

A COMPLEX SURFACE OF GENERAL TYPE WITH $p_g = 0$, $K^2 = 2$ AND $H_1 = \mathbb{Z}/4\mathbb{Z}$

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ABSTRACT. We construct a new minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$ (in fact, $\pi_1^{\text{alg}} = \mathbb{Z}/4\mathbb{Z}$), which settles the existence question for numerical Campedelli surfaces with all possible algebraic fundamental groups. The main techniques involved in the construction are a rational blow-down surgery and a \mathbb{Q} -Gorenstein smoothing theory.

1. INTRODUCTION

One of the fundamental problems in the classification of complex surfaces is to find a new family of complex surfaces of general type with $p_g = 0$. In this paper we construct a new minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$.

In order to classify complex surfaces of general type with $p_g = 0$, it seems to be natural to first classify them up to their topological types. For instance, if S is a *numerical Godeaux surface*, that is, a minimal complex surface of general type with $p_g = 0$ and $K^2 = 1$, then it is known that $\pi_1^{\text{alg}}(S) = \mathbb{Z}/m\mathbb{Z}$ for some $1 \leq m \leq 5$, where $\pi_1^{\text{alg}}(S)$ is the algebraic fundamental group of S ; cf. Reid [21]. It is conjectured that the moduli space of numerical Godeaux surfaces has exactly five irreducible components corresponding to each $\pi_1 = \mathbb{Z}/m\mathbb{Z}$ for all $1 \leq m \leq 5$. The conjecture is proved for $m \geq 3$; cf. Reid [21]. Furthermore, for each $1 \leq m \leq 5$ there are numerical Godeaux surfaces with $\pi_1 = \mathbb{Z}/m\mathbb{Z}$; cf. Bauer-Catanese-Pignatelli [2].

In the case of *numerical Campedelli surfaces*, that is, minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 2$, it has been known by Reid [20] and Xiao [23] that the algebraic fundamental group of a numerical Campedelli surface is a finite group of order ≤ 9 . Furthermore, the topological fundamental groups π_1 for any numerical Campedelli surfaces are also of order ≤ 9 in as far as they have been determined. Hence it is a natural conjecture that $|\pi_1| \leq 9$ for all numerical Campedelli surfaces.

Conversely, one may ask whether every group of order ≤ 9 occurs as the topological fundamental group or as the algebraic fundamental group of a numerical Campedelli surface. It is proved that the dihedral groups D_3 of order 6 or D_4 of order 8 cannot be fundamental groups of numerical Campedelli surfaces; see Naie [15], Mendes Lopes-Pardini [13], Mendes Lopes-Pardini-Reid [14], Reid [20].

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However, all groups of order ≤ 9 , *except* D_3 , D_4 , $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, occur as the topological fundamental groups of numerical Campedelli surfaces. The first example of numerical Campedelli surfaces was constructed by Campedelli himself, and his example has $\pi_1 = (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. Furthermore, Mendes Lopes, Pardini, and Reid ([13], [14], [20]) constructed and classified all numerical Campedelli surfaces with $|\pi_1^{\text{alg}}| = 8, 9$ and showed that the topological fundamental group equals the algebraic fundamental group. An example with $\pi_1 = \mathbb{Z}/7\mathbb{Z}$ was constructed by Reid [22]. Catanese [4] constructed an example with $\pi_1 = \mathbb{Z}/5\mathbb{Z}$ by taking a $\mathbb{Z}/5\mathbb{Z}$ -quotient of a certain double cover of a linearly symmetric quintic. An example with $\pi_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ was constructed by Inoue [7] as the quotients of the hypersurfaces of the product of three elliptic curves. Such an example was also constructed by J. Keum [8]. Recently, several examples of numerically Campedelli surfaces with $|\pi_1| \geq 3$ were constructed by the so-called product-quotient method, that is, first take a product of two curves and then take a quotient of the product by a group action; see Bauer-Catanese-Grunewald-Pignatelli [1], Bauer-Pignatelli [3]. Classically, many examples of numerical Campedelli surfaces with $|\pi_1| \geq 3$ are constructed by the method of taking quotients of group action. On the other hand, numerical Campedelli surfaces with small fundamental groups were constructed by the so-called \mathbb{Q} -Gorenstein smoothing method developed in Y. Lee-J. Park [11]. For instance, a simply connected numerical Campedelli surface was constructed by Y. Lee-J. Park [11] and an example with $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ was constructed by J. Keum-Y. Lee-H. Park [9].

