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PRINCIPAL LYAPUNOV EXPONENTS AND PRINCIPAL FLOQUET SPACES OF POSITIVE RANDOM DYNAMICAL SYSTEMS. I. GENERAL THEORY

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Abstract. This series of papers is concerned with principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces. The current part of the series focuses on the development of general theory. First, the notions of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations for general positive random dynamical systems in ordered Banach spaces are introduced, which extend the classical notions of principal Floquet subspaces, principal Lyapunov exponents, and exponential separations for strongly positive deterministic systems in strongly ordered Banach to general positive random dynamical systems in ordered Banach spaces. Under some quite general assumptions, it is then shown that a positive random dynamical system in an ordered Banach space admits a family of generalized principal Floquet subspaces, a generalized principal Lyapunov exponent, and a generalized exponential separation. We will consider in the forthcoming part(s) the applications of the general theory developed in this part to positive random dynamical systems arising from a variety of random mappings and differential equations, including random Leslie matrix models, random cooperative systems of ordinary differential equations, and random parabolic equations.

1. Introduction

This is the first part of a series of papers. The series is devoted to the study of principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces. This first part focuses on the development of general theory of principal Lyapunov exponents and principal Floquet subspaces of general positive random dynamical systems in ordered Banach spaces. The forthcoming part(s) of the series will concern the applications of the general theory developed in Part I to positive random dynamical systems arising from a variety of random mappings and differential equations, including random Leslie

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matrix models, random cooperative systems of ordinary differential equations, and random parabolic equations.

Lyapunov exponents play an important role in the study of asymptotic dynamics of linear and nonlinear random evolution systems. The study of Lyapunov exponents traces back to Lyapunov [16]. Oseledets in [26] obtained some important results on Lyapunov exponents for finite-dimensional systems, which is now called the Oseledets multiplicative ergodic theorem. Since then, a huge amount of research has been carried out toward alternative proofs of the Osedelets multiplicative ergodic theorem and extensions of the Osedelets multiplicative theorem for finite-dimensional systems to certain infinite-dimensional ones (see [1], [10], [13], [14], [17], [25], [30], [31], [33], [35], and the references therein). In the recent work [14], Lian and Lu studied Lyapunov exponents of general infinite-dimensional random dynamical systems in a Banach space and established a multiplicative ergodic theorem for such systems.

The largest finite Lyapunov exponents (or top Lyapunov exponents) and the associated invariant subspaces of both deterministic and random dynamical systems play special roles in the applications to nonlinear systems. Classically, the top finite Lyapunov exponent of a positive deterministic or random dynamical system in an ordered Banach space is called the *principal Lyapunov exponent* if its associated invariant subspace is one dimensional and is spanned by a positive vector (in such a case, the invariant subspace is called the *principal Floquet subspace*). Principal Lyapunov exponents and principal Floquet subspaces are the analog of principal eigenvalues and principal eigenfunctions of elliptic and time periodic parabolic operators. Numerous works have also been carried out toward principal Lyapunov exponents and principal Floquet subspaces for certain positive deterministic as well as random dynamical systems in ordered Banach spaces, in particular, for deterministic and random dynamical systems generated by nonautonomous and random parabolic equations with bounded coefficients (see [6], [7], [8], [9], [18], [19], [21], [22], [23], [27], [37], [38], and the references therein).

Many strongly positive deterministic as well as random dynamical systems in strongly ordered Banach space are shown to have principal Lyapunov exponents (and hence principal Floquet subspaces and entire positive orbits). Moreover, the so-called exponential separations are admitted in such systems. For example, let $(Z, (\sigma_t)_{t\in\mathbb{R}})$ be a compact uniquely ergodic minimal flow and X be a strongly ordered Banach space with the positive cone X^+ (see Section 2.3 for details). Let $\Pi = (\Pi_t)_{t\geq 0}$, $\Pi_t\colon X \times Z \to X \times Z$ be a skew-product semiflow over $(Z, (\sigma_t)_{t\in\mathbb{R}})$, and

$$\Pi_t(x, z) = (\Phi(t, z)x, \sigma_t z),$$

where $\Phi(t, z) \in \mathcal{L}(X, X)$. If Π is strongly positive (i.e. $\Phi(t, z)x \in \operatorname{Int}(X^+)$ for any $t > 0$, $z \in Z$, and $x \in X^+ \setminus \{0\}$) and completely continuous (i.e., $\{\, \Phi(t, z)B : z \in Z \,\}$ is a relatively compact subset of X for any $t > 0$ and any bounded subset B of X), then there are $\lambda_1 \in \mathbb{R}$, $M, \gamma > 0$, a subspace $E(z) \subset X$ with $E(z) = \operatorname{span}\{v(z)\}$ for some $v(z) \in \operatorname{Int}(X^+)$, $\|v(z)\| = 1$, and a subspace $F(z) \subset X$ with $F(z) \cap X^+ = \{0\}$ such that $X = E(z) \oplus F(z)$ for any $z \in Z$, $E(z)$ and $F(z)$ are continuous in $z \in Z$, and

(i) $\Phi(t, z)E(z) = E(\sigma_t z)$ for any $t > 0$ and $z \in Z$;
(ii) $\Phi(t, z)F(z) \subset F(\sigma_t z)$ for any $t > 0$ and $z \in Z$;

(iii) $\lim_{t\to\infty} \frac{\ln \|\Phi(t,z)v(z)\|}{t} = \lambda_1$;

(iv) $\frac{\|\Phi(t,z)w\|}{\|\Phi(t,z)v(z)\|} \le Me^{-\gamma t}$ for any $w \in F(z)$ with $\|w\| = 1$, $t > 0$, and $z \in Z$

(see [21], [29]). Here λ_1 and $\{E(z)\}_{z\in Z}$ are the principal Lyapunov exponent and principal Floquet subspaces of Π, respectively, and property (iv) is referred to as the *exponential separation* of Π. Note that the above results extend the classical Kreĭn–Rutman theorem for strongly positive and compact operators in strongly ordered Banach spaces to strongly positive and compact deterministic skew-product semiflows in strongly ordered Banach spaces.

For a general positive random dynamical system, there may be no finite Lyapunov exponents (and hence no principal Lyapunov exponent in the classical sense); if the top Lyapunov exponent is finite, its associated invariant subspace may not be one dimensional (and hence there is no principal Lyapunov exponent in the classical sense either). It is not known whether a general positive random dynamical system admits positive entire orbits and/or invariant subspaces spanned by positive vectors.

The objective of the current part of the series is to investigate the extent to which the principal Lyapunov exponents and principal Floquet subspaces theory for strongly positive and compact deterministic dynamical systems may be generalized to general positive random dynamical systems. The classical Kreĭn–Rutman theorem for strongly positive and compact operators in strongly ordered Banach spaces is extended to quite general positive random dynamical systems in ordered Banach spaces. In particular, the existence of entire positive orbits is shown without the assumption of strong positivity (see Theorem 3.5), the existence of one dimensional invariant measurable subspaces which are spanned by positive vectors and whose associated Lyapunov exponent is the largest (such invariant subspaces and the associated Lyapunov exponent are called *generalized principal Floquet subspaces* and *generalized principal Lyapunov exponents*, respectively; see Definition 3.2) is proved without the assumption of the existence of finite Lyapunov exponents (see Theorem 3.6), and the existence of a *generalized exponential separation* (see Definition 3.3) is also proved without the assumption of the existence of finite Lyapunov exponents (see Theorem 3.8).

In the forthcoming part(s) of this series we will study the applications of the general results established in this part to random Leslie matrix models, random cooperative systems of ordinary differential equations, and random parabolic equations.

The rest of the current part is organized as follows. In Section 2, we introduce standing notions and assumptions. We introduce the concepts of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations and state the main results of this part in Section 3. In Section 4, we present some preliminary materials to be used in the proofs of the main results, including some classical ergodic theorems and fundamental properties of the Hilbert projective metric. We prove the main results in the last section.

2. Standing notions and assumptions

In this section, we introduce standing notions and assumptions.

If $f\colon A \to \mathbb{R}$ we define

$$f^{+}(a) := \begin{cases} f(a) & \text{if } f(a) \ge 0, \\ 0 & \text{if } f(a) < 0 \end{cases} \qquad \text{and} \qquad f^{-}(a) := \begin{cases} 0 & \text{if } f(a) \ge 0, \\ -f(a) & \text{if } f(a) < 0. \end{cases}$$

We have $f = f^{+} - f^{-}$ and $|f| = f^{+} + f^{-}$.

For a metric space Y, $\mathfrak{B}(Y)$ stands for the σ-algebra of all Borel subsets of Y.

2.1. **Metric dynamical systems.** By a *probability space* we understand a triple $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is a set, $\mathfrak{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure defined for all $F \in \mathfrak{F}$.

Let $\mathbb{T}$ stand for either $\mathbb{Z}$ or $\mathbb{R}$.

A *measurable dynamical system* on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is a $(\mathfrak{B}(\mathbb{T}) \otimes \mathfrak{F}, \mathfrak{F})$-measurable mapping $\theta\colon \mathbb{T} \times \Omega \to \Omega$ such that

- $\theta(0, \omega) = \omega$ for any $\omega \in \Omega$,
- $\theta(t_1 + t_2, \omega) = \theta(t_2, \theta(t_1, \omega))$ for any $t_1, t_2 \in \mathbb{T}$ and any $\omega \in \Omega$.

We write $\theta(t, \omega)$ as $\theta_t \omega$. Also, we usually denote measurable dynamical systems by $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{T}})$ or simply by $(\theta_t)_{t \in \mathbb{T}}$.

A *metric dynamical system* is a measurable dynamical system $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{T}})$ such that for each $t \in \mathbb{T}$ the mapping $\theta_t\colon \Omega \to \Omega$ is $\mathbb{P}$-preserving (i.e., $\mathbb{P}(\theta_t^{-1}(F)) = \mathbb{P}(F)$ for any $F \in \mathfrak{F}$ and $t \in \mathbb{T}$).

When $\mathbb{T} = \mathbb{R}$ we call a (measurable, metric) dynamical system a (measurable, metric) *flow*. To emphasize the situation when $\mathbb{T} = \mathbb{Z}$, we speak of a (measurable, metric) *discrete-time* dynamical system.

When we use the symbol "$\lim_{n\to\infty}$" it is implied that n is considered for (perhaps sufficiently large) $n \in \mathbb{N}$. Similarly, when we use the symbol "$\lim_{n\to-\infty}$" it is implied that n is considered for (perhaps sufficiently large) negative integers n.

For a measurable dynamical system $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{T}})$, $\Omega' \subset \Omega$ is *invariant* if $\theta_t(\Omega') = \Omega'$ for all $t \in \mathbb{T}$. An $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable function defined on an invariant $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ is called *invariant* if $f(\theta_t \omega) = f(\omega)$ for any $\omega \in \Omega'$ and any $t \in \mathbb{T}$.

$((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{T}})$ is said to be *ergodic* if for any invariant $F \in \mathfrak{F}$, either $\mathbb{P}(F) = 1$ or $\mathbb{P}(F) = 0$.

2.2. **Measurable linear skew-product semidynamical systems.** Let X be a real Banach space with norm $\|\cdot\|$. Let $\mathcal{L}(X)$ stand for the Banach space of bounded linear mappings from X into X. The standard norm in $\mathcal{L}(X)$ will also be denoted by $\|\cdot\|$.

For a Banach space X, we will denote by X^* its dual and by $\langle \cdot, \cdot \rangle$ the standard duality pairing (that is, for $u \in X$ and $u^* \in X^*$ the symbol $\langle u, u^* \rangle$ denotes the value of the bounded linear functional u^* at u). Without further mention, we understand that the norm in X^* is given by $\|u^*\| = \sup\{\, |\langle u, u^* \rangle| : \|u\| \le 1 \,\}$.

For $\mathbb{T} = \mathbb{R}$ we write $\mathbb{T}^+$ for $[0, \infty)$. For $\mathbb{T} = \mathbb{Z}$ we write $\mathbb{T}^+$ for $\{0, 1, 2, 3, \dots\}$.

Let $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{T}})$ be a measurable dynamical system. By a *measurable linear skew-product semidynamical system* $\Phi = ((U_\omega(t))_{\omega \in \Omega, t \in \mathbb{T}^+}, (\theta_t)_{t \in \mathbb{T}})$ on a Banach space X covering $(\theta_t)_{t \in \mathbb{T}}$, we understand a $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F} \otimes \mathfrak{B}(X), \mathfrak{B}(X))$-measurable mapping

$$[\, \mathbb{T}^+ \times \Omega \times X \ni (t, \omega, u) \mapsto U_\omega(t) u \in X \,]$$

satisfying the following:

- $$U_\omega(0) = \mathrm{Id}_X \quad \forall \omega \in \Omega, \tag{2.1}$$
  $$U_{\theta_s\omega}(t) \circ U_\omega(s) = U_\omega(t+s) \qquad \forall \omega \in \Omega,\ t, s \in \mathbb{T}^+; \tag{2.2}$$
- for each $\omega \in \Omega$ and $t \in \mathbb{T}^+$, $[\, X \ni u \mapsto U_\omega(t)u \in X \,] \in \mathcal{L}(X)$.

When $\mathbb{T}^+ = [0, \infty)$ we call a measurable linear skew-product semidynamical system a (measurable linear skew-product) *semiflow*. To emphasize the situation when $\mathbb{T}^+ = \{0, 1, 2, \dots\}$, we speak of a (measurable linear skew-product) *discrete-time* semidynamical system.

If the Banach space X is separable, by Pettis' theorem (see, e.g., [36, Theorem 1.1.6]) the measurability of the mapping $[\,(t, \omega, u) \mapsto U_\omega(t)u\,]$ is equivalent to the fact that for each $u^* \in X^*$ the mapping

$$[\, \mathbb{T}^+ \times \Omega \times X \ni (t, \omega, u) \mapsto \langle U_\omega(t)u, u^* \rangle \in \mathbb{R} \,]$$

is $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F} \otimes \mathfrak{B}(X), \mathfrak{B}(\mathbb{R}))$-measurable.

For $\omega \in \Omega$, $t \in \mathbb{T}^+$ and $u^* \in X^*$ we define $U^*_\omega(t)u^*$ by

$$\langle u, U^*_\omega(t)u^* \rangle = \langle U_{\theta_{-t}\omega}(t)u, u^* \rangle \qquad \text{for each } u \in X \tag{2.3}$$

(in other words, $U^*_\omega(t)$ is the mapping dual to $U_{\theta_{-t}\omega}(t)$). It is straightforward that

$$U^*_\omega(0) = \mathrm{Id}_{X^*} \quad \text{for any } \omega \in \Omega \tag{2.4}$$

and

$$U^*_{\theta_{-s}\omega}(t) \circ U^*_\omega(s) = U^*_\omega(t+s) \qquad \text{for any } \omega \in \Omega \text{ and any } t, s \in \mathbb{T}^+. \tag{2.5}$$

In the case where the mapping

$$[\, \mathbb{T}^+ \times \Omega \times X^* \ni (t, \omega, u^*) \mapsto U^*_\omega(t)u^* \in X^* \,]$$

is $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F} \otimes \mathfrak{B}(X^*), \mathfrak{B}(X^*))$-measurable, we will call the measurable linear skew-product semidynamical system $\Phi^* = ((U^*_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_{-t})_{t\in\mathbb{T}})$ on X^* covering $(\theta_{-t})_{t\in\mathbb{T}}$ the *dual* of Φ.

For instance, if we assume that X is separable and reflexive, then X^* is separable. Hence, by Pettis' theorem, the measurability of the mapping $[\,(t, \omega, u^*) \mapsto U^*_\omega(t)u^*\,]$ is equivalent to the fact that for each $u \in X$ the mapping

$$[\, \mathbb{T}^+ \times \Omega \times X^* \ni (t, \omega, u^*) \mapsto \langle u, U^*_\omega(t)u^* \rangle = \langle U_{\theta_{-t}\omega}(t)u, u^* \rangle \in \mathbb{R} \,]$$

is $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F} \otimes \mathfrak{B}(X^*), \mathfrak{B}(\mathbb{R}))$-measurable, which in turn follows from the fact that the composition $[\,(t, \omega) \mapsto (t, \theta_{-t}\omega) \mapsto U_{\theta_{-t}\omega}(t)u\,]$ is $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F}, \mathfrak{B}(X))$-measurable and that the $\langle\cdot,\cdot\rangle$ operation is continuous.

From now until the end of the subsection we assume $((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}^+})$ is a measurable linear skew-product semidynamical system.

Let l be a positive integer. By a *family of l-dimensional vector subspaces of X* we understand a mapping E, defined on some $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$, assigning to each $\omega \in \Omega_0$ an l-dimensional vector subspace $E(\omega)$ of X. Similarly, by a *family of l-codimensional closed vector subspaces of X* we understand a mapping F, defined on some $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$, assigning to each $\omega \in \Omega_0$ an l-codimensional closed vector subspace $F(\omega)$ of X.

We will usually denote families of vector subspaces by $\{E(\omega)\}_{\omega\in\Omega_0}$, etc.

Regarding the measurability of families of finite-dimensional vector subspaces, we will use the following definition: A family $\{E(\omega)\}_{\omega\in\Omega_0}$ of l-dimensional vector subspaces of X is *measurable* if there are $(\mathfrak{F},\mathfrak{B}(X))$-measurable functions $v_1,\dots,v_l\colon \Omega_0\to X$ such that $(v_1(\omega),\dots,v_l(\omega))$ forms a basis of $E(\omega)$ for each $\omega\in\Omega_0$ (see [14, Lemma 5.6 and Corollary 7.3]).

Let $\{E(\omega)\}_{\omega\in\Omega_0}$ be a family of l-dimensional vector subspaces of X, and let $\{F(\omega)\}_{\omega\in\Omega_0}$ be a family of l-codimensional closed vector subspaces of X such that $E(\omega)\oplus F(\omega)=X$ for all $\omega\in\Omega_0$. We define the *family of projections associated with the decomposition* $E(\omega)\oplus F(\omega)=X$ as $\{P(\omega)\}_{\omega\in\Omega_0}$, where $P(\omega)$ is the linear projection of X onto $F(\omega)$ along $E(\omega)$, for each $\omega\in\Omega_0$.

