is a lattice set while $\{(0,0),(1,1),(2,2)\}$ is not. Next, we say a finite set $\Gamma \subset \mathbb{N}_0^d$ is *lower inclusive* when for any $\gamma = (g_1, \ldots, g_d)$ and $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ with $m_j \leq g_j$, $1 \leq j \leq d$, we have that $m \in \Gamma$. For example, $\{(0,0),(0,1),(1,0)\}$ is a lower inclusive subset of $\mathbb{N}_0^2$ while $\{(0,0),(0,1),(1,1)\}$ is not. Note that every lower inclusive set is a lattice set, but the converse is not true. E.g., $\{(0,0),(0,1),(1,1),(1,2)\}$ is a lattice set but is not a lower inclusive set. Throughout this paper, we will assume that any lattice set or lower inclusive $\Lambda \subset \mathbb{N}_0^d$ possesses a total ordering, and also that K is a non-empty set.

Let $\Gamma \subset \mathbb{N}_0^d$ be a lattice set, $\{S_\gamma\}_{\gamma\in\Gamma}$ be an $\mathcal{H}_p$-valued sequence, and $K \subseteq \mathbb{R}^d$. The *truncated matrix-valued K-moment problem on* $\mathbb{R}^d$ consists of determining whether or not there exists a positive $\mathcal{H}_p$-valued Borel measure σ so that $\int_{\mathbb{R}^d} \xi^m d\sigma(\xi)$ exists for all $m \in \mathbb{N}_0^d$,

$$S_\gamma = \int_{\mathbb{R}^d} \xi^\gamma d\sigma(\xi), \ \gamma \in \Gamma, \tag{1.1}$$

and

$$\operatorname{supp} \sigma \subseteq K. \tag{1.2}$$

When a positive $\mathcal{H}_p$-valued measure, with convergent moments, has been shown to exist which satisfies (1.1), we say that $\{S_\gamma\}_{\gamma\in\Gamma}$ has a *representing measure*. In addition, when a positive $\mathcal{H}_p$-valued measure has been shown to exist which satisfies (1.1) and (1.2), we say that $\{S_\gamma\}_{\gamma\in\Gamma}$ has a *K-representing measure*. Given $w \in \mathbb{R}^d$, we will let δ_w denote the *Dirac mass* with respect to w, i.e.,

$$\delta_w(\Delta) = \begin{cases} 1 & \text{if } w \in \Delta, \\ 0 & \text{if } w \notin \Delta, \end{cases}$$

where Δ is a Borel set in $\mathbb{R}^d$ (denoted by $\Delta \in \mathcal{B}(\mathbb{R}^d)$). A measure of the form

$$\sigma = \sum_{q=1}^{k} T_q \delta_{w_q}, \tag{1.3}$$

where $T_1, \ldots, T_k \geq 0$ and where $w_1, \ldots, w_k$ are distinct points in $\mathbb{R}^d$, is called *finitely-atomic*. If we write $T_q = c_1^{(q)} c_1^{(q)*} + \cdots + c_{r_q}^{(q)} c_{r_q}^{(q)*}$, where $c_j^{(q)} \in \mathbb{C}^p$, $1 \leq j \leq r_q$, then $\left(c_j^{(q)} c_j^{(q)*}, w_q\right)$ is called an *atom* and $\{T_q\}_{q=1}^k$ are called *densities* of σ. If we let $l = \sum_{q=1}^k \operatorname{rank} T_q$, then we say σ is *l-atomic*.

We will now elaborate on the existing literature with regard to scalar-valued K-moment problems on $\mathbb{R}^d$ and $\mathbb{C}^d$ (see section 3). The case when $K \subseteq \mathbb{R}^d$, $p = 1$, $d \geq 1$, and $\Gamma = \mathbb{N}_0^d$, known as the *full K-moment problem on* $\mathbb{R}^d$, has been solved in [15]. In particular, when K is compact and semialgebriac, a solution can be found in [21] which relies on real algebraic geometry. Subsequently, [18] improved upon the approach in [21]. The case when K is a closed subset of $\mathbb{R}^d$, $p = 1$, $d \geq 1$, $\Gamma = \{\gamma \in \mathbb{N}_0^d \colon |\gamma| \leq m\}$, which we will call the *truncated total degree K-moment problem on* $\mathbb{R}^d$, has been considered in [9]. In [23], a link between the full and truncated total degree K-moment problem on $\mathbb{R}^d$ and $\mathbb{C}^d$ was established. We remark that [8] and [9] have results when m is even. However, general results, independent of flat extension theory, for the case when m is odd, do not exist to the best of the authors' knowledge. In addition, [9] provides an explicit link between the truncated total degree K-moment problem on $\mathbb{C}^d$ and the truncated total degree K-moment problem on $\mathbb{R}^{2d}$.

Next, let us explore the existing literature for matrix-valued K-moment problems on $\mathbb{R}$ and matrix-valued moments problems on $\mathbb{T}^d$ (see section 4). The case when $K = \mathbb{R}$, $p \geq 1$, and $d = 1$ was considered in [10] and [3]. In [2], when $p \geq 1$ and $d = 1, 2$, the cases of $K = \mathbb{R}^d$ and $K = \mathbb{T}^d$ were explored. It should be noted that in the case of $K = \mathbb{T}^d$, the measure which is sought is a *Bernstein-Szegő measure* (see [13] and [2]). Matrix factorization techniques were used to achieve a solution in both cases in [2]. We will use matrix factorizations in this paper to analyze the matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$.

This paper is organized as follows. In section 2, we will demonstrate a solution to the truncated matrix-valued K-moment problem on $\mathbb{R}^d$ for a particular family of lattice sets in $\mathbb{N}_0^d$. We establish conditions, which can easily be checked, for the Hermitian matrices to be the moments of a Hermitian matrix-valued measure with support in $K \subseteq \mathbb{R}^d$, which is minimal (see Remark 2.5). Minimal representing measures supported on prescribed sets arise when one wants to construct a minimal cubature formula. For a moment matrix approach see [12], [11]. For a different approach see [17]. Moreover, we establish that these conditions are necessary when an appropriate indexing set is chosen. We will also show that we can use our solution to prove a result established by Curto and Fialkow via their flat extension theory (see [5], [6], and [7]). In sections 3 and 4, we will develop analogous sufficient conditions for a given $\mathbb{C}^{p\times p}$-valued sequence to be the corresponding moments of a positive $\mathcal{H}_p$-valued measure with support in $K \subseteq \mathbb{C}^d$ or $\mathbb{T}^d$, respectively. However, unlike the real case, we only have the necessity of these conditions when $d = 1$ (in the case when $K \subseteq \mathbb{C}^d$) and $d = 1, 2$ (in the case when $K \subseteq \mathbb{T}^d$). Finally, we will see that our solution is a very natural tool with respect to the odd case of the truncated K-moment problem on $\mathbb{R}^d$ and $\mathbb{C}^d$ considered in [8] and [9], when a minimal representing measure is desired.

2. The truncated matrix-valued K-moment problem on $\mathbb{R}^d$

Before we can consider the truncated matrix-valued K-moment problem on $\mathbb{R}^d$, we must introduce some matrix-valued measure and integration theory. A function $\sigma\colon \mathcal{B}(\mathbb{R}^d) \to \mathcal{H}_p$, is called a *positive $\mathcal{H}_p$-valued measure* on $\mathbb{R}^d$ if for each $y \in \mathbb{C}^p$, $\langle \sigma(\Delta)y, y\rangle$ defines a positive measure on $\mathbb{R}^d$ for all sets $\Delta \in \mathcal{B}(\mathbb{R}^d)$. For a measurable function $f\colon \mathbb{R}^d \to \mathbb{C}$, its integral $\int f(\xi)d\sigma(\xi) := \int_{\mathbb{R}^d} f(\xi_1, \ldots, \xi_d)d\sigma(\xi_1, \ldots, \xi_d) \in \mathcal{H}_p$ is defined by the formula

$$\left\langle \int f(\xi)d\sigma(\xi)y, \tilde{y} \right\rangle = \int f(\xi)\langle d\sigma(\xi)y, \tilde{y}\rangle,$$

for all $y, \tilde{y} \in \mathbb{C}^p$, provided all integrals on the right-hand side converge.

The power moments of a positive $\mathcal{H}_p$-valued measure σ on $\mathbb{R}^d$ are defined by the formula,

$$\hat{\sigma}(m) = \int \xi^m d\sigma(\xi), \quad m \in \mathbb{N}_0^d, \tag{2.1}$$

provided the integrals converge. The truncated matrix-valued K-moment problem on $\mathbb{R}^d$ can now be posed. Given $K \subseteq \mathbb{R}^d$ and the $\mathcal{H}_p$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma \subset \mathbb{N}_0^d$ is a lattice set, we look for a positive $\mathcal{H}_p$-valued measure σ on $\mathbb{R}^d$ so that $\hat{\sigma}(m)$ exists for all $m \in \mathbb{N}_0^d$,

$$\hat{\sigma}(\gamma) = S_\gamma, \text{ for all } \gamma \in \Gamma, \tag{2.2}$$

and

$$\operatorname{supp} \sigma \subseteq K. \tag{2.3}$$

Given a set $\Lambda \subset \mathbb{N}_0^d$, we define $\Lambda + \Lambda = \{\lambda + \mu \colon \lambda, \mu \in \Lambda\}$ and $\Lambda + \Lambda + \mathrm{e}_j = \{\lambda + \mu + \mathrm{e}_j : \lambda, \mu \in \Lambda\}$, $1 \leq j \leq d$. We put

$$\Gamma = (\Lambda \cup \Lambda) \cup (\Lambda + \Lambda + \mathrm{e}_1) \cup \cdots \cup (\Lambda + \Lambda + \mathrm{e}_d), \tag{2.4}$$

which will serve as an indexing set for the given Hermitian matrices $\{S_\gamma\}_{\gamma \in \Gamma}$.

Remark 2.1. Let $\Lambda = \{\lambda \in \mathbb{N}_0^d \colon |\lambda| \leq n\}$; then Γ, as defined in (2.4), is $\{\gamma \in \mathbb{N}_0^d \colon |\gamma| \leq 2n+1\}$. Given $\Gamma = \{\gamma \in \mathbb{N}_0^d \colon |\gamma| \leq 2n\}$, there does not exist a lattice set $\Lambda \subset \mathbb{N}_0^d$ so that (2.4) yields Γ.

We will now introduce the matrices $\Phi, \Phi_1, \ldots, \Phi_d$. Index the rows and columns of a matrix Φ, Φ_j, $1 \leq j \leq d$, by Λ. For Φ, let the entry in the row indexed by λ and the column indexed by μ be given by $S_{\lambda+\mu}$. That is,

$$\Phi = (S_{\lambda+\mu})_{\lambda,\mu \in \Lambda}.$$

For Φ_j, let the entry in the row indexed by λ and the column indexed by μ be given by $S_{\lambda+\mu+\mathrm{e}_j}$. That is,

$$\Phi_j = (S_{\lambda+\mu+\mathrm{e}_j})_{\lambda,\mu \in \Lambda}, \ 1 \leq j \leq d.$$

Let us consider the following example, which illustrates how Φ, $\Phi_1, \ldots, \Phi_d$ are constructed with respect to a particular lattice set $\Lambda \subset \mathbb{N}_0^2$.

Example 2.2. Let $\Lambda = \{(0,0),(0,1),(1,0)\}$. Then

$$\Lambda + \Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0)\},$$

$$\Lambda + \Lambda + \mathrm{e}_1 = \{(1,0),(1,1),(1,2),(2,0),(2,2),(3,0)\},$$

and

$$\Lambda + \Lambda + \mathrm{e}_2 = \{(0,1),(0,2),(0,3),(1,1),(1,3),(2,1)\}.$$

Hence

$$\Gamma = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0)\},$$

and we get the following matrices:

$$\Phi = \begin{pmatrix} S_{00} & S_{01} & S_{10} \\ S_{01} & S_{02} & S_{11} \\ S_{10} & S_{11} & S_{20} \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} S_{10} & S_{11} & S_{20} \\ S_{11} & S_{12} & S_{21} \\ S_{20} & S_{21} & S_{30} \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} S_{01} & S_{02} & S_{11} \\ S_{02} & S_{03} & S_{12} \\ S_{11} & S_{12} & S_{21} \end{pmatrix}.$$

Let us catalog some basic necessary conditions for given data to have a representing measure.

Lemma 2.3. *Let* $L = \{l_1, \dots, l_k\} \subset \mathbb{N}_0^d$ *and suppose the* $\mathcal{H}_p$*-valued sequence* $\{S_\gamma\}_{\gamma\in\Gamma}$*, where* $\Gamma = L + L$*, is given. Let* $\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in L}$*. If* $\{S_\gamma\}_{\gamma\in\Gamma}$ *has a representing measure* σ*, then* $\Phi \geq 0$*.*

Proof. For any vector $y = \text{col}(y_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^{kp}$, we have

$$\int \langle d\sigma(\xi) \sum_{\lambda\in L} y_\lambda \xi^\lambda, \sum_{\mu\in L} y_\mu \xi^\mu \rangle \geq 0. \tag{2.5}$$

If we use the sesquilinearity of $\langle\cdot,\cdot\rangle$, then (2.5) becomes

$$\sum_{\lambda,\mu\in L} \int \xi^{\lambda+\mu} \langle d\nu(\xi) y_\lambda, y_\mu \rangle \geq 0.$$

Use the fact that $S_\gamma = \int \xi^\gamma d\sigma(\xi)$, for all $\gamma \in \Gamma$, so then (2.5) becomes

$$\sum_{\lambda,\mu\in L} \langle S_{\lambda+\mu} y_\lambda, y_\mu \rangle \geq 0,$$

i.e., $\Phi \geq 0$. □

Lemma 2.4. *Let* L*,* $\{S_\gamma\}_{\gamma\in\Gamma}$ *and* Φ *be as in Lemma 2.3. If* $\{S_\gamma\}_{\gamma\in\Gamma}$ *has a finitely atomic representing measure* $\sigma = \sum_{q=1}^k T_q \delta_{w_q}$*, where* $T_1, \dots, T_k \geq 0$ *and* $w_1, \dots, w_k$ *are distinct points in* $\mathbb{R}^d$*, then*

$$\text{rank } \Phi \leq \sum_{q=1}^k \text{rank } T_q. \tag{2.6}$$

Proof. For $m \in \mathbb{N}_0^d$, note that

$$\begin{aligned} S_m &= \int \xi^m d\sigma(\xi) \\ &= \sum_{q=1}^k T_q w_q^m. \end{aligned}$$

One can check that

$$\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in L} = (V \otimes I_p)^T R (V \otimes I_p), \tag{2.7}$$

where $V = \begin{pmatrix} w_1^{l_1} & \dots & w_1^{l_k} \\ \vdots & & \vdots \\ w_k^{l_1} & \dots & w_k^{l_k} \end{pmatrix}$, $\otimes$ denotes the Kronecker product, I_p is the identity matrix in $\mathbb{C}^{p\times p}$, and $R = T_1 \oplus \cdots \oplus T_k := \begin{pmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_k \end{pmatrix} \in \mathbb{C}^{kp\times kp}$. It is now clear that $\text{rank } \Phi \leq \text{rank } R$, whence we arrive at (2.6). □

Remark 2.5. In Lemma 2.3, when a representing measure σ is finitely atomic, (2.7) allows us to give a different proof of Lemma 2.3. As a consequence of Lemma 2.4, given Φ and σ as in Lemma 2.4 we say σ is *minimal* when $\text{rank } \Phi = \sum_{q=1}^k \text{rank } T_q$.

Let $W_1, \ldots, W_d \in \mathbb{C}^{n\times n}$. We say that $W_1, \ldots, W_d$ *commute with respect to the subspace* $\mathfrak{M} \subseteq \mathbb{C}^n$ when $\tilde{W}_1, \ldots \tilde{W}_d$ commute, where $\tilde{W}_j$ is given by

$$W_j = \begin{pmatrix} \tilde{W}_j & 0 \\ * & * \end{pmatrix} : \begin{matrix} \mathfrak{M} \\ \bigoplus \\ \mathfrak{M}^\perp \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \bigoplus \\ \mathfrak{M}^\perp \end{matrix}, \quad 1 \le j \le d.$$

We say $W_1, \ldots, W_d \in \mathbb{C}^{n\times n}$ have the *K-inclusive eigenvalue property with respect to the subspace* $\mathfrak{M} \subseteq \mathbb{C}^n$, with $\dim \mathfrak{M} = k$, if the following conditions are satisfied:

(1) W_j^* is $\mathfrak{M}$-invariant, i.e.,

$$W_j = \begin{pmatrix} \tilde{W}_j & 0 \\ * & * \end{pmatrix} : \begin{matrix} \mathfrak{M} \\ \bigoplus \\ \mathfrak{M}^\perp \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \bigoplus \\ \mathfrak{M}^\perp \end{matrix}, \quad 1 \le j \le d.$$

(2) There exists an invertible S so that

$$S^{-1}\tilde{W}_j S = \operatorname{diag}(x_j^{(1)}, \ldots, x_j^{(k)}), 1 \le j \le d.$$

(3) $(x_1^{(q)}, \ldots, x_d^{(q)}) \in K, 1 \le q \le k$.

Remark 2.6. Note that condition (2) above implies that $W_1, \ldots, W_d$ commute with respect to $\mathfrak{M}$.

Example 2.7. Let $K = \{(1,0),(0,1)\} \subset \mathbb{R}^2$. Then $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ have the K-inclusive eigenvalue property with respect to $\mathfrak{M} = \mathbb{C}^2$. Indeed,

$$W_1 = \tilde{W}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$W_2 = \tilde{W}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So then $\{(1,0),(0,1)\} = K$.

We will now formulate our main result, which provides necessary and sufficient conditions on a set of given Hermitian matrices, indexed as in (2.4), to admit a minimal K-representing measure.

Theorem 2.8. *Let $K \subseteq \mathbb{R}^d$, $\Lambda \subset \mathbb{N}_0^d$ be a lattice set and suppose the $\mathcal{H}_p$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma = (\Lambda+\Lambda)\cup(\Lambda+\Lambda+\mathrm{e}_1)\cup\cdots\cup(\Lambda+\Lambda+\mathrm{e}_d)$, is given. Let $\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in\Lambda}$, $\Phi_j = (S_{\lambda+\mu+\mathrm{e}_j})_{\lambda,\mu\in\Lambda}$, $1 \le j \le d$. There exists a solution to the truncated matrix-valued K-moment problem on $\mathbb{R}^d$, i.e., there exists a measure so that $\hat{\sigma}(m)$ exists for all $m \in \mathbb{N}_0^d$, and* (2.2) *and* (2.3) *hold if the following conditions are satisfied:*

(1) $\Phi \ge 0$.

(2) There exist $\Theta_1, \ldots, \Theta_d$ so that
- *(i)* $\Phi\Theta_j = \Phi_j$, $1 \le j \le d$;
- *(ii) $\Theta_1, \ldots, \Theta_d$ have the K-inclusive eigenvalue property with respect to $\mathfrak{M} = \operatorname{Ran}\Phi$.*

In that case, we can find σ of the following form:

$$\sigma = \sum_{q=1}^{k} T_q \delta_{w_q}, \tag{2.8}$$

where $T_1, \dots, T_k \geq 0$ and $w_1, \dots, w_k$ are different points in K. Moreover, we have that $\sum_{q=1}^{k} \operatorname{rank} T_q = \operatorname{rank} \Phi$, i.e., σ is minimal.