Unlike the case of a topological fundamental group, there is a numerical Campedelli surface with $H_1 = \mathbb{Z}/6\mathbb{Z}$. Recently Neves and Papadakis [17] constructed a numerical Campedelli surface with $H_1 = \mathbb{Z}/6\mathbb{Z}$ (in fact $\pi_1^{\text{alg}} = \mathbb{Z}/6\mathbb{Z}$) by constructing the canonical ring of the étale six to one cover using serial unprojection together with a suitable base point free action of $\mathbb{Z}/6\mathbb{Z}$. Therefore all abelian groups of order ≤ 9 *except* $\mathbb{Z}/4\mathbb{Z}$ occur as the first homology groups (and algebraic fundamental groups) of numerical Campedelli surfaces. Nevertheless, until now, the question on the existence of numerical Campedelli surfaces with a given topological type was completely open for $\mathbb{Z}/4\mathbb{Z}$.

The main result of this paper is the following.

Theorem. *There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$. Furthermore, its algebraic fundamental group π_1^{alg} is also $\mathbb{Z}/4\mathbb{Z}$.*

To our knowledge there were no such examples previously known, and it settles the existence question for numerical Campedelli surfaces with $|H_1| \leq 9$ and furthermore the existence question for $|\pi_1^{\text{alg}}| \leq 9$ because it can be shown that the algebraic fundamental group of the numerical Campedelli surface in the main theorem is also equal to $\mathbb{Z}/4\mathbb{Z}$. The reason is the following argument used in Bauer-Catanese-Pignatelli [2] (Pignatelli [19]): As mentioned above, by Reid [20] and Xiao [23], any numerical Campedelli surface has $|\pi_1^{\text{alg}}| \leq 9$. Then it follows that the algebraic fundamental group is the quotient of the topological fundamental group by the intersection N of all normal subgroups of finite index. Since the first homology group H_1 of the main example is finite, the intersection N is contained in the commutator subgroup of the topological fundamental group. It follows that H_1 , the abelian quotient of the topological fundamental group, is also the abelian

quotient of the algebraic fundamental group. It has been known by Reid [20] and Xiao [23] again that the only possible non-abelian group of algebraic fundamental groups for a numerical Campedelli surface is the quaternion group Q_8 ; however, its abelian quotient is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore the algebraic fundamental group of the surface in the main theorem above is abelian, and hence it is equal to $\mathbb{Z}/4\mathbb{Z}$.

In order to construct such a surface we use the \mathbb{Q} -Gorenstein smoothing method of Y. Lee-J. Park [11]. We blow up an elliptic Enriques surface Y in a suitable set of points so that we obtain a surface Z with special linear chains of \mathbb{P}^1 's. Then, by contracting these linear chains, we obtain a singular surface X with permissible singular points in Section 3. Each of these singular points admits a local \mathbb{Q} -Gorenstein smoothing. In Section 4 we show that the obstruction space of a global \mathbb{Q} -Gorenstein smoothing of the singular surface X is zero by a similar strategy as in the work of J. Keum-Y. Lee-H. Park [9]. Hence these local smoothings can be glued to a global \mathbb{Q} -Gorenstein smoothing of the whole singular surface X . Finally, we show that a general fiber X_t of a \mathbb{Q} -Gorenstein smoothing of X is the desired surface.

The main ingredient of this paper is the proof of $H_1(X_t) = \mathbb{Z}/4\mathbb{Z}$ in Section 5. To prove it, instead of a general fiber X_t , we consider a rational blow-down 4-manifold \bar{Z} obtained by replacing certain small neighborhoods of the linear chains in Z with the corresponding Milnor fibers. The rational blow-down 4-manifold \bar{Z} is known to be diffeomorphic to a general fiber X_t by Milnor fiber theory. We first prove $|H_1(\bar{Z}; \mathbb{Z})| = 4$ by analyzing the long exact sequence of homologies induced from a pair $(\bar{Z}, \text{the Milnor fibers})$. We then carefully construct a certain loop β lying on a certain curve of genus 2 in the blown-up surface Z and prove that $H_1(\bar{Z}; \mathbb{Z})$ is generated by the loop β , which finishes the proof. In order to show that the loop β is a generator of $H_1(\bar{Z}; \mathbb{Z})$, we lift the loop 2β , a loop twice of β , up to a certain blown-up K3 surface W which is an unramified double cover of the blown-up Enriques surface Z , and in W we construct a real 2-dimensional surface U_T such that the lifting of 2β is one of the boundary components of U_T . By analyzing the boundaries of U_T we can show that the lifting of 2β into W is not homologous to zero, and this fact will imply that β is a generator of $H_1(\bar{Z}; \mathbb{Z})$.