The family of projections associated with the decomposition $E(\omega)\oplus F(\omega)=X$ is called *strongly measurable* if for each $u\in X$ the mapping $[\,\Omega_0\ni\omega\mapsto P(\omega)u\in X\,]$ is $(\mathfrak{F},\mathfrak{B}(X))$-measurable.

We say that the decomposition $E(\omega)\oplus F(\omega)=X$, with $\{E(\omega)\}_{\omega\in\Omega_0}$ finite-dimensional, is *invariant* if Ω_0 is invariant, $U_\omega(t)E(\omega)=E(\theta_t\omega)$ and $U_\omega(t)F(\omega)\subset F(\theta_t\omega)$, for each $t\in\mathbb{T}^+$.

A strongly measurable family of projections associated with the invariant decomposition $E(\omega)\oplus F(\omega)=X$ is referred to as *tempered* if

$$\lim_{\substack{t\to\pm\infty\\ t\in\mathbb{T}}}\frac{\ln\|P(\theta_t\omega)\|}{t}=0\qquad \mathbb{P}\text{-a.s. on }\Omega_0.$$

2.3. **Ordered Banach spaces.** Let X be a real Banach space with norm $\|\cdot\|$. By a *cone* in X we understand a closed convex set X^+ such that

(C1) $\alpha\ge 0$ and $u\in X^+$ imply $\alpha u\in X^+$, and
(C2) $X^+\cap(-X^+)=\{0\}$.

A pair (X,X^+), where X is a Banach space and X^+ is a cone in X, is referred to as an *ordered Banach space*.

If (X,X^+) is an ordered Banach space, for $u,v\in X$ we write $u\le v$ if $v-u\in X^+$ and $u<v$ if $u\le v$ and $u\ne v$. The symbols $\ge$ and $>$ are used in an analogous way.

We say that $u,v\in X^+\setminus\{0\}$ are *comparable*, written $u\sim v$, if there are positive numbers $\underline{\alpha},\overline{\alpha}$ such that $\underline{\alpha}v\le u\le\overline{\alpha}v$. The $\sim$ relation is clearly an equivalence relation. For a nonzero $u\in X^+$ we call the *component* of u, denoted by C_u, the equivalence class of u, $C_u=\{\,v\in X^+\setminus\{0\}:v\sim u\,\}$.

A cone X^+ in a Banach space X is called

- *solid* if the interior X^{++} of X^+ is nonempty,
- *reproducing* if $X^+-X^+=X$, and
- *total* if X^+-X^+ is dense in X.

An ordered Banach space (X,X^+) is called *strongly ordered* if X^+ is solid.

A cone X^+ in a Banach space X is called *normal* if there exists $K>0$ such that for any $u,v\in X$ satisfying $0\le u\le v$ there holds $\|u\|\le K\|v\|$.

If X^+ is a normal cone we say that (X,X^+) is a *normally ordered Banach space*. In such a case, the Banach space X can be renormed so that for any $u,v\in X$, $0\le u\le v$ implies $\|u\|\le\|v\|$ (see [34, V.3.1, p. 216]). Such a norm is called *monotonic*.

From now on, when speaking of a normally ordered Banach space we assume that the norm on X is monotonic.

For an ordered Banach space (X, X^+) denote by $(X^*)^+$ the set of all $u^* \in X^*$ such that $\langle u, u^* \rangle \geq 0$ for all $u \in X^+$. The closed subset $(X^*)^+$ of X^* is convex and satisfies (C1), however it need not satisfy (C2). Nevertheless, if the cone X^+ is total, then $(X^*)^+$ satisfies (C2) (therefore is a cone).

It is a classical result that X^+ is normal if and only if $(X^*)^+$ is reproducing and that X^+ is reproducing if and only if $(X^*)^+$ is normal; see e.g. [34, V.3.5].

Sometimes an ordered Banach space (X, X^+) is a lattice: any two $u, v \in X$ have a least upper bound $u \vee v$ and a greatest lower bound $u \wedge v$. In such a case we write $u^+ := u \wedge 0$, $u^- := (-u) \vee 0$, and $|u| := u^+ + u^-$. We have $u = u^+ - u^-$ for any $u \in X$.

An ordered Banach space (X, X^+) being a lattice is a *Banach lattice* if there is a norm $\|\cdot\|$ on X (a *lattice norm*) such that for any $u, v \in X$, if $|u| \leq |v|$, then $\|u\| \leq \|v\|$. From now on, when speaking of a Banach lattice we assume that the norm on X is a lattice norm.

It is straightforward that in a Banach lattice the cone is normal and reproducing. Moreover, if (X, X^+) is a Banach lattice, then $(X^*, (X^*)^+)$ is a Banach lattice, too (see [34, V.7.4]).

The reader is referred to forthcoming papers in the current series for a variety of examples of ordered Banach spaces and ordered Banach lattices.

2.4. **Assumptions.** We now list assumptions we will make at various points in the sequel.

(A0) (Ordered Banach space)

(i) (X, X^+) *is an ordered separable Banach space with* $\dim X \geq 2$.
(ii) (X, X^+) *is a normally ordered separable Banach space with* $\dim X \geq 2$.
(iii) (X, X^+) *is a separable Banach lattice with* $\dim X \geq 2$.

(A0)* (Ordered Banach space)

(i) $(X^*, (X^*)^+)$ *is an ordered separable Banach space with* $\dim X^* \geq 2$.
(ii) $(X^*, (X^*)^+)$ *is a normally ordered separable Banach space with* $\dim X^* \geq 2$.
(iii) $(X^*, (X^*)^+)$ *is a separable Banach lattice with* $\dim X^* \geq 2$.

(A1) (Integrability/injectivity/complete continuity) $\Phi = ((U_\omega(t))_{\omega \in \Omega, t \in \mathbb{T}^+}, (\theta_t)_{t \in \mathbb{T}})$ *is a measurable linear skew-product semidynamical system on a separable Banach space X covering an ergodic metric dynamical system $(\theta_t)_{t \in \mathbb{T}}$ on $(\Omega, \mathfrak{F}, \mathbb{P})$, with the complete measure $\mathbb{P}$ in the case of $\mathbb{T} = \mathbb{R}$, satisfying the following:*

(i) (Integrability)

- *In the discrete-time case: The function*

$$[\, \Omega \ni \omega \mapsto \ln^+ \|U_\omega(1)\| \in [0, \infty) \,]$$

belongs to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$.

- *In the continuous-time case: The functions*

$$[\, \Omega \ni \omega \mapsto \sup_{0 \leq s \leq 1} \ln^+ \|U_\omega(s)\| \in [0, \infty) \,]$$

and

$$[\, \Omega \ni \omega \mapsto \sup_{0 \leq s \leq 1} \ln^+ \|U_{\theta_s \omega}(1 - s)\| \in [0, \infty) \,]$$

belong to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$.

(ii) (Injectivity) *The linear operator* $U_\omega(1)$ *is injective almost surely on* Ω.
(iii) (Complete continuity) *The linear operator* $U_\omega(1)$ *is completely continuous almost surely on* Ω.

In the sequel, by (A1)*(i), (A1)*(ii) and (A1)*(iii) we will understand the counterparts of (A1)(i), (A1)(ii) and (A1)(iii) for the dual measurable linear skew-product semidynamical system Φ^*. More precisely, for example (A1)*(ii) means the following: "the mapping $[\,\mathbb{T}^+ \times \Omega \times X^* \ni (t, \omega, u^*) \mapsto U^*_\omega(t)u^* \in X^*\,]$ is $(\mathfrak{B}(\mathbb{T}^+) \otimes \mathfrak{F} \otimes \mathfrak{B}(X^*), \mathfrak{B}(X^*))$-measurable, and the linear operator $U^*_\omega(1)$ is injective almost surely on Ω".

Assuming the measurability in the definition of Φ^* holds, observe that if (A1)(i) is satisfied, then (A1)*(i) is satisfied, too. Similarly, if (A1)(iii) is satisfied, then (A1)(iii)* is satisfied.

(A2) (Positivity) *X satisfies* (A0)(i) *and* $\Phi = ((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ *is a measurable linear skew-product semidynamical system on X covering an ergodic metric dynamical system* $(\theta_t)_{t\in\mathbb{T}}$ *on* $(\Omega, \mathfrak{F}, \mathbb{P})$*, satisfying the following:*

$$U_\omega(t)u_1 \le U_\omega(t)u_2$$

for any $\omega \in \Omega$*,* $t \in \mathbb{T}^+$ *and* $u_1, u_2 \in X$ *with* $u_1 \le u_2$.

Similarly, we write

(A2)* (Positivity) *X^* satisfies* (A0)*(i) *and* $\Phi^* = ((U^*_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_{-t})_{t\in\mathbb{T}})$ *is a measurable linear skew-product semidynamical system on X^* covering an ergodic metric dynamical system* $(\theta_{-t})_{t\in\mathbb{T}}$ *on* $(\Omega, \mathfrak{F}, \mathbb{P})$*, satisfying the following:*

$$U^*_\omega(t)u^*_1 \le U^*_\omega(t)u^*_2$$

for any $\omega \in \Omega$*,* $t \in \mathbb{T}^+$ *and* $u^*_1, u^*_2 \in X^*$ *with* $u^*_1 \le u^*_2$.

Observe that if (A0)*(i) is satisfied and the measurability in the definition of Φ^* holds, then (A2) implies (A2)*.

(A3) (Focusing) (A2) *is satisfied and there are* $\mathbf{e} \in X^+$ *with* $\|\mathbf{e}\| = 1$ *and an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* $\varkappa\colon \Omega \to [1, \infty)$ *with* $\ln^+ \ln \varkappa \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ *such that for any* $\omega \in \Omega$ *and any nonzero* $u \in X^+$ *there is* $\beta(\omega, u) > 0$ *with the property that*

$$\beta(\omega, u)\mathbf{e} \le U_\omega(1)u \le \varkappa(\omega)\beta(\omega, u)\mathbf{e}.$$

(A3)* (Focusing) (A2)* *is satisfied and there are* $\mathbf{e}^* \in (X^*)^+$ *with* $\|\mathbf{e}^*\| = 1$ *and an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* $\varkappa^*\colon \Omega \to [1, \infty)$ *with* $\ln^+ \ln \varkappa^* \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ *such that for any* $\omega \in \Omega$ *and any nonzero* $u^* \in (X^*)^+$ *there is* $\beta^*(\omega, u^*) > 0$ *with the property that*

$$\beta^*(\omega, u^*)\mathbf{e}^* \le U^*_\omega(1)u^* \le \varkappa^*(\omega)\beta^*(\omega, u^*)\mathbf{e}^*.$$

(A4) (Strong focusing) (A3), (A3)* *are satisfied and* $\ln \varkappa \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*,* $\ln \varkappa^* \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*, and* $\langle \mathbf{e}, \mathbf{e}^* \rangle > 0$.

(A5) (Strong positivity in one direction) *There are* $\bar{\mathbf{e}} \in X^+$ *with* $\|\bar{\mathbf{e}}\| = 1$ *and an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* $\nu\colon \Omega \to (0, \infty)$*, with* $\ln^- \nu \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*, such that*

$$U_\omega(1)\bar{\mathbf{e}} \ge \nu(\omega)\bar{\mathbf{e}} \quad \forall\, \omega \in \Omega.$$

(A5)* (Strong positivity in one direction) *There are* $\bar{\mathbf{e}}^* \in (X^*)^+$ *with* $\|\bar{\mathbf{e}}^*\| = 1$ *and an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* $\nu^*\colon \Omega \to (0, \infty)$*, with* $\ln^- \nu^* \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*,* *such that*

$$U_\omega^*(1)\overline{\mathbf{e}}^* \geq \nu^*(\omega)\overline{\mathbf{e}}^* \quad \forall\, \omega \in \Omega.$$

Remark 2.1. We can replace time 1 with some nonzero T belonging to $\mathbb{T}^+$ in (A1), (A3), (A4), (A5), and (A1)*, (A3)*, (A5)*.

Remark 2.2. The focusing property in (A3) (resp. (A3)*) is an extension of the so-called u_0-positivity for a deterministic linear operator (see [11], [12]) to a measurable linear skew-product semidynamical system.

3. Definitions and main results

In this section, we state the definitions and main results of the paper. We first state the definitions in Section 3.1, then recall an Oseledets-type theorem proved in [14] in Section 3.2, and finally state the main results in Section 3.3. Throughout this section, we assume that $((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}^+})$ is a measurable linear skew-product semidynamical system on a Banach space X covering $(\theta_t)_{t\in\mathbb{T}}$.

3.1. Definitions.

In this subsection, we introduce the concepts of entire solutions and extend the notions of principal eigenfunction and exponential separation of strongly positive and compact operators. Throughout this subsection, we assume (A0)(i) and (A2).

Definition 3.1 (Entire orbit)**.** For $\omega \in \Omega$, by an *entire orbit* of U_ω we understand a mapping $v_\omega\colon \mathbb{T} \to X$ such that $v_\omega(s+t) = U_{\theta_s\omega}(t)v_\omega(s)$ for any $s \in \mathbb{T}$ and $t \in \mathbb{T}^+$. The function constantly equal to zero is referred to as the *trivial entire orbit*.

Entire orbits of Φ^* are defined in a similar way.

Definition 3.2 (Generalized principal Floquet subspaces and a principal Lyapunov exponent)**.** A family of one-dimensional subspaces $\{\tilde{E}(\omega)\}_{\omega\in\tilde{\Omega}}$ of X is called a family of *generalized principal Floquet subspaces* of $\Phi = ((U_\omega(t))_{\omega\in\Omega,t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ if $\tilde{\Omega} \subset \Omega$ is invariant, $\mathbb{P}(\tilde{\Omega}) = 1$, and

(i) $\tilde{E}(\omega) = \operatorname{span}\{w(\omega)\}$, with $w\colon \tilde{\Omega} \to X^+ \setminus \{0\}$ being $(\mathfrak{F}, \mathfrak{B}(X))$-measurable,
(ii) $U_\omega(t)\tilde{E}(\omega) = \tilde{E}(\theta_t\omega)$ for any $\omega \in \tilde{\Omega}$ and any $t \in \mathbb{T}^+$,
(iii) there is $\tilde{\lambda} \in [-\infty, \infty)$ such that

$$\tilde{\lambda} = \lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)w(\omega)\|$$

for any $\omega \in \tilde{\Omega}$, and

(iv)

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \leq \tilde{\lambda}$$

for any $\omega \in \tilde{\Omega}$ and any $u \in X \setminus \{0\}$.

$\tilde{\lambda}$ is called the *generalized principal Lyapunov exponent* of Φ associated to the generalized principal Floquet subspaces $\{\tilde{E}(\omega)\}_{\omega\in\tilde{\Omega}}$.

Note that the notions of generalized principal Floquet subspaces and a principal Lyapunov exponent are the extensions of principal eigenspaces and principal eigenvalues of strongly positive and compact operators. If $\Phi = ((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ admits a family of generalized Floquet subspaces $\{\tilde{E}(\omega)\}_{\omega\in\tilde{\Omega}}$, then $[\mathbb{R} \ni t \mapsto U_\omega(t)w(\omega)]$ is a nontrivial entire positive orbit, where $U_\omega(t)w(\omega)$ is, for $t < 0$, understood as $(U_{\theta_t\omega}(-t)|_{\tilde{E}_1(\theta_t\omega)})^{-1}w(\omega)$.

Definition 3.3 (Generalized exponential separation)**.** $\Phi = ((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ is said to admit a *generalized exponential separation* if there are a family of generalized principal Floquet subspaces $\{\tilde{E}(\omega)\}_{\omega\in\tilde{\Omega}}$ and a family of one-codimensional subspaces $\{\tilde{F}(\omega)\}_{\omega\in\tilde{\Omega}}$ of X satisfying the following:

(i) $\tilde{F}(\omega) \cap X^+ = \{0\}$ for any $\omega \in \tilde{\Omega}$,
(ii) $X = \tilde{E}(\omega) \oplus \tilde{F}(\omega)$ for any $\omega \in \tilde{\Omega}$, where the decomposition is invariant, and the family of projections associated with this decomposition is strongly measurable and tempered,
(iii) there exists $\tilde{\sigma} \in (0, \infty]$ such that

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \frac{\|U_\omega(t)|_{\tilde{F}(\omega)}\|}{\|U_\omega(t)w(\omega)\|} = -\tilde{\sigma}$$

for each $\omega \in \tilde{\Omega}$.

We say that $\{\tilde{E}(\cdot), \tilde{F}(\cdot), \tilde{\sigma}\}$ *generates a generalized exponential separation.*

We remark that in general the generalized principal Lyapunov exponent $\tilde{\lambda}$ associated to the generalized principal Floquet subspaces $\{\tilde{E}(\omega)\}_{\omega\in\tilde{\Omega}}$ may be $-\infty$. The limit in Definition 3.3 may not be uniform in $\omega \in \tilde{\Omega}$. The generalized exponential separation is the extension of the classical exponential separation.

3.2. **Oseledets-type theorem.** In this subsection, we recall an Oseledets-type theorem proved in [14].