Conversely, if σ is of the form (2.8)*, where $T_1, \dots, T_k \geq 0$ and $w_1, \dots, w_k$ are different points in K, there exists a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0^d$ with k points so that $\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in\tilde{\Lambda}}$, $\Phi_j = (S_{\lambda+\mu+\mathrm{e}_j})_{\lambda,\mu\in\tilde{\Lambda}}$, $1 \leq j \leq d$, satisfy conditions* (1) *and* (2)*, where $S_m = \hat{\sigma}(m)$, $m \in \mathbb{N}_0^d$. Moreover, we get that $\sum_{q=1}^{k} \operatorname{rank} T_q = \operatorname{rank} \Phi$.*

Remark 2.9. Since the measure σ constructed in Theorem 2.8 has finite support, all moments $\hat{\sigma}(m)$, $m \in \mathbb{N}_0^d$, automatically exist.

Remark 2.10. When $K = \mathbb{R}^d$ in Theorem 2.8, condition (2) reduces to checking that $\Theta_1, \dots, \Theta_d$ commutes with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$. When $K = [0, d)^\infty$, condition (2) reduces to checking that $\tilde{\Theta}_1, \dots, \tilde{\Theta}_d$ have non-negative eigenvalues, where $\tilde{\Theta}_j$ is given by

$$\Theta_j = \begin{pmatrix} \tilde{\Theta}_j & 0 \\ * & * \end{pmatrix} : \begin{matrix} \operatorname{Ran} \Phi \\ \bigoplus \\ \operatorname{Ker} \Phi \end{matrix} \to \begin{matrix} \operatorname{Ran} \Phi \\ \bigoplus \\ \operatorname{Ker} \Phi \end{matrix}, 1 \leq j \leq d.$$

Note that the choice of Λ in the converse part of Theorem 2.8 indeed depends on the given measure, unlike the one-variable case, as the following example will show.

Example 2.11. Let $\sigma = \rho_1 \delta_{(1,0)} + \rho_2 \delta_{(2,0)}$, where ρ_1 and ρ_2 are positive real numbers. If we choose the lower inclusive set $\Lambda = \{(0,0),(0,1)\}$, then

$$\Phi = \begin{pmatrix} \rho_1 + \rho_2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} \rho_1 + 2\rho_2 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By this choice of Λ we get that $\operatorname{rank} \Phi = 1 < 2 = \operatorname{rank} \rho_1 + \operatorname{rank} \rho_2$. The correct choice for Λ is $\{(0,0),(1,0)\}$.

In order to prove Theorem 2.8, we must introduce *multivariable Vandermonde* matrices. Given a sequence of distinct points $w_1, \dots, w_k \in \mathbb{R}^d$ and the lattice set $\Lambda = \{\lambda_1, \dots, \lambda_k\} \subset \mathbb{N}_0^d$, we define

$$V(w_1, \dots, w_m; \Lambda) = \begin{pmatrix} w_1^{\lambda_1} & \cdots & w_1^{\lambda_k} \\ \vdots & & \vdots \\ w_k^{\lambda_1} & \cdots & w_k^{\lambda_k} \end{pmatrix}.$$

Consider the following example.

Example 2.12. If $w_1=(1,1)$, $w_2=(2,1)$, $w_3=(3,1)$ and $\Lambda=\{(0,0),(0,1),(1,0)\}$, then

$$V(w_1,w_2,w_3;\Lambda)=\begin{pmatrix}1&1&1\\1&1&2\\1&1&3\end{pmatrix}.$$

Note that Example 4.7 illustrates that unlike square one-variable Vandermonde matrices, distinctness of points in two or more variables is not enough to guarantee invertibility. A natural question is the following. Given distinct points $w_1,\dots,w_k\in\mathbb{R}^d$, can one construct a lattice set $\Lambda\subset\mathbb{N}_0^d$, with card $\Lambda=k$, so that $V(w_1,\dots,w_k;\Lambda)$ is invertible? A construction of T. Sauer, given in [20], provides an answer in the affirmative.

In [20], the minimal degree interpolation problem is explored, which is motivated by the fact that interpolating polynomials with small total degree are easier to store and compute. First, let us introduce some preliminary notation and definitions. Let Π_n^d be the space of polynomials in $\mathbb{R}^d$ with total degree less than or equal to n. Given a set of distinct points $W=\{w_1,\dots,w_k\}$ in $\mathbb{R}^d$, the Lagrange interpolation problem with respect to W is *poised* in the subspace $\mathcal{P}(W)\subseteq\Pi_n^d$ if given any $f\colon\mathbb{R}^d\to\mathbb{R}$, then there exists a unique polynomial $p:=L_{\mathcal{P}(W)}(f)\in\mathcal{P}(W)$ such that

$$p(x_q)=f(x_q),\quad q=1,\dots,k.$$

We use the notation $\mathcal{P}(W)$ to indicate that the subspace depends on W. The uniqueness requirement on p will occur if and only if dim $\mathcal{P}=k$. The minimal degree interpolation problem entails finding a subspace $\mathcal{P}(W)\subset\Pi_n^d$ with n as small as possible so that the Lagrange interpolation problem, with respect to the set of distinct points W, is poised. Moreover, we require that $\mathcal{P}(W)$ is *degree reducing*; i.e., for $m\le n$, whenever $q\in\Pi_m^d$ we have that $L_{\mathcal{P}(W)}(q)\in\Pi_m^d$.

Given a distinct set of points $W=\{w_1,\dots,w_k\}$ in $\mathbb{R}^d$, [20] provides an algorithm (see Algorithm 1) that generates a unique minimal degree interpolation subspace $\mathcal{P}^*(W)\subset\Pi_n^d$, upon imposing further requirements to the minimal degree interpolation problem. The set of indices for the monomials generated by the algorithm, say $\Lambda=\{\lambda_1,\dots,\lambda_k\}\subset\mathbb{N}_0^d$, corresponds to a lower inclusive set, which is called *lower* in [20], with k points. Given a function $f\colon\mathbb{R}^d\to\mathbb{R}$, the coefficients $c_1,\dots,c_k$ of the unique interpolating polynomial $p\in\mathcal{P}^*(W)$ are given by the following equation:

$$\begin{pmatrix}w_1^{\lambda_1}&\cdots&w_1^{\lambda_k}\\\vdots&&\vdots\\w_k^{\lambda_1}&\cdots&w_k^{\lambda_k}\end{pmatrix}\begin{pmatrix}y_1\\\vdots\\y_k\end{pmatrix}=\begin{pmatrix}f(w_1)\\\vdots\\f(w_k)\end{pmatrix}. \tag{2.9}$$

Realize that $\begin{pmatrix}w_1^{\lambda_1}&\cdots&w_1^{\lambda_k}\\\vdots&&\vdots\\w_k^{\lambda_1}&\cdots&w_k^{\lambda_k}\end{pmatrix}$ is the invertible multivariable Vandermonde matrix $V(w_1,\dots,w_k;\Lambda)$. Thus the construction of $\mathcal{P}^*(W)$ may be viewed as a way of producing a lower inclusive set so that the multivariable Vandermonde matrix $V(w_1,\dots,w_k;\Lambda)$ is invertible. We will formulate this observation as follows.

Theorem 2.13 ([20]). *Given distinct points $w_1,\dots,w_k$ in $\mathbb{R}^d$, there exists a lower inclusive set $\Lambda\subset\mathbb{N}_0^d$, with* card $\Lambda=k$, *so that $V(w_1,\dots,w_k;\Lambda)$ is invertible.*

We will now prove matrix factorizations which will be useful when proving Theorem 2.8.

Lemma 2.14. *Let $A \geq 0$ and $B_j = B_j^*$ be $p \times p$ matrices so that there exist matrices $W_1, \ldots, W_d$ which commute with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$ and satisfy $AW_j = B_j$, $1 \leq j \leq d$. Put $k = \operatorname{rank} A$. Then there exist $k \times k$ real diagonal matrices $D_1, \ldots, D_d$ and an injective $p \times k$ matrix C so that*

$$A = CC^* \text{ and } B_j = CD_jC^*, \ 1 \leq j \leq d. \tag{2.10}$$

Proof. Decompose $\mathbb{C}^p = \operatorname{Ran} A \oplus \operatorname{Ker} A$. Then with respect to this decomposition we have

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix}, \ W_j = \begin{pmatrix} \tilde{W}_j & 0 \\ * & * \end{pmatrix} \text{ and } B_j = \begin{pmatrix} \tilde{B}_j & 0 \\ 0 & 0 \end{pmatrix},$$

where we used the fact that $AW_j = B_j$, $1 \leq j \leq d$. Note also that $\tilde{W}_1, \ldots, \tilde{W}_d$ commute. Realize $AW_j = B_j$ yields $\tilde{A}\tilde{W}_j = \tilde{B}_j$. Since $\tilde{A}$ is invertible, we get $\tilde{W}_j = \tilde{A}^{-1}\tilde{B}_j$, $1 \leq j \leq d$. Hence $\tilde{A}^{-\frac{1}{2}}\tilde{B}_j\tilde{A}^{-\frac{1}{2}} = \tilde{A}^{\frac{1}{2}}\tilde{W}_j\tilde{A}^{-\frac{1}{2}}$ and $\{\tilde{A}^{-\frac{1}{2}}\tilde{B}_j\tilde{A}^{-\frac{1}{2}}\}_{j=1}^d$ is a commuting family of Hermitian matrices, where we used the fact that $\tilde{W}_1, \ldots, \tilde{W}_d$ commute. So there must exist a unitary U and real diagonal matrices $D_1, \ldots, D_d$ so that

$$\tilde{A}^{\frac{1}{2}}\tilde{W}_j\tilde{A}^{-\frac{1}{2}} = UD_jU^*, \ 1 \leq j \leq d. \tag{2.11}$$

Put $C = \begin{pmatrix} \tilde{A}^{\frac{1}{2}} \\ 0 \end{pmatrix} U$, and then (2.10) holds. □

Theorem 2.15. *Let $\Lambda \subset \mathbb{N}_0^d$, with $\operatorname{card} \Lambda = k$, be a lattice set and $\Gamma = (\Lambda + \Lambda) \cup (\Lambda + \Lambda + e_1) \cup \cdots \cup (\Lambda + \Lambda + e_d)$. Suppose the $\mathcal{H}_p$-valued sequence $\{S_\gamma\}_{\gamma \in \Gamma}$ is given. Set $\Phi = (S_{\lambda+\mu})_{\lambda,\mu \in \Lambda}$, $\Phi_j = (S_{\lambda+\mu+e_j})_{\lambda,\mu \in \Lambda}$, $1 \leq j \leq d$, and $r = \operatorname{rank} \Phi$. Suppose that $\Phi \geq 0$ and there is a family of $kp \times kp$ matrices $\{\Theta_j\}_{j=1}^d$, which commute with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$, so that $\Phi\Theta_j = \Phi_j$, $1 \leq j \leq d$. Then there exists a $kp \times r$ matrix C_0 and real $r \times r$ diagonal matrices $D_1, \ldots, D_d$ so that*

$$S_\gamma = C_0(D_1^{g_1} \cdots D_d^{g_d})C_0^*, \ \text{ for all } \gamma = (g_1, \ldots, g_d) \in \Gamma. \tag{2.12}$$

Proof. Consider $A = \Phi$, $W_j = \Theta_j$ and $B_j = \Phi_j$, $1 \leq j \leq d$. Then $A \geq 0, B_j = B_j^*$, and $W_1, \ldots, W_d$ are matrices so that $\tilde{W}_1, \ldots, \tilde{W}_d$ commute and satisfy $AW_j = B_j$, $1 \leq j \leq d$. We can apply Lemma 2.14 to obtain an injective matrix C and real diagonal matrices $D_1, \ldots, D_d$ so that

$$\Phi = CC^* \text{ and } \Phi_j = CD_jC^*, \ 1 \leq j \leq d.$$

Write $C = \operatorname{col}(C_\lambda)_{\lambda \in \Lambda}$, and when λ, μ, and $\mu + e_j \in \Lambda$, we get

$$S_{\lambda+\mu+e_j} = C_\lambda C_{\mu+e_j}^* = C_\lambda D_j C_\mu^*. \tag{2.13}$$

Notice that (2.13) implies

$$C(C_{\mu+e_j}^* - D_jC_\mu^*) = 0. \tag{2.14}$$

Since C is injective, (2.14) yields

$$C_{\mu+e_j} = C_\mu D_j, \tag{2.15}$$

whenever $\mu, \mu + e_j \in \Lambda$.

Consider $\lambda = (l_1, \ldots, l_d) \in \Lambda$ with $|\lambda| = s$. Since Λ is a lattice set, there must exist $\lambda_1 = 0_d, \lambda_2, \ldots, \lambda_s \in \Lambda$ and $j_1, \ldots, j_s \in \{1, \ldots, d\}$ so that $\lambda_1 = 0_d + \mathrm{e}_{j_1}, \ldots, \lambda = \lambda_s + \mathrm{e}_{j_s}$. Choose $\mu = \lambda_s = \lambda - \mathrm{e}_{j_s}$ so that (2.15) gives

$$C_{\mu + \mathrm{e}_{j_s}} = C_\mu D_j.$$

So then we get $C_\lambda = C_{\lambda - \mathrm{e}_{j_s}} D_{j_s}$. Continuing this way we arrive at

$$C_\lambda = C_0 D_1^{l_1} \cdots D_d^{l_d}. \tag{2.16}$$

But then $\Phi = CC^*$ and $\Phi_j = CD_jC^*$, $1 \le j \le d$, give (2.12). □

Remark 2.16. In the proof of Theorem 2.15 the fact that Λ is a lattice set is used to achieve (2.12). As we will see, the factorization given in (2.12) will be useful when proving Theorem 2.8.

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Suppose $\Phi \ge 0$ and there exists a family of matrices $\{\Theta_j\}_{j=1}^d$ which satisfy the K-inclusive property with respect to $\mathfrak{M} = \operatorname{Ran}\Phi$ and $\Phi\Theta_j = \Phi_j$, $1 \le j \le d$. Write

$$\Theta_j = \begin{pmatrix} \tilde{\Theta}_j & 0 \\ * & * \end{pmatrix} : \begin{matrix} \operatorname{Ran}\Phi \\ \bigoplus \\ \operatorname{Ker}\Phi \end{matrix} \to \begin{matrix} \operatorname{Ran}\Phi \\ \bigoplus \\ \operatorname{Ker}\Phi \end{matrix},$$

and so we must have the existence of an invertible matrix S so that $S^{-1}\tilde{\Theta}_j S = \operatorname{diag}(x_j^{(1)}, \ldots, x_j^{(r)})$, $1 \le j \le d$, where $r = \operatorname{rank}\Phi$ and $(x_1^{(q)}, \ldots, x_d^{(q)}) \in K$, $1 \le q \le r$. Use Theorem 2.15 to obtain an injective matrix $C := \operatorname{col}(C_\lambda)_{\lambda\in\Lambda}$ and real diagonal matrices $D_1, \ldots, D_d$ so that $\Phi = CC^*$ and $\Phi_j = CD_jC^*$, $1 \le j \le d$. By (2.11) we have that $D_j = \operatorname{diag}(x_j^{(1)}, \ldots, x_j^{(r)})$, $1 \le j \le d$. Write $C_{0_d} = \begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix}$, where $c_1, \ldots, c_r \in \mathbb{C}^p$. Then (2.12) holds. Put $w_q = (x_1^{(q)}, \ldots, x_d^{(q)}) \in K, 1 \le q \le r$. Without loss of generality, assume $w_1, \ldots, w_k$ are distinct, where $k \le r$. Fix $q \in \{1, \ldots, k\}$ and consider the set $\mathcal{I}_q = \{i \in \{1, \ldots, r\} \colon w_i = w_q\}$. Note that

$$\operatorname{card} \bigcup_{q=1,\ldots,r} \mathcal{I}_q = r.$$

Let $T_q = \sum_{i\in\mathcal{I}_q} c_i c_i^* \ge 0$ and note that (2.16) gives the fact that $\{c_i\}_{i\in\mathcal{I}_q}$ is linearly independent since C is injective. Hence $\operatorname{rank} T_q = \operatorname{card}\mathcal{I}_q$, and so we get

$$\operatorname{rank} T_q = \operatorname{card}\mathcal{I}_q, \ 1 \le q \le k.$$

Thus $\sum_{q=1}^k \operatorname{rank} T_q = r$, and so $\operatorname{rank}\Phi = \sum_{q=1}^k \operatorname{rank} T_q$. Put σ as in the statement of Theorem 2.8. One can directly verify (2.2) and (2.3). Note that the existence of all subsequent moments follows from Remark 2.9.

Conversely, let $\sigma = \sum_{q=1}^k T_q \delta_{w_q}$, where $T_1, \ldots, T_k \ge 0$ and $w_1, \ldots, w_k$ are distinct points in K. Write $w_q = (x_1^{(q)}, \ldots, x_d^{(q)}), 1 \le q \le k$. For $m \in \mathbb{N}_0^d$, put $S_m := \int \xi^m d\sigma(\xi) = \sum_{q=1}^k T_q w_q^m$. Use Theorem 2.13 to produce a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0^d$, with $\operatorname{card}\tilde{\Lambda} = k$, so that $V = V(w_1, \ldots, w_k; \tilde{\Lambda})$ is invertible. One can check that $\Phi = (V \otimes I_p)^T R (V \otimes I_p) \ge 0$, $\Phi_j = (V \otimes I_p)^T R X_j (V \otimes I_p)$, where $R = T_1 \oplus \cdots \oplus T_k$ and $X_j = x_j^{(1)} I_p \oplus \cdots \oplus x_j^{(k)} I_p, 1 \le j \le d$. Choosing

$$\Theta_j = (V \otimes I_p)^{-1} X_j (V \otimes I_p), \ \ 1 \le j \le d,$$

yields a family of matrices $\Theta_1, \dots, \Theta_d$ which satisfy the K-inclusive property with respect to $\mathfrak{M} = \text{Ran } \Phi$ and $\Phi\Theta_j = \Phi_j$, $1 \le j \le d$. Thus conditions (1) and (2) are satisfied. Moreover, by construction we have that $\sum_{q=1}^{k} \text{rank } T_q = \text{rank } \Phi$. □

Remark 2.17. Given $\sigma = \sum_{q=1}^{k} T_q \delta_{w_q}$, where $T_1, \dots, T_k \ge 0$ and $w_1, \dots, w_k$ are distinct points in $K \subseteq \mathbb{R}^d$, the proof of Theorem 2.8 reveals the following observation. Any lattice set $L \subseteq \mathbb{N}_0^d$ so that $V(w_1, \dots, w_k; L)$ is invertible will lead to a construction of $\Phi, \Phi_1, \dots, \Phi_d$, which admits $\Theta_1, \dots, \Theta_d$ which satisfy the K-inclusive eigenvalue property with respect to $\mathfrak{M} = \text{Ran } \Phi$ and $\Phi\Theta_j = \Phi_j$, $1 \le j \le d$. Moreover, we also have the fact that $\text{rank } \Phi = \sum_{q=1}^{k} \text{rank } T_q$.

We will now exhibit a series of examples which utilize Theorem 2.8.