In this paper we construct four different families of numerical Campedelli surfaces with torsion $\mathbb{Z}/4\mathbb{Z}$. We will show that there is a \mathbb{Q} -Gorenstein smoothing of the singular surface X in the main construction such that a general fiber X_t of the \mathbb{Q} -Gorenstein smoothing of X has a non-ample canonical divisor; see Proposition 3.2. In §6.1, by modifying the configuration used in the main construction, we construct another numerical Campedelli surface with $H_1 = \mathbb{Z}/4\mathbb{Z}$ whose canonical divisor is ample. Furthermore, we construct two more examples of a numerical Campedelli surface with $H_1 = \mathbb{Z}/4\mathbb{Z}$ using different configurations in §6.2. Finally, in the Appendix we also construct a numerical Campedelli surface with $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ using a similar method.

Note. A few months after this paper was announced, Frapporti [6] constructed a numerical Campedelli surface with $\pi_1 = \mathbb{Z}/4\mathbb{Z}$ by a completely different method.

2. AN ENRIQUES SURFACE

We start with an elliptic Enriques surface constructed by Kondo [10, Example V]. We briefly summarize the construction for the convenience of the reader. We will follow the notation of Kondo [10].

Let $\mathbb{P}^1 \times \mathbb{P}^1 = \{(u_0 : u_1, v_0 : v_1) \mid [u_0 : u_1], [v_0 : v_1] \in \mathbb{P}^1\}$. Let us consider the following smooth rational curves:

$$\begin{aligned} C_+ &: (v_0 + v_1)(u_0 + u_1) = -2(u_0 - u_1)v_0, \\ C_- &: (v_0 - v_1)(u_0 - u_1) = -2(u_0 + u_1)v_0, \\ L_{1\pm} &: u_0 = \pm u_1, \quad L_{2\pm} : v_0 = \pm v_1, \\ L_1 &: u_0 = 0, \quad L_2 : v_0 = 0, \quad L_3 : u_0v_1 = u_1v_0, \\ F_{1\pm} &: u_0 = \pm \frac{1}{\sqrt{-3}}u_1, \quad F_{2\pm} : v_0 = \pm \frac{1}{\sqrt{-3}}v_1. \end{aligned}$$

The configurations of these curves are given in Figure 1.

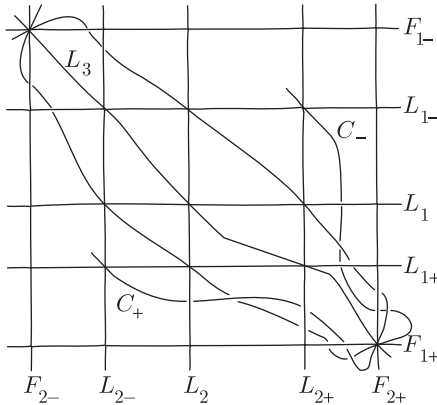


FIGURE 1. $\mathbb{P}^1 \times \mathbb{P}^1$; cf. Kondo [10, Fig. 5.1]

Put $B = C_+ + C_- + L_{1+} + L_{1-} + L_{2+} + L_{2-}$, which is a $(4, 4)$ -divisor in $\mathbb{P}^1 \times \mathbb{P}^1$. We first blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 10 singular points of B . Let H_1 and H_2 be the exceptional curves over $(-1 : 1, 1 : 1)$ and $(1 : 1, -1 : 1)$, respectively. We again blow up at the 6 intersection points of H_1, H_2 and the proper transforms of $L_{1\pm}, L_{2\pm}$, and C_{\pm} . Denote by $R = (\mathbb{P}^1 \times \mathbb{P}^1) \# 16\mathbb{P}^2$ the blown-up rational surface; Figure 2. Let $C'_{\pm}, L'_{1\pm}, L'_{2\pm}, L'_1, L'_2, L'_3, F'_{1\pm}, F'_{2\pm}, H'_1, H'_2$ be the proper transforms of $C_{\pm}, L_{1\pm}, L_{2\pm}, L_1, L_2, L_3, F_{1\pm}, F_{2\pm}, H_1, H_2$, respectively. The curves H'_3 and H'_4 denote the proper transforms of the exceptional curves over $(-1/\sqrt{-3} : 1, -1/\sqrt{-3} : 1)$ and $(1/\sqrt{-3} : 1, 1/\sqrt{-3} : 1)$, respectively. The configurations of the proper transforms of the curves and the exceptional curves in R are given in Figure 2.

2.1. An Enriques surface. Let V be the double covering of R branched along the divisor $B' = C'_+ + C'_- + L'_{1+} + L'_{1-} + L'_{2+} + L'_{2-} + H'_1 + H'_2$. Then V is a K3 surface. Let $\phi : V \rightarrow R$ be the covering map. The configuration of the inverse images of the rational curves of R by the covering ϕ is given in Figure 3.