Theorem 3.4. *Let X be a separable Banach space. Let Φ be a measurable linear skew-product semidynamical system satisfying* (A1)(i)–(iii). *Then there exists an invariant $\Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$, with the property that one of the following* (*mutually exclusive*) *cases,* (1), (2) *or* (3), *holds:*

(1) *For each $\omega \in \Omega_0$,*

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)\| = -\infty.$$

(2) *There are k $(k \geq 1)$ real numbers $\lambda_1 > \cdots > \lambda_k$, k measurable families $\{E_1(\omega)\}_{\omega\in\Omega_0}$, ..., $\{E_k(\omega)\}_{\omega\in\Omega_0}$ of vector subspaces of finite dimensions, and a family $\{F_\infty(\omega)\}_{\omega\in\Omega_0}$ of closed vector subspaces of finite codimension such that*
- *$U_\omega(t)E_i(\omega) = E_i(\theta_t\omega)$ $(i = 1, 2, \ldots, k)$ and $U_\omega(t)F_\infty(\omega) \subset F_\infty(\theta_t\omega)$ for any $\omega \in \Omega_0$ and $t \in \mathbb{T}^+$,*
- *$E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_\infty(\omega) = X$ for any $\omega \in \Omega_0$; moreover, the family of projections associated with the decomposition $\Big(\bigoplus_{j=1}^{i} E_j(\omega)\Big) \oplus$*

$\Big(\bigoplus_{j=i+1}^{k} E_j(\omega) \oplus F_\infty(\omega)\Big) = X$ $(i = 1, 2, \dots, k)$ *is strongly measurable and tempered,*

- $$\lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} \ln \|U_\omega(t)|_{E_i(\omega)}\| = \lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} \ln \|U_\omega(t)u\| = \lambda_i$$
  for any $\omega \in \Omega_0$ *and any nonzero* $u \in E_i(\omega)$ $(i = 1, \dots, k)$,
- $$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| = \lambda_i$$
  for any $\omega \in \Omega_0$ *and any* $u \in (E_i(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_\infty(\omega)) \setminus (E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_\infty(\omega))$ $(i = 1, \dots, k-1)$,
- $$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| = \lambda_k$$
  for any $\omega \in \Omega_0$ *and any* $u \in (E_k(\omega) \oplus F_\infty(\omega)) \setminus F_\infty(\omega)$, *and*
- $$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)|_{F_\infty(\omega)}\| = -\infty$$
  for any $\omega \in \Omega_0$.

(3) *There are a sequence of real numbers* $\lambda_1 > \cdots > \lambda_i > \lambda_{i+1} > \cdots$ *having limit* $-\infty$, *countably many measurable families* $\{E_1(\omega)\}_{\omega\in\Omega_0}$, $\{E_2(\omega)\}_{\omega\in\Omega_0}$, $\dots$ *of vector subspaces of finite dimensions, and countably many families* $\{F_1(\omega)\}_{\omega\in\Omega_0}$, $\{F_2(\omega)\}_{\omega\in\Omega_0}$, $\dots$ *of closed vector subspaces of finite codimensions such that*
- $U_\omega(t)E_i(\omega) = E_i(\theta_t\omega)$ *and* $U_\omega(t)F_i(\omega) \subset F_i(\theta_t\omega)$ $(i = 1, 2, \dots)$ *for any* $\omega \in \Omega_0$ *and* $t \in \mathbb{T}^+$,
- $E_1(\omega)\oplus\cdots\oplus E_i(\omega)\oplus F_i(\omega) = X$ *and* $F_i(\omega) = E_{i+1}(\omega)\oplus F_{i+1}(\omega)$ *for any* $\omega \in \Omega_0$ $(i = 1, 2, \dots)$; *moreover, the family of projections associated with the decomposition* $\Big(\bigoplus_{j=1}^{i} E_j(\omega)\Big) \oplus F_i(\omega) = X$ $(i = 1, 2, \dots)$ *is strongly measurable and tempered,*
- $$\lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} \ln \|U_\omega(t)|_{E_i(\omega)}\| = \lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} \ln \|U_\omega(t)u\| = \lambda_i$$
  for any $\omega \in \Omega_0$ *and any nonzero* $u \in E_j(\omega)$ $(i = 1, 2, \dots)$,
- $$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| = \lambda_i$$
  for any $\omega \in \Omega_0$ *and any* $u \in (E_i(\omega) \oplus F_i(\omega)) \setminus F_i(\omega)$ $(i = 1, 2, \dots)$, *and*
- $$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)|_{F_i(\omega)}\| = \lambda_{i+1}$$
  for any $\omega \in \Omega_0$ $(i = 1, 2, \dots)$.

In the above, for $t = -s$ for some $s \in \mathbb{T}^+$ and $u \in E_i(\omega)$, the symbol $U_\omega(t)u$ stands for $v \in E_i(\theta_t\omega)$ such that $U_{\theta_t\omega}(s)v = u$. In view of the fact that $U_{\theta_t\omega}(s)E_i(\theta_t\omega) = E_i(\omega)$ and the injectivity (A1)(ii), such a v is well defined.

In case (2), we write $F_i(\omega)$ for $E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_\infty(\omega)$, $i = 1, 2, \ldots, k$.

In literature, λ_i's in the cases (2) and (3) are called *Lyapunov exponents* and $E_i(\omega)$'s are called the *Oseledets spaces* associated to λ_i's.

3.3. **Main results.** We state the main results of the paper in this subsection. The first theorem is on the existence of entire positive orbits.

Theorem 3.5 (Entire positive orbits)**.** *Assume Φ is a continuous measurable linear skew-product semidynamical system satisfying* (A0)(i)*,* (A1)(i)–(iii) *and* (A2)*. If Theorem* 3.4*(2) or (3) occurs and X^+ is total, then the set Ω_1 of those $\omega \in \Omega_0$ such that $E_1(\omega) \cap X^+ \supsetneqq \{0\}$ has $\mathbb{P}$-measure one, and for each $\omega \in \Omega_1$ there exists an entire positive orbit $v_\omega \colon \mathbb{T} \to X^+$ of U_ω such that*

$$v_\omega(t) \in (E_1(\theta_t\omega) \cap X^+) \setminus \{0\} \quad \forall t \in \mathbb{T}.$$

The above theorem shows the existence of an entire positive orbit of U_ω for a.e. $\omega \in \Omega$ without the assumption that U_ω is strongly positive, which extends the principal eigenfunction theory for strongly positive and compact operators. Note that in general $E_1(\omega) \neq \operatorname{span}\{v_\omega(0)\}$ in the case where Theorem 3.4(2) or (3) occurs.

The next theorem shows the existence of generalized Floquet subspaces and principal Lyapunov exponent and the uniqueness of entire positive orbits.

Theorem 3.6 (Generalized principal Floquet subspace and Lyapunov exponent)**.** *Assume* (A0)(ii)*,* (A1)(i)*,* (A2) *and* (A3)*. Then there exist an invariant set $\tilde{\Omega}_1 \subset \Omega$, $\mathbb{P}(\tilde{\Omega}_1) = 1$, and an $(\mathfrak{F}, \mathfrak{B}(X))$-measurable function $w \colon \tilde{\Omega}_1 \to X$, $w(\omega) \in C_{\mathbf{e}}$ and $\|w(\omega)\| = 1$ for all $\omega \in \tilde{\Omega}_1$, having the following properties:*

(1)
$$w(\theta_t\omega) = \frac{U_\omega(t)w(\omega)}{\|U_\omega(t)w(\omega)\|}$$
for any $\omega \in \tilde{\Omega}_1$ and $t \in \mathbb{T}^+$.

(2) *For some $\omega \in \tilde{\Omega}_1$ let a function $v_\omega \colon \mathbb{T} \to X^+ \setminus \{0\}$ be an entire orbit of U_ω. Then $v_\omega(t) = \|v_\omega(0)\| w_\omega(t)$ for all $t \in \mathbb{T}$, where*
$$w_\omega(t) := \begin{cases} (U_{\theta_t\omega}(-t)|_{\tilde{E}_1(\theta_t\omega)})^{-1} w(\omega) & \text{for } t \in \mathbb{T},\ t < 0, \\ U_\omega(t)w(\omega) & \text{for } t \in \mathbb{T}^+, \end{cases}$$
with $\tilde{E}_1(\omega) = \operatorname{span}\{w(\omega)\}$.

(3) *There exists $\tilde{\lambda}_1 \in [-\infty, \infty)$ such that*
$$\tilde{\lambda}_1 = \lim_{\substack{t \to \pm\infty \\ t \in \mathbb{T}}} \frac{1}{t} \ln \rho_t(\omega) = \int_\Omega \ln \rho_1 \, d\mathbb{P}$$
for each $\omega \in \tilde{\Omega}_1$, where
$$\rho_t(\omega) := \begin{cases} \|U_\omega(t)w(\omega)\| & \text{for } t \geq 0, \\ 1/\|U_{\theta_t\omega}(-t)w(\theta_t\omega)\| & \text{for } t < 0. \end{cases}$$

(4) *Assume, moreover, that* (A1)(ii)–(iii) *hold and that* X^+ *is total. If Theorem* 3.4*(1) occurs, then* $\tilde{\lambda}_1 = -\infty$. *If Theorem* 3.4*(2) or (3) occurs, then* $\tilde{\lambda}_1 = \lambda_1$. *Hence for any* $u \in X \setminus \{0\}$,

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \le \tilde{\lambda}_1,$$

and then $\{\tilde{E}_1(\omega)\}_{\omega\in\tilde{\Omega}_1}$ *is a family of generalized Floquet subspaces.*

(5) *Assume, moreover, that* (A0)(iii) *holds. Then for any* $u \in X \setminus \{0\}$,

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \le \tilde{\lambda}_1,$$

and then $\{\tilde{E}_1(\omega)\}_{\omega\in\tilde{\Omega}_1}$ *is a family of generalized Floquet subspaces.*

Observe that $U_\omega(t)\tilde{E}_1(\omega) = \tilde{E}_1(\theta_t\omega)$ for any $\omega \in \tilde{\Omega}_1$ and any $t \in \mathbb{T}^+$. Since $w\colon \tilde{\Omega}_1 \to X$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable, $\{\tilde{E}_1(\omega)\}_{\omega\in\tilde{\Omega}_1}$ is a measurable family of one-dimensional subspaces of X. For $\omega \in \tilde{\Omega}_1$, the function $w_\omega\colon \mathbb{T} \to X^+$ is a nontrivial entire orbit of U_ω. By Theorem 3.6(2), a nontrivial entire orbit of U_ω is unique up to multiplication by positive scalar, which extends the fundamental property on the existence and uniqueness of positive eigenvectors of compact u_0-positive linear operators (see [11] and [12]). Note that Theorem 3.4(1) may occur under the assumptions of Theorem 3.6.

The theorem below is a counterpart of Theorem 3.6 for the dual system.

Theorem 3.7 (Generalized principal Floquet subspace and Lyapunov exponent)**.** *Assume* (A0)*(ii), (A1)*(i), (A2)* *and* (A3)*. *Then there exist an invariant set* $\tilde{\Omega}_1^* \subset \Omega$, $\mathbb{P}(\tilde{\Omega}_1^*) = 1$, *and an* $(\mathfrak{F}, \mathfrak{B}(X^*))$*-measurable function* $w^*\colon \tilde{\Omega}_1^* \to X^*$, $w^*(\omega) \in C_{\mathbf{e}^*}$ *and* $\|w^*(\omega)\| = 1$ *for all* $\omega \in \tilde{\Omega}_1^*$, *having the following properties:*

(1)
$$w^*(\theta_{-t}\omega) = \frac{U_\omega^*(t)w^*(\omega)}{\|U_\omega^*(t)w^*(\omega)\|}$$

for any $\omega \in \tilde{\Omega}_1^*$ *and* $t \in \mathbb{T}^+$.

(2) *For some* $\omega \in \tilde{\Omega}_1^*$ *let a function* $v_\omega^*\colon \mathbb{T} \to (X^*)^+ \setminus \{0\}$ *be an entire orbit of* U_ω^*. *Then* $v_\omega^*(t) = \|v_\omega^*(0)\| w_\omega^*(t)$ *for all* $t \in \mathbb{T}$, *where*

$$w_\omega^*(t) := \begin{cases} (U^*_{\theta_{-t}\omega}(-t)|_{\tilde{E}_1^*(\theta_{-t}\omega)})^{-1} w^*(\omega) & \text{for } t \in \mathbb{T},\ t < 0, \\ U_\omega^*(t)w^*(\omega) & \text{for } t \in \mathbb{T}^+, \end{cases}$$

where $\tilde{E}_1^*(\omega) = \operatorname{span}\{w^*(\omega)\}$.

(3) *There exists* $\tilde{\lambda}_1^* \in [-\infty, \infty)$ *such that*

$$\tilde{\lambda}_1^* = \lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} \ln \rho_t^*(\omega) = \int_\Omega \ln \rho_1^* \, d\mathbb{P}$$

for each $\omega \in \tilde{\Omega}_1$, *where*

$$\rho_t^*(\omega) := \begin{cases} \|U_\omega^*(t)w^*(\omega)\| & \text{for } t \ge 0, \\ 1/\|U^*_{\theta_{-t}\omega}(-t)w^*(\theta_{-t}\omega)\| & \text{for } t < 0. \end{cases}$$

(4) *If* (A0)(ii), (A1)(i), (A2) *and* (A3) *are satisfied, then* $\tilde{\lambda}_1 = \tilde{\lambda}_1^*$.

For $\omega \in \tilde{\Omega}_1^*$, define $\tilde{F}_1(\omega) := \{ u \in X : \langle u, w^*(\omega) \rangle = 0 \}$. Then $\{\tilde{F}_1(\omega)\}_{\omega \in \tilde{\Omega}_1^*}$ is a family of one-codimensional subspaces of X such that $U_\omega(t)\tilde{F}_1(\omega) \subset \tilde{F}_1(\theta_t\omega)$ for any $\omega \in \tilde{\Omega}_1^*$ and any $t \in \mathbb{T}^+$.

Assume, for the moment, that (A1)(ii)–(iii). For $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$, let $\hat{F}_1(\omega)$ be defined by

$$\hat{F}_1(\omega) = \begin{cases} F_\infty(\omega) & \text{if (2) in Theorem 3.4 holds with } k = 1, \\ \bigoplus_{j=2}^{k} E_j(\omega) \oplus F_\infty(\omega) & \text{if (2) in Theorem 3.4 holds with } k > 1, \\ F_1(\omega) & \text{if (3) in Theorem 3.4 holds.} \end{cases} \tag{3.1}$$

Further, let $\hat{\lambda}_2$ be defined by

$$\hat{\lambda}_2 = \begin{cases} \lambda_2 & \text{if (3) in Theorem 3.4 holds or if} \\ & \text{(2) in Theorem 3.4 holds with } k > 1, \\ \lim\limits_{\substack{t\to\infty \\ t\in\mathbb{T}}} \dfrac{1}{t} \ln \|U_\omega(t)|_{\hat{F}_1(\omega)}\| & \text{if (2) in Theorem 3.4 holds with } k = 1. \end{cases} \tag{3.2}$$

The next theorem shows the existence of a generalized exponential separation.

Theorem 3.8 (Generalized exponential separation)**.** *Assume* (A0)(iii), (A1)(i), (A2), (A0)*(iii), (A1)*(i), (A2)*, *and* (A4). *Then there is an invariant set* $\tilde{\Omega}_0$, $\mathbb{P}(\tilde{\Omega}_0) = 1$, *having the following properties:*

(1) *The family* $\{\tilde{P}(\omega)\}_{\omega\in\tilde{\Omega}_0}$ *of projections associated with the invariant decomposition* $\tilde{E}_1(\omega) \oplus \tilde{F}_1(\omega) = X$ *is strongly measurable and tempered.*

(2) $\tilde{F}_1(\omega) \cap X^+ = \{0\}$ *for any* $\omega \in \tilde{\Omega}_0$.

(3) *For any* $\omega \in \tilde{\Omega}_0$ *and any* $u \in X \setminus \tilde{F}_1(\omega)$ (*in particular, for any nonzero* $u \in X^+$) *there holds*

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)\| = \lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| = \tilde{\lambda}_1.$$

(4) *There exist* $\tilde{\sigma} \in (0, \infty]$ *and* $\tilde{\lambda}_2 \in [-\infty, \infty)$, $\tilde{\lambda}_2 = \tilde{\lambda}_1 - \tilde{\sigma}$ *such that*

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \frac{\|U_\omega(t)|_{\tilde{F}(\omega)}\|}{\|U_\omega(t)w(\omega)\|} = -\tilde{\sigma}$$

and

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)|_{\tilde{F}_1(\omega)}\| = \tilde{\lambda}_2$$

for each $\omega \in \tilde{\Omega}_0$. *Hence* Φ *admits a generalized exponential separation.*

(5) *Assume moreover* (A1)(ii)–(iii) *and* (A1)*(ii)–(iii). *If Theorem* 3.4*(2) or (3) occurs, then* $\tilde{\lambda}_2 = \hat{\lambda}_2 < \tilde{\lambda}_1$ *and* $E_1(\omega) = \tilde{E}_1(\omega)$ *and* $\hat{F}_1(\omega) = \tilde{F}_1(\omega)$ *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega_0$.

(6) *If* (A5) *or* (A5)* *holds, then* $\tilde{\lambda}_1 > -\infty$. *If additionally* (A1)(ii)–(iii) *and* (A1)*(ii)–(iii) *hold, then Theorem* 3.4*(2) or (3) occurs.*

The last theorem is about comparison of principal Lyapunov exponents.

Theorem 3.9 (Monotonicity)**.** *Let two measurable linear skew-product semidynamical systems* $\Phi^{(1)} = ((U^{(1)}_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ *and* $\Phi^{(2)} = ((U^{(2)}_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ *have the property that*

$$U^{(1)}_\omega(t_0)u \le U^{(2)}_\omega(t_0)u$$

for some $t_0 \in \mathbb{T}^+ \setminus \{0\}$, $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and all* $u \in X^+$*. Assume that both* $\Phi^{(1)}$ *and* $\Phi^{(2)}$ *satisfy* (A0)(iii), (A1)(i), (A2), (A0)*(iii), (A1)*(i), (A2)*, *and* (A4)*. Then*

$$\tilde{\lambda}^{(1)}_1 \le \tilde{\lambda}^{(2)}_1,$$

where $\tilde{\lambda}^{(i)}_1$, $i = 1, 2$, *denotes the generalized principal Lyapunov exponent for* $\Phi^{(i)}$.

4. Preliminaries

In this section, we present some preliminary materials for use in the proofs of the main results, including the Birkhoff Ergodic Theorem, Kingman Subadditive Ergodic Theorem, Hilbert projective metric in ordered Banach spaces and basic properties, oscillation ratio, Birkhoff contraction ratio, and projective diameter of positive operators in ordered Banach spaces and basic properties.