Example 2.18. Let $K = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon \xi_1^2 + \xi_2^2 \le 2\}$. Suppose one is given the $\mathcal{H}_2$-valued sequence $\{S_\gamma\}_{\gamma \in \Gamma}$, with $\Gamma = \{m \in \mathbb{N}_0^2 \colon |m| \le 3\}$, where $S_{00} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $S_{21} = S_{03} = S_{01} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $S_{12} = S_{30} = S_{10} = -S_{01}$, $S_{20} = S_{02} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and $S_{11} = -S_{02}$. Let

$$\Phi = \begin{pmatrix} S_{00} & S_{01} & S_{10} \\ S_{01} & S_{02} & S_{11} \\ S_{10} & S_{11} & S_{20} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & -1 & -1 \\ 1 & 2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 2 & -1 & -2 \\ -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -2 & 1 & 2 \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} S_{10} & S_{11} & S_{20} \\ S_{11} & S_{12} & S_{21} \\ S_{20} & S_{21} & S_{30} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & -1 & -1 \\ 1 & 2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 2 & -1 & -2 \\ -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -2 & 1 & 2 \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} S_{01} & S_{02} & S_{11} \\ S_{02} & S_{03} & S_{12} \\ S_{11} & S_{12} & S_{21} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 2 & 1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -2 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\text{rank } \Phi = 3$ and we have

$$Q^*\Phi Q = \begin{pmatrix} 0.7898 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.7393 & -0.0000 & 0 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 6.4709 & -0.0000 & 0 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \end{pmatrix},$$

$$Q^*\Phi_1 Q = \begin{pmatrix} -0.0263 & -0.0553 & -0.4953 & 0.0000 & -0.0000 & -0.0000 \\ -0.0553 & 2.3413 & 2.0942 & 0 & 0.0000 & 0.0000 \\ -0.4953 & 2.0942 & -5.3150 & 0 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \end{pmatrix},$$

and

$$Q^*\Phi_2 Q = \begin{pmatrix} 0.0263 & 0.0553 & 0.4953 & -0.0000 & 0.0000 & 0.0000 \\ 0.0553 & -2.3413 & -2.0942 & 0 & -0.0000 & -0.0000 \\ 0.4953 & -2.0942 & 5.3150 & 0 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \end{pmatrix},$$

where

$$Q = \begin{pmatrix} 0.8660 & 0.2376 & -0.4401 & -0.0000 & 0.0000 & 0.0000 \\ -0.3347 & 0.7295 & -0.2649 & 0.0382 & -0.5331 & -0.0026 \\ -0.2305 & 0.1508 & -0.3721 & -0.0923 & 0.5327 & -0.7025 \\ -0.1262 & -0.4278 & -0.4792 & -0.6841 & -0.3172 & 0.0527 \\ 0.2305 & -0.1508 & 0.3721 & -0.0159 & -0.5336 & -0.7076 \\ 0.1262 & 0.4278 & 0.4792 & -0.7223 & 0.2159 & 0.0553 \end{pmatrix}.$$

Let

$$\tilde{\Phi} = \begin{pmatrix} 0.7898 & 0.0000 & -0.0000 \\ 0.0000 & 2.7393 & -0.0000 \\ -0.0000 & -0.0000 & 6.4709 \end{pmatrix},$$

$$\tilde{\Phi}_1 = \begin{pmatrix} -0.0263 & -0.0553 & -0.4953 \\ -0.0553 & 2.3413 & 2.0942 \\ -0.4953 & 2.0942 & -5.3150 \end{pmatrix},$$

and

$$\tilde{\Phi}_2 = \begin{pmatrix} 0.0263 & 0.0553 & 0.4953 \\ 0.0553 & -2.3413 & -2.0942 \\ 0.4953 & -2.0942 & 5.3150 \end{pmatrix}.$$

Hence

$$\tilde{\Theta}_1 = \tilde{\Phi}^{-1}\tilde{\Phi}_1 = \begin{pmatrix} -0.0334 & -0.0700 & -0.6271 \\ -0.0202 & 0.8547 & 0.7645 \\ -0.0765 & 0.3236 & -0.8214 \end{pmatrix}$$

and

$$\tilde{\Theta}_2 = \tilde{\Phi}^{-1}\tilde{\Phi}_2 = \begin{pmatrix} 0.0334 & 0.0700 & 0.6271 \\ 0.0202 & -0.8547 & -0.7645 \\ 0.0765 & -0.3236 & 0.8214 \end{pmatrix}.$$

Note that $\tilde{\Theta}_1\tilde{\Theta}_2 = \tilde{\Theta}_2\tilde{\Theta}_1$ and $\tilde{\Phi}^{\frac{1}{2}}\tilde{\Theta}_1\tilde{\Phi}^{-\frac{1}{2}} = U\ \mathrm{diag}(-1,0,1)\ U^*$ and $\tilde{\Phi}^{\frac{1}{2}}\tilde{\Theta}_2\tilde{\Phi}^{-\frac{1}{2}} = U\ \mathrm{diag}(1,0,-1)\ U^*$, where $U = \begin{pmatrix} -0.2048 & -0.9744 & 0.0927 \\ 0.2497 & -0.1435 & -0.9576 \\ -0.9464 & 0.1730 & -0.2727 \end{pmatrix}$. We now know that the support of the K-representing measure for $\{S_\gamma\}_{\gamma\in\Gamma}$ is given by $\{(-1,1),(0,0),(1,-1)\}$, and we may put $w_1=(-1,1), w_2=(0,0), w_3=(1,-1)\in K$. Put

$$C = Q\begin{pmatrix}\tilde{\Phi}^{\frac{1}{2}}U \\ 0\end{pmatrix} = \begin{pmatrix} 1.0000 & -1.0000 & 0.0000 \\ 1.0000 & 0.0000 & -1.0000 \\ 1.0000 & 0.0000 & 0.0000 \\ 1.0000 & -0.0000 & 1.0000 \\ -1.0000 & -0.0000 & -0.0000 \\ -1.0000 & 0.0000 & -1.0000 \end{pmatrix}.$$

We will now calculate the densities. Write $C=\operatorname{col}(C_\lambda)_{\lambda\in\Lambda}$ and note that $C_{0_2} = \begin{pmatrix} 1.0000 & -1.0000 & 0.0000 \\ 1.0000 & 0.0000 & -1.0000 \end{pmatrix}$, whence we get $T_1 = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \geq 0, T_2 = \begin{pmatrix}-1\\0\end{pmatrix}\begin{pmatrix}-1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \geq 0$ and $T_3 = \begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \geq 0$. Thus a minimal K-representing measure for $\{S_\gamma\}_{\gamma\in\Gamma}$ with 3-atoms is given by

$$\sigma = \sum_{q=1}^{3} T_q\delta_{w_q} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\delta_{(-1,1)} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\delta_{(0,0)} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\delta_{(1,-1)}.$$

Our next example will provide insight into the case when one is given data which does not satisfy condition (2) of Theorem 2.8. As we will see, Theorem 2.8 can be used to rule out the existence of a minimal K-representing measure.

Example 2.19. Let $K=\{(\xi_1,\xi_2)\in\mathbb{R}^2\colon \xi_1>0,\xi_2>0\}$. Suppose one is given the real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma=(\Lambda+\Lambda)\cup(\Lambda+\Lambda+\mathrm{e}_1)\cup(\Lambda+\Lambda+\mathrm{e}_2)$, with $\Lambda=\{\lambda\in\mathbb{N}_0^2\colon |\lambda|\leq 1\}$, so that

$$\Phi(\Lambda) = \begin{pmatrix} s_{00} & s_{01} & s_{10} \\ s_{01} & s_{02} & s_{11} \\ s_{10} & s_{11} & s_{20}\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix},$$

$$\Phi_1(\Lambda) = \begin{pmatrix} s_{10} & s_{11} & s_{20} \\ s_{11} & s_{12} & s_{21} \\ s_{20} & s_{21} & s_{30}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix},$$

and

$$\Phi_2(\Lambda) = \begin{pmatrix} s_{01} & s_{02} & s_{11} \\ s_{02} & s_{03} & s_{12} \\ s_{11} & s_{12} & s_{21}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

We claim that $\{s_\gamma\}_{\gamma\in\Gamma}$ does not have a K-representing measure with 3-atoms. Suppose it did. Then there must exist a positive Borel measure $\sigma=\sum_{q=1}^{3}\rho_q\delta_{w_q}$, where ρ_1,ρ_2,ρ_3 are positive real numbers and w_1,w_2,w_3 are distinct points in K. Put $s_m=\hat{\sigma}(m)$, for all $m\in\mathbb{N}_0^2$. By Theorem 2.8, there must exist a lower inclusive set $\tilde{\Lambda}$ with 3 points so that conditions (1) and (2) hold in Theorem 2.8.

There are only 3 lower-inclusive sets in $\mathbb{N}_0^2$ which consist of only 3 points. First, consider $\Lambda_1=\{(0,0),(0,1),(1,0)\}$ and then

$$\Theta_1(\Lambda_1) = \Phi^{-1}(\Lambda_1)\Phi_1(\Lambda_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix}$$

and

$$\Theta_2(\Lambda_1) = \Phi^{-1}(\Lambda_1)\Phi_2(\Lambda_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\Theta_1(\Lambda_1)$ and $\Theta_2(\Lambda_1)$ do not commute. Hence condition (2) of Theorem 2.8 will not be satisfied.

Next, consider $\Lambda_2 = \{(0,0),(0,1),(0,2)\}$. Then

$$\Phi(\Lambda_2) = \begin{pmatrix} s_{00} & s_{01} & s_{02} \\ s_{01} & s_{02} & s_{03} \\ s_{02} & s_{03} & s_{04} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & s_{04} \end{pmatrix},$$

$$\Phi_1(\Lambda_2) = \begin{pmatrix} s_{10} & s_{11} & s_{12} \\ s_{11} & s_{12} & s_{13} \\ s_{12} & s_{13} & s_{14} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{13} \\ 0 & s_{13} & s_{14} \end{pmatrix},$$

and

$$\Phi_2(\Lambda_2) = \begin{pmatrix} s_{01} & s_{02} & s_{03} \\ s_{02} & s_{03} & s_{04} \\ s_{03} & s_{04} & s_{05} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & s_{04} \\ 0 & s_{04} & s_{05} \end{pmatrix}.$$

Note that we may assume that $\Phi(\Lambda_2) > 0$, i.e., $s_{04} > 1$, since σ has 3-atoms. So then

$$\Theta_1(\Lambda_2) = \Phi^{-1}(\Lambda_2)\Phi_1(\Lambda_2) = \begin{pmatrix} 0 & -\frac{s_{13}}{s_{04}-1} & -\frac{s_{14}}{s_{04}-1} \\ 0 & 0 & s_{13} \\ 0 & \frac{s_{13}}{s_{04}-1} & \frac{s_{14}}{s_{04}-1} \end{pmatrix}$$

and

$$\Theta_2(\Lambda_2) = \Phi^{-1}(\Lambda_2)\Phi_2(\Lambda_2) = \begin{pmatrix} 0 & 0 & -\frac{s_{05}}{s_{04}-1} \\ 1 & 0 & s_{04} \\ 0 & 1 & \frac{s_{05}}{s_{04}-1} \end{pmatrix}.$$

Note that $\Theta_1(\Lambda_2)$ and $\Theta_2(\Lambda_2)$ commute only when $s_{13} = s_{14} = 0$. So then $\Theta_1(\Lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and hence the K-inclusive eigenvalue property will not be satisfied.

Finally, consider $\tilde{\Lambda} = \Lambda_3 = \{(0,0),(1,0),(2,0)\}$. Then

$$\Phi(\Lambda_3) = \begin{pmatrix} s_{00} & s_{10} & s_{20} \\ s_{10} & s_{20} & s_{30} \\ s_{20} & s_{30} & s_{40} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & s_{40} \end{pmatrix},$$

$$\Phi_1(\Lambda_3) = \begin{pmatrix} s_{10} & s_{20} & s_{30} \\ s_{20} & s_{30} & s_{40} \\ s_{30} & s_{40} & s_{50} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & s_{40} \\ 0 & s_{40} & s_{50} \end{pmatrix},$$

and

$$\Phi_2(\Lambda_3) = \begin{pmatrix} s_{01} & s_{11} & s_{21} \\ s_{11} & s_{21} & s_{31} \\ s_{21} & s_{31} & s_{41} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{31} \\ 0 & s_{31} & s_{41} \end{pmatrix}.$$

Note that we may assume that $\Phi(\Lambda_3) > 0$, i.e., $s_{40} > 1$, since σ has 3-atoms. So then

$$\Theta_1(\Lambda_3) = \Phi^{-1}(\Lambda_3)\Phi_1(\Lambda_3) = \begin{pmatrix} 0 & 0 & -\frac{s_{50}}{s_{40}-1} \\ 1 & 0 & s_{40} \\ 0 & 1 & \frac{s_{50}}{s_{40}-1} \end{pmatrix}$$

and

$$\Theta_2(\Lambda_3) = \Phi^{-1}(\Lambda_3)\Phi_2(\Lambda_3) = \begin{pmatrix} 0 & -\frac{s_{31}}{s_{40}-1} & -\frac{s_{41}}{s_{40}-1} \\ 0 & 0 & s_{31} \\ 0 & \frac{s_{31}}{s_{40}-1} & \frac{s_{41}}{s_{40}-1} \end{pmatrix}.$$

Note that $\Theta_1(\Lambda_3)$ and $\Theta_2(\Lambda_3)$ commute only when $s_{31} = s_{41} = 0$. So then $\Theta_2(\Lambda_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and hence the K-inclusive eigenvalue property will not be satisfied.

Remark 2.20. In point of fact, one can show that $\{s_\gamma\}_{\gamma\in\Gamma}$ in Example 2.19 does not have a representing measure with 3-atoms. Note that although a choice of Λ_2 and Λ_3 lead to the existence of commuting Θ_1 and Θ_2, it is not hard to see that the representing measures that are produced, in both cases, are not representing measures for $\{s_\gamma\}_{\gamma\in\Gamma}$. It is helpful to point out that if one was given the sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where $S_\gamma \in \mathcal{H}_p$ with $p > 1$, then the procedure outlined in Example 2.19 is more complicated. One would have to check all lower inclusive sets with cardinality at most rank Φ.

Suppose one is given data indexed by $\Lambda + \Lambda$, where $\Lambda \subset \mathbb{N}_0^d$ is a lattice set. An immediate corollary of Theorem 2.8 is the following.

Corollary 2.21. *Let $K \subseteq \mathbb{R}^d$ and $\Lambda \subset \mathbb{N}_0^d$ be a lattice set. Suppose the $\mathcal{H}_p$-valued sequence $\{S_{\lambda+\mu}\}_{\lambda,\mu\in\Lambda}$ is given so that $\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in\Lambda} \geq 0$. Put $\Gamma = (\Lambda+\Lambda)\cup(\Lambda+\Lambda+\mathrm{e}_1)\cup\cdots\cup(\Lambda+\Lambda+\mathrm{e}_d)$. If an $\mathcal{H}_p$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma\setminus(\Lambda+\Lambda)}$ exists so that $\Phi = (S_{\lambda+\mu})_{\lambda,\mu\in\Lambda}$ and $\Phi_j = (S_{\lambda+\mu+\mathrm{e}_j})_{\lambda,\mu\in\Lambda}$, $1 \leq j \leq d$, satisfy conditions (1) and (2) in Theorem* 2.8*, then $\{S_\gamma\}_{\gamma\in\Gamma}$ has a K-representing measure which is of the form* (2.8).

Proof. Apply Theorem 2.8 in a straightforward manner. □

We will now consider an example using Corollary 2.21, which corresponds to the even case of the total-degree K-moment problem on $\mathbb{R}^d$ considered in [9].

Example 2.22. Let $K = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon \xi_1^2 + \xi_2^2 < 1\}$ and $\Lambda = \{(0,0), (0,1), (1,0)\}$. Suppose one is given $\{s_{\lambda+\mu}\}_{\lambda,\mu\in\Lambda}$, where $s_{00} = 3, s_{10} = s_{01} = 0, s_{20} = s_{02} = 1$, and $s_{11} = -s_{02}$. Put $\Gamma = (\Lambda+\Lambda)\cup(\Lambda+\Lambda+\mathrm{e}_1)\cup(\Lambda+\Lambda+\mathrm{e}_2) = \{m \in \mathbb{N}_0^2 \colon |m| \leq 3\}$ and let

$$\Phi = \begin{pmatrix} s_{00} & s_{01} & s_{10} \\ s_{01} & s_{02} & s_{11} \\ s_{10} & s_{11} & s_{20} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} s_{10} & s_{11} & s_{20} \\ s_{11} & s_{12} & s_{21} \\ s_{20} & s_{21} & s_{30} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & s_{12} & s_{21} \\ 1 & s_{21} & s_{30} \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} s_{01} & s_{02} & s_{11} \\ s_{02} & s_{03} & s_{12} \\ s_{11} & s_{12} & s_{21} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & s_{03} & s_{12} \\ -1 & s_{12} & s_{21} \end{pmatrix}.$$

Note that $\Phi \geq 0$ and rank $\Phi = 2$. We must choose s_{12}, s_{21}, s_{03}, and s_{30} so that condition (2) of Theorem 2.8 is satisfied. In order for Φ, Φ_1, and Φ_2 to admit matrices Θ_1, Θ_2 which commute with respect to $\mathfrak{M} = \operatorname{Ran}\Phi$ so that $\Phi\Theta_j = \Phi_j$, we must have Ran $\Phi_j \subseteq$ Ran Φ, $j = 1, 2$. We wish to find all possible $s_{12}, s_{21}, s_{03}, s_{30} \in \mathbb{R}$ so that with respect to the decomposition $\mathbb{C}^3 = \text{Ran } \Phi \oplus \text{Ker } \Phi$ we may write

$$\Phi = \begin{pmatrix} \tilde{\Phi} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} \tilde{\Phi}_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} \tilde{\Phi}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice $Q^*\Phi Q = \text{diag}\,(3, 2, 0)$,

$$Q^*\Phi_1 Q = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & \frac{1}{2}s_{12} - s_{21} + \frac{1}{2}s_{30} & -\frac{1}{2}s_{30} + \frac{1}{2}s_{30} \\ 0 & -\frac{1}{2}s_{12} + \frac{1}{2}s_{30} & \frac{1}{2}s_{12} + s_{21} + \frac{1}{2}s_{30} \end{pmatrix},$$

and

$$Q^*\Phi_2 Q = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ -\sqrt{2} & \frac{1}{2}s_{03} - s_{12} + \frac{1}{2}s_{21} & -\frac{1}{2}s_{03} + \frac{1}{2}s_{21} \\ 0 & -\frac{1}{2}s_{03} + \frac{1}{2}s_{21} & \frac{1}{2}s_{03} + s_{12} + \frac{1}{2}s_{21} \end{pmatrix},$$

where $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Realize that Ran $\Phi_1 \subseteq$ Ran Φ yields that $-\frac{1}{2}s_{12} + \frac{1}{2}s_{30} = 0$ and $\frac{1}{2}s_{12} + s_{21} + \frac{1}{2}s_{30} = 0$, i.e., $s_{12} = s_{30}$ and $s_{21} = -s_{30}$, with s_{30} being free. Next, Ran $\Phi_2 \subseteq$ Ran Φ yields that $-\frac{1}{2}s_{03} + \frac{1}{2}s_{21} = 0$ and $\frac{1}{2}s_{03} + s_{12} + \frac{1}{2}s_{21} = 0$, i.e., $s_{21} = s_{03}$ and $s_{12} = -s_{03}$. But then we get $-s_{30} = s_{03}$, with s_{30} remaining free. Hence we get

$$Q^*\Phi_1 Q = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 2s_{30} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Q^*\Phi_2 Q = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ -\sqrt{2} & -2s_{30} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Put $\tilde{\Phi} = \text{diag}(3, 2), \tilde{\Phi}_1 = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 2s_{30} \end{pmatrix}$ and $\tilde{\Phi}_2 = -\tilde{\Phi}_1$. Then $\tilde{\Theta}_1 = \tilde{\Phi}^{-1}\tilde{\Phi}_1 = \begin{pmatrix} 0 & \frac{\sqrt{2}}{3} \\ \sqrt{2}2 & -s_{30} \end{pmatrix}$ and $\tilde{\Theta}_2 = \tilde{\Phi}^{-1}\tilde{\Phi}_2 = -\tilde{\Theta}_1$. Clearly $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ commute, for any choice of s_{30}. Finally, to satisfy condition (2) of Theorem 2.8, note that the eigenvalues of $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are $\{\frac{1}{2}s_{30} + \frac{1}{6}\sqrt{9s_{30}^2 + 12}, \frac{1}{2}s_{30} - \frac{1}{6}\sqrt{9s_{30}^2 + 12}\}$ and $\{-\frac{1}{2}s_{30} - \frac{1}{6}\sqrt{9s_{30}^2 + 12}, -\frac{1}{2}s_{30} + \frac{1}{6}\sqrt{9s_{30}^2 + 12}\}$, respectively. Thus choosing $|s_{30}| < \frac{\sqrt{2}}{6}$ will lead to condition (2) in Theorem 2.8 being satisfied. In particular, when $s_{30} = 0$, a K-representing measure σ with 2-atoms is given by $\sigma = \frac{3}{2}\delta_{(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})} + \frac{3}{2}\delta_{(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})}$.