The K3 surface V has a fixed point free involution $\sigma : V \rightarrow V$ defined by the composition of the involution induced by the map $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1, (u_0 : u_1, v_0 : v_1) \mapsto (-u_0 : u_1, -v_0 : v_1)$ and the covering involution $\tau : V \rightarrow V$ induced by the covering $\phi : V \rightarrow R$. Then the quotient $Y = V/\langle\sigma\rangle$ is an Enriques surface. Let $\pi : V \rightarrow Y$ be the unramified double covering. Since σ acts on the rational curves E_i^{\pm} as $\sigma(E_i^+) = E_i^-$, we have $\pi(E_i^+) = \pi(E_i^-)$. Set $E_i = \pi(E_i^+) = \pi(E_i^-)$.

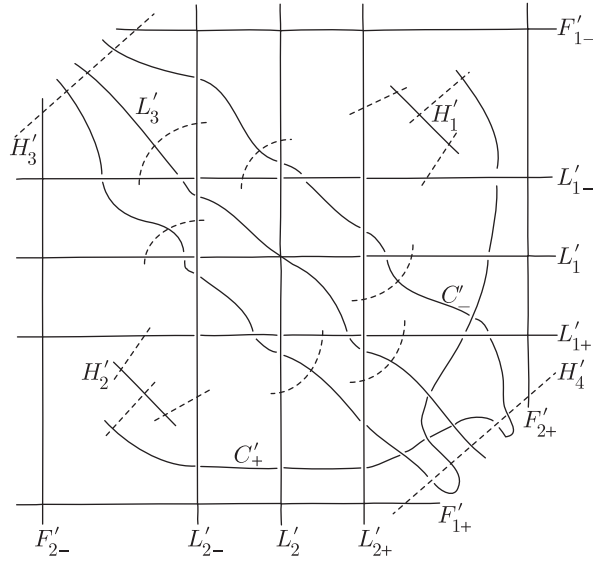


FIGURE 2. $R = (\mathbb{P}^1 \times \mathbb{P}^1) \# 16\overline{\mathbb{P}^2}$; cf. Kondo [10, Fig. 5.2]

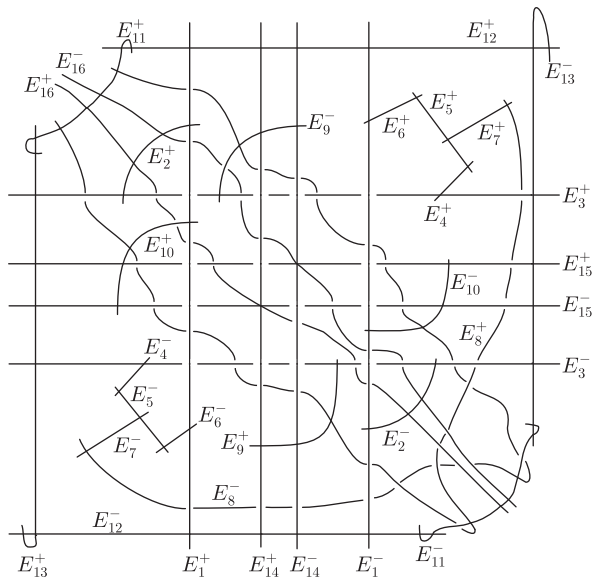


FIGURE 3. A K3 surface V ; cf. Kondo [10, Fig. 5.3]

According to Kondo [10, Example V], there are exactly 20 rational curves in Y . The dual graph of all rational curves on Y is given in Figure 4.

2.2. An elliptic fibration. In the Enriques surface Y the linear system $|E_1 + E_6 + E_5 + E_4 + E_3 + E_9 + E_8 + E_{10}| = |E_{16} + E_{19}|$ defines an elliptic fibration with one I_8 -singular fiber consisting of $E_1 + E_6 + E_5 + E_4 + E_3 + E_9 + E_8 + E_{10}$ and one I_2 -singular fiber consisting of $E_{16} + E_{19}$. The two rational curves E_2 and E_{11} are