4.1. **Ergodic theorems.**

In this subsection, we recall the Birkhoff Ergodic Theorem and Kingman Subadditive Ergodic Theorem.

Theorem 4.1 (Birkhoff Ergodic Theorem)**.**

(i) (Discrete-time case) *Assume that* $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ *is a metric discrete-time dynamical system. Let* $f\colon \Omega \to \mathbb{R}$ *be an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function with* $f^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*. Then there exist*
- *an invariant set* $\Omega_1 \subset \Omega$ *with* $\mathbb{P}(\Omega_1) = 1$ *and*
- *an invariant* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* f_{av} *with* $(f_{\mathrm{av}})^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$

such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\theta_i\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\theta_{-i}\omega) = f_{\mathrm{av}}(\omega)$$

for all $\omega \in \Omega_1$*. Moreover,*

$$\int_\Omega f_{\mathrm{av}}\, d\mathbb{P} = \int_\Omega f\, d\mathbb{P} \in [-\infty, \infty).$$

If $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ *is ergodic, then* f_{av} *is constantly equal to* $\int_\Omega f\, d\mathbb{P}$.

(ii) (Continuous-time case) *Assume that* $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ *is a metric flow. Let* $f\colon \Omega \to \mathbb{R}$ *be an* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function with* $f^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$*. Then there exist*
- *an invariant set* $\Omega_1 \subset \Omega$ *with* $\mathbb{P}(\Omega_1) = 1$ *and*
- *an invariant* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* f_{av} *with* $(f_{\mathrm{av}})^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$

such that

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_{0}^{t} f(\theta_s\omega)\, ds = \lim_{t\to\infty} \frac{1}{t} \int\limits_{-t}^{0} f(\theta_s\omega)\, ds = f_{\mathrm{av}}(\omega)$$

for all $\omega \in \Omega_1$*. Moreover,*

$$\int\limits_{\Omega} f_{\mathrm{av}}\, d\mathbb{P} = \int\limits_{\Omega} f\, d\mathbb{P} \in [-\infty, \infty).$$

If $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ *is ergodic, then* f_{av} *is constantly equal to* $\int_\Omega f\, d\mathbb{P}$*.*

Lemma 4.2. *Assume that* $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{T}})$ *is an ergodic metric dynamical system. Then for each* $f \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ *the set of those* $\omega \in \Omega$ *for which*

$$\lim_{\substack{t\to\pm\infty \\ t\in\mathbb{T}}} \frac{1}{t} f(\theta_t\omega) = 0$$

has $\mathbb{P}$*-measure one.*

Theorem 4.3 (Kingman Subadditive Ergodic Theorem)**.** *Assume that* $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ *is a metric discrete-time dynamical system. Let* $(f_n)_{n=1}^{\infty}$*,* $f_n \colon \Omega \to \mathbb{R}$*, be a sequence of* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable functions with* $(f_1)^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ *such that*

$$f_{m+n}(\omega) \le f_m(\omega) + f_n(\theta_m\omega) \qquad \text{for any } m, n \in \mathbb{N} \text{ and any } \omega \in \Omega.$$

Then there exist

- *an invariant set* $\Omega_1 \subset \Omega$ *with* $\mathbb{P}(\Omega_1) = 1$ *and*
- *an invariant* $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable function* f_{av} *with* $(f_{\mathrm{av}})^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$

such that

$$\lim_{n\to\infty} \frac{1}{n} f_n(\omega) = f_{\mathrm{av}}(\omega)$$

for all $\omega \in \Omega_1$*. Moreover,*

$$\int\limits_{\Omega} f_{\mathrm{av}}\, d\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int\limits_{\Omega} f_n\, d\mathbb{P} = \inf_{n\in\mathbb{N}} \frac{1}{n} \int\limits_{\Omega} f_n\, d\mathbb{P} \in [-\infty, \infty).$$

If $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ *is ergodic, then* f_{av} *is constantly equal to* $\lim_{n\to\infty} (1/n) \int_\Omega f_n\, d\mathbb{P}$*.*

4.2. **Hilbert projective metric.** Throughout this subsection, we assume that (X, X^+) is an ordered Banach space. We recall the concept of the Hilbert projective metric and present some basic properties.

Definition 4.4. (1) For given $u, v \in X$, if $\{\, \underline{\alpha} \in \mathbb{R} : \underline{\alpha} v \le u \,\}$ is nonempty, define

$$m(u/v) := \sup\{\, \underline{\alpha} \in \mathbb{R} : \underline{\alpha} v \le u \,\}.$$

If $\{\overline{\alpha} \in \mathbb{R} : u \le \overline{\alpha} v\,\}$ is nonempty, define

$$M(u/v) := \inf\{\, \overline{\alpha} \in \mathbb{R} : u \le \overline{\alpha}\, v\}.$$

(2) For given $u, v \in X$, if both $m(u/v)$ and $M(u/v)$ exist, define

$$\operatorname{osc}(u/v) := M(u/v) - m(u/v) \ \text{ and } \ d(u, v) := \ln \frac{M(u/v)}{m(u/v)}.$$

$\operatorname{osc}(u/v)$ is called the *oscillation* of u over v, and $d(u, v)$ is called the *projective distance* between u and v.

It should be noted that for comparable $u, v \in X^+ \setminus \{0\}$, we have the following alternative:

- either
$$m(u/v)v < u < M(u/v)v$$
- or there is $\alpha > 0$ such that $v = \alpha u$.

The following lemma follows easily.

Lemma 4.5. (1) *$d(u,v) = d(v,u)$ if $u, v \in X^+ \setminus \{0\}$ and $u \sim v$.*
(2) *$d(\lambda u, \mu v) = d(u,v)$ if $u, v \in X^+ \setminus \{0\}$, $u \sim v$, and $\lambda, \mu > 0$.*
(3) *$d(u,v) \le d(u,w) + d(w,v)$ if $u, v, w \in X^+ \setminus \{0\}$ and $u \sim v \sim w$.*
(4) *For any two $u, v \in X^+ \setminus \{0\}$, $d(u,v) = 0$ implies the existence of $\alpha > 0$ such that $v = \alpha u$.*
(5) *$\operatorname{osc}(\lambda u/\mu v) = |\lambda|\mu^{-1} \operatorname{osc}(u/v)$ if $\operatorname{osc}(u/v)$ exists, $u, v \in X^+ \setminus \{0\}$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mu > 0$.*
(6) *$\operatorname{osc}(u + v/w) \le \operatorname{osc}(u/w) + \operatorname{osc}(v/w)$ if $u, v, w \in X^+ \setminus \{0\}$ and $u \sim w$, $v \sim w$.*

Lemma 4.6. *Assume that X^+ is normal. Then for any $u, v \in X^+$, $u \sim v$, with $\|u\| = \|v\| = 1$, there holds*
$$\|u - v\| \le 3\big(e^{d(u,v)} - 1\big).$$
If (X, X^+) is a Banach lattice, then for any $u, v \in X^+$, $u \sim v$, with $\|u\| = \|v\| = 1$, there holds
$$\|u - v\| \le e^{d(u,v)} - 1.$$

Proof. See [5, Proposition 1.2.1]. □

Lemma 4.7. *Suppose that $u, v, u_k, v_k \in X^+ \setminus \{0\}$ for $k = 1, 2, \dots$. If $u_k \sim v_k$ for $k = 1, 2, \dots$, $(1/m(u_k/v_k))_{m=1}^{\infty}$ and $(M(u_k/v_k))_{m=1}^{\infty}$ are bounded sequences, and $u_k \to u$, $v_k \to v$ as $k \to \infty$, then $u \sim v$ and $d(u,v) \le \liminf_{k\to\infty} d(u_k, v_k)$.*

Proof. First, let $M > 1$ be such that
$$m(u_k/v_k) \ge \frac{1}{M-1} \quad \text{and} \quad M(u_k/v_k) \le M - 1 \quad \text{for} \quad k = 1, 2, \dots.$$
Then there are $\frac{1}{M} \le \underline{\alpha}_k, \overline{\alpha}_k \le M$, such that
$$\underline{\alpha}_k v_k \le u_k \le \overline{\alpha}_k v_k.$$
By (C2), $\underline{\alpha}_k \le \overline{\alpha}_k$. Without loss of generality, we may assume that $\underline{\alpha}_k \to \underline{\alpha}$ and $\overline{\alpha}_k \to \overline{\alpha}$ as $n \to \infty$. Then $0 < \underline{\alpha} \le \overline{\alpha}$ and
$$\underline{\alpha} v \le u \le \overline{\alpha} v.$$
Hence $u \sim v$.

For any $\epsilon > 0$ there are $\frac{1}{M} \le \underline{\alpha}_k \le \overline{\alpha}_k \le M$ such that
$$\underline{\alpha}_k v_k \le u_k \le \overline{\alpha}_k v_k \quad \text{and} \quad d(u_k, v_k) \ge \ln \frac{\overline{\alpha}_k}{\underline{\alpha}_k} - \epsilon.$$
Assume that $k_l \to \infty$ is such that
$$\lim_{k\to\infty} d(u_{k_l}, v_{k_l}) = \liminf_{k\to\infty} d(u_k, v_k)$$
and
$$\underline{\alpha}_{k_l} \to \underline{\alpha}, \quad \overline{\alpha}_{k_l} \to \overline{\alpha} \quad \text{as} \quad k \to \infty.$$

Then

$$\underline{\alpha} v \le u \le \overline{\alpha} v \quad \text{and} \quad d(u,v) \le \ln \frac{\overline{\alpha}}{\underline{\alpha}}.$$

Therefore

$$d(u,v) \le \liminf_{k\to\infty} d(u_k, v_k) + \epsilon$$

for any $\epsilon > 0$, and hence

$$d(u,v) \le \liminf_{k\to\infty} d(u_k, v_k).$$

□

4.3. **Oscillation ratio, Birkhoff contraction ratio, and projective diameter.** Throughout this subsection, we assume that (X, X^+) is an ordered Banach space and that $\Phi = ((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ is a measurable linear skew-product semidynamical system on X covering $(\theta_t)_{t\in\mathbb{T}}$, satisfying (A2). At some places (A3) will be assumed.

Definition 4.8. (1) For $\omega \in \Omega$ define

$$p(\omega) := \sup_{\substack{u,v\in X^+ \\ u\sim v \\ u\ne\alpha v}} \frac{\operatorname{osc}(U_\omega(1)u/U_\omega(1)v)}{\operatorname{osc}(u/v)}.$$

$p(\omega)$ is called the *oscillation ratio* of $U_\omega(1)$.

(2) For $\omega \in \Omega$ define

$$q(\omega) := \sup_{\substack{u,v\in X^+ \\ u\sim v \\ u\ne\alpha v}} \frac{d(U_\omega(1)u, U_\omega(1)v)}{d(u,v)}.$$

$q(\omega)$ is called the *Birkhoff contraction ratio* of $U_\omega(1)$.

(3) For $\omega \in \Omega$ define

$$\tau(\omega) := \sup_{\substack{u,v\in X^+ \\ U_\omega(1)u\sim U_\omega(1)v}} d(U_\omega(1)u, U_\omega(1)v).$$

$\tau(\omega)$ is called the *projective diameter* of $U_\omega(1)$.

The functions p^*, q^* and τ^* for the dual Φ^* are defined in an analogous way.

Lemma 4.9. *For any $\omega \in \Omega$, any $t \in \mathbb{T}^+$ and any $u, v \in X^+ \setminus \{0\}$ with $u \sim v$, there holds*

$$\begin{aligned} m(U_\omega(t)u/U_\omega(t)v) &\ge m(u/v), \\ M(U_\omega(t)u/U_\omega(t)v) &\le M(u/v), \\ \operatorname{osc}(U_\omega(t)u/U_\omega(t)v) &\le \operatorname{osc}(u/v), \\ d(U_\omega(t)u, U_\omega(t)v) &\le d(u,v). \end{aligned}$$

Proof. For any $\underline{\alpha}, \overline{\alpha} > 0$ with

$$\underline{\alpha} v \le u \le \overline{\alpha} v,$$

by (A2),

$$\underline{\alpha} U_\omega(t)v \le U_\omega(t)u \le \overline{\alpha} U_\omega(t)v \quad \forall\, \omega \in \Omega,\ t \in \mathbb{T}^+.$$

This implies that

$$m(U_\omega(t)u/U_\omega(t)v) \ge m(u/v) \quad \text{and} \quad M(U_\omega(t)u/U_\omega(t)v) \le M(u/v),$$

thus

$$\operatorname{osc}(U_\omega(t)u/U_\omega(t)v) \le \operatorname{osc}(u/v) \quad \text{and} \quad d(U_\omega(t)u, U_\omega(t)v) \le d(u,v).$$

□

The next four results will be formulated for both Φ and its dual Φ^*; we will however formulate their proofs for Φ only.

Lemma 4.10. *Assume moreover* (A3) *and* (A3)*. *For each* $\omega \in \Omega$ *and each* $u \in X^+ \setminus \{0\}$, $u^* \in (X^*)^+ \setminus \{0\}$, *there holds* $U_\omega(1)u \sim \mathbf{e}$, $U^*_\omega(1)u^* \sim \mathbf{e}^*$, *and*

$$d(U_\omega(1)u, \mathbf{e}) \le \ln \varkappa(\omega),\ d(U_\omega(1)u^*, \mathbf{e}^*) \le \ln \varkappa^*(\omega).$$

Consequently, $\tau(\omega) \le 2\ln\varkappa(\omega)$ *and* $\tau^*(\omega) \le 2\ln\varkappa^*(\omega)$ *for any* $\omega \in \Omega$.

Proof. By (A3), for any $\omega \in \Omega$ and $u \in X^+ \setminus \{0\}$ we have

$$m(U_\omega(1)u/\mathbf{e}) \ge \beta(\omega,u), \qquad M(U_\omega(1)u/\mathbf{e}) \le \varkappa(\omega)\beta(\omega,u).$$

It now suffices to apply the definition of $d(\cdot,\cdot)$ and Lemma 4.5(3). □

Lemma 4.11. *Assume moreover* (A3) *and* (A3)*. *For each* $\omega \in \Omega$,

$$\tau(\omega) < \infty,\ \tau^*(\omega) < \infty$$

and

$$p(\omega) = q(\omega) = \tanh\frac{1}{4}\tau(\omega)(<1),\ p^*(\omega) = q^*(\omega) = \tanh\frac{1}{4}\tau^*(\omega)(<1).$$

Proof. By [5, Theorem 2.1.1], for any $\omega \in \Omega$, either

$$\tau(\omega) = \infty, \quad p(\omega) = 1, \quad q(\omega) = 1 \tag{4.1}$$

or

$$p(\omega) = q(\omega) = \tanh\frac{1}{4}\tau(\omega). \tag{4.2}$$

By Lemma 4.10, $\tau(\omega) < \infty$. The lemma then follows. □

Lemma 4.12. *Assume moreover* (A3) *and* (A3)*. *For any* $\omega \in \Omega$, *if* $u, v \in X^+ \setminus \{0\}$ *are such that* $u \sim v$ *but* $v \ne \alpha u$ *for any positive real* α, *then*

$$m(U_\omega(1)u/U_\omega(1)v) > m(u/v) \quad \textit{and} \quad M(U_\omega(1)u/U_\omega(1)v) < M(u/v).$$

Similarly, for any $\omega \in \Omega$, *if* $u^*, v^* \in (X^*)^+ \setminus \{0\}$ *are such that* $u^* \sim v^*$ *but* $v^* \ne \alpha u^*$ *for any positive real* α, *then*

$$m(U^*_\omega(1)u^*/U^*_\omega(1)v^*) > m(u^*/v^*) \quad \textit{and} \quad M(U^*_\omega(1)u^*/U^*_\omega(1)v^*) < M(u^*/v^*).$$

Proof. For u, v as in the assumption we have

$$m(u/v)v < u < M(u/v)v.$$

By (A3),

$$U_\omega(1)(u - m(u/v)v) \ge \beta(\omega, u - m(u/v)v)\mathbf{e}$$

and

$$\mathbf{e} \ge \frac{1}{\varkappa(\omega)\beta(\omega,v)}U_\omega(1)v,$$

which gives that

$$U_\omega(1)u \ge \left(m(u/v) + \frac{\beta(\omega, u - m(u/v)v)}{\varkappa(\omega)\beta(\omega,v)}\right)U_\omega(1)v.$$

The first inequality follows immediately. The proof of the second inequality is similar. □

Lemma 4.13. *Let the cones X^+, $(X^*)^+$ be normal and X, X^* be separable. Assume moreover* (A3) *and* (A3)**. Then the functions*

$$[\,\Omega \ni \omega \mapsto \tau(\omega) \in \mathbb{R}\,],\ [\,\Omega \ni \omega \mapsto \tau^*(\omega) \in \mathbb{R}\,],$$
$$[\,\Omega \ni \omega \mapsto p(\omega) \in \mathbb{R}\,],\ [\,\Omega \ni \omega \mapsto p^*(\omega) \in \mathbb{R}\,],$$
$$[\,\Omega \ni \omega \mapsto q(\omega) \in \mathbb{R}\,],\ [\,\Omega \ni \omega \mapsto q^*(\omega) \in \mathbb{R}\,]$$

are $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$*-measurable.*

Proof. By Lemma 4.11, it suffices to prove that $[\,\Omega \ni \omega \mapsto \tau(\omega) \in \mathbb{R}\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable.