We will end this section by demonstrating a connection between Theorem 2.8 and a result in [5]. Curto and Fialkow have considered scalar-valued K-moment problems on $\mathbb{R}^d$ in the following setting. Given $K \subseteq \mathbb{R}^d$ and the real-valued sequence

$\{s_\gamma\}_{\gamma\in\Gamma(n)}$, where $\Gamma(n) = \{\gamma \in \mathbb{N}_0^d \colon |\gamma| \leq n\}$, we wish to find a positive Borel measure on $\mathbb{R}^d$ so that $\hat{\sigma}(m)$ exists for all $m \in \mathbb{N}_0^d$,

$$\hat{\sigma}(\gamma) = s_\gamma \quad \gamma \in \Gamma,$$

and

$$\operatorname{supp} \sigma \subseteq K.$$

Curto and Fialkow have developed the theory of *flat extensions* (see [5], [6], and [7]). Define the lower inclusive set $\Lambda(n) = \{\lambda \in \mathbb{N}_0^d \colon |\lambda| \leq n\}$. We can order $\Lambda(n)$ using the graded lexicographic order, i.e., first order the elements of $\Lambda(n)$ by total degree, and when two elements of $\Lambda(n)$ have the same total degree, we order them by the lexicographic ordering. For example, when $d = 2$ and $n = 3$,

$$\Lambda(3) = \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2),(2,1),(3,0)\}.$$

If we are given the real-valued sequence $\{s_\gamma\}_{\gamma\in\Gamma(2n)}$, then we can build the matrix

$$M(n) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n)}.$$

We say that $M(n) \geq 0$ has a flat extension if there exists a real-valued sequence $\{s_{\tilde{\gamma}}\}_{\tilde{\gamma}\in\Gamma(2n+2)\setminus\Gamma(2n)}$ so that $M(n+1) \geq 0$ and $\operatorname{rank} M(n) = \operatorname{rank} M(n+1)$. The following lemma, due to Šmul′jan, is a useful tool for analyzing flat extensions.

Lemma 2.23 ([22])**.** *Let $A \geq 0$. Then*

$$A_{\text{ext}} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \tag{2.17}$$

and

$$\operatorname{rank} A_{\text{ext}} = \operatorname{rank} A, \tag{2.18}$$

if and only if there exists a matrix W so that $AW = B$ and $C = W^*AW$.

Proof. Suppose (2.17) and (2.18) hold. Let $\begin{pmatrix} P \\ Q \end{pmatrix}$ be the square root of A_{ext}, i.e.,

$$\begin{pmatrix} P \\ Q \end{pmatrix} \begin{pmatrix} P^* & Q^* \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{2.19}$$

Hence we get $PP^* = A^{\frac{1}{2}}A^{\frac{1}{2}}, B = A^{\frac{1}{2}}Q$, and $QQ^* = C^{\frac{1}{2}}C^{\frac{1}{2}}$. Since $\operatorname{Ran} A = \operatorname{Ran} A^{\frac{1}{2}}$, we have the existence of a matrix W so that $AW = B$. By (2.18) we then must have $B^*W = C$, i.e., $C = W^*AW$.

Next, suppose a matrix W exists so that $AW = B$ and $C = W^*AW$. Then it is easy to check that

$$A_{\text{ext}} = \begin{pmatrix} A^{\frac{1}{2}} \\ W^*A^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & A^{\frac{1}{2}}W \end{pmatrix} \geq 0.$$

□

A result from [5] is the following, which we will prove independently.

Theorem 2.24 ([5])**.** *Let $\Lambda(n) = \{\lambda \in \mathbb{N}_0^d \colon |\lambda| \leq n\}$. Suppose the real-valued sequence $\{s_{\lambda+\mu}\}_{\lambda,\mu\in\Lambda(n)}$ is given. The sequence $\{s_{\lambda+\mu}\}_{\lambda,\mu\in\Lambda(n)}$ has a representing measure of the form*

$$\sigma = \sum_{q=1}^{k} \rho_q \delta_{w_q}, \tag{2.20}$$

where $k = \operatorname{rank} M(n)$*,* $\rho_1, \dots, \rho_k$ *are positive real numbers, and* $w_1, \dots, w_k$ *are distinct points in* $\mathbb{R}^d$ *if and only if* $M(n) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n)} \geq 0$ *and there exists a sequence* $(s_{\tilde{\lambda}+\tilde{\mu}})_{\tilde{\lambda},\tilde{\mu}\in\Lambda(n+1)\setminus\Lambda(n)}$ *so that* $M(n+1) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n+1)} \geq 0$ *and* $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$*.*

We shall ultimately prove Theorem 2.24 by using Theorem 2.8. However, first we need two auxillary results.

Lemma 2.25. *Let* $\Lambda(n) = \{\lambda \in \mathbb{N}_0^d \colon |\lambda| \leq n\}$*. Suppose the real-valued sequence* $\{s_{\lambda+\mu}\}_{\lambda,\mu\in\Lambda(n)}$ *is given so that* $M(n) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n)} \geq 0$*. Suppose there exists a sequence* $\{s_{\tilde{\lambda}+\tilde{\mu}}\}_{\tilde{\lambda},\tilde{\mu}\in\Lambda(n+1)\setminus\Lambda(n)}$ *so that the following hold:*

$$M(n+1) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n+1)} = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \geq 0, \tag{2.21}$$

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n), \tag{2.22}$$

where $B = (s_{\lambda+\mu})_{\lambda\in\Lambda(n),\, \mu\in\Lambda(n+1)\setminus\Lambda(n)}$ *and* $C = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n+1)\setminus\Lambda(n)}$*. Let* $\Phi = M(n)$ *and* $\Phi_i = (s_{\lambda+\mu+\mathrm{e}_i})_{\lambda,\mu\in\Lambda(n)}$*. Then there exist matrices* $\{\Theta_i\}_{i=1}^d$ *so that* $\Phi\Theta_i = \Phi_i$ *and*

$$\Phi_i\Theta_j = \Phi_j\Theta_i, \; 1 \leq i, j \leq d. \tag{2.23}$$

Proof. Use (2.21) and (2.22) to invoke Lemma 2.23 to obtain a matrix W so that $B = \Phi W$ and $C = W^*\Phi W$. Let $\mu \in \Lambda(n)$ and $\tilde{\mu} \in \Lambda(n+1) \setminus \Lambda(n)$. If we let $\varphi_\mu = \operatorname{col}(s_{\lambda+\mu})_{\lambda\in\Lambda(n)}$ and $\tilde{\varphi}_{\tilde{\mu}} = \operatorname{col}(s_{\lambda+\mu})_{\lambda\in\Lambda(n+1)\setminus\Lambda(n)}$, then we can write

$$\begin{aligned} \Phi &= \operatorname{row}\left(\varphi_\lambda\right)_{\lambda\in\Lambda(n)}, \\ W &= \operatorname{row}(w_\mu)_{\mu\in\Lambda(n+1)\setminus\Lambda(n)}, \\ B &= \operatorname{row}(\varphi_\mu)_{\mu\in\Lambda(n+1)\setminus\Lambda(n)}, \\ C &= \operatorname{row}(\tilde{\varphi}_{\tilde{\mu}})_{\tilde{\mu}\in\Lambda(n+1)\setminus\Lambda(n)}, \end{aligned}$$

and

$$\Phi_i = \operatorname{row}(\varphi_\mu + \mathrm{e}_i)_{\mu\in\Lambda(n+1)\setminus\Lambda(n)}. \tag{2.24}$$

Notice that $\Phi W = B$ gives

$$\Phi w_\mu = \varphi_\mu, \; \mu \in \Lambda(n+1) \setminus \Lambda(n). \tag{2.25}$$

Also, $W^*\Phi W = C$ gives $W^*B = C$, i.e.,

$$s_{\mu+\tilde{\mu}} = w_\mu^*\varphi_{\tilde{\mu}}, \; \mu, \tilde{\mu} \in \Lambda(n+1) \setminus \Lambda(n), \tag{2.26}$$

upon consideration of (2.25).

Realize for any $i \in \{1, \dots, d\}$ that $\Lambda(n)$ has the property that $\lambda + \mathrm{e}_i \in \Lambda(n)$ only when $\lambda \in \Lambda(n-1)$. If we choose

$$\Theta_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \end{pmatrix},$$

where $M_{11}^{(i)} = \operatorname{row}(\mathrm{e}_{\lambda+\mathrm{e}_i})_{\lambda\in\Lambda(n-1)}$ and $M_{12}^{(i)} = \operatorname{row}(w_{\mu+\mathrm{e}_i})_{\mu\in\Lambda(n)\setminus\Lambda(n-1)}$, then

$$\Phi\Theta_i = \begin{pmatrix} N_{11}^{(i)} & N_{12}^{(i)} \end{pmatrix},$$

where

$$N_{11}^{(i)} = \operatorname{row}(\varphi_{\lambda+\mathrm{e}_i})_{\lambda\in\Lambda(n-1)}$$

and

$$N_{12}^{(i)} = \operatorname{row}(\varphi_{\mu+\mathrm{e}_i})_{\mu\in\Lambda(n)\setminus\Lambda(n-1)},$$

where we used (2.25). Then it is clear that $\Phi\Theta_i = \Theta_i, 1 \le i \le d$, where we used (2.24).

Next, we must check that $\Phi_i\Theta_j = \Phi_j\Theta_i$, $1 \le i, j \le d$. Since Φ_i, Θ_i are matrices of size card $\Lambda(n)$, it suffices to show that $\mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu = \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu$, for all $\lambda, \mu \in \Lambda(n)$. Let us remark that if $\lambda + \mathrm{e}_i \in \Lambda(n)$ for any fixed $i \in \{1, \ldots, d\}$, then $\lambda + \mathrm{e}_j \in \Lambda(n)$ for all $1 \le j \le d$. Similarly, if $\lambda + \mathrm{e}_i \notin \Lambda(n)$, then $\lambda + \mathrm{e}_j \notin \Lambda(n)$ for all $1 \le j \le d$. Thus the following cases emerge for consideration. First let $\lambda, \mu \in \Lambda(n)$ so that $\mu + \mathrm{e}_i \in \Lambda(n)$. Then

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu &= (\Phi_i\mathrm{e}_\lambda)^*\Theta_j\mathrm{e}_\mu \\ &= (\varphi_{\lambda+\mathrm{e}_i})^*\mathrm{e}_{\mu+\mathrm{e}_j} \\ &= s_{\lambda+\mu+\mathrm{e}_i+\mathrm{e}_j}, \end{aligned}$$

where we used (2.25) in the last equality. Similarly,

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu &= (\Phi_j\mathrm{e}_\lambda)^*\Theta_i\mathrm{e}_\mu \\ &= \varphi_{\lambda+\mathrm{e}_j}^* e_{\mu+\mathrm{e}_j} \\ &= s_{\lambda+\mu+\mathrm{e}_j+\mathrm{e}_i}, \end{aligned}$$

where we used (2.25) in the last equality. Thus we get $\mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu = \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu$ when $\mu + \mathrm{e}_i \in \Lambda(n)$. Next, let $\lambda, \mu \in \Lambda(n)$ so that $\lambda + \mathrm{e}_i \in \Lambda(n)$ and $\mu + \mathrm{e}_i \notin \Lambda(n)$. Then

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu &= (\Phi_i\mathrm{e}_\lambda)^*\Theta_j\mathrm{e}_\mu \\ &= (\varphi_{\lambda+\mathrm{e}_i})^* w_{\mu+\mathrm{e}_j} \\ &= s_{\lambda+\mu+\mathrm{e}_i+\mathrm{e}_j}, \end{aligned}$$

where we used (2.25) in the last equality. Similarly,

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu &= (\Phi_j\mathrm{e}_\lambda)^*\Theta_i\mathrm{e}_\mu \\ &= \varphi_{\lambda+\mathrm{e}_j}^* w_{\mu+\mathrm{e}_i} \\ &= s_{\lambda+\mu+\mathrm{e}_j+\mathrm{e}_i}, \end{aligned}$$

where we used (2.25) in the last equality. Thus we get $\mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu = \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu$ when $\lambda + \mathrm{e}_i \in \Lambda(n)$ and $\mu + \mathrm{e}_i \notin \Lambda(n)$. Finally, let $\lambda, \mu \in \Lambda(n)$ so that $\lambda + \mathrm{e}_i \notin \Lambda(n)$ and $\mu + \mathrm{e}_i \notin \Lambda(n)$. Then

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu &= (\Phi_i\mathrm{e}_\lambda)^*\Theta_j\mathrm{e}_\mu \\ &= \varphi_{\lambda+\mathrm{e}_i}^* w_{\mu+\mathrm{e}_j} \\ &= s_{\lambda+\mu+\mathrm{e}_i+\mathrm{e}_j}, \end{aligned}$$

where we used (2.26). Similarly,

$$\begin{aligned} \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu &= (\Phi_j\mathrm{e}_\lambda)^*\Theta_i\mathrm{e}_\mu \\ &= \varphi_{\lambda+\mathrm{e}_j}^* w_{\mu+\mathrm{e}_i} \\ &= s_{\lambda+\mu+\mathrm{e}_j+\mathrm{e}_i}, \end{aligned}$$

where again we used (2.26). Thus we get $\mathrm{e}_\lambda^*\Phi_i\Theta_j\mathrm{e}_\mu = \mathrm{e}_\lambda^*\Phi_j\Theta_i\mathrm{e}_\mu$ when $\lambda+\mathrm{e}_i \notin \Lambda(n)$ and $\mu + \mathrm{e}_i \notin \Lambda(n)$. Finally, we arrive at $\Phi\Theta_j = \Phi_j$, $1 \le j \le d$. □

Lemma 2.26. *Suppose the positive Borel measure* $\sigma = \sum_{q=1}^k \rho_j\delta_{w_j}$ *is given, where* $\rho_1, \ldots, \rho_k$ *are positive real numbers and* $w_1, \ldots, w_k$ *are distinct points in* $\mathbb{R}^d$*. For*

$m \in \mathbb{N}_0^d$, put

$$s_m = \int \xi^m d\sigma(\xi).$$

There exists a lower inclusive set $\Lambda \subset \mathbb{N}_0^d$, and matrices $\Phi = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda}$, $\Theta_1, \ldots, \Theta_d$ so that $\Phi\Theta_1^{m_1} \cdots \Theta_d^{m_d}$ generates the infinite sequence $\{s_m\}_{m\in\mathbb{N}_0^d}$, i.e., for any $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, s_m is an entry of the matrix $\Phi\Theta_1^{m_1} \cdots \Theta_1^{m_d}$.

Proof. Put $w_q = (x_1^{(q)}, \ldots, x_d^{(q)})$, $1 \le q \le k$. For $m \in \mathbb{N}_0^d$, we have $s_m = \sum_{q=1}^k \rho_j w_j^m$. By [20], we can find a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0^d$, with card $\tilde{\Lambda} = k$, so that $V := V(w_1, \ldots, w_q; \Lambda)$ is invertible. Put $\Phi = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda}$ and $\Phi_m = (s_{\lambda+\mu+m})_{\lambda,\mu\in\Lambda}$ and notice that $\Phi = V^T RV > 0$ (where we used the fact that V is invertible) and

$$\Phi_m = V^T R X_1^{m_1} \cdots X_d^{m_d} V, \tag{2.27}$$

where $R = \mathrm{diag}(\rho_1, \ldots, \rho_k)$ and $X_j = \mathrm{diag}(x_j^{(1)}, \ldots, x_j^{(k)})$, $1 \le j \le d$. Note that s_m can be found in the (1,1) entry of the matrix $\Phi_m = (s_{\lambda+\mu+m})_{\lambda,\mu\in\Lambda}$. Realize that $\Phi\Theta_j = \Phi_j$ has a unique solution given by

$$\Theta_j = \Phi^{-1}\Phi_j = V^{-1}X_j V, \ \ 1 \le j \le d.$$

To complete the proof, realize that

$$\begin{aligned}\Phi\Theta_1^{m_1} \cdots \Theta_d^{m_d} &= V^T RV(V^{-1}X_1^{m_1}V) \cdots (V^{-1}X_d^{m_d}V)\\ &= V^T R X_1^{m_1} \cdots X_d^{m_d} V\\ &= \Phi_m,\end{aligned}$$

where we used (2.27) to get the last equality. □

We are now ready to prove Theorem 2.24.