First, fix u and v in $X^+ \setminus \{0\}$. We prove that $[\,\omega \mapsto M(U_\omega(1)u/U_\omega(1)v)\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable. To this end, take a countable set $\{\rho_k\}$ which is dense in $\mathbb{R}^+$. Let

$$\Omega_k := \{\, \omega \in \Omega : U_\omega(1)u \le \rho_k U_\omega(1)v \,\}.$$

Then Ω_k is the inverse image of X^+ under the function $[\,\omega \mapsto \rho_k U_\omega(1)v - U_\omega(1)u \in X\,]$. By the measurability of $U_\omega(1)$ in ω, Ω_k is a measurable subset of Ω. Let

$$M_k(\omega) := \begin{cases} \rho_k & \text{for} \quad \omega \in \Omega_k, \\ \infty & \text{for} \quad \omega \in \Omega \setminus \Omega_k. \end{cases}$$

Then $[\,\Omega \ni \omega \mapsto M_k(\omega) \in \mathbb{R}\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable and

$$M(U_\omega(1)u/U_\omega(1)v) = \inf_{k \ge 1} M_k(\omega).$$

It then follows that $[\,\omega \mapsto M(U_\omega(1)u/U_\omega(1)v)\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable.

Similarly, it can be proved that $[\,\omega \mapsto m(U_\omega(1)u/U_\omega(1)v)\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable, for fixed u and v in $X^+ \setminus \{0\}$.

Hence $[\,\omega \mapsto d(U_\omega(1)u, U_\omega(1)v)\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable, for fixed u and v in $X^+ \setminus \{0\}$.

Now let $\{u_k\}$ and $\{v_l\}$ be two dense countable sets in $X^+ \setminus \{0\}$. We claim that

$$\tau(\omega) = \sup_{k,l \ge 1} d(U_\omega(1)u_k, U_\omega(1)v_l), \tag{4.3}$$

and hence $\tau(\omega)$ is measurable in ω. In fact, for any $u, v \in X^+ \setminus \{0\}$ there are $(u_{k_m})_{m=1}^\infty$ and $(v_{l_m})_{k=1}^\infty$ such that $u_{k_m} \to u$ and $v_{l_m} \to v$ as $m \to \infty$. Observe that it follows from (A3) that

$$M(U_\omega(1)u_{k_m}/U_\omega(1)v_{l_m}) \le \varkappa^2(\omega) \cdot m(U_\omega(1)u_{k_m}/U_\omega(1)v_{l_m}) \tag{4.4}$$

for $m = 1, 2, \dots$.

We prove that $\{\, m(U_\omega(1)u_{k_m}/U_\omega(1)v_{k_m}) : m = 1, 2, \dots \}$ is bounded away from zero. Indeed, if not, then there is a subsequence $m_j \to \infty$ as $j \to \infty$ such that $m(U_\omega(1)u_{k_{m_j}}/U_\omega(1)v_{l_{m_j}}) \to 0$ as $j \to \infty$, from which it follows via (4.4) that $U_\omega(1)v_{l_{m_j}} \to 0$, which contradicts the fact that $U_\omega(1)v_{l_{m_j}} \to U_\omega(1)v \ne 0$. Now we prove that $\{\, M(U_\omega(1)u_{k_m}/U_\omega(1)v_{l_m}) : m = 1, 2, \dots \}$ is bounded. Indeed, if not, then there is a subsequence $m_j \to \infty$ as $j \to \infty$ such that $M(U_\omega(1)u_{k_{m_j}}/U_\omega(1)v_{l_{m_j}}) \to \infty$ as $j \to \infty$, from which it follows via (4.4) that $m(U_\omega(1)u_{k_{m_j}}/U_\omega(1)v_{l_{m_j}}) \to \infty$ as $j \to \infty$. It follows via the normality of the cone X^+ that $\|U_\omega(1)u_{k_{m_j}}\| \to \infty$, which contradicts the fact that $U_\omega(1)u_{k_{m_j}} \to U_\omega(1)u$.

Hence $\{M(U_\omega(1)u_{k_m}/U_\omega(1)v_{l_m})\}$ and $\{\frac{1}{m(U_\omega(1)u_{k_m}/U_\omega(1)v_{l_m})}\}$ are bounded. By Lemma 4.7,

$$d(U_\omega(1)u, U_\omega(1)v) \le \liminf_{m\to\infty} d(U_\omega(1)u_{k_m}, U_\omega(1)v_{l_m}).$$

This implies that (4.3) holds and then $[\,\omega \mapsto \tau(\omega)\,]$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable. □

5. Proofs of the main results

Throughout this entire section we assume (A0)(i) and that $\Phi = ((U_\omega(t))_{\omega\in\Omega, t\in\mathbb{T}^+}, (\theta_t)_{t\in\mathbb{T}})$ is a measurable linear skew-product semidynamical system on X covering $(\theta_t)_{t\in\mathbb{T}}$, satisfying (A1)(i) and (A2).

For a closed $E \subset X^+$, $E \ne \{0\}$, such that $u \in E$ and $\alpha \ge 0$ implies $\alpha u \in E$, the symbol $\mathcal{S}_1(E)$ will denote the intersection of E with the unit sphere in X, $\mathcal{S}_1(E) := \{\, u \in E : \|u\| = 1 \,\}$.

Under the assumption (A1)(ii) or (A3) we define, for $t \in \mathbb{T}^+$ and $\omega \in \Omega$, a function $\mathcal{U}_\omega(t) \colon \mathcal{S}_1(X^+) \to \mathcal{S}_1(X^+)$ by the formula

$$\mathcal{U}_\omega(t)u := \frac{U_\omega(t)u}{\|U_\omega(t)u\|}, \quad u \in \mathcal{S}_1(X^+).$$

The function $\mathcal{U}_\omega(t)$ is well defined. This is obvious under (A1)(ii). Then assume (A3). If $U_\omega(t)u = 0$ for some $t \in \mathbb{T}^+$ and $u \in \mathcal{S}_1(X^+)$, then there exists $n_0 \in \mathbb{N}\cup\{0\}$ such that $U_\omega(n_0)u \in X^+ \setminus \{0\}$ but $U_\omega(n_0+1)u = 0$. As $U_\omega(n_0+1)u = U_{\theta_{n_0}\omega}(1)(U_\omega(n_0)u)$, this contradicts (A3).

The function $\mathcal{U}_\omega(t)$ is clearly continuous. Furthermore, as a consequence of (2.2) we have

$$\mathcal{U}_\omega(s+t) = \mathcal{U}_{\theta_s\omega}(t) \circ \mathcal{U}_\omega(s), \quad s,t \in \mathbb{T}^+,\ \omega \in \Omega. \tag{5.1}$$

5.1. **Entire positive orbits and the proof of Theorem 3.5.** Throughout this subsection we assume moreover (A1)(ii)–(iii) and that X^+ is total. Further, we assume that Theorem 3.4(2) or (3) occurs. We investigate the existence of entire positive orbits of Φ and prove Theorem 3.5.

Let Ω_0 be as in Theorem 3.4. Recall that $\{E_1(\omega)\}_{\omega\in\Omega_0}$ is a measurable family of finite-dimensional vector subspaces of X such that for each $\omega \in \Omega_0$ and each $t \in \mathbb{T}^+$, the mapping $U_\omega(t)|_{E_1(\omega)} \colon E_1(\omega) \to E_1(\theta_t\omega)$ is a linear isomorphism.

Let $\hat{F}_1(\omega)$ be as in (3.1). Let $\{P(\omega)\}_{\omega\in\Omega_0}$ be the family of projections associated with the decomposition $E_1(\omega) \oplus \hat{F}_1(\omega) = X$. For $\omega \in \Omega_0$ put

$$\delta(\omega) := \inf\left\{ \frac{\|P(\omega)u\|}{\|u - P(\omega)u\|} : u \in X^+ \setminus \{0\} \right\}.$$

Observe that for any nonzero $u \in X^+$, $P(\omega)u$ and $u - P(\omega)u$ cannot both be zero, so $\|P(\omega)u\|/\|u - P(\omega)u\|$ is well defined (perhaps equal to ∞).

We claim that if X^+ is total, then $\delta(\omega)$ is a nonnegative real number for each $\omega \in \Omega_0$. Indeed, $\delta(\omega) = \infty$ for some $\omega \in \Omega_0$ means that $X^+ \subset \hat{F}_1(\omega)$. As the cone X^+ is total, we have that $X = \mathrm{cl}\,(X^+ - X^+) \subset \hat{F}_1(\omega)$, which is impossible.

Lemma 5.1. *For any $\omega \in \Omega_0$, $\delta(\omega) = 0$ if and only if $E_1(\omega) \cap X^+ \supsetneq \{0\}$.*

Proof. The "if" part is straightforward. Assume that $\delta(\omega) = 0$ for some $\omega \in \Omega_0$. It follows that for each $k = 1, 2, \dots$ we can choose $u_k \in X^+$, $\|u_k\| = 1$, such that $\|P(\omega)u_k\| < \frac{1}{k}\|u_k - P(\omega)u_k\|$. Since the set $\{\, (\mathrm{Id}_X - P(\omega))u_k : k \in \mathbb{N} \,\}$, being a bounded subset of a finite-dimensional vector subspace $E_1(\omega)$, has compact closure, we can extract a subsequence $((\mathrm{Id}_X - P(\omega))u_{k_l})_{l=1}^{\infty}$ convergent to some $v \in E_1(\omega)$. Observe that $P(\omega)u_{k_l} \to 0$, consequently $u_{k_l} \to v$ as $l \to \infty$. We have $v \in X^+$ and $\|v\| = 1$. □

Lemma 5.2. *The function $\delta\colon \Omega_0 \to [0,\infty)$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable.*

Proof. Take $\{z_k\}_{k=1}^{\infty}$ to be a countable dense subset of $\mathcal{S}_1(X^+)$. Notice that for any $\omega \in \Omega_0$ and any positive real r, "$\delta(\omega) \ge r$" is equivalent to "$\|P(\omega)z_k\|/\|z_k - P(\omega)z_k\| \ge r$ for all $k = 1, 2, 3, \dots$". As $\{P(\omega)\}_{\omega\in\Omega_0}$ is strongly measurable, for any positive real r and any $k = 1, 2, 3, \dots$ the set $\{\, \omega \in \Omega_0 : \|P(\omega)z_k\|/\|z_k - P(\omega)z_k\| \ge r \,\}$ belongs to $\mathfrak{F}$. This implies that $\delta\colon \Omega_0 \to [0,\infty)$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable. □

Proof of Theorem 3.5. We first prove that $E_1(\omega) \cap X^+ \supsetneq \{0\}$ for a.e. $\omega \in \Omega$.

Fix two real numbers $\underline{\lambda} < \overline{\lambda}$ in the following way. If (2) in Theorem 3.4 holds with $k = 1$, we stipulate only that $\underline{\lambda} < \overline{\lambda} < \lambda_1$. Otherwise we take $\lambda_2 < \underline{\lambda} < \overline{\lambda} < \lambda_1$.

For each $\omega \in \Omega_0$ there are $\overline{c}(\omega) \in (0, 1]$ and $\underline{c}(\omega) \ge 1$ such that

$$\begin{aligned} &\|U_\omega(t)u\| \ge \overline{c}(\omega)e^{\overline{\lambda}t}\|u\| \qquad \text{for any } u \in E_1(\omega) \text{ and } t \in \mathbb{T}^+, \\ &\|U_\omega(t)u\| \le \underline{c}(\omega)e^{\underline{\lambda}t}\|u\| \qquad \text{for any } u \in \hat{F}_1(\omega) \text{ and } t \in \mathbb{T}^+. \end{aligned} \tag{5.2}$$

Consequently,

$$\begin{aligned} \frac{\|P(\theta_t\omega)U_\omega(t)u\|}{\|(\mathrm{Id}_X - P(\theta_t\omega))U_\omega(t)u\|} &= \frac{\|U_\omega(t)P(\omega)u\|}{\|U_\omega(t)(\mathrm{Id}_X - P(\omega))u\|} \\ &\le \frac{\underline{c}(\omega)}{\overline{c}(\omega)}\, e^{(\underline{\lambda}-\overline{\lambda})t} \frac{\|P(\omega)u\|}{\|u - P(\omega)u\|} \end{aligned}$$

for each $u \in X \setminus \hat{F}_1(\omega)$ and each $t \in \mathbb{T}^+$. Therefore we have

$$\delta(\theta_t\omega) \le \frac{\underline{c}(\omega)}{\overline{c}(\omega)}\, e^{(\underline{\lambda}-\overline{\lambda})t}\, \delta(\omega)$$

for all $\omega \in \Omega_0$ and $t \in \mathbb{T}^+$, which implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta(\theta_i\omega) = 0$$

for all $\omega \in \Omega_0$. We apply the Birkhoff Ergodic Theorem (Theorem 4.1(i)) to $((U_\omega(n))_{\omega\in\Omega, n\in\mathbb{Z}^+}, (\theta_n)_{n\in\mathbb{Z}})$ and the function $-\delta$ to conclude that $\int_\Omega \delta\, d\mathbb{P} = 0$, from which it follows that $\delta(\omega) = 0$ for ω belonging to Ω_1 with $\mathbb{P}(\Omega_1) = 1$ and $\theta_1(\Omega_1) = \Omega_1$. An application of Lemma 5.1 gives that $E_1(\theta_n\omega) \cap X^+ \supsetneq \{0\}$ for all $\omega \in \Omega_1$ and all $n \in \mathbb{Z}$. This finishes the proof in the discrete-time case. In the continuous-time case, let $t \in \mathbb{R} \setminus \mathbb{Z}$. Pick a nonzero $u \in E_1(\theta_{\lfloor t \rfloor}(\omega)) \cap X^+$. We have a nonzero $U_{\theta_{\lfloor t \rfloor}\omega}(t - \lfloor t \rfloor)u \in E_1(\theta_t\omega) \cap X^+$.

Next we prove that for each $\omega \in \Omega_1$ there exists an entire positive orbit $v_\omega\colon \mathbb{T} \to X^+$ of U_ω such that

$$v_\omega(t) \in (E_1(\theta_t\omega) \cap X^+) \setminus \{0\} \quad \forall t \in \mathbb{T}.$$

Fix $\omega \in \Omega_1$. For $n = 1, 2, 3, \dots$ the sets $\mathcal{U}_{\theta_{-n}\omega}(n)(\mathcal{S}_1(E_1(\theta_{-n}(\omega)) \cap X^+))$ are compact and nonempty. Further, it follows from (5.1) that they form a nonincreasing family; consequently their intersection, G_0, is a nonempty compact set. It now suffices to pick one $u \in G_0$ and put $v_\omega(t) := U_\omega(t)u$, $t \in \mathbb{T}$, where $U_\omega(t)$ is, for $t < 0$, understood as $(U_{\theta_{-t}\omega}(-t)|_{E_1(\theta_{-t}\omega)})^{-1}$. □

5.2. **Principal Floquet subspaces and proofs of Theorems 3.6 and 3.7.** In this subsection, we investigate the existence of generalized principal Floquet subspaces and principal Lyapunov exponents and prove Theorems 3.6 and 3.7. Throughout this subsection, we assume additionally (A0)(ii) and (A3).

Before proving Theorems 3.6 and 3.7, we first prove some propositions.

Proposition 5.3. (i) *Let* $\omega \in \Omega$, $t \in \mathbb{T}^+$, $2 \le t \le t_1$ *and* $u, \tilde{u} \in \mathcal{S}_1(X^+)$. *Then*

$$\|\mathcal{U}_{\theta_{-t}\omega}(t_1)(u-\tilde{u})\| \le 6\varkappa^2(\theta_{-1}\omega)\left(\ln\varkappa(\theta_{-\lfloor t\rfloor}\omega)\right) q(\theta_{-\lfloor t\rfloor+1}\omega)\cdots q(\theta_{-1}\omega). \tag{5.3}$$

In particular, for $n = 2, 3, \dots$ *one has*

$$\|\mathcal{U}_{\theta_{-n}\omega}(n)(u-\tilde{u})\| \le 6\varkappa^2(\theta_{-1}\omega)\left(\ln\varkappa(\theta_{-n}\omega)\right) q(\theta_{-n+1}\omega)\cdots q(\theta_{-1}\omega). \tag{5.4}$$

(ii) *Let* $\omega \in \Omega$, $t \in \mathbb{T}^+$, $t \ge 2$ *and* $u, \tilde{u} \in \mathcal{S}_1(X^+)$. *Then*

$$\|\mathcal{U}_{\omega}(t)(u-\tilde{u})\| \le 6\varkappa^2(\theta_{\lfloor t\rfloor-1}\omega)\left(\ln\varkappa(\omega)\right) q(\theta_1\omega)\cdots q(\theta_{\lfloor t\rfloor-1}\omega). \tag{5.5}$$

In particular, for $n = 2, 3, \dots$ *one has*

$$\|\mathcal{U}_{\omega}(n)(u-\tilde{u})\| \le 6\varkappa^2(\theta_{n-1}\omega)\left(\ln\varkappa(\omega)\right) q(\theta_1\omega)\cdots q(\theta_{n-1}\omega). \tag{5.6}$$

Proof. (i) Observe that, by (5.1), we have

$$\mathcal{U}_{\theta_{-t}\omega}(t) = \mathcal{U}_{\theta_{-1}\omega}(1)\circ\cdots\circ\mathcal{U}_{\theta_{-\lfloor t\rfloor+1}\omega}(1)\circ\mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1).$$

Consequently, by the definition of q,

$$\begin{aligned} &d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}) \\ &\quad \le q(\theta_{-\lfloor t\rfloor+1}\omega)\cdots q(\theta_{-1}\omega)\, d(\mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)u, \mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)\tilde{u}). \end{aligned} \tag{5.7}$$

Since both $\mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)u$ and $\mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)\tilde{u}$ belong to the image of $\mathcal{S}_1(X^+)$ under $\mathcal{U}_{\theta_{-\lfloor t\rfloor}\omega}(1)$, it follows from Lemma 4.10 that

$$d(\mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)u, \mathcal{U}_{\theta_{-t}\omega}(t-\lfloor t\rfloor+1)\tilde{u}) \le 2\ln\varkappa(\theta_{-\lfloor t\rfloor}\omega). \tag{5.8}$$

As both $\mathcal{U}_{\theta_{-t}\omega}(t)u$ and $\mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}$ belong to the image of $\mathcal{S}_1(X^+)$ under $\mathcal{U}_{\theta_{-1}\omega}(1)$, there holds

$$d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}) \le \tau(\theta_{-1}\omega).$$

By Lemma 4.6,

$$\|\mathcal{U}_{\theta_{-t}\omega}(t)(u-\tilde{u})\| \le 3(\exp d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}) - 1),$$

which is, by standard calculus, $\le 3d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}) \exp\tau(\theta_{-1}\omega)$. Putting the above inequalities together and applying Lemmas 4.9 and 4.10, we obtain

$$\begin{aligned} \|\mathcal{U}_{\omega}(t_1)(u-\tilde{u})\| &\le 3d(\mathcal{U}_{\theta_{-t}\omega}(t_1)u, \mathcal{U}_{\theta_{-t}\omega}(t_1)\tilde{u}) \exp\tau(\theta_{-1}\omega) \\ &\le 3d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u}) \exp\tau(\theta_{-1}\omega) && \text{by Lemma 4.9} \\ &\le 3d(\mathcal{U}_{\theta_{-t}\omega}(t)u, \mathcal{U}_{\theta_{-t}\omega}(t)\tilde{u})\, \varkappa^2(\theta_{-1}\omega) && \text{by Lemma 4.10.} \end{aligned} \tag{5.9}$$

(5.7)–(5.9) give (5.3).

(ii) Observe that we have

(5.10)
$$\begin{aligned}
&d(\mathcal{U}_\omega(\lfloor t\rfloor)u,\mathcal{U}_\omega(\lfloor t\rfloor)\tilde{u}) \\
&\qquad \le q(\theta_1\omega)\cdots q(\theta_{\lfloor t\rfloor-1}\omega)\, d(\mathcal{U}_\omega(1)u,\mathcal{U}_\omega(1)\tilde{u}) \qquad \text{by the definition of } q \\
&\qquad \le \tau(\omega)\, q(\theta_1\omega)\cdots q(\theta_{\lfloor t\rfloor-1}\omega) \\
&\qquad \le 2\ln\varkappa(\omega)\, q(\theta_1\omega)\cdots q(\theta_{\lfloor t\rfloor-1}\omega) \qquad \text{by Lemma 4.10.}
\end{aligned}$$

As both $\mathcal{U}_\omega(\lfloor t\rfloor)u$ and $\mathcal{U}_\omega(\lfloor t\rfloor)\tilde{u}$ belong to the image of $\mathcal{S}_1(X^+)$ under $\mathcal{U}_{\theta_{\lfloor t\rfloor-1}\omega}(1)$, we obtain, applying Lemma 4.9, that

$$d(\mathcal{U}_\omega(t)u,\mathcal{U}_\omega(t)\tilde{u}) \le d(\mathcal{U}_\omega(\lfloor t\rfloor)u,\mathcal{U}_\omega(\lfloor t\rfloor)\tilde{u}) \le \tau(\theta_{\lfloor t\rfloor-1}\omega).$$

By Lemma 4.6,

$$\|\mathcal{U}_\omega(t)(u-\tilde{u})\| \le 3(\exp d(\mathcal{U}_\omega(t)u,\mathcal{U}_\omega(t)\tilde{u})-1),$$

which is, by standard calculus, $\le 3d(\mathcal{U}_\omega(t)u,\mathcal{U}_\omega(t)\tilde{u})\exp\tau(\theta_{\lfloor t\rfloor-1}\omega)$. An application of Lemma 4.10 yields

$$\|\mathcal{U}_\omega(t)(u-\tilde{u})\| \le 3d(\mathcal{U}_\omega(t)u,\mathcal{U}_\omega(t)\tilde{u})\,\varkappa^2(\theta_{\lfloor t\rfloor-1}\omega). \tag{5.11}$$

(5.10) and (5.11) give (5.5). ☐

Proposition 5.4. *Let $I := \int_\Omega \ln q\, d\mathbb{P}$. Then there exists an invariant $\bar{\Omega}_1 \subset \Omega$, $\mathbb{P}(\bar{\Omega}_1) = 1$, with the property that*

(1) *for any $I < J < 0$ and any $\omega \in \bar{\Omega}_1$, there is $C_1(J,\omega) > 0$ such that*
$$\|\mathcal{U}_{\theta_{-t}\omega}(t)(u-\tilde{u})\| \le C_1(J,\omega)e^{Jt}$$
for any $t \in \mathbb{T}^+$, $t \ge 3$, and any $u,\tilde{u} \in \mathcal{S}_1(X^+)$.

(2) *for any $I < J < 0$ and any $\omega \in \bar{\Omega}_1$, there is $C_2(J,\omega) > 0$ such that*
$$\|\mathcal{U}_\omega(t)(u-\tilde{u})\| \le C_2(J,\omega)e^{Jt}$$
for any $t \in \mathbb{T}^+$, $t \ge 2$, and any $u,\tilde{u} \in \mathcal{S}_1(X^+)$.

Proof. (1) Consider first the discrete-time case.

It follows from the Birkhoff Ergodic Theorem (Theorem 4.1(i)) applied to the $(\mathfrak{F},\mathfrak{B}(\mathbb{R}))$-measurable function $\ln q\colon \Omega \to (-\infty,0)$ that the invariant set Ω' consisting of those $\omega \in \Omega$ for which

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln q(\theta_{-i}\omega) = I \tag{5.12}$$

has $\mathbb{P}$-measure one. Since $\ln^+\ln\varkappa \in L_1((\Omega,\mathfrak{F},\mathbb{P}))$, Lemma 4.2 establishes the existence of an invariant $\Omega'' \subset \Omega$, $\mathbb{P}(\Omega'') = 1$ such that

$$\limsup_{n\to\infty}\frac{\ln\ln\varkappa(\theta_{-n}\omega)}{n} \le 0 \qquad \text{for all } \omega \in \Omega''. \tag{5.13}$$

Let $\bar{\Omega}_1^{(1)} := \Omega' \cap \Omega''$. The set $\bar{\Omega}_1^{(1)}$ is invariant, with $\mathbb{P}(\tilde{\Omega}_1^{(1)}) = 1$. By (5.3), (5.12) and (5.13),

$$\limsup_{n\to\infty}\sup_{u,\tilde{u}\in\mathcal{S}_1(X^+)}\frac{\ln\|\mathcal{U}_{\theta_{-n}\omega}(n)(u-\tilde{u})\|}{n} \le I$$

for each $\omega \in \bar{\Omega}_1^{(1)}$. Therefore, for any $J \in (I,0)$ and any $\omega \in \bar{\Omega}_1^{(1)}$ there is $N = N(J,\omega) \in \mathbb{N}$ such that

$$\ln\|\mathcal{U}_{\theta_{-n}\omega}(n)(u-\tilde{u})\| \le Jn$$

for all $n = N, N+1, \dots$ and any $u, \tilde{u} \in \mathcal{S}_1(X^+)$. It now suffices to apply the estimate (5.4) to $n = 3, 4, \dots, N-1$ to get the desired result.

We now proceed to the continuous-time case.

It follows from the Birkhoff Ergodic Theorem (Theorem 4.1(i)) applied to the discrete-time metric dynamical system $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ and the $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable function $\ln q\colon \Omega \to (-\infty, 0)$ that the set Ω' of those $\omega \in \Omega$ for which the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln q(\theta_{-i}\omega) =: (\ln q)_{\mathrm{av}}(\omega) \tag{5.14}$$

exists has $\mathbb{P}$-measure one. Since $\ln^+ \ln \varkappa \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ (see (A3)), Lemma 4.2 establishes the existence of $\Omega'' \subset \Omega$, $\theta_1(\Omega') = \Omega'$, $\mathbb{P}(\Omega'') = 1$ such that

$$\limsup_{n\to\infty} \frac{\ln\ln \varkappa(\theta_{-n}\omega)}{n} \le 0 \qquad \text{for all } \omega \in \Omega''. \tag{5.15}$$

Let $\bar{\Omega}_1^{(1)} := \bigcup_{0\le T\le 1} \theta_T(\Omega' \cap \Omega'')$. The set $\bar{\Omega}_1^{(1)}$ is invariant and contains the set $\Omega' \cap \Omega''$ of full $\mathbb{P}$-measure. Since $\mathbb{P}$ is complete, $\bar{\Omega}_1^{(1)} \in \mathfrak{F}$ and $\mathbb{P}(\bar{\Omega}_1^{(1)}) = 1$. By (5.3), we estimate, for any $\omega \in \bar{\Omega}_1^{(1)}$ of the form $\theta_T\tilde{\omega}$, $T \in [0,1)$, $\tilde{\omega} \in \Omega' \cap \Omega''$,

$$\begin{aligned} \|\mathcal{U}_{\theta_{-t}\omega}(t)(u-\tilde{u})\| &= \|\mathcal{U}_{\theta_{-t+T}\tilde{\omega}}(t)(u-\tilde{u})\| \\ &\le 6\varkappa^2(\theta_{-1}\tilde{\omega})\, \ln \varkappa(\theta_{-\lfloor t-T\rfloor}\tilde{\omega})\, q(\theta_{-\lfloor t-T\rfloor+1}\tilde{\omega}) \cdots q(\theta_{-1}\tilde{\omega}), \end{aligned} \tag{5.16}$$

where $t \ge 3$ and $u, \tilde{u} \in \mathcal{S}_1(X^+)$.

It follows from (5.14), (5.15) and (5.16) that

$$\limsup_{t\to\infty} \sup_{u,\tilde{u}\in\mathcal{S}_1(X^+)} \frac{\ln\|\mathcal{U}_{\theta_{-t}\omega}(t)(u-\tilde{u})\|}{t} \le (\ln q)_{\mathrm{av}}(\tilde{\omega})$$

for all $\omega \in \bar{\Omega}_1^{(1)}$. Since $(\ln q)_{\mathrm{av}}(\theta_1\tilde{\omega}) = (\ln q)_{\mathrm{av}}(\tilde{\omega})$, we have that the left-hand side of the above inequality is, for all $\omega \in \bar{\Omega}_1^{(1)}$, bounded above by a constant whose integral over Ω is not larger than I. Therefore, for any $J \in (I, 0)$ and any $\omega \in \bar{\Omega}_1^{(1)}$ there is $\tau = \tau(J, \omega) > 0$ such that

$$\ln\|\mathcal{U}_{\theta_{-t}\omega}(t)(u-\tilde{u})\| \le Jt$$

for all $t \ge \tau$ and any $u, \tilde{u} \in \mathcal{S}_1(X^+)$. It now suffices to apply the estimate (5.4) to $t \in [3, \tau)$ to get the desired result.

A proof of part (2) is similar: we find an invariant set $\bar{\Omega}_1^{(2)}$ with $\mathbb{P}(\bar{\Omega}_1^{(2)}) = 1$ having the corresponding properties.

The required set $\bar{\Omega}_1$ is defined as $\bar{\Omega}_1^{(1)} \cap \bar{\Omega}_1^{(2)}$. □

Let $J \in (I, 0)$, where I is as in Proposition 5.4. For any $\omega \in \bar{\Omega}_1$ and $t \in \mathbb{T}^+$, $3 \le s \le t$, we obtain, via the equality $\mathcal{U}_{\theta_{-t}\omega}(t) = \mathcal{U}_{\theta_{-s}\omega}(s) \circ \mathcal{U}_{\theta_{-t}\omega}(t-s)$, that

$$\|\mathcal{U}_{\theta_{-s}\omega}(s)\mathbf{e} - \mathcal{U}_{\theta_{-t}\omega}(t)\mathbf{e}\| = \|\mathcal{U}_{\theta_{-s}\omega}(s)(\mathbf{e} - \mathcal{U}_{\theta_{-t}\omega}(t-s)\mathbf{e})\| \le C_1(J,\omega)e^{Js},$$

which allows us to define

$$w(\omega) := \lim_{s\to\infty} \mathcal{U}_{\theta_{-s}\omega}(s)\mathbf{e}, \tag{5.17}$$

where the limit is taken in the X-norm.

Since $w(\omega) = \lim_{n\to\infty} \mathcal{U}_{\theta_{-n}\omega}(n)\mathbf{e}$ and $\mathcal{U}_{\theta_{-n}\omega}(n)\mathbf{e} \in X^+$, we have that $w(\omega) \in X^+$. Moreover, as the functions $[\omega \mapsto \mathcal{U}_{\theta_{-n}\omega}(n)\mathbf{e}]$ are $(\mathfrak{F}, \mathfrak{B}(X))$-measurable, $w\colon \bar{\Omega}_1 \to X$ is measurable. We will prove that $w(\cdot)$ satisfies Theorem 3.6.

Proposition 5.5. (1) *There is a $\tilde{\sigma}_1 > 0$ such that for each $\omega \in \bar{\Omega}_1$,*

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \sup \left\{ \left\| \frac{U_{\theta_{-t}\omega}(t)u}{\|U_{\theta_{-t}\omega}(t)u\|} - w(\omega) \right\| : u \in X^+,\ \|u\| = 1 \right\} \le -\tilde{\sigma}_1.$$

(2) *For each $\omega \in \bar{\Omega}_1$,*

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \sup \left\{ \left\| \frac{U_{\omega}(t)u}{\|U_{\omega}(t)u\|} - w(\theta_t\omega) \right\| : u \in X^+,\ \|u\| = 1 \right\} \le -\tilde{\sigma}_1.$$

Proof. (1) It follows from Proposition 5.4(1) and the definition of $w(\omega)$ that

$$\|\mathcal{U}_{\theta_{-t}\omega}(t)\mathbf{e} - w(\omega)\| \le C_1(J,\omega)e^{Jt} \quad \text{and} \quad \|\mathcal{U}_{\theta_{-t}\omega}(t)(\mathbf{e} - u)\| \le C_1(J,\omega)e^{Jt}.$$

Consequently

$$\|\mathcal{U}_{\theta_{-t}\omega}(t)u - w(\omega)\| \le 2C_1(J,\omega)e^{Jt}$$

for any $\omega \in \bar{\Omega}_1$, any $t \in \mathbb{T}^+$, $t \ge 3$, and any $u \in \mathcal{S}_1(X^+)$. Therefore

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \sup \{ \|\mathcal{U}_{\theta_{-t}\omega}(t)u - w(\omega)\| : u \in \mathcal{S}_1(X^+) \} \le J,$$

and, since $J > I$ can be taken arbitrarily close to I,

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \sup \{ \|\mathcal{U}_{\theta_{-t}\omega}(t)u(t) - w(\omega)\| : u \in \mathcal{S}_1(X^+) \} \le \int_\Omega \ln q \, d\mathbb{P}$$

for any $\omega \in \bar{\Omega}_1$. Thus (1) holds.

(2) is proved just as (1) is, with Proposition 5.4(1) replaced by Proposition 5.4(2). ☐

Proof of Theorem 3.6. (1) Observe that

$$\begin{aligned} \mathcal{U}_\omega(t)w(\omega) &= \mathcal{U}_\omega(t)(\lim_{s\to\infty} \mathcal{U}_{\theta_{-s}\omega}(s)\mathbf{e}) \\ &= \lim_{s\to\infty} \mathcal{U}_{\theta_{-s}\omega}(s+t)\mathbf{e} \\ &= \lim_{s\to\infty} \mathcal{U}_{\theta_{-s}(\theta_t\omega)}(s)(\mathcal{U}_{\theta_{-s}\omega}(t)\mathbf{e}) \end{aligned}$$

for any $\omega \in \bar{\Omega}_1$ and $t \in \mathbb{T}$. By Proposition 5.4(1),

$$\|\mathcal{U}_{\theta_{-s}(\theta_t\omega)}(s)(\mathbf{e} - \mathcal{U}_{\theta_{-s}\omega}(t)\mathbf{e})\| \le C_1(J,\theta_t\omega)e^{Js},$$

from which it follows that $\mathcal{U}_\omega(t)w(\omega) = w(\theta_t\omega)$. (1) is thus proved with $\tilde{\Omega}_1 = \bar{\Omega}_1$.

For any $\omega \in \bar{\Omega}_1$, as $w(\omega) = \mathcal{U}_{\theta_{-1}\omega}(1)w(\theta_{-1}\omega)$, Lemma 4.10 implies that $w(\omega) \in C_{\mathbf{e}}$.

(2) It follows from Proposition 5.5(1) that

$$\lim_{\substack{s\to\infty\\ s\in\mathbb{T}^+}} \left\| \mathcal{U}_{\theta_{-s}(\theta_t\omega)}(s) \frac{v_\omega(-s+t)}{\|v_\omega(-s+t)\|} - w(\theta_t\omega) \right\| = 0$$

for each $t \in \mathbb{T}$ and $\omega \in \bar{\Omega}_1$.

But $\mathcal{U}_{\theta_{-s}\omega}(t)\frac{v_\omega(-s+t)}{\|v_\omega(-s+t)\|}$ is equal, for each $s \in \mathbb{T}^+$, to $\frac{v_\omega(t)}{\|v_\omega(t)\|}$. Thus we have $v_\omega(t) = \|v_\omega(t)\| w(\theta_t\omega)$ for all $t \in \mathbb{T}$. As, for each $t \in \mathbb{T}$, both $v_\omega(t)$ and $w_\omega(t)$ belong to the one-dimensional subspace $\tilde{E}_1(\theta_t\omega)$, we must have that $\|v_\omega(t)\|/\|w_\omega(t)\|$ is constant and then $v_\omega(t) = \|v_\omega(0)\|w_\omega(t)$. Hence (2) holds with $\tilde{\Omega}_1 = \bar{\Omega}_1$.