Proof of Theorem 2.24. For sufficiency, suppose $M(n) \ge 0, M(n+1) \ge 0$, and rank $M(n+1) =$ rank $M(n)$. Then if we let $\Phi = M(n)$ and $\Phi_i = (s_{\lambda+\mu+e_i})_{\lambda,\mu\in\Lambda(n)}$, then Lemma 2.25 yields the existence of matrices $\{\Theta_i\}_{i=1}^d$ so that $\Phi\Theta_i = \Phi_i$ and $\Phi_i\Theta_j = \Phi_j\Theta_i$, $1 \le i, j \le d$. If we write $\mathbb{C}^r =$ Ran $\Phi \oplus$ Ker Φ, then with respect to this decomposition we have $\Phi = \begin{pmatrix}\tilde{\Phi} & 0\\ 0 & 0\end{pmatrix}$, $\Phi_i = \begin{pmatrix}\tilde{\Phi}_i & 0\\ 0 & 0\end{pmatrix}$, and $\Theta_i = \begin{pmatrix}\tilde{\Theta}_i & 0\\ * & *\end{pmatrix}$, where we used the fact that $\Phi\Theta_i = \Phi_i$, $1 \le i \le d$. Thus $\Phi_i\Theta_j = \Phi_j\Theta_i$ yields

$$\tilde{\Phi}\tilde{\Theta}_i\tilde{\Theta}_j = \tilde{\Phi}\tilde{\Theta}_j\tilde{\Theta}_j,$$

i.e., $\tilde{\Theta}_i\tilde{\Theta}_j = \tilde{\Phi}\tilde{\Theta}_j\tilde{\Theta}_i$, $1 \le i, j \le d$, since $\tilde{\Phi}$ is invertible. We then deduce that $\Theta_1, \ldots, \Theta_d$ commute with respect to $\mathfrak{M} =$ Ran Φ. Thus Theorem 2.8 yields that there exists a measure of the form (2.20).

For necessity, suppose that $\sigma = \sum_{q=1}^k \rho_q \delta_{w_q}$ is given, where $\rho_1, \ldots, \rho_k$ are positive real numbers and $w_1, \ldots, w_k$ are distinct points in $\mathbb{R}^d$. Put

$$s_m = \int \xi^m d\sigma(\xi) = \sum_{q=1}^k \rho_q w_q^m, \ \ m \in \mathbb{N}_0^d.$$

Lemma 2.26 gives the existence of a lower inclusive set $\tilde{\Lambda}$ so that we can construct $\Phi = (s_{\tilde{\lambda}+\tilde{\mu}})_{\tilde{\lambda},\tilde{\mu}\in\tilde{\Lambda}} > 0$ and $\Theta_1, \ldots, \Theta_d$, so that $\{s_m\}_{m\in\mathbb{N}_0^d}$ is generated by $\Phi\Theta_1^{m_1} \cdots \Theta_d^{m_d}$. We will now construct the infinite matrix $H := (H_m)_{m\in\mathbb{N}_0^d} \ge$

0, where $H_m = \Phi\Theta_1^{m_1}\cdots\Theta_d^{m_d}$. Note that by construction we get rank H = rank Φ. Notice that $M(n) = (s_{\lambda+\mu})_{\lambda,\mu\in\Lambda(n)}$, $B = (s_{\beta+\tilde\beta})_{\beta\in\Lambda(n),|\tilde\beta|=n+1}$ and $C = (s_{\gamma+\tilde\gamma})_{|\gamma|=|\tilde\gamma|=n+1}$ form a block matrix $M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C\end{pmatrix}$, which is a principal submatrix of H. Hence $M(n+1) \geq 0$, and we also get rank $M(n+1)$ = rank $M(n)$ since rank H = rank Φ. □

3. The truncated matrix-valued K-moment problem on $\mathbb{C}^d$

We will now consider the truncated matrix-valued K-moment problem on $\mathbb{C}^d$. First, we will introduce some preliminary notions and definitions. Let $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and $\bar z = (\bar z_1, \dots, \bar z_d) \in \mathbb{C}^d$. A function $\nu\colon \mathcal{B}(\mathbb{C}^d) \to \mathbb{C}^{p\times p}$ is called a positive $\mathcal{H}_p$-valued measure on $\mathbb{C}^d$ if for each $y \in \mathbb{C}^p$, $\langle \nu(\cdot)y, y\rangle$ defines a positive Borel measure on $\mathbb{C}^d$. For a measurable function $f\colon \mathbb{C}^d \to \mathbb{C}$, its integral $\int_{\mathbb{C}^d} f(z)d\nu(\bar z, z) := \int f(z_1,\dots,z_d)d\nu(z_1,\dots,z_d)$ is defined by the formula

$$\left\langle \int f(z)d\nu(z)y, \tilde y\right\rangle = \int f(z)\langle d\nu(z)y, \tilde y\rangle,$$

for all $y, \tilde y \in \mathbb{C}^p$, provided the integrals on the right-hand side converge.

The power moments of a positive $\mathcal{H}_p$-valued measure ν on $\mathbb{C}^d$ are defined by the formula,

$$\hat\nu(m,n) = \int \bar z^m z^n d\nu(z), \tag{3.1}$$

where $m, n \in \mathbb{N}_0^d$, provided the integrals converge. Given a set $K \subseteq \mathbb{C}^d$ and a lattice set $\Gamma \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$, let $\{S_{(\lambda,\mu)}\}_{(\lambda,\mu)\in\Gamma}$ be a given $\mathbb{C}^{p\times p}$-valued sequence. We look for a positive $\mathcal{H}_p$-valued measure ν on $\mathbb{C}^d$ such that $\hat\nu(m,n)$ exists for all $m, n \in \mathbb{N}_0^d$,

$$\hat\nu(\lambda,\mu) = S_{(\lambda,\mu)}, \text{ for all } (\lambda,\mu) \in \Gamma, \tag{3.2}$$

and

$$\operatorname{supp} \nu \subseteq K. \tag{3.3}$$

Notice that $\mathbb{N}_0^d \times \mathbb{N}_0^d = \mathbb{N}_0^{2d}$. We will denote the $2d$-tuple with all zeros except for one in the j-th, $1 \leq j \leq d$, entry by $(\mathrm{e}_j, 0_d)$. Next, we will denote the $2d$-tuple with all zeros except for one in the $(d+j)$-th, $1 \leq j \leq d$, entry by $(0_d, \mathrm{e}_j)$. We will say a finite set $\Lambda \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$ is a lattice set when for all $(\lambda,\mu) \in \Lambda$ there exist $(\lambda_1,\mu_1) = (0_d, 0_d), \dots, (\lambda_k,\mu_k) \in \Lambda$, $i_1, \dots i_k \in \{0,1\}$ and $j_1, \dots, j_k \in \{1,\dots,d\}$ so that

$$\begin{aligned}(\lambda_2,\mu_2) &= (\lambda_1,\mu_1) + (i_1\mathrm{e}_{j_1}, (1-i_1)\mathrm{e}_{j_1})\\ &\vdots\\ (\lambda,\mu) &= (\lambda_k,\mu_k) + (i_k\mathrm{e}_{j_k}, (1-i_k)\mathrm{e}_{j_k}),\end{aligned}$$

where $k = |(\lambda,\mu)|$. Given a set $\Lambda \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$, we define the set $\Lambda^T = \{(\mu,\lambda)\colon (\lambda,\mu) \in \Lambda\}$ and put

$$\Gamma = (\Lambda + \Lambda^T) \cup \bigcup_{j=1}^d (\Lambda + \Lambda^T + (\mathrm{e}_j, 0_d)) \cup (\Lambda + \Lambda^T + (0_d, \mathrm{e}_j)), \tag{3.4}$$

which will serve as an indexing set for the given $p \times p$ matrices $\{S_{(\lambda,\nu)}\}_{(\lambda,\mu)\in\Gamma}$. Introduce the matrix Φ as follows. Index the rows and of the matrix Φ by Λ and the columns of Φ by Λ^T. Let the entry in the row indexed by $(\alpha,\beta)\in\Lambda$ and the column indexed by $(\mu,\lambda)\in\Lambda^T$ be given by $S_{(\alpha+\beta)+(\mu,\lambda)}$. That is,

$$\Phi = (S_{(\alpha+\mu,\beta+\lambda)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T}.$$

Next, introduce the matrix Φ_{z_j} as follows. Index the rows of Φ_{z_j} by Λ and columns of Φ_{z_j} by Λ^T. Let the entry in the row indexed by $(\alpha,\beta)\in\Lambda$ and the column indexed by $(\mu,\lambda)\in\Lambda^T$ be given by $S_{(\alpha+\beta)+(\mu,\lambda)+(0_d,\mathrm{e}_j)}$, $1\le j\le d$. That is,

$$\Phi_{z_j} = (S_{(\alpha+\mu,\beta+\lambda+\mathrm{e}_j)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T},\ 1\le j\le d.$$

Finally, introduce the matrix $\Phi_{\bar z_j}$ as follows. Index the rows of $\Phi_{\bar z_j}$ by Λ and the columns of $\Phi_{\bar z_j}$ by Λ^T. Let the entry in the row indexed by $(\alpha,\beta)\in\Lambda$ and the column indexed by $(\mu,\lambda)\in\Lambda^T$ be given by $S_{(\alpha,\beta)+(\mu,\lambda)+(\mathrm{e}_j,0_d)}$, $1\le j\le d$. That is,

$$\Phi_{\bar z_j} = (S_{(\alpha+\mu+\mathrm{e}_j,\beta+\lambda)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T},\ 1\le j\le d.$$

Note that since $\hat\nu(m,n)=\hat\nu(n,m)^*$, we necessarily have that $\Phi=\Phi^*$ and $\Phi_{z_j}^*=\Phi_{\bar z_j}, 1\le j\le d$.

Let us consider the following example, which illustrates how $\Phi, \Phi_{z_1}, \Phi_{\bar z_1},\ldots,\Phi_{z_d}$, $\Phi_{\bar z_d}$ are constructed with respect to a particular lattice set $\Lambda\subset\mathbb{N}_0\times\mathbb{N}_0$.

Example 3.1. Let $\Lambda=\{(0,0),(0,1),(1,0)\}$. Then $\Lambda^T=\{(0,0),(1,0),(0,1)\}$ and $\Gamma=(\Lambda+\Lambda^T)\cup(\Lambda+\Lambda^T+(1,0))\cup(\Lambda+\Lambda^T+(0,1))=\{(m,n)\in\mathbb{N}_0^2\colon 0\le m+n\le 3\}$. We get the following matrices:

$$\Phi=\begin{pmatrix} S_{00} & S_{10} & S_{01}\\ S_{01} & S_{11} & S_{02}\\ S_{10} & S_{20} & S_{11}\end{pmatrix},$$

$$\Phi_{\overline{z}_1}=\begin{pmatrix} S_{10} & S_{20} & S_{11}\\ S_{11} & S_{21} & S_{12}\\ S_{20} & S_{30} & S_{21}\end{pmatrix},$$

and

$$\Phi_{z_1}=\begin{pmatrix} S_{01} & S_{11} & S_{02}\\ S_{02} & S_{12} & S_{12}\\ S_{11} & S_{21} & S_{12}\end{pmatrix}.$$

Let us catalog some basic necessary conditions for the given data to have a representing measure.

Lemma 3.2. *Let $L\subset\mathbb{N}_0^d\times\mathbb{N}_0^d$, with* card $L=k$, *and suppose the $\mathbb{C}^{p\times p}$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where $\Gamma=L+L^T$, is given. Let $\Phi=(S_{(\alpha,\beta)+(\mu,\lambda)})_{(\alpha,\beta)\in L,\ (\mu,\lambda)\in L^T}$. If $\{S_\gamma\}_{\gamma\in\Gamma}$ has a representing measure ν, then $\Phi\succeq 0$.*

Proof. For any vector $y=\mathrm{col}(y_{(\alpha,\beta)})_{(\alpha,\beta)\in\Lambda}\in\mathbb{C}^{kp}$, we have

$$\int_{\mathbb{C}^d}\left\langle d\nu(z)\sum_{(\alpha,\beta)\in L}\bar z^\alpha z^\beta y_{(\alpha,\beta)},\ \sum_{(\lambda,\mu)\in L}\bar z^\lambda z^\mu y_{(\lambda,\mu)}\right\rangle\ge 0. \tag{3.5}$$

If we use the sesquilinearity of $\langle\cdot,\cdot\rangle$, then (2.5) becomes

$$\sum_{(\alpha,\beta)\in L}\int_{\mathbb{C}^d}\bar z^\alpha z^\beta\langle d\nu(z)y_{(\alpha,\beta)},\ \sum_{(\lambda,\mu)\in L}\bar z^\lambda, z^\mu y_{(\lambda,\mu)}\rangle\ge 0.$$

Antilinearity and sequilinearity of $\langle \cdot, \cdot \rangle$ yield

$$\sum_{(\alpha,\beta),(\lambda,\mu)\in L} \int_{\mathbb{C}^d} \bar{z}^{\alpha+\mu} z^{\beta+\lambda} \langle d\nu(z) y_{(\lambda,\mu)}, y_{(\lambda,\mu)} \rangle \geq 0.$$

Use the fact that $S_{\gamma,\tilde{\gamma}} = \int_{\mathbb{C}^d} \bar{z}^\gamma z^{\tilde{\gamma}} d\nu(z)$ for any $(\gamma, \tilde{\gamma}) \in \Gamma$; then we get

$$\sum_{(\alpha,\beta),(\lambda,\mu)\in L} \langle S_{(\alpha+\mu,\beta+\lambda)} y_{(\alpha,\beta)}, y_{(\lambda,\mu)} \rangle \geq 0,$$

i.e., $\Phi \geq 0$. □

Lemma 3.3. *Let L, $\{S_\gamma\}_{\gamma\in\Gamma}$ and Φ be as in Lemma 3.2. If $\{S_\gamma\}_{\gamma\in\Gamma}$ has a finitely atomic representing measure ν, i.e., $\nu = \sum_{q=1}^k T_q \delta_{u_q}$, where $T_1, \ldots, T_k \geq 0$ and $u_1, \ldots, u_k$ are distinct points in $\mathbb{C}^d$, then*

$$\text{rank } \Phi \leq \sum_{q=1}^k \text{rank } T_q. \tag{3.6}$$

Proof. For $(m, n) \in \mathbb{N}_0^d \times \mathbb{N}_0^d$,

$$\begin{aligned} S_{(m,n)} &= \int \bar{z}^m z^n d\nu(z) \\ &= \sum_{q=1}^k T_q \bar{u}_q^m u_q^n. \end{aligned}$$

One can check that

$$\Phi = (S_{(\alpha,\beta)+(\mu,\lambda)})_{(\alpha,\beta)\in L,\ (\mu,\lambda)\in L^T} = (V \otimes I_p)^* R (V \otimes I_p), \tag{3.7}$$

where $V = \begin{pmatrix} \bar{u}_1^{\lambda_1} u_1^{\mu_1} & \ldots & \bar{u}_1^{\lambda_k} u_1^{\mu_k} \\ \vdots & & \vdots \\ \bar{u}_k^{\lambda_1} u_k^{\mu_1} & \ldots & \bar{u}_k^{\lambda_k} u_k^{\mu_k} \end{pmatrix}$ and $R = T_1 \oplus \cdots \oplus T_k$. We then get rank $\Phi \leq$ rank R, whence we arrive at (2.6). □

We will now formulate the main result of section 3, which provides conditions on a set of given square matrices, indexed by a particular family of lattice sets whose construction was given in (3.4), to admit a minimal K-representing measure on $\mathbb{C}^d$.

Theorem 3.4. *Let $K \subseteq \mathbb{C}^d$, $\Lambda \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$ be a lattice set and suppose the $\mathbb{C}^{p\times p}$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where*

$$\Gamma = (\Lambda + \Lambda^T) \cup \bigcup_{j=1}^d (\Lambda + \Lambda^T + (\mathrm{e}_j, 0)) \cup (\Lambda + \Lambda^T + (0, \mathrm{e}_j))$$

is given. Let

$$\Phi = (S_{(\alpha,\beta)+(\mu,\lambda)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T},$$

$$\Phi_{z_j} = (S_{(\alpha,\beta)+(\mu,\lambda)+(0,\mathrm{e}_j)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T},\ 1 \leq j \leq d,$$

and

$$\Phi_{\bar{z}_j} = (S_{(\alpha,\beta)+(\mu,\lambda)+(\mathrm{e}_j,0)})_{(\alpha,\beta)\in\Lambda,\ (\mu,\lambda)\in\Lambda^T}, 1 \leq j \leq d.$$

There exists a solution to the truncated matrix-valued K-moment problem on $\mathbb{C}^d$, i.e., there exists a positive $\mathcal{H}_p$-valued measure so that (3.2) and (3.3) hold if the following conditions are satisfied:

(1) $\Phi \geq 0$ *and* $\Phi_{z_j}^* = \Phi_{\bar{z}_j}$.

(2) There exist matrices $\Theta_{z_1}, \ldots, \Theta_{z_d}, \Theta_{\bar{z}_1}, \ldots, \Theta_{\bar{z}_d}$ *which commute with respect to* $\mathfrak{M} = \operatorname{Ran} \Phi$ *so that*

  (i) $\Phi\Theta_{z_j} = \Phi_{z_j}$, *and* $\Phi\Theta_{\bar{z}_j} = \Phi_{\bar{z}_j}$, $1 \leq j \leq d$;

  (ii) $\Theta_{z_1}, \ldots, \Theta_{z_d}$ *satisfy the K-inclusive eigenvalue property with respect to* $\mathfrak{M} = \operatorname{Ran} \Phi$.

In that case, we can find ν of the following form:

$$\nu = \sum_{q=1}^{k} T_q \delta_{u_q}, \tag{3.8}$$

where $u_1, \ldots, u_k$ *are different points in* $\mathbb{C}^d$ *and* $T_1, \ldots, T_k \geq 0$ *with* $\sum_{q=1}^k \operatorname{rank} T_q = \operatorname{rank} \Phi$, *i.e.,* ν *is minimal.*

Remark 3.5. Since the measure ν constructed in Theorem 3.4 has finite support, all moments $\hat{\nu}(m,n)$, $(m,n) \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, automatically exist.

Remark 3.6. When $K = \mathbb{C}^d$ in Theorem 3.4, condition (2)(ii) in Theorem 3.4 can be omitted.

We will now prove useful matrix factorization results which are analogous to Lemma 2.14 and Theorem 2.15, respectively.

Lemma 3.7. *Let $A \geq 0$ and B_j be $p \times p$ matrices so that there are matrices $W_1, \ldots, W_d, Y_1, \ldots, Y_d$ which commute with respect to $\mathfrak{M} = \operatorname{Ran} A$ and satisfy $AW_j = B_j$ and $AY_j = B_j^*$, $1 \leq j \leq d$. Put $k = \operatorname{rank} A$. Then there exist $k \times k$ diagonal matrices $D_1, \ldots, D_d$ and a $p \times k$ injective matrix C so that*

$$A = CC^* \ \text{ and } \ B_j = CD_jC^*, \ \ 1 \leq j \leq d. \tag{3.9}$$

Proof. Decompose $\mathbb{C}^p = \operatorname{Ran} A \oplus \operatorname{Ker} A$. We claim that with respect to this decomposition we have

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix}, \ W_j = \begin{pmatrix} \tilde{W}_j & 0 \\ * & * \end{pmatrix}, \ Y_j = \begin{pmatrix} \tilde{Y}_j & 0 \\ * & * \end{pmatrix}, \ \text{ and } B_j = \begin{pmatrix} \tilde{B}_j & 0 \\ 0 & 0 \end{pmatrix}, 1 \leq j \leq d.$$

A priori, with respect to the decomposition $\mathbb{C}^p = \operatorname{Ran} A \oplus \operatorname{Ker} A$, we have

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_j = \begin{pmatrix} \tilde{B}_j & B_{12}^{(j)} \\ B_{21}^{(j)} & B_{22}^{(j)} \end{pmatrix},$$

$$W_j = \begin{pmatrix} \tilde{W}_j & W_{12}^{(j)} \\ * & * \end{pmatrix},$$

and

$$Y_j = \begin{pmatrix} \tilde{Y}_j & Y_{12}^{(j)} \\ * & * \end{pmatrix}.$$

Then $AW_j = B_j$ gives the fact that $W_{12}^{(j)} = \tilde{A}^{-1} B_{12}^{(j)}$ and $B_{21}^{(j)} = B_{22}^{(j)} = 0$. Next, $AY_j = B_j^*$ gives the fact that $Y_{12}^{(j)} = \tilde{A}^{-1} B_{21}^{(j)} = B_{12}^{(j)} = 0$ and hence $W_{12}^{(j)} = 0$,

$1 \le j \le d$, and so we have the claim. We also have that $\tilde{W}_j^* = \tilde{Y}_j$, since $AW_j = B$ and $AY_j = B^*$, $1 \le j \le d$.