(3) The mapping

$$[\,(t,\omega) \ni \mathbb{T} \times \bar{\Omega}_1 \mapsto \ln \rho_t(\omega) \in (-\infty,\infty)\,]$$

is $(\mathfrak{B}(\mathbb{T}) \otimes \mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable. We have

$$\ln \rho_{s+t}(\omega) = \ln \rho_t(\theta_s\omega) + \ln \rho_s(\omega) \quad \text{for any } s,t \in \mathbb{T} \text{ and any } \omega \in \tilde{\Omega}_1.$$

In the discrete-time case, the Birkhoff Ergodic Theorem (Theorem 4.1(i)) applied to $\ln \rho_1$ (observe that, by (A1)(i), $\ln^+ \rho_1 \in L_1((\Omega, \mathfrak{F}, \mathbb{P})))$ guarantees the existence of $\tilde{\lambda}_1 \in [-\infty,\infty)$ and an invariant $\Omega_1^{'} \subset \bar{\Omega}_1$, $\mathbb{P}(\hat{\Omega}) = 1$ such that

$$\tilde{\lambda}_1 = \lim_{n\to\pm\infty} \frac{1}{n} \ln \rho_n(\omega) = \int\limits_\Omega \ln \rho_1 \, d\mathbb{P}.$$

for all $\omega \in \Omega_1^{'}$. (3) then holds with $\tilde{\Omega}_1 = \Omega_1^{'}$.

In the continuous-time case, applying the Birkhoff Ergodic Theorem (Theorem 4.1(i)) to $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ and $\ln \rho_1$, we obtain the existence of $\Omega_1^{''} \subset \bar{\Omega}_1$, $\theta_1(\Omega_1^{''}) = \Omega_1^{''}$, $\mathbb{P}(\Omega_1^{''}) = 1$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \rho_n(\omega) = (\ln \rho_1)_{\mathrm{av}}(\omega)$$

for all $\omega \in \Omega_1^{''}$, where $\int_\Omega \ln \rho_1 \, d\mathbb{P} = \int_\Omega (\ln \rho_1)_{\mathrm{av}} \, d\mathbb{P}$. Put $\Omega_1^{'''} := \bigcup\limits_{T\in[0,1)} \theta_T(\Omega_1^{''})$. As $\mathbb{P}$ is complete, $\Omega_1^{'''} \in \mathfrak{F}$ and $\mathbb{P}(\Omega_1^{'''}) = 1$. Using (A1)(i) we prove that

$$\lim_{t\to\pm\infty} \frac{1}{t} \ln \rho_t(\omega) = \lim_{n\to\pm\infty} \frac{1}{n} \ln \rho_n(\tilde{\omega})$$

for any $\omega \in \Omega_1^{'''}$ such that $\omega = \theta_T\tilde{\omega}$ for some $\tilde{\omega} \in \Omega_1^{'''}$ and $T \in [0,1)$ (cf. the proof of [14, Lemma 3.4]). Consequently, $(\ln \rho_1)_{\mathrm{av}}$ is $\mathbb{P}$-a.e. constant, so it must be equal to $\int_\Omega \ln \rho_1 \, d\mathbb{P}$. (3) then holds with $\tilde{\Omega}_1 = \Omega_1^{'''}$.

(4) Assume moreover (A1)(ii)–(iii) and that the cone X^+ is total. If Theorem 3.4(1) holds, it is clear that $\tilde{\lambda}_1 = -\infty$. If Theorem 3.4(2) or (3) holds, by Theorem 3.5 and (2), we must have $w(\omega) \in E_1(\omega)$, and hence $\tilde{\lambda}_1 = \lambda_1$. In any case, it follows that for any $u \in X \setminus \{0\}$,

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \le \tilde{\lambda}_1$$

for any $\omega \in \Omega_0 \cap \tilde{\Omega}_1$, where $\tilde{\Omega}_1$ is as in (3).

(5) Assume moreover (A0)(iii). By (A0)(iii) and (A2), for any $u \in X \setminus \{0\}$,

$$\|U_\omega(t)u\| \le \|U_\omega(t)|u|\| \quad \forall\, \omega \in \Omega,\ t > 0,\ t \in \mathbb{T}^+.$$

It suffices to prove that

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \le \tilde{\lambda}_1$$

for any $u \in X^+ \setminus \{0\}$ and $\omega \in \tilde{\Omega}_1$, where $\tilde{\Omega}_1$ is as in (3). By (A3), for any $u \in X^+ \setminus \{0\}$ and $\omega \in \tilde{\Omega}_1$,

$$U_\omega(1)u \le \frac{\varkappa(\omega)\beta(\omega,u)}{\beta(\omega,w(\omega))} U_\omega(1)w(\omega),$$

and then, by (A2),

$$U_\omega(t)u \le \frac{\varkappa(\omega)\beta(\omega,u)}{\beta(\omega,w(\omega))} U_\omega(t)w(\omega) \quad \forall t \ge 1,\ t \in \mathbb{T}^+.$$

This together with (A0)(iii) implies that

$$\|U_\omega(t)u\| \le \frac{\varkappa(\omega)\beta(\omega,u)}{\beta(\omega,w(\omega))} \|U_\omega(t)w(\omega)\| \quad \forall t \ge 1,\ t \in \mathbb{T}^+,$$

and hence

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)u\| \le \tilde{\lambda}_1.$$

□

Proof of Theorem 3.7. (1)–(3) can be proved by arguments similar to those in the proofs of Theorem 3.6(1)–(3).

(4) For given $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$ and $t > 0$, $t \in \mathbb{T}^+$, we have

$$\begin{aligned} \rho_t^*(\omega)\langle w(\theta_{-t}\omega), w^*(\theta_{-t}\omega)\rangle &= \langle w(\theta_{-t}\omega), U_\omega^*(t)w^*(\omega)\rangle \\ &= \langle U_{\theta_{-t}\omega}(t)w(\theta_{-t}\omega), w^*(\omega)\rangle \\ &= \rho_t(\theta_{-t}\omega)\langle w(\omega), w^*(\omega)\rangle \\ &= \frac{1}{\rho_{-t}(\omega)}\langle w(\omega), w^*(\omega)\rangle \end{aligned}$$

and

$$\begin{aligned} \rho_t(\omega)\langle w(\theta_t\omega), w^*(\theta_t\omega)\rangle &= \langle U_\omega(t)w(\omega), w^*(\omega)\rangle \\ &= \langle w(\omega), U_{\theta_t\omega}^*(t)w^*(\theta_t\omega)\rangle \\ &= \rho_t^*(\theta_t\omega)\langle w(\omega), w^*(\omega)\rangle \\ &= \frac{1}{\rho_{-t}^*(\omega)}\langle w(\omega), w^*(\omega)\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{\lambda}_1 &= \lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \left(-\frac{1}{t}\ln\rho_{-t}(\omega) + \frac{1}{t}\ln\langle w(\omega), w^*(\omega)\rangle \right) \\ &= \lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \left(\frac{1}{t}\ln\rho_t^*(\omega) + \frac{1}{t}\ln\langle w(\theta_{-t}\omega), w^*(\theta_{-t}\omega)\rangle \right) \\ &\le \lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t}\ln\rho_t^*(\omega) + \limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t}\ln\langle w(\theta_{-t}\omega), w^*(\theta_{-t}\omega)\rangle \\ &= \tilde{\lambda}_1^* + \limsup_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t}\ln\langle w(\theta_{-t}\omega), w^*(\theta_{-t}\omega)\rangle \end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}_1^* &= \lim_{\substack{t\to\infty\\ t\in\mathbb{T}^+}}\Big(-\frac{1}{t}\ln\rho^*_{-t}(\omega)+\frac{1}{t}\ln\langle w(\omega),w^*(\omega)\rangle\Big)\\ &= \lim_{\substack{t\to\infty\\ t\in\mathbb{T}^+}}\Big(\frac{1}{t}\ln\rho_t(\omega)+\frac{1}{t}\ln\langle w(\theta_t\omega),w^*(\theta_t\omega)\rangle\Big)\\ &\le \lim_{\substack{t\to\infty\\ t\in\mathbb{T}^+}}\frac{1}{t}\ln\rho_t(\omega)+\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}}\frac{1}{t}\ln\langle w(\theta_t\omega),w^*(\theta_t\omega)\rangle\\ &= \tilde{\lambda}_1+\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}}\frac{1}{t}\ln\langle w(\theta_t\omega),w^*(\theta_t\omega)\rangle.\end{aligned}$$

It is enough now to note that $0\le\langle w(\omega),w^*(\omega)\rangle\le 1$ for any $\omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*$. □

5.3. **Generalized exponential separation and the proof of Theorem 3.8.** In this subsection, we study the attractivity properties of generalized principal Floquet subspaces and prove Theorem 3.8. To this end, we first prove some auxiliary results. Throughout this subsection, we assume (A0)(iii), (A0)*(iii), (A1)*(i), (A2)*, and (A4).

The next result gives the formula for the projection of X on $\tilde{F}_1(\omega)$ along $\tilde{E}_1(\omega)$.

Lemma 5.6. *The family* $\{\tilde{P}(\omega)\}_{\omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*}$ *of projections associated with the decomposition* $\tilde{E}_1(\omega)\oplus\tilde{F}_1(\omega)=X$ *is given by the formula*

$$\tilde{P}(\omega)u=u-\frac{\langle u,w^*(\omega)\rangle}{\langle w(\omega),w^*(\omega)\rangle}w(\omega),\qquad \omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*. \tag{5.18}$$

Proof. Fix $\omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*$. After simple computation it is clear that $u\in\tilde{E}_1(\omega)$ if and only if $\tilde{P}(\omega)u=0$, and that $\tilde{P}(\omega)u=u$ if and only if $u\in\tilde{F}(\omega)$. □

Lemma 5.7. *For each* $\omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*$ *there holds*

$$w(\omega)\ge\frac{\mathbf{e}}{\varkappa(\theta_{-1}\omega)}\qquad\text{and}\qquad w^*(\omega)\ge\frac{\mathbf{e}^*}{\varkappa^*(\theta_1\omega)}.$$

Proof. By (A3) and (A3)*, for each $\omega\in\tilde{\Omega}_1\cap\tilde{\Omega}_1^*$ there are $\tilde{\beta},\tilde{\beta}^*>0$ such that

$$\tilde{\beta}\mathbf{e}\le w(\omega)\le\varkappa(\theta_{-1}\omega)\tilde{\beta}\mathbf{e}\qquad\text{and}\qquad \tilde{\beta}^*\mathbf{e}^*\le w^*(\omega)\le\varkappa^*(\theta_1\omega)\tilde{\beta}^*\mathbf{e}^*.$$

Since $\|w(\omega)\|=\|w^*(\omega)\|=\|\mathbf{e}\|=\|\mathbf{e}^*\|=1$, we have $\tilde{\beta}\le 1\le\tilde{\beta}\varkappa(\theta_{-1}\omega)$ and $\tilde{\beta}^*\le 1\le\tilde{\beta}^*\varkappa^*(\theta_1\omega)$. Hence

$$w(\omega)\ge\frac{\mathbf{e}}{\varkappa(\theta_{-1}\omega)}\qquad\text{and}\qquad w^*(\omega)\ge\frac{\mathbf{e}^*}{\varkappa^*(\theta_1\omega)}.$$

□

For $\omega\in\tilde{\Omega}_1^*$ we define

$$W^+(\omega):=\{\,u\in X:\langle u,w^*(\omega)\rangle>0\},$$
$$W^-(\omega):=\{\,u\in X:\langle u,w^*(\omega)\rangle<0\,\}.$$

Observe that, by (2.3),

$$\langle U_\omega(t)u,v^*\rangle=\langle u,U^*_{\theta_t\omega}(t)v^*\rangle$$

for each $t\ge 0$. Hence $U_\omega(t)W^+(\omega)\subset W^+(\theta_t\omega)$ and $U_\omega(t)W^-(\omega)\subset W^-(\theta_t\omega)$.

Lemma 5.8. *For each $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$ there holds $X^+ \setminus \{0\} \subset W^+(\omega)$.*

Proof. Let $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$ and let $u \in X^+ \setminus \{0\}$. We have

$$\begin{aligned}
\langle u, w^*(\omega)\rangle &= \frac{1}{\|U^*_{\theta_1\omega}(1)w^*(\theta_1\omega)\|}\langle u, U^*_{\theta_1\omega}(1)w^*(\theta_1\omega)\rangle && \text{by Theorem 3.7(1)}\\
&= \frac{1}{\rho_1^*(\theta_1\omega)}\langle u, U^*_{\theta_1\omega}(1)w^*(\theta_1\omega)\rangle && \text{by the definition of } \rho_1^*\\
&= \frac{1}{\rho_1^*(\theta_1\omega)}\langle U_\omega(1)u, w^*(\theta_1\omega)\rangle && \text{by (2.3)}\\
&\ge \frac{\beta(\omega,u)}{\rho_1^*(\theta_1\omega)}\langle \mathbf{e}, w^*(\theta_1\omega)\rangle && \text{by (A3)}\\
&\ge \frac{\beta(\omega,u)}{\varkappa^*(\theta_2\omega)\,\rho_1^*(\theta_1\omega)}\langle \mathbf{e}, \mathbf{e}^*\rangle && \text{by Lemma 5.7}\\
&> 0 && \text{by (A4).}
\end{aligned}$$

□

Proposition 5.9. *The function*

$$[\,\tilde{\Omega}_1 \cap \tilde{\Omega}_1^* \ni \omega \mapsto \ln\langle w(\omega), w^*(\omega)\rangle \in (-\infty, 0]\,]$$

belongs to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$.

Proof. By Lemma 5.7,

$$0 < \frac{\langle \mathbf{e}, \mathbf{e}^*\rangle}{\varkappa(\theta_{-1}\omega)\varkappa^*(\theta_1\omega)} \le \langle w(\omega), w^*(\omega)\rangle \le 1,$$

which implies that

$$\ln\langle \mathbf{e}, \mathbf{e}^*\rangle - \ln\varkappa(\theta_{-1}\omega) - \ln\varkappa^*(\theta_1\omega) \le \ln\langle w(\omega), w^*(\omega)\rangle \le 0$$

for all $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$. It suffices to apply the fact that both $\ln\varkappa$ and $\ln\varkappa^*$ belong to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$. □

Proposition 5.10. *There exists an invariant $\tilde{\Omega}_2 \subset \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$, $\mathbb{P}(\tilde{\Omega}_2) = 1$ with the property that for each J, $\int_\Omega \ln p\, d\mathbb{P} < J < 0$, and each $\omega \in \tilde{\Omega}_2$ there is $C_3(\omega, J) > 0$ such that*

$$\operatorname{osc}\Big(\tfrac{U_\omega(t)u}{\rho_t(\omega)}/w(\theta_t\omega)\Big) \le C_3(J,\omega)e^{Jt}$$

for all $u \in \mathcal{S}_1(X^+)$ and all $t \ge 1$, $t \in \mathbb{T}^+$.

Proof. By (A3),

$$\beta(\omega,u)\mathbf{e} \le U_\omega(1)u \le \varkappa(\omega)\beta(\omega,u)\mathbf{e}$$

and

$$\beta(\omega,w(\omega))\mathbf{e} \le U_\omega(1)w(\omega) \le \varkappa(\omega)\beta(\omega,w(\omega))\mathbf{e}.$$

Consequently

$$\operatorname{osc}(U_\omega(1)u/U_\omega(1)w(\omega)) \le \left(\varkappa(\omega) - \frac{1}{\varkappa(\omega)}\right)\frac{\beta(\omega,u)}{\beta(\omega,w(\omega))}$$

for each nonzero $u \in X^+$. Observe that $\beta(\omega,u)\mathbf{e} \le U_\omega(1)u$ and $U_\omega(1)w(\omega) \le \varkappa(\omega)\beta(\omega,w(\omega))\mathbf{e}$, which implies that $\beta(\omega,u) \le \|U_\omega(1)u\| \le \|U_\omega(1)\|$ for all $u \in \mathcal{S}_1(X^+)$, and $\rho_1(\omega) \le \varkappa(\omega)\beta(\omega,w(\omega))$. Therefore

$$\operatorname{osc}(U_\omega(1)u/U_\omega(1)w(\omega)) \le (\varkappa^2(\omega)-1)\frac{\|U_\omega(1)\|}{\rho_1(\omega)}$$

for all $u \in \mathcal{S}_1(X^+)$. The remainder of the proof goes, with the help of Lemma 4.9 and the equality

$$\operatorname{osc}\Big(\tfrac{U_\omega(t)u}{\rho_t(\omega)}/w(\theta_t\omega)\Big) = \operatorname{osc}(U_\omega(t)u/U_\omega(t)w(\omega)), \qquad u \in \mathcal{S}_1(X^+),\ t \ge 1,\ t \in \mathbb{T}^+,$$

along the lines of the proof of Proposition 5.4. □

Proposition 5.11. *There is a $\tilde\sigma_2 > 0$ such that for each $\omega \in \tilde\Omega_2$ ($\tilde\Omega_2$ is as in Proposition* 5.10) *there holds*

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{T}^+}} \frac{1}{t}\ln\sup\left\{\left\|\frac{U_\omega(t)u}{\rho_t(\omega)} - \frac{\langle u,w^*(\omega)\rangle}{\langle w(\omega),w^*(\omega)\rangle}w(\theta_t\omega)\right\| : u \in X,\ \|u\|=1\right\} \le -\tilde\sigma_2.$$

Proof. Denote $\tilde U_\omega(t)u := \frac{U_\omega(t)u}{\rho_t(\omega)}$. By Proposition 5.10, there exists an invariant $\tilde\Omega_2 \subset \tilde\Omega_1 \cap \tilde\Omega_1^*$, $\mathbb{P}(\tilde\Omega_2) = 1$ such that for any $J \in (I,0)$ and any $\omega \in \tilde\Omega_2$ there is $C_3(J,\omega) > 0$ such that

$$\operatorname{osc}(\tilde U_\omega(t)u/w(\theta_t\omega)) \le C_3(J,\omega)e^{Jt}$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, and all $u \in \mathcal{S}_1(X^+)$. Since

$$\operatorname{osc}(\tilde U_\omega(t)u/w(\theta_t\omega)) = M(\tilde U_\omega(t)u/w(\theta_t\omega)) - m(\tilde U_\omega(t)u/w(\theta_t\omega)),$$

it follows via Lemma 4.9 that $m(\tilde U_\omega(t)u/w(\theta_t\omega))$ converges in a nondecreasing way, as $t\to\infty$, and $M(\tilde U_\omega(t)u/w(\theta_t\omega))$ converges in a nonincreasing way, as $t \to \infty$, to a common limit (denoted by $\mu(u,\omega)$). Further, we have

$$\mu(u,\omega) - m(\tilde U_\omega(t)u/w(\theta_t\omega)) \le C_3(J,\omega)e^{Jt},$$
$$M(\tilde U_\omega(t)u/w(\theta_t\omega)) - \mu(u,\omega) \le C_3(J,\omega)e^{Jt}$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, and all $u \in \mathcal{S}_1(X^+)$.

As

$$\begin{aligned}(m(\tilde U_\omega(t)u/w(\theta_t\omega)) - \mu(u,\omega))w(\theta_t\omega) &\le \tilde U_\omega(t)u - \mu(u,\omega)w(\theta_t\omega)\\ &\le (M(\tilde U_\omega(t)u/w(\theta_t\omega)) - \mu(u,\omega))w(\theta_t\omega),\end{aligned}$$

there holds

$$\left\|\frac{U_\omega(t)u}{\rho_t(\omega)} - \mu(u,\omega)w(\theta_t\omega)\right\| \le C_3(J,\omega)e^{Jt}$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, and all $u \in \mathcal{S}_1(X^+)$.

Fix $u \in \mathcal{S}_1(X^+)$ and $\omega \in \tilde\Omega_2$. We apply the functionals $w^*(\theta_n\omega)$ to the exponentially decaying sequence (in X),

$$\frac{U_\omega(n)u}{\rho_n(\omega)} - \mu(u,\omega)w(\theta_n\omega),$$

to obtain an exponentially decaying sequence (in $\mathbb{R}$)

$$\begin{aligned}(5.19)\qquad &\left\langle \frac{U_\omega(n)u}{\rho_n(\omega)} - \mu(u,\omega)w(\theta_n\omega), w^*(\theta_n\omega)\right\rangle \\ &= \frac{\langle U_\omega(n)u, w^*(\theta_n\omega)\rangle}{\rho_n(\omega)} - \mu(u,\omega)\langle w(\theta_n\omega), w^*(\theta_n\omega)\rangle.\end{aligned}$$

Observe that

$$\langle U_\omega(n)u, w^*(\theta_n\omega)\rangle = \langle u, U^*_{\theta_n\omega}(n)w^*(\theta_n\omega)\rangle = \rho^*_n(\theta_n\omega)\langle u, w^*(\omega)\rangle$$

and

$$\begin{aligned}\rho_n(\omega)\langle w(\theta_n\omega), w^*(\theta_n\omega)\rangle &= \langle U_\omega(n)w(\omega), w^*(\theta_n\omega)\rangle \\ = \langle w(\omega), U^*_{\theta_n\omega}(n)w^*(\theta_n\omega)\rangle &= \rho^*_n(\theta_n\omega)\langle w(\omega), w^*(\omega)\rangle,\end{aligned}$$

hence the exponentially decaying sequence in (5.19) equals

$$\left(\frac{\langle u, w^*(\omega)\rangle}{\langle w(\omega), w^*(\omega)\rangle} - \mu(u,\omega)\right)\langle w(\theta_n\omega), w^*(\theta_n\omega)\rangle.$$

It follows from Proposition 5.9 that

$$\lim_{n\to\infty}\frac{1}{n}\ln\langle w(\theta_n\omega), w^*(\theta_n\omega)\rangle = 0;$$

consequently

$$\frac{\langle u, w^*(\omega)\rangle}{\langle w(\omega), w^*(\omega)\rangle} = \mu(u,\omega).$$

We have thus proved that

$$\left\|\frac{U_\omega(t)u}{\rho_t(\omega)} - \frac{\langle u, w^*(\omega)\rangle}{\langle w(\omega), w^*(\omega)\rangle}w(\theta_t\omega)\right\| \le C_3(J,\omega)\|u\|e^{Jt}$$

for $\omega \in \tilde{\Omega}_2$, $t \ge 1$, $t \in \mathbb{T}^+$, and $u \in X^+$.

Next, let $u \in \mathcal{S}_1(X)$ be arbitrary. As (X, X^+) is a Banach lattice, there holds $\|u^+\| \le \|u\| = 1$ and $\|u^-\| \le \|u\| = 1$. Therefore we have

$$\left\|\frac{U_\omega(t)u}{\rho_t(\omega)} - \mu(u,\omega)w(\theta_t\omega)\right\| \le 2C_3(J,\omega)e^{Jt}$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, and $\omega \in \tilde{\Omega}_2$. □

Proof of Theorem 3.8. (1) The strong measurability follows, through formula (5.18), by the measurability of w and w^*.

For $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}^*_1$ define

$$\tilde{\tilde{P}}(\omega)u := \frac{\langle u, w^*(\omega)\rangle}{\langle w(\omega), w^*(\omega)\rangle}w(\omega), \qquad u \in X.$$

$\tilde{\tilde{P}}(\omega) = \mathrm{Id}_X - \tilde{P}(\omega)$, that is, it equals the projection of X onto $\tilde{E}_1(\omega)$ along $\tilde{F}_1(\omega)$. There holds

$$1 \le \|\tilde{\tilde{P}}(\omega)\| \le \frac{1}{\langle w(\omega), w^*(\omega)\rangle};$$

consequently

$$0 \le \ln\|\tilde{\tilde{P}}(\omega)\| \le -\ln\langle w(\omega), w^*(\omega)\rangle.$$

It follows from Proposition 5.9 that $\ln \|\tilde{\tilde{P}}(\cdot)\|$ belongs to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$. Hence, by Lemma 4.2,

$$\lim_{\substack{t \to \pm\infty \\ t \in \mathbb{T}}} \frac{\ln \|\tilde{\tilde{P}}(\theta_t \omega)\|}{t} = 0,$$

$\mathbb{P}$-a.s. on $\tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$. Therefore, for each $\epsilon > 0$ and $\mathbb{P}$-a.e. $\omega \in \tilde{\Omega}_1 \cap \tilde{\Omega}_1^*$ there is $D(\epsilon, \omega) > 0$ such that

$$\|\tilde{\tilde{P}}(\theta_t \omega)\| \le D(\epsilon, \omega) e^{\epsilon |t|}$$

for all $t \in \mathbb{T}$. Since $\tilde{P}(\omega) = \mathrm{Id}_X - \tilde{\tilde{P}}(\omega)$, we estimate

$$\|\tilde{P}(\theta_t \omega)\| \le 1 + D(\epsilon, \omega) e^{\epsilon |t|} \le (1 + D(\epsilon, \omega)) e^{\epsilon |t|}$$

for all $t \in \mathbb{T}$. Consequently, $\limsup\limits_{\substack{t \to \pm\infty \\ t \in \mathbb{T}}} (1/t) \ln \|\tilde{P}(\theta_t \omega)\| \le \epsilon$, but as $\epsilon > 0$ is arbitrary, we have $\limsup\limits_{\substack{t \to \pm\infty \\ t \in \mathbb{T}}} (1/t) \ln \|\tilde{P}(\theta_t \omega)\| \le 0$. The inequality $\liminf\limits_{\substack{t \to \pm\infty \\ t \in \mathbb{T}}} (1/t) \ln \|\tilde{P}(\theta_t \omega)\| \ge 0$ follows by the fact that $\|\tilde{P}(\omega)\| \ge 1$.

(2) It is a consequence of Lemma 5.8.

(3) Fix $\omega \in \tilde{\Omega}_2$. It is clear that $\liminf\limits_{\substack{t \to \infty \\ t \in \mathbb{T}^+}} (1/t) \ln \|U_\omega(t)\| \ge \tilde{\lambda}_1$.

For a nonzero $u \in X$, put $u_1 := \tilde{P}(\omega) u$, $u_2 := u - \tilde{P}(\omega) u$. We have $u_2 = \frac{\langle u, w^*(\omega) \rangle}{\langle w(\omega), w^*(\omega) \rangle} w(\omega)$. Take some J, $\int_\Omega \ln p \, d\mathbb{P} < J < 0$. It follows from Proposition 5.11 that

$$\begin{aligned} \|U_\omega(t) u_1\| &= \left\| U_\omega(t) u - \frac{\langle u, w^*(\omega) \rangle}{\langle w(\omega), w^*(\omega) \rangle} \rho_t(\omega) w(\theta_t \omega) \right\| \\ &\le C_4(J, \omega) \|u\| \rho_t(\omega) e^{Jt}, \end{aligned}$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, where $C_4(J, \omega) > 0$. Consequently,

$$\|U_\omega(t) u\| \le \|U_\omega(t) u_2\| + \|U_\omega(t) u_1\| \le \rho_t(\omega) \|u\| (\|\mathrm{Id}_X - \tilde{P}(\omega)\| + C_4(J, \omega) e^{Jt}).$$

This implies that $\limsup\limits_{\substack{t \to \infty \\ t \in \mathbb{T}^+}} (1/t) \ln \|U_\omega(t)\| \le \tilde{\lambda}_1$. Hence $\lim\limits_{\substack{t \to \infty \\ t \in \mathbb{T}^+}} (1/t) \ln \|U_\omega(t)\| = \tilde{\lambda}_1$.

Now assume $u \in X \setminus \tilde{F}_1(\omega)$. This means that $\langle u, w^*(\omega) \rangle \ne 0$, from which it follows that $\|U_\omega(t) u_2\| = \rho_t(\omega) \|u_2\| > 0$ for all $t \in \mathbb{T}^+$. We thus estimate

$$\|U_\omega(t) u\| \ge \|U_\omega(t) u_2\| - \|U_\omega(t) u_1\| \ge \rho_t(\omega) (\|u_2\| - C_4(J, \omega) \|u\| e^{Jt})$$

for all $t \in \mathbb{T}^+$, $t \ge 1$, which gives $\liminf\limits_{\substack{t \to \infty \\ t \in \mathbb{T}^+}} (1/t) \ln \|U_\omega(t) u\| \ge \tilde{\lambda}_1$.

If $u \in X^+ \setminus \{0\}$ we apply Lemma 5.8 to conclude that $u \in X \setminus \tilde{F}_1(\omega)$. Then by Proposition 5.5(2), $\lim\limits_{\substack{t \to \infty \\ t \in \mathbb{T}^+}} (1/t) \ln \|U_\omega(t) u\| = \tilde{\lambda}_1$ for any $u \in X^+ \setminus \{0\}$.

(4) For each $n \in \mathbb{N}$ put

$$f_n(\omega) := \ln \frac{\|U_\omega(n)|_{\tilde{F}_1(\omega)}\|}{\|U_\omega(n) w(\omega)\|}.$$

The functions f_n are $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable with $(f_1)^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ (by (A1)(i) and Theorem 3.6(3)). Moreover,

$$f_{m+n}(\omega) \le f_m(\omega) + f_n(\theta_m \omega), \qquad m, n \in \mathbb{N},\ \omega \in \tilde{\Omega}_2.$$

In the discrete-time case an application of the Kingman Subadditive Ergodic Theorem (Theorem 4.3) to $((\tilde{\Omega}_2, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ and (f_n) gives the existence of an invariant $\tilde{\Omega}_0 \subset \tilde{\Omega}_2$, $\mathbb{P}(\tilde{\Omega}_0) = 1$, and of $\tilde{\sigma} \in (-\infty, \infty]$ with the property that

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\|U_\omega(n)|_{\tilde{F}_1(\omega)}\|}{\|U_\omega(n)w(\omega)\|} = -\tilde{\sigma}$$

for any $\omega \in \tilde{\Omega}_0$.

In the continuous-time case an application of the Kingman Subadditive Ergodic Theorem (Theorem 4.3) to $((\tilde{\Omega}_2, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ and (f_n) gives the existence of $\Omega' \subset \tilde{\Omega}_2$, $\theta_1(\Omega') = \Omega'$, $\mathbb{P}(\Omega') = 1$, and of an $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable function $g \colon \Omega' \to \mathbb{R}$ satisfying $g^+ \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ and $\int_\Omega g \, d\mathbb{P} = \lim_{n\to\infty} (1/n) \int_\Omega f_n \, d\mathbb{P}$. It follows from (A1)(i) (see the proof of [14, Lemma 3.4]) combined with Theorem 3.6(3) that

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{\|U_\omega(t)|_{\tilde{F}_1(\omega)}\|}{\|U_\omega(t)w(\omega)\|}$$

is constant for all $\omega \in \tilde{\Omega}_0 := \bigcup_{T\in[0,1]} \theta_T(\Omega')$. The set $\tilde{\Omega}_0$ is clearly invariant. Since $\mathbb{P}$ is complete, $\tilde{\Omega}_0 \in \mathfrak{F}$ with $\mathbb{P}(\tilde{\Omega}_0) = 1$.

As a consequence of Proposition 5.11, $\tilde{\sigma} \ge \tilde{\sigma}_2 > 0$.

The equality

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{1}{t} \ln \|U_\omega(t)|_{\tilde{F}_1(\omega)}\| = \tilde{\lambda}_2 = \tilde{\lambda}_1 - \tilde{\sigma}, \qquad \omega \in \tilde{\Omega}_0,$$

is straightforward.

(5) First, if Theorem 3.4(2) or (3) occurs, then by Theorem 3.6(4), $\lambda_1 = \tilde{\lambda}_1 > -\infty$.

Observe that, by parts (3) and (4), $\tilde{F}_1(\omega) \setminus \{0\}$ is, for each $\omega \in \tilde{\Omega}_0$, characterized as the set of those nonzero $u \in X$ for which $\limsup_{t\to\infty, t\in\mathbb{T}^+} (1/t) \ln \|U_\omega(t)u\| < \tilde{\lambda}_1$. By Theorem 3.4, $\hat{F}_1(\omega) \setminus \{0\}$ is, for each $\omega \in \Omega_0$, characterized as the set of those nonzero $u \in X$ for which $\limsup_{t\to\infty, t\in\mathbb{T}^+} (1/t) \ln \|U_\omega(t)u\| < \lambda_1$. Consequently, $\tilde{F}_1(\omega) = \hat{F}_1(\omega)$ for all $\omega \in \Omega_0 \cap \tilde{\Omega}_0$. Further, from the above characterizations it follows that $\hat{\lambda}_2 = \tilde{\lambda}_2$.

As $\operatorname{codim} \hat{F}_1(\omega) = \operatorname{codim} \tilde{F}_1(\omega) = 1$, we have $\dim E_1(\omega) = \dim \tilde{E}_1(\omega) = 1$, for any $\omega \in \Omega_0 \cap \tilde{\Omega}_0$.

By Theorem 3.6(2), for $\mathbb{P}$-a.e. $\omega \in \Omega_0$ there exists an entire trajectory v_ω of U_ω such that $v_\omega(t) \in (E_1(\theta_t\omega) \cap X^+) \setminus \{0\}$ for all $t \in \mathbb{T}$. Therefore $v_\omega(t) \in \tilde{E}_1(\theta_t\omega) \setminus \{0\}$ for all $t \in \mathbb{T}$. But $E_1(\theta_t\omega) = \operatorname{span}\{v_\omega(t)\}$, hence $E_1(\theta_t\omega) = \tilde{E}_1(\theta_t\omega)$.

(6) Observe that, by Theorem 3.6(4), $\tilde{\lambda}_1 = \tilde{\lambda}_1^*$. Without loss of generality, we assume that (A5) holds and prove $\tilde{\lambda}_1 > -\infty$.

For each $\omega \in \Omega$ and each $n = 1, 2, 3, \ldots$ we have

$$\|U_\omega(n)\bar{\mathbf{e}}\| \ge \nu(\omega) \cdots \nu(\theta_{n-1}\omega).$$

In the discrete-time case the Birkhoff Ergodic Theorem (Theorem 4.1(i)) applied to $-\ln \nu$ gives the fact that for $\mathbb{P}$-a.e. $\omega \in \Omega$ there holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \nu(\theta_i\omega) = \int_\Omega \ln \nu \, d\mathbb{P} > -\infty.$$

In the continuous-time case the Birkhoff Ergodic Theorem (Theorem 4.1(i)) applied to $((\Omega, \mathfrak{F}, \mathbb{P}), (\theta_n)_{n\in\mathbb{Z}})$ and to $-\ln\nu$, together with (A1)(i) (again cf. the proof of [14, Lemma 3.4]), give the fact that for $\mathbb{P}$-a.e. $\omega \in \Omega$ one has

$$\lim_{t\to\infty} \frac{1}{t} \ln \|U_\omega(t)\bar{\mathbf{e}}\| \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \nu(\theta_i\omega) =: (\ln\nu)_{\mathrm{av}}(\omega).$$

As the left-hand side is $\mathbb{P}$-a.e. constant, it must be $\geq \int_\Omega (\ln\nu)_{\mathrm{av}}\, d\mathbb{P} = \int_\Omega \ln\nu\, d\mathbb{P} > -\infty$. Then by Theorem 3.6(4), we must have $\tilde{\lambda}_1 > -\infty$. □

5.4. **Monotonicity.** In this subsection, we prove Theorem 3.9, which shows that the monotonicity of two measurable skew-product semiflows at some time implies the monotonicity of the associated generalized principal Lyapunov exponents. We assume

Proof of Theorem 3.9. Let $\tilde{\Omega}_0^{(i)}$, $i = 1, 2$, be a set of full measure such that

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}^+}} \frac{\ln \|U_\omega^{(i)}(t)u\|}{t} = \tilde{\lambda}_1^{(i)}$$

holds for any $\omega \in \tilde{\Omega}_0^{(i)}$ and any $u \in X^+ \setminus \{0\}$ (see Theorem 3.8(3)). Pick $\omega \in \tilde{\Omega}_0^{(1)} \cap \tilde{\Omega}_0^{(2)}$ such that $U_\omega^{(1)}(t_0)u \leq U_\omega^{(2)}(t_0)u$ for all $u \in X^+$. We have, by the monotonicity of the norm,

$$\tilde{\lambda}_1^{(1)} = \lim_{n\to\infty} \frac{\ln \|U_\omega^{(1)}(nt_0)u\|}{n} \leq \lim_{n\to\infty} \frac{\ln \|U_\omega^{(2)}(nt_0)u\|}{n} = \tilde{\lambda}_1^{(2)}.$$

□

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