Since $W_1, \dots, W_d, Y_1, \dots, Y_d$ commute with respect to $\mathfrak{M} = \operatorname{Ran} A$, we get that $\tilde{W}_1, \dots, \tilde{W}_d, \tilde{Y}_1, \dots, \tilde{Y}_d$ commute. Thus we have that $\tilde{A}^{\frac{1}{2}} \tilde{W}_j \tilde{A}^{-\frac{1}{2}}$ and $\tilde{A}^{\frac{1}{2}} \tilde{Y}_j \tilde{A}^{-\frac{1}{2}}$, $1 \le j \le d$, are commuting normal matrices. Note that normality follows from the fact that $(\tilde{A}^{\frac{1}{2}} \tilde{W}_j \tilde{A}^{-\frac{1}{2}})^* = \tilde{A}^{\frac{1}{2}} \tilde{Y}_j \tilde{A}^{-\frac{1}{2}}$, where we used the fact that $\tilde{W}_j^* = \tilde{A} \tilde{Y}_j \tilde{A}^{-1}, 1 \le j \le d$.

Thus there must exist a unitary U and diagonal D_j so that

$$\tilde{A}^{\frac{1}{2}} \tilde{W}_j \tilde{A}^{-\frac{1}{2}} = U D_j U^* \tag{3.10}$$

and

$$\tilde{A}^{\frac{1}{2}} \tilde{Y}_j \tilde{A}^{-\frac{1}{2}} = U D_j^* U^*. \tag{3.11}$$

Put $C = \begin{pmatrix} \tilde{A}^{\frac{1}{2}} \\ 0 \end{pmatrix} U$, so then $A = CC^*$ and $B_j = C D_j C^*, 1 \le j \le d$. □

Theorem 3.8. *Let* $\Lambda \subset \mathbb{N}_0^d \times \mathbb{N}_0^d$, *with* card $\Lambda = a$, *be a lattice set and*

$$\Gamma = (\Lambda + \Lambda^T) \cup \bigcup_{j=1}^{d} (\Lambda + \Lambda^T + (\mathrm{e}_j, 0)) \cup (\Lambda + \Lambda^T + (0, \mathrm{e}_j)).$$

Suppose the $\mathbb{C}^{p \times p}$*-valued sequence* $\{S_{(\gamma,\tilde{\gamma})}\}_{(\gamma,\tilde{\gamma}) \in \Gamma}$ *is given. Set*

$$\Phi = (S_{(\lambda,\mu)+(\beta,\alpha)})_{(\lambda,\mu)\in\Lambda,(\beta,\alpha)\in\Lambda^T},$$

$$\Phi_{z_j} = (S_{(\lambda,\mu)+(\beta,\alpha)+(0_d,\mathrm{e}_j)})_{(\lambda,\mu)\in\Lambda,(\beta,\alpha)\in\Lambda^T},\ 1 \le j \le d,$$

$$\Phi_{\bar{z}_j} = (S_{(\lambda,\mu)+(\beta,\alpha)+(\mathrm{e}_j,0_d)})_{(\lambda,\beta)\in\Lambda,(\beta,\alpha)\in\Lambda^T}, 1 \le j \le d,$$

and put $k = \operatorname{rank} \Phi$. *Suppose that* $\Phi \ge 0$ *and* $\Theta_{z_1}, \dots, \Theta_{z_d}, \Theta_{\bar{z}_1}, \dots, \Theta_{\bar{z}_d}$ *are* $ap \times ap$ *matrices which commute with respect to* $\mathfrak{M} = \operatorname{Ran} \Phi$ *and satisfy* $\Phi \Theta_{z_j} = \Phi_j$ *and* $\Phi \Theta_{\bar{z}_j} = \Phi_j^*, 1 \le j \le d$. *Then there exist a* $p \times k$ *matrix* $C_{(0_d,0_d)}$ *and* $k \times k$ *diagonal matrices* $D_1, \dots, D_d$ *so that*

$$S_{(\gamma,\tilde{\gamma})} = C_{(0_d,0_d)} (D_1^{g_1} \cdots D_d^{g_d})^* D_1^{\tilde{g}_1} \cdots D_d^{\tilde{g}_d} C_{(0_d,0_d)}^*, \tag{3.12}$$

for all $\gamma = (g_1, \dots, g_d), \tilde{\gamma} = (\tilde{g}_1, \dots, \tilde{g}_d)$ *so that* $(\gamma, \tilde{\gamma}) \in \Gamma$.

Proof. Consider $A = \Phi$, $W_j = \Theta_{z_j}$, $Y_j = \Theta_{\bar{z}_j}$, $B_j = \Phi_{z_j}$, $1 \le j \le d$. Then $A \ge 0$ and $W_1, \dots, W_d, Y_1, \dots, Y_d$ are matrices which commute with respect to $\mathfrak{M} = \operatorname{Ran} A$ and satisfy $AW_j = B_j$ and $AY_j = B_j^*, 1 \le i \le d$. We can apply Lemma 3.7 to obtain an injective matrix C and diagonal matrices $D_1, \dots, D_d$ so that

$$\Phi = CC^* \text{ and } \Phi_{z_j} = C D_j C^*,\ 1 \le j \le d.$$

Write $C = \operatorname{col}(C_{(\alpha,\beta)})_{(\alpha,\beta)\in\Lambda}$. When $(\alpha, \beta) \in \Lambda$ and $(\mu, \lambda), (\mu, \lambda) + (\mathrm{e}_j, 0_d) \in \Lambda^T$, we get $(\lambda, \mu), (\lambda, \mu) + (0_d, \mathrm{e}_j) \in \Lambda$ and $(\beta, \alpha) \in \Lambda^T$. Realize

$$\begin{aligned} S_{(\alpha,\beta)+(\mu,\lambda)+(\mathrm{e}_j,0_d)} &= C_{(\alpha,\beta)} C_{(\lambda,\mu)+(0_d,\mathrm{e}_j)}^* \\ &= C_{(\alpha,\beta)} D_j^* C_{(\lambda,\mu)}^*. \end{aligned} \tag{3.13}$$

Notice that (3.13) implies

$$C(C_{(\lambda,\mu)+(0_d,\mathrm{e}_j)}^* - D_j^* C_{(\lambda,\mu)}^*) = 0. \tag{3.14}$$

Since C is injective, (3.14) yields

$$C_{(\lambda,\mu)+(0_d,\mathrm{e}_j)} = C_{(\lambda,\mu)} D_j \tag{3.15}$$

whenever $(\lambda,\mu),(\lambda,\mu)+(0_d,\mathrm{e}_j)\in\Lambda$. It is also not hard to see that we also have

$$C_{(\lambda,\mu)+(\mathrm{e}_j,0_d)} = C_{(\lambda,\mu)}D_j^* \tag{3.16}$$

whenever $(\lambda,\mu),(\lambda,\mu)+(\mathrm{e}_j,0_d)\in\Lambda$.

Consider $(\lambda,\mu)\in\Lambda$, where $\lambda=(l_1,\dots,l_d)$ and $\mu=(m_1,\dots,m_d)$. Since Λ is a lattice set, there exist $(\lambda_1,\mu_1),\dots,(\lambda_k,\mu_k)\in\Lambda$, $i_1,\dots,i_k\in\{0,1\}$, and $j_1,\dots,j_k\in\{1,\dots,d\}$ so that

$$\begin{aligned}(\lambda_1,\mu_1) &= (i_1\mathrm{e}_{j_1},(1-i_1)\mathrm{e}_{j_1})\\ &\vdots\\ (\lambda,\mu) &= (\lambda_k,\mu_k)+(i_k\mathrm{e}_{j_k},(1-i_k)\mathrm{e}_{j_k}).\end{aligned}$$

Choose $(\alpha,\beta)=(\lambda,\mu)-(i_k\mathrm{e}_{j_k},(1-i_k)\mathrm{e}_{j_k})$ so that (3.15) and (3.16) give

$$C_{(\lambda,\mu)} = \begin{cases} C_{\alpha,\beta}D_{j_k} & \text{if } i_k=1,\\ C_{\alpha,\beta}D_{j_k}^* & \text{if } i_k=0.\end{cases}$$

Continuing this way we arrive at

$$C_{(\lambda,\mu)} = C_{(0_d,0_d)}(D_1^{l_1})^*\cdots(D_d^{l_d})^*D_1^{m_1}\cdots D_d^{m_d}. \tag{3.17}$$

But then $\Phi=CC^*$ and $\Phi_{z_j}=CD_jC^*$, $1\le j\le d$, give (2.12). □

Remark 3.9. Similar to Theorem 2.15, the fact that Λ was a lattice was used to achieve the factorization in (3.12). As we will see, (3.12) will be useful when proving Theorem 3.4.

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Suppose $\Phi\ge 0$ and there exist matrices $\Theta_{z_1},\dots,\Theta_{z_d},\Theta_{\bar z_1},\dots,\Theta_{\bar z_d}$ which commute with respect to $\mathfrak{M}=\operatorname{Ran}\Phi$ and $\Theta_{z_1},\dots,\Theta_{z_d}$ which satisfy the K-inclusive eigenvalue property, $\Phi\Theta_{z_j}=\Theta_{z_j}$, and $\Phi\Theta_{\bar z_j}=\Phi_{\bar z_j}$, $1\le j\le d$. Write

$$\Theta_{z_j}=\begin{pmatrix}\tilde\Theta_{z_j} & 0\\ * & *\end{pmatrix}:\begin{matrix}\operatorname{Ran}\Phi\\ \bigoplus\\ \operatorname{Ker}\Phi\end{matrix}\to\begin{matrix}\operatorname{Ran}\Phi\\ \bigoplus\\ \operatorname{Ker}\Phi\end{matrix},\ \ 1\le j\le d,$$

and so we must have the existence of an invertible matrix S so that $S^{-1}\tilde\Theta_{z_j}S=\operatorname{diag}(u_j^{(1)},\dots,u_j^{(r)})$, $1\le j\le d$, where $r=\operatorname{rank}\Phi$ and $(u_1^{(q)},\dots,u_d^{(q)})\in K$, $1\le q\le r$. Use Theorem 3.8 to obtain an injective matrix $C=\operatorname{col}(C_{(\lambda,\mu)})_{(\lambda,\mu)\in\Lambda}$ and diagonal matrices $D_1,\dots,D_d$ so that $\Phi=CC^*$, $\Phi_{z_j}=CD_jC^*$, and $\Phi_{\bar z_j}=CD_j^*C^*$, $1\le j\le d$. By (3.10), we have that $D_j=\operatorname{diag}(u_j^{(1)},\dots,u_j^{(r)})$, $1\le j\le d$. Write $C_{(0_d,0_d)}=\begin{pmatrix}c_1&\cdots&c_r\end{pmatrix}$, where $c_1,\dots,c_r\in\mathbb{C}^p$. Then (3.12) holds. Without loss of generality, assume $c_1,\dots,c_k$ are distinct. Fix $q\in\{1,\dots,k\}$ and consider the set $\mathcal{I}_q=\{\alpha\in\{1,\dots,r\}\colon c_\alpha=c_j\}$. Note that

$$\operatorname{card}\bigcup_{q=1,\dots,r}\mathcal{I}_q=r.$$

Let $T_q=\sum_{\alpha\in\mathcal{I}_q}c_\alpha c_\alpha^*\ge 0$ and note that (3.17) gives the fact that $\{c_\alpha\}_{\alpha\in\mathcal{I}_q}$ is linearly independent since C is injective. Hence $\operatorname{rank}T_q=\operatorname{card}\mathcal{I}_q$, and so we get

$$\operatorname{rank}T_q=\operatorname{card}\mathcal{I}_q,\ 1\le q\le k,$$

i.e., $\sum_{q=1}^k \operatorname{rank} T_j = r$, and so $\operatorname{rank} \Phi = \sum_{q=1}^k \operatorname{rank} T_q$. Put ν as in the statement of Theorem 3.4. One can directly verify (3.2) and (3.3). Note that the existence of all subsequent moments follows from Remark 3.5. $\square$

Finally we will state a converse to Theorem 3.4 when $d = 1$.

Theorem 3.10. *Let*

$$\nu = \sum_{q=1}^{k} T_q \delta_{u_q}$$

be given, where $T_1, \dots, T_k \geq 0$ and $u_1, \dots, u_k$ are different points in $K \subseteq \mathbb{C}$. Put $S_{(m,n)} = \int \bar{z}^m z^n d\nu(z)$. There exists a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0$ with k points so that we can build

$$\Phi = (S_{(\alpha,\beta)+(\mu,\lambda)})_{(\alpha,\beta)\in\tilde{\Lambda},\ (\mu,\lambda)\in\tilde{\Lambda}^T},$$

$$\Phi_{z_1} = (S_{\lambda+\mu+(0,\mathrm{e}_1)})_{(\alpha,\beta)\in\tilde{\Lambda},\ (\mu,\lambda)\in\tilde{\Lambda}^T},$$

and

$$\Phi_{\bar{z}_1} = (S_{\lambda+\mu+(\mathrm{e}_1,0)})_{(\alpha,\beta)\in\tilde{\Lambda},\ (\mu,\lambda)\in\tilde{\Lambda}^T}$$

so that conditions (1) *and* (2) *in Theorem* 3.4 *hold. Moreover, we have that* $\operatorname{rank} \Phi = \sum_{q=1}^k \operatorname{rank} T_q$.

Proof. Simply choose $(0,0), \dots, (0, k-1)$ and notice that the Vandermonde matrix

$$V = \begin{pmatrix} 1 & \cdots & u_1^{k-1} \\ \vdots & & \vdots \\ 1 & \cdots & u_k^{k-1} \end{pmatrix}$$

is invertible, due to $u_1, \dots, u_k \in \mathbb{C}$ being distinct. One can check that $\Phi = V^* R V \geq 0$, $\Phi_{z_1} = (V \otimes I_p)^* R Z (V \otimes I_p)$ and $\Phi_{\bar{z}_1} = (V \otimes I_p)^* R Z^* (V \otimes I_p)$, where $R = T_1 \oplus \cdots \oplus T_k$ and $Z = u_1 I_p \oplus \cdots \oplus u_k I_p$. Choosing

$$\Theta_{z_1} = (V^*)^{-1} Z V$$

and

$$\Theta_{\bar{z}_1} = (V^*)^{-1} Z^* V$$

yield Θ_{z_1} and $\Theta_{\bar{z}_1}$, which commute with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$. Note that Θ_{z_1} satisfies the K-inclusive property with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$, $\Phi\Theta_{z_1} = \Phi_{z_1}$, and $\Phi\Theta_{\bar{z}_1} = \Phi_{\bar{z}_1}$. Moreover, by construction we have that $\sum_{q=1}^k \operatorname{rank} T_q = \operatorname{rank} \Phi$. $\square$

We will end this section by posing a question which would lead to a generalization of Theorem 3.10 when $d > 1$. Given distinct points $u_1, \dots, u_k \in \mathbb{C}^d$, can one choose $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$, which form a lattice set in $\mathbb{N}_0^d$, so that

$$\begin{pmatrix} \bar{u}_1^{\alpha_1} u_1^{\beta_1} & \cdots & \bar{u}_1^{\alpha_1} u_1^{\beta_k} \\ \vdots & & \vdots \\ \bar{u}_k^{\alpha_1} u_k^{\beta_1} & \cdots & \bar{u}_k^{\alpha_k} u_k^{\beta_k} \end{pmatrix}$$

is invertible?

4. The truncated matrix-valued K-moment problem on $\mathbb{T}^d$

We will now consider the truncated matrix-valued K-moment problem on $\mathbb{T}^d$. First, we will introduce some preliminary notions and definitions. The power moments of a positive $\mathcal{H}_p$-valued measure τ on $\mathbb{T}^d$ are defined by the formula

$$\hat{\tau}(m) := \int z^m d\tau(z) := \int_{\mathbb{T}^d} z^m d\tau(z), \tag{4.1}$$

where $m \in \mathbb{Z}^d$, provided the integrals converge. Note that $\hat{\tau}(m)^* = \hat{\tau}(-m)$. For $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$, we define the length $|m| = |m_1| + \cdots + |m_d|$. We will say that a finite subset $\Lambda \subset \mathbb{Z}^d$ is a lattice set when for all $\lambda \in \Lambda$, there exist $i_1, \dots, i_k \in \{-1, 1\}$, $j_1, \dots, j_k \in \{1, \dots, d\}$, and $\lambda_1 = 0_d, \dots, \lambda_k$ so that

$$\begin{aligned} \lambda_2 &= \lambda_1 + i_1 \mathrm{e}_{j_1} \\ &\vdots \\ \lambda &= \lambda_k + i_k \mathrm{e}_{j_k}, \end{aligned}$$

where $|\lambda| = k$. We will say a lattice set $\Lambda \subset \mathbb{Z}^d$ is *symmetric* when $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$. Given a symmetric lattice set $\Gamma \subset \mathbb{Z}^d$, let $\{S_m\}_{m \in \Gamma}$ be a given $\mathbb{C}^{p\times p}$-valued sequence. We look for a positive $\mathbb{C}^{p\times p}$-valued measure τ so that $\hat{\tau}(m)$ exists for all $m \in \mathbb{Z}^d$,

$$\hat{\tau}(m) = S_m \text{ for all } m \in \Gamma, \tag{4.2}$$

and

$$\operatorname{supp} \tau \subseteq K. \tag{4.3}$$

Given a set $\Lambda \subset \mathbb{N}_0^d$, we define $\Lambda - \Lambda = \{\lambda - \mu \colon \lambda, \mu \in \Lambda\}$, $\Lambda - \Lambda + \mathrm{e}_j = \{\lambda - \mu + \mathrm{e}_j \colon \lambda, \mu \in \Lambda\}$, and $\Lambda - \Lambda - \mathrm{e}_j = \{\lambda - \mu - \mathrm{e}_j \colon \lambda, \mu \in \Lambda\}$, $1 \le j \le d$. Given a lattice set $\Lambda \subset \mathbb{N}_0^d$, put

$$\Gamma = (\Lambda - \Lambda) \bigcup_{j=1}^{d} ((\Lambda - \Lambda + \mathrm{e}_j) \cup (\Lambda - \Lambda - \mathrm{e}_j)). \tag{4.4}$$

Note that Γ in (4.4) is a symmetric lattice set. In addition, Γ will serve as an indexing set for the given $\mathbb{C}^{p\times p}$-valued sequence $\{S_\gamma\}_{\gamma \in \Gamma}$. Introduce the matrix Φ as follows. Index the rows of the matrix Φ by Λ and the columns of Φ by Λ. Let the entry in the row indexed by λ and the column indexed by μ be given by $S_{\lambda-\mu}$. That is,

$$\Phi = (S_{\lambda-\mu})_{\lambda,\mu\in\Lambda}.$$

Next, introduce the matrix Φ_j as follows. Index the rows of Φ_j by Λ and the columns of Φ_j by Λ. Let the entry in the row indexed by λ and the column indexed by μ be given by $S_{\lambda-\mu+\mathrm{e}_j}$, $1 \le j \le d$. That is,

$$\Phi_j = (S_{\lambda-\mu+\mathrm{e}_j})_{\lambda,\mu\in\Lambda},\ 1 \le j \le d.$$

Finally, introduce the matrix Φ_{-j} as follows. Index the rows of Φ_{-j} by Λ and the columns of Φ_{-j} by Λ. Let the entry in the row indexed by λ and the column indexed by μ be given by $S_{\lambda-\mu-\mathrm{e}_j}$, $1 \le j \le d$. That is,

$$\Phi_{-j} = (S_{\lambda-\mu-\mathrm{e}_j})_{\lambda,\mu\in\Lambda},\ 1 \le j \le d.$$

Note that since $\hat{\tau}(m)^* = \hat{\tau}(-m)$, we necessarily have $\Phi = \Phi^*$ and also $\Phi_j^* = \Phi_{-j}, 1 \le j \le d$.

Let us consider the following example, which illustrates how $\Gamma, \Phi, \Phi_1, \Phi_{-1}, \dots, \Phi_d, \Phi_{-d}$ are constructed with respect to a particular lattice set $\Lambda \subset \mathbb{N}_0^2$.

Example 4.1. Let $\Lambda = \{(0,0),(0,1),(1,0)\}$. Then

$$\begin{aligned}\Gamma &= (\Lambda - \Lambda) \cup (\Lambda - \Lambda + (1,0)) \cup (\Lambda + \Lambda - (1,0)) \\ &\quad \cup (\Lambda - \Lambda + (0,1)) \cup (\Lambda - \Lambda - (0,1)) \\ &= \{m \in \mathbb{Z}^2 \colon 0 \le |m| \le 3\} \setminus \{(3,0),(0,3),(-3,0),(0,-3)\}.\end{aligned}$$

We get the following matrices:

$$\Phi = \begin{pmatrix} S_{(0,0)} & S_{(0,-1)} & S_{(-1,0)} \\ S_{(0,1)} & S_{(0,0)} & S_{(-1,1)} \\ S_{(1,0)} & S_{(1,-1)} & S_{(0,0)} \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} S_{(1,0)} & S_{(1,-1)} & S_{(0,0)} \\ S_{(1,1)} & S_{(1,0)} & S_{(0,1)} \\ S_{(2,0)} & S_{(2,-1)} & S_{(1,0)} \end{pmatrix}, \quad \Phi_{-1} = \begin{pmatrix} S_{(-1,0)} & S_{(-1,-1)} & S_{(-2,0)} \\ S_{(-1,1)} & S_{(-1,0)} & S_{(-2,1)} \\ S_{(0,0)} & S_{(0,-1)} & S_{(-1,0)} \end{pmatrix},$$

$$\Phi_2 = \begin{pmatrix} S_{(0,1)} & S_{(0,0)} & S_{(-1,1)} \\ S_{(0,2)} & S_{(0,1)} & S_{(-1,2)} \\ S_{(1,1)} & S_{(1,0)} & S_{(0,1)} \end{pmatrix}, \quad \text{and} \quad \Phi_{-2} = \begin{pmatrix} S_{(0,-1)} & S_{(0,-2)} & S_{(-1,-1)} \\ S_{(0,0)} & S_{(0,-1)} & S_{(-1,0)} \\ S_{(1,-1)} & S_{(1,-2)} & S_{(0,-1)} \end{pmatrix}.$$

Let us catalog some basic necessary conditions for given data to have a representing measure.

Lemma 4.2. *Let $L = \{l_1, \dots, l_k\} \subset \mathbb{N}_0^d$ and suppose the $\mathbb{C}^{p \times p}$-valued sequence $\{S_\gamma\}_{\gamma \in \Gamma}$, where $\Gamma = L - L$, is given. Let $\Phi = (S_{\lambda - \mu})_{\lambda, \mu \in L}$. If $\{S_\gamma\}_{\gamma \in \Gamma}$ has a representing measure τ supported on $\mathbb{T}^d$, then $\Phi \ge 0$.*

Proof. For any vector $y = \operatorname{col}(y_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^{kp}$, we have

$$\int \langle d\tau(z) \sum_{\lambda \in L} y_\lambda z^\lambda, \sum_{\mu \in L} y_\mu z^\mu \rangle \ge 0. \tag{4.5}$$

If we use the sesquilinearity of $\langle \cdot, \cdot \rangle$, then (4.5) becomes

$$\sum_{\lambda, \mu \in L} \int z^{\lambda - \mu} \langle d\tau(z) y_\lambda, y_\mu \rangle \ge 0,$$

where we used the fact that $z \in \mathbb{T}^d$. Also use the fact that $S_\gamma = \int z^\gamma d\tau(z)$, for any $\gamma \in \Gamma$, so then (4.5) becomes

$$\sum_{\lambda, \mu \in L} \langle S_{\lambda - \mu} y_\lambda, y_\mu \rangle \ge 0,$$

i.e., $\Phi \ge 0$. $\square$

Lemma 4.3. *Let L, $\{S_\gamma\}_{\gamma \in \Gamma}$, and Φ be as in Lemma 2.3. If $\{S_\gamma\}_{\gamma \in \Gamma}$ has a finitely atomic representing measure $\tau = \sum_{q=1}^k T_q \delta_{b_q}$, where $T_1, \dots, T_k \ge 0$ and $b_1, \dots, b_k$ are distinct points in $\mathbb{T}^d$, then*

$$\operatorname{rank} \Phi \le \sum_{q=1}^k \operatorname{rank} T_q. \tag{4.6}$$

Proof. For $m \in \mathbb{Z}^d$,

$$S_m = \int z^m d\tau(z) = \sum_{q=1}^{k} T_q b_q^m.$$

One can check that

$$\Phi = (S_{\lambda-\mu})_{\lambda,\mu\in L} = (V \otimes I_p)^* R (V \otimes I_p), \tag{4.7}$$

where $V = \begin{pmatrix} \overline{b}_1^{l_1} & \dots & \overline{b}_k^{l_k} \\ \vdots & & \vdots \\ \overline{b}_k^{l_1} & \dots & \overline{b}_k^{l_k} \end{pmatrix}$ and $R = T_1 \oplus \cdots \oplus T_k$. We then get $\operatorname{rank} \Phi \le \operatorname{rank} R$, whence we arrive at (4.6). □

We will now formulate the main result of this section, which provides conditions on a set of given square matrices, indexed by a particular family of symmetric lattice sets whose construction was given in (4.4), to admit a minimal representing measure supported on a given set $K \subseteq \mathbb{T}^d$.

Theorem 4.4. *Let $K \subseteq \mathbb{T}^d$, $\Lambda \subset \mathbb{N}_0^d$ be a lattice set, and suppose the $\mathbb{C}^{p\times p}$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$, where*

$$\Gamma = (\Lambda - \Lambda) \cup \bigcup_{j=1}^{d} (\Lambda - \Lambda + \mathrm{e}_j) \cup (\Lambda - \Lambda - \mathrm{e}_j)$$

is given. Let

$$\Phi = (S_{\lambda-\mu})_{\lambda,\mu\in\Lambda},$$
$$\Phi_j = (S_{\lambda-\mu+\mathrm{e}_j})_{\lambda,\mu\in\Lambda},\ 1 \le j \le d,$$

and

$$\Phi_{-j} = (S_{\lambda-\mu-\mathrm{e}_j})_{\lambda,\mu\in\Lambda}, 1 \le j \le d.$$

There exists a solution to the truncated matrix-valued K-moment problem on $\mathbb{T}^d$, i.e., there exists a positive $\mathcal{H}_p$-valued measure τ so that (4.2) and (4.3) hold, if the following conditions are satisfied:

(1) *$\Phi \ge 0$ and $\Phi_j^* = \Phi_{-j}$, $1 \le j \le d$.*
(2) *There exist matrices $\Theta_1, \dots, \Theta_d, \Theta_{-1}, \dots, \Theta_{-d}$ which commute with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$ so that*
   (i) *$\Phi\Theta_j = \Phi_j$ and $\Phi\Theta_{-j} = \Phi_{-j}$, $1 \le j \le d$;*
   (ii) *$\Theta_1, \dots, \Theta_d$ have the K-inclusive eigenvalue property with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$.*

In that case, we can find τ of the following form:

$$\tau = \sum_{q=1}^{k} T_q \delta_{b_q}, \tag{4.8}$$

where $b_1, \dots, b_k$ are different points in K and $T_1, \dots, T_k \ge 0$ with $\sum_{q=1}^{k} \operatorname{rank} T_q = \operatorname{rank} \Phi$.

Conversely, when $d = 1, 2$, if τ is of the form (4.8), where $T_1, \dots, T_k \ge 0$ and $b_1, \dots, b_k$ are different points in $K \subseteq \mathbb{T}^d$, there exists a lower inclusive set

$\tilde{\Lambda} \subset \mathbb{N}_0^d$, *with* k *points, so that* $\Phi = (S_{\lambda-\mu})_{\lambda,\mu\in\tilde{\Lambda}}$, $\Phi_j = (S_{\lambda-\mu+e_j})_{\lambda,\mu\in\tilde{\Lambda}}$, *and* $\Phi_{-j} = (S_{\lambda-\mu-e_j})_{\lambda,\mu\in\Lambda}$, $1 \le j \le d$, *satisfying conditions* (1) *and* (2), *where* $S_m = \int z^m d\tau(z)$, $m \in \mathbb{Z}^d$. *Moreover, we get that* $\operatorname{rank}\Phi = \sum_{q=1}^k \operatorname{rank} T_q$.

Remark 4.5. Since the measure τ constructed in Theorem 4.4 has finite support, all moments $\hat{\tau}(m)$, $m \in \mathbb{Z}^d$, automatically exist.

Remark 4.6. When $K = \mathbb{T}^d$ in Theorem 4.4, condition (2) simplifies to $\tilde{\Phi}^{\frac{1}{2}}\tilde{\Theta}_j\tilde{\Phi}^{-\frac{1}{2}}$ and $\tilde{\Phi}^{\frac{1}{2}}\tilde{\Theta}_{-j}\tilde{\Phi}^{-\frac{1}{2}}$, $1 \le j \le d$, which are unitary.

In order to prove Theorem 4.4 we must prove a result for two-variable Vandermonde matrices and prove a factorization result analogous to Theorem 2.15 and Theorem 3.8. Given a sequence of distinct points $b_1, \dots, b_n$ in $\mathbb{T}^2$ and a lower inclusive set $\{\lambda_1, \dots, \lambda_m\} \subset \mathbb{N}_0^2$, we define

$$V(b_1, \dots, b_n; \Lambda) := (\overline{b}_i^{\lambda_j})_{i=1,\dots,n;\, j=1,\dots,m}.$$

Consider the following example.

Example 4.7. If $b_1 = (1,1)$, $b_2 = (-1,1)$, $w_3 = (-i,1)$ and $\Lambda = \{(0,0),(0,1),(1,0)\}$, then

$$V(b_1, b_2, b_3; \Lambda) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & i \end{pmatrix}.$$

Note that Example 4.7 illustrates that unlike square Vandermonde matrices in one variable, distinctness of points in two or more variables is not enough to guarantee invertibility. A natural question is the following. Given distinct points $b_1, \dots, b_n$ in $\mathbb{T}^2$, can one construct a lower inclusive set $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{N}_0^2$ so that $V(b_1, \dots, b_n; \Lambda)$ is invertible? The following theorem resolves this question in the affirmative.

Theorem 4.8. *Given distinct points* $b_1 = (e^{ix_1}, e^{iy_1}), \dots, b_n = (e^{ix_n}, e^{iy_n})$ *in* $\mathbb{T}^2$, *there exists a lower inclusive set* $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{N}_0^2$ *so that* $V(b_1, \dots, b_n; \Lambda)$ *is invertible. Without loss of generality, we can reorder* $b_1, \dots, b_n$ *via*

$$\begin{aligned} b_1^{(1)} &:= (e^{ix_1}, e^{iy_1^{(1)}}), \dots, b_{n_1}^{(1)} := (e^{ix_1}, e^{iy_{n_1}^{(1)}}) \\ &\vdots \\ b_1^{(k)} &:= (e^{ix_k}, e^{iy_1^{(k)}}), \dots, b_{n_k}^{(k)} := (e^{ix_k}, e^{iy_{n_k}^{(k)}}), \end{aligned}$$

where $x_i \ne x_j$, *when* $i \ne j$, $n_1 \ge \dots \ge n_k \ge 1$, *and* $n_1 + \dots + n_k = n$. *Then we have*

$$\det V(b_1, \dots, b_n; \Lambda) = \prod_{1 \le i < j \le k} (x_j - x_i)^{n_i} \prod_{1 \le p < q \le n_i} (y_q^{(i)} - y_p^{(i)}). \tag{4.9}$$

Given distinct points $z_0, \dots, z_n \in \mathbb{C}$ and a function $f\colon \mathbb{C} \to \mathbb{C}$, the zeroth divided difference for the function f with respect to z_0 is

$$f[z_0] := f(z_0).$$

The other divided differences can be defined inductively via

$$f[z_i, \dots, z_{i+j}] = \frac{f[z_{i+1}, \dots, z_{i+j}] - f[z_i, \dots, z_{i+j-1}]}{z_{i+j} - z_i},\ 0 \le i \le n-j.$$

When $f(z) = z^n$ we will define $[z_i, \ldots, z_{i+j}]^n = f[z_i, \ldots, z_{i+j}]$. Divided differences arise naturally as coefficients of Newton interpolating polynomials. Suppose $z_1, \ldots, z_n$ are distinct complex numbers and we give $(z_0, f(z_0)), \ldots, (z_n, f(z_n))$. If we wish to construct the unique n-th degree: by Newton interpolation, then we must solve the following system:

$$\begin{pmatrix} 1 & & & 0 \\ 1 & z_1 - z_0 & & \\ \vdots & \vdots & \ddots & \\ 1 & z_n - z_0 & \cdots\cdots & \prod_{i=0}^{n-1}(z_n - z_i) \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f(z_0) \\ \vdots \\ f(z_n) \end{pmatrix}. \tag{4.10}$$

One easily sees that $y_0 = f[z_0], y_1 = f[z_0, z_1], \ldots, y_n = f[z_0, \ldots, z_n]$, and hence the interpolating polynomial, via Newton interpolation, has the form

$$P(z) = f[z_0] + f[z_0, z_1](z - z_0) + \cdots + f[z_0, \ldots, z_n] \prod_{i=0}^{n-1} (z - z_i).$$

From this point of view, we get the following.

Lemma 4.9. *If $\{i_0, \ldots, i_n\}$ is a permutation of the set $\{0, \ldots, n\}$ then $f[z_0, \ldots, z_n] = f[z_{i_0}, \ldots, z_{i_n}]$.*

Proof. Since $f[z_0, \ldots, z_n]$ is the leading coefficient of the unique interpolating polynomial, it suffices to look at the leading coefficient of the interpolating polynomial, via Lagrange interpolation, which is given by

$$\sum_{i=1}^{n} \frac{f(z_i)}{\prod_{\substack{j=1 \\ i \neq j}}^{n} (z_i - z_j)}. \tag{4.11}$$

Since (4.11) is invariant upon permutation of $\{0, \ldots, n\}$, we get $f[z_0, \ldots, z_n] = f[z_{i_0}, \ldots, z_{i_n}]$. □

We will use the following basic lemma.

Lemma 4.10. *If $f(x) = z^n$, then $f[z_0, \ldots, z_n] = 1$.*

Proof. Use (4.10). □

We are now ready to prove the two-variable Vandermonde result.

Proof of Theorem 4.8. First reorder $b_1, \ldots, b_n$ with respect to redundancy in the first coordinate as follows. Let

$$\begin{aligned} b_1^{(1)} &= (e^{ix_1}, e^{iy_1^{(1)}}), \ldots, b_{n_1}^{(1)} = (e^{ix_1}, e^{y_{n_1}^{(1)}}), \\ b_1^{(2)} &= (e^{ix_2}, e^{iy_1^{(2)}}), \ldots, b_{n_2}^{(2)} = (e^{ix_2}, e^{y_{n_2}^{(2)}}) \\ &\vdots \\ b_1^{(k)} &= (e^{ix_k}, e^{iy_1^{(k)}}), \ldots, b_{n_k}^{(k)} = (e^{ix_k}, e^{iy_{n_k}^{(k)}}), \end{aligned}$$

where $x_i \neq x_j$, when $i \neq j$, $n_1 \geq \cdots \geq n_k \geq 1$, and $n_1 + \cdots + n_k = n$. We claim that the lower inclusive set $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$, where $\Lambda_i = \{(i-1, 0), \ldots, (i-1, n_i-1)\}$, $1 \leq i \leq k$, will produce an invertible $V(b_1, \ldots, b_n; \Lambda)$ whose determinant is given by

(4.9). For notational convenience we will prove the result when $k = 3$. The other cases follow in the same manner. Realize $V(b_1, \dots, b_n; \Lambda)$ is

$$\begin{pmatrix} V(b_1^{(1)}, \dots, b_{n_1}^{(1)}; \Lambda_1) & V(b_1^{(1)}, \dots, b_{n_1}^{(1)}; \Lambda_2) & V(b_1^{(1)}, \dots, b_{n_1}^{(1)}; \Lambda_3) \\ V(b_1^{(2)}, \dots, b_{n_2}^{(2)}; \Lambda_1) & V(b_1^{(2)}, \dots, b_{n_2}^{(2)}; \Lambda_2) & V(b_1^{(2)}, \dots, b_{n_2}^{(2)}; \Lambda_3) \\ V(b_1^{(3)}, \dots, b_{n_3}^{(3)}; \Lambda_1) & V(b_1^{(3)}, \dots, w_{n_3}^{(3)}; \Lambda_2) & V(b_1^{(3)}, \dots, b_{n_3}^{(3)}; \Lambda_3) \end{pmatrix},$$

which can be rewritten as

$$\begin{pmatrix} V_{11} & e^{ix_1} V_{12} & e^{i2x_1} V_{13} \\ V_{21} & e^{ix_2} V_{22} & e^{ix_2} V_{23} \\ V_{31} & e^{ix_3} V_{32} & e^{ix_3} V_{33} \end{pmatrix},$$

where $V_{ij} = V(e^{iy_1^{(i)}}, \dots, e^{iy_{n_i}^{(i)}}; \{0, \dots, n_j - 1\})$, $1 \le i, j \le 3$. Using column operations we see that $V(b_1, \dots, b_n; \Lambda)$ is column equivalent to

$$\begin{pmatrix} V_{11} & 0 & 0 \\ V_{21} & (e^{ix_2} - e^{ix_1}) V_{22} & (e^{i2x_2} - e^{i2x_1}) V_{23} \\ V_{31} & (e^{ix_3} - e^{ix_1}) V_{32} & (e^{i2x_3} - e^{i2x_1}) V_{33} \end{pmatrix}, \tag{4.12}$$

where we used the fact that the first n_j columns of $V_{i,1}$ are precisely $V_{i,j}$, $i = 1, 2, 3$ and $j = 2, 3$. Upon row scaling, note that a factor of $(e^{ix_2} - e^{ix_1})^{n_2} (e^{ix_3} - e^{ix_1})^{n_3}$ must now be taken into account for $\det V(b_1, \dots, b_n; \Lambda)$; (4.12) then becomes

$$\begin{pmatrix} V_{11} & 0 & 0 \\ * & V_{22} & \frac{e^{i2x_2} - e^{i2x_1}}{e^{ix_2} - e^{ix_1}} V_{23} \\ * & V_{32} & \frac{e^{i2x_3} - e^{i2x_1}}{e^{ix_3} - e^{ix_1}} V_{33} \end{pmatrix}. \tag{4.13}$$

Rewriting (4.13) in divided difference notation yields

$$\begin{pmatrix} V_{11} & 0 & 0 \\ * & V_{22} & [e^{ix_1}, e^{ix_2}]^2 V_{23} \\ * & V_{32} & [e^{ix_1}, e^{ix_3}]^2 V_{33} \end{pmatrix}. \tag{4.14}$$

Using column operations, (4.14) becomes

$$\begin{pmatrix} V_{11} & 0 & 0 \\ * & V_{22} & 0 \\ * & * & ([e^{ix_1}, e^{ix_3}]^2 - [e^{ix_1}, e^{ix_2}]^2) V_{33} \end{pmatrix}. \tag{4.15}$$

Scale the third block rows by $e^{ix_3} - e^{ix_2}$ and note that

$$\frac{[e^{ix_1}, e^{ix_3}]^2 - [e^{ix_1}, e^{ix_2}]^2}{e^{ix_3} - e^{ix_2}} = [e^{ix_2}, e^{ix_1}, e^{ix_3}]^2 = [e^{ix_1}, e^{ix_2}, e^{ix_3}]^2 = 1,$$

where Lemma 4.9 was used to reorder the divided difference and Lemma 4.10 was used to achieve the last equality. Note that a factor of $(e^{ix_3} - e^{ix_2})^{n_3}$ must now be taken into account for $\det V(b_1, \dots, b_n; \Lambda)$, and (4.15) becomes

$$\begin{pmatrix} V_{11} & 0 & 0 \\ * & V_{22} & 0 \\ * & * & V_{33} \end{pmatrix}.$$

Notice that V_{ii}, $i = 1, 2, 3$, are invertible Vandermonde matrices, and so we have the invertibility of $V(b_1, \dots, b_n, \Lambda)$. Moreover, for $i = 1, 2, 3$, since

$$\det V_{ii} = \prod_{1 \le p < q \le n_i} (e^{iy_q^{(i)}} - e^{iy_p^{(i)}}),$$

it is easy to arrive at (4.9), when $k = 3$, upon consideration of the row scaling that was performed. For $k > 3$ note that the coefficient of V_{ii} can always be scaled to be 1 by recognizing the appropriate divided difference and then using Lemma 4.9 followed by Lemma 4.10. □

Next, we will prove a useful factorization result similar to Theorem 2.15 and Theorem 3.8.

Theorem 4.11. *Let $\Lambda \subset \mathbb{N}_0^d$, with* card $\Lambda = m$, *be a lattice set and*

$$\Gamma = (\Lambda - \Lambda) \cup \bigcup_{j=1}^{d} ((\Lambda - \Lambda + \mathrm{e}_j) \cup (\Lambda - \Lambda - \mathrm{e}_j)).$$

Let the $\mathbb{C}^{p\times p}$-valued sequence $\{S_\gamma\}_{\gamma\in\Gamma}$ be given. Set

$$\Phi = (S_{\lambda-\mu})_{\lambda,\mu\in\Lambda},$$
$$\Phi_j = (S_{\lambda-\mu+\mathrm{e}_j})_{\lambda,\mu\in\Lambda},\ 1 \le j \le d,$$
$$\Phi_{-j} = (S_{\lambda-\mu-\mathrm{e}_j})_{\lambda,\mu\in\Lambda}, 1 \le j \le d,$$

and put $k = \operatorname{rank} \Phi$. Suppose that $\Phi \ge 0$, $\Phi_j^* = \Phi_{-j}$, and $\Theta_1, \dots, \Theta_d, \Theta_{-1}, \dots, \Theta_{-d}$ are $mp \times mp$ matrices which commute and satisfy $\Phi\Theta_j = \Phi_j$ and $\Phi\Theta_{-j} = \Phi_j^*, 1 \le j \le d$. Then there exist a $p \times k$ matrix C_{0_d} and $k \times k$ diagonal matrices D_j, $1 \le j \le d$, so that

$$S_\gamma = C_{0_d}\left(D_1^{\frac{|g_1|+g_1}{2}} \cdots D_d^{\frac{|g_d|+g_d}{2}}\right)\left(D_1^{-\frac{|g_1|+g_1}{2}} \cdots D_d^{-\frac{|g_d|+g_d}{2}}\right)^* C_{0_d}^*, \tag{4.16}$$

for all $\gamma = (g_1, \dots, g_d) \in \Gamma$.

Proof. Consider $A = \Phi$, $W_j = \Theta_j$, $Y_j = \Theta_{-j}$, $B_j = \Phi_j$, and $B_{-j} = \Phi_{-j}$, $1 \le j \le d$. Then $A \ge 0$, $W_1, \dots, W_d$, and $Y_1, \dots, Y_d$ are matrices which commute with respect to $\mathfrak{M} = \operatorname{Ran} \Phi$, which satisfy $AW_j = B_j$ and $AY_j = B_{-j} = B_j^*, 1 \le i \le d$. We can apply Lemma 3.7 to obtain an injective matrix C and diagonal matrices D_j so that

$$\Phi = CC^* \ \text{ and }\ \Phi_{z_j} = CD_jC^*,\ 1 \le j \le d.$$

Write $C = \operatorname{col}(C_\lambda)_{\lambda\in\Lambda}$. When $\lambda \in \Lambda$ and $\mu, \mu + \mathrm{e}_j \in \Lambda$, we realize

$$S_{\lambda-(\mu+\mathrm{e}_j)} = S_{\lambda-\mu-\mathrm{e}_j} = C_\lambda C_{\mu+\mathrm{e}_j}^* = C_\lambda D_j^* C_\mu^*. \tag{4.17}$$

Notice that (4.17) implies

$$C(C_\lambda^* - D_j^* C_\mu^*) = 0. \tag{4.18}$$

Since C is injective, (4.18) yields

$$C_{\mu+\mathrm{e}_j} = C_\mu D_j. \tag{4.19}$$

Similarly, when $\lambda \in \Lambda$ and $\mu, \mu - \mathrm{e}_j \in \Lambda$, we realize

$$S_{\lambda-(\mu-\mathrm{e}_j)} = S_{\lambda-\mu+\mathrm{e}_j} = C_\lambda C_{\mu-\mathrm{e}_j}^* = C_\lambda D_j C_\mu^*. \tag{4.20}$$

Notice that (4.20) implies

$$C(C_\lambda^* - D_j C_\mu^*) = 0. \tag{4.21}$$

Since C is injective, (4.21) yields

$$C_{\mu-\mathrm{e}_j} = C_\mu D_j^*. \tag{4.22}$$

Consider $\lambda \in \Lambda$, where $\lambda = (l_1, \dots, l_d)$. Since Λ is a lattice set, for all $\lambda \in \Lambda$, we have the existence of $\lambda_1 \dots, \lambda_k \in \Lambda$, $i_1, \dots, i_k \in \{-1, 1\}$, and $j_1, \dots, j_k \in \{1, \dots, d\}$

so that $\lambda_1 = i_1 e_{j_1}, \dots, \lambda = \lambda_k + i_k e_{j_k}$. Choose $\mu = \lambda - i_k e_{j_k}$ so that (4.19) and (4.22) give

$$C_\lambda = \begin{cases} C_\mu D_{j_k}^* & \text{if } i_k = -1, \\ C_\mu D_{j_k} & \text{if } i_k = 1. \end{cases}$$

Continuing this way we arrive at

$$C_\lambda = C_{0_d} \left(D_1^{\frac{|l_1|+l_1}{2}} \cdots D_d^{\frac{|l_d|+l_d}{2}} \right) \left(D_1^{\frac{-|l_1|+l_1}{2}} \cdots D_d^{-\frac{|l_d|+l_d}{2}} \right)^*. \tag{4.23}$$

But then $\Phi = CC^*$ and $\Phi_j = CD_jC^*$, $1 \le j \le d$, give (4.16). □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. Suppose $\Phi \ge 0$ and there exist matrices $\Theta_1, \dots, \Theta_d, \Theta_{-1} \dots, \Theta_{-d}$ which commute with respect to $\mathfrak{M} = \text{Ran } \Phi$ and satisfy $\Phi\Theta_j = \Phi_j$ and $\Phi\Theta_{-j} = \Phi_{-j}$. Write

$$\Theta_j = \begin{pmatrix} \tilde{\Theta}_j & 0 \\ * & * \end{pmatrix} : \begin{matrix} \text{Ran } \Phi \\ \bigoplus \\ \text{Ker } \Phi \end{matrix} \to \begin{matrix} \text{Ran } \Phi \\ \bigoplus \\ \text{Ker } \Phi \end{matrix}, \quad 1 \le j \le d,$$

and so we must have the existence of an invertible matrix S so that $S^{-1}\tilde{\Theta}_j S = \text{diag}\,(b_j^{(1)}, \dots, b_j^{(r)})$, $1 \le j \le d$, where $r = \text{rank } \Phi$ and $(b_1^{(q)}, \dots, b_d^{(q)}) \in K$, $1 \le q \le r$. Use Theorem 4.11 to obtain an injective matrix $C := \text{col}(C_\lambda)_{\lambda \in \Lambda}$ and diagonal matrices $D_1, \dots, D_d$ so that $\Phi = CC^*$ and $\Phi_j = CD_jC^*$, $1 \le j \le d$. By (3.10), we have that $D_j = \text{diag}(x_j^{(1)}, \dots, x_j^{(r)})$, $1 \le j \le d$. Write $C_{0_d} = \begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix}$, where $c_1, \dots, c_r \in \mathbb{C}^p$. Then (4.16) holds. Put $b_q = (b_1^{(q)}, \dots, b_d^{(q)}) \in K, 1 \le q \le k$. Without loss of generality, assume $b_1, \dots, b_k$ are distinct, where $k \le r$. Fix $q \in \{1, \dots, k\}$ and consider the set $\mathcal{I}_q = \{i \in \{1, \dots, r\} : b_i = b_q\}$. Note that

$$\text{card} \bigcup_{q=1,\dots,r} \mathcal{I}_q = r.$$

Let $T_q = \sum_{i \in \mathcal{I}_q} c_i c_i^* \ge 0$ and note that (2.16) gives the fact that $\{c_i\}_{i \in \mathcal{I}_q}$ is linearly independent since C is injective. Hence rank $T_q = \text{card } \mathcal{I}_q$, and so we get

$$\text{rank } T_q = \text{card } \mathcal{I}_q, \ 1 \le q \le k.$$

Thus $\sum_{q=1}^k \text{rank } T_q = r$, and so rank $\Phi = \sum_{q=1}^k \text{rank } T_q$. Put τ as in the statement of Theorem 4.4. One can directly verify (4.2) and (4.3). Note that the existence of all subsequent moments follows from Remark 4.5.

Conversely, when $d = 1$, put $\tilde{\Lambda} = \{0, \dots, k-1\}$ to produce a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0$, with card $\tilde{\Lambda} = k$, so that $V := V(b_1, \dots, b_k; \tilde{\Lambda})$ is invertible. One can check that $\Phi = (V \otimes I_p)^* R (V \otimes I_p) \ge 0$, $\Phi_1 = (V \otimes I_p)^* R B_1 (V \otimes I_p)$, and $\Phi_{-1} = (V \otimes I_p)^* R B_1^* (V \otimes P)$, where $R = T_1 \oplus \dots \oplus T_k$ and $B_1 = b_1 I_p \oplus \cdots \oplus b_k I_p$. Choosing

$$\Theta_1 = (V \otimes I_p)^{-1} B_1 (V \otimes I_p) \ \text{ and } \ \Theta_{-1} = (V \otimes I_p)^{-1} B_1^* (V \otimes I_p)$$

yield matrices Θ_1, Θ_{-1}, which commute with respect to $\mathfrak{M} = \text{Ran } \Phi$. Note that Θ_1 satisfies K-inclusive property with respect to $\mathfrak{M} = \text{Ran } \Phi$, $\Phi\Theta_1 = \Phi_1$, and $\Phi\Theta_{-1} = \Phi_{-1}$. Moreover, we get rank $\Phi = \sum_{q=1}^k \text{rank } T_q$.

When $d = 2$, let $b_q = (e^{ix_q}, e^{iy_q})$, $1 \le q \le k$. Use Theorem 4.8 to produce a lower inclusive set $\tilde{\Lambda} \subset \mathbb{N}_0^2$, with card $\tilde{\Lambda} = k$, so that $V := V(b_1, \dots, b_k; \tilde{\Lambda})$ is invertible.

One can check that $\Phi = (V \otimes I_p)^* R (V \otimes I_p) \geq 0$, $\Phi_j = (V \otimes I_p)^* R B_j (V \otimes I_p)$, and $\Phi_{-j} = (V \otimes I_p)^* R B_j^* (V \otimes I_p)$, where $R = T_1 \oplus \cdots \oplus T_k$, $B_1 = e^{ix_1} I_p \oplus \cdots \oplus e^{ix_k} I_p$, and $B_2 = e^{iy_1} I_p \oplus \cdots \oplus e^{iy_k} I_p$. Choosing

$$\Theta_1 = (V \otimes I_p)^{-1} B_1 (V \otimes I_p), \;\; \Theta_1 = (V \otimes I_p)^{-1} B_1^* (V \otimes I_p)$$

and

$$\Theta_2 = (V \otimes I_p)^{-1} B_2 (V \otimes I_p), \;\; \Theta_2 = (V \otimes I_p)^{-1} B_2^* (V \otimes I_p)$$

yield matrices $\Theta_1, \Theta_2, \Theta_{-1}, \Theta_{-2}$, which commute with respect to $\mathfrak{M} = \text{Ran}\,\Phi$. Note that Θ_1, Θ_2 satisfy K-inclusive property with respect to $\mathfrak{M} = \text{Ran}\,\Phi$, $\Phi\Theta_j = \Phi_j$, and $\Phi\Theta_{-j} = \Phi_{-j}$, $1 \leq j \leq 2$. Moreover, we get rank $\Phi = \sum_{q=1}^{k}$ rank T_q. $\square$

We will end the section by posing the following question which would resolve the case when $d > 2$ in the converse statement in Theorem 4.4. Given distinct points $b_1, \ldots, b_k \in \mathbb{T}^d$, can one choose $\lambda_1, \ldots, \lambda_k$, which form a lattice set (or lower inclusive set) in $\mathbb{N}_0^d$ so that

$$\begin{pmatrix} b_1^{\lambda_1} & \cdots & b_1^{\lambda_k} \\ \vdots & & \vdots \\ b_k^{\lambda_1} & \cdots & b_k^{\lambda_k} \end{pmatrix}$$

is invertible?

References

1. N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965. MR0184042 (32:1518)
2. Mihály Bakonyi and Hugo J. Woerdeman, *Matrix completions, moments, and sums of hermitian squares*, Princeton University Press, Princeton, New Jersey, 2011. MR2807419 (2012d:47003)
3. Vladimir Bolotnikov, *On degenerate Hamburger moment problem and extensions of nonnegative Hankel block matrices*, Integral Equations Operator Theory **25** (1996), no. 3, 253–276. MR1395706 (97k:44011)
4. Raúl E. Curto and Lawrence A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), no. 4, 603–635. MR1147276 (93a:47016)
5. ______, *Solution of the truncated complex moment problem for flat data*, Mem. Amer. Math. Soc. **119** (1996), no. 568, x+52. MR1303090 (96g:47009)
6. ______, *Flat extensions of positive moment matrices: Recursively generated relations*, Mem. Amer. Math. Soc. **136** (1998), no. 648, x+56. MR1445490 (99d:47015)
7. ______, *Flat extensions of positive moment matrices: Relations in analytic or conjugate terms*, Nonselfadjoint operator algebras, operator theory, and related topics, Oper. Theory Adv. Appl., vol. 104, Birkhäuser, Basel, 1998, pp. 59–82. MR1639649 (99i:47026)
8. ______, *The truncated complex K-moment problem*, Trans. Amer. Math. Soc. **352** (2000), no. 6, 2825–2855. MR1661305 (2000j:47027)
9. ______, *Truncated K-moment problems in several variables*, J. Operator Theory **54** (2005), no. 1, 189–226. MR2168867 (2006e:47032)
10. Harry Dym, *On Hermitian block Hankel matrices, matrix polynomials, the Hamburger moment problem, interpolation and maximum entropy*, Integral Equations Operator Theory **12** (1989), no. 6, 757–812. MR1018213 (91c:30065)
11. Chirakkal Easwaran, Lawrence Fialkow, and Srdjan Petrovic, *Can a minimal degree* 6 *cubature rule for the disk have all points inside?* J. Comput. Appl. Math. **185** (2006), no. 1, 144–165. MR2165358 (2006d:65023)
12. Lawrence Fialkow and Srdjan Petrovic, *A moment matrix approach to multivariable cubature*, Integral Equations Operator Theory **52** (2005), no. 1, 85–124. MR2138699 (2005k:47032)
13. Jeffrey S. Geronimo and Hugo J. Woerdeman, *Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables*, Ann. of Math. (2) **160** (2004), no. 3, 839–906. MR2144970 (2006b:42036)

14. Hans Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. **82** (1921), no. 3-4, 168–187. MR1511981
15. E. K. Haviland, *On the Momentum Problem for Distribution Functions in More Than One Dimension. II*, Amer. J. Math. **58** (1936), no. 1, 164–168. MR1507139
16. M. G. Kreĭn and A. A. Nudel′man, *The Markov moment problem and extremal problems*, American Mathematical Society, Providence, R.I., 1977, Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50. MR0458081 (56:16284)
17. M. V. Noskov and H. J. Schmid, *Minimal cubature formulae of degree* 3 *for integrals over the surface of the torus*, Computing **57** (1996), no. 3, 213–223. MR1415006 (97h:65028)
18. Mihai Putinar and Florian-Horia Vasilescu, *Solving moment problems by dimensional extension*, Ann. of Math. (2) **149** (1999), no. 3, 1087–1107. MR1709313 (2001c:47023b)
19. Marcel Riesz, *Sur le probléme des moments*, Arkiv för Matematik, Astronomi och Fysik **17** (1923), no. 16, 1–52.
20. Thomas Sauer, *Polynomial interpolation of minimal degree*, Numer. Math. **78** (1997), no. 1, 59–85. MR1483569 (99f:41042)
21. Konrad Schmüdgen, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), no. 2, 203–206. MR1092173 (92b:44011)
22. Ju. L. Šmul′jan, *An operator Hellinger integral*, Mat. Sb. (N.S.) **49 (91)** (1959), 381–430. MR0121662 (22:12396)
23. Jan Stochel, *Solving the truncated moment problem solves the full moment problem*, Glasg. Math. J. **43** (2001), no. 3, 335–341. MR1878578 (2002j:47026)

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