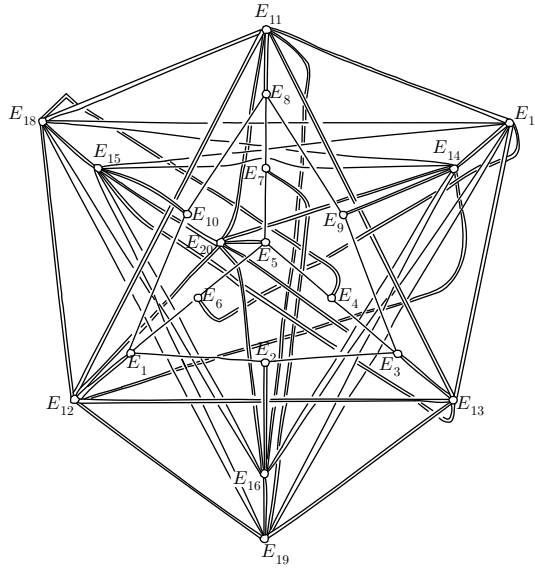


FIGURE 4. A dual graph of rational curves on Y ; cf. Kondo [10, Fig. 5.5]

bisections of the elliptic fibration; see Figure 5. Note that E_2 intersects transversely E_{16} at two different points. Also E_{11} intersects E_8 and E_{16} at two different points, respectively; cf. Figure 2.

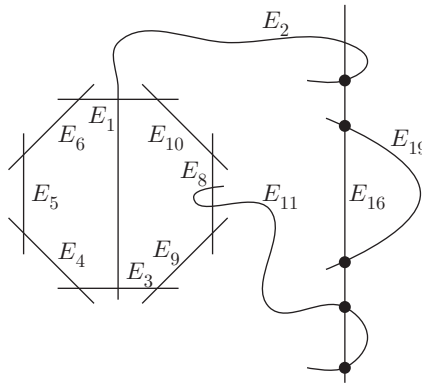


FIGURE 5. An elliptic fibration on Y

Remark 2.1. All the rational curves except E_{19} in the singular fibers $I_8 + I_2$ of the elliptic fibration of Y originate from the curves given in Figure 2. The curves E_3 and E_{16} were the rational curves $L_{1\pm}$ and L_3 (respectively) in $\mathbb{P}^1 \times \mathbb{P}^1$, and the other curves are exceptional curves contained in $R = (\mathbb{P}^1 \times \mathbb{P}^1) \# 16\overline{\mathbb{P}^2}$; cf. Figure 1.

On the other hand, the rational curve E_{19} is induced from the curve $G : u_0v_0 + u_1v_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The curve G intersects with the other curves $C_{\pm}, L_{1\pm}, L_{2\pm}, L_1, L_2, F_{1\pm}, F_{2\pm}$ as follows:

$$G \cap C_{\pm} = G \cap L_{1\pm} = G \cap L_{2\mp} = \{(\pm 1 : 1, \mp 1 : 1)\},$$

$$\begin{aligned}
 G \cap L_1 &= \{(0 : 1, 1 : 0)\}, \quad G \cap L_2 = \{(1 : 0, 0 : 1)\}, \\
 G \cap L_3 &= \{(\sqrt{-1} : 1, \sqrt{-1} : 1), (-\sqrt{-1} : 1, -\sqrt{-1} : 1)\}, \\
 G \cap F_{1\pm} &= \left\{ \left(\pm \frac{1}{\sqrt{-3}} : 1, \mp \sqrt{-3} : 1 \right) \right\}, \\
 G \cap F_{2\pm} &= \left\{ \left(\mp \sqrt{-3} : 1, \pm \frac{1}{\sqrt{-3}} : 1 \right) \right\}.
 \end{aligned}$$

Note that G is tangent to C_+ and C_- at the intersection points, respectively. Since G intersects with L_3 at two different points, their images E_{16} and E_{19} in the Enriques surface Y form an I_2 -singular fiber.

3. MAIN CONSTRUCTION

In this section we first construct a singular surface X with three permissible singularities. We then apply a \mathbb{Q} -Gorenstein smoothing theory to the singular surface X so that we obtain the desired surface.

We blow up at the five marked points \bullet on the Enriques surface Y ; cf. Figure 5. Then we get a surface $Z = Y \# 5\overline{\mathbb{P}^2}$; Figure 6. We denote the five exceptional curves by e_1, \dots, e_5 . There exist three disjoint linear chains of \mathbb{P}^1 's in Z , which are denoted by the following dual graphs:

$$\begin{aligned}
 C_1 &: \begin{array}{cccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ u_1 & & u_2 & & u_3 & & u_4 & & u_5 & & u_6 \end{array}, \\
 C_2 &: \begin{array}{c} \circ \\ u_7 \end{array}, \quad C_3 : \begin{array}{c} \circ \\ u_8 \end{array},
 \end{aligned}$$

where u_i denotes the corresponding rational curves. The configuration C_1 consists of the proper transforms of $E_{16}, E_2, E_3, E_4, E_5, E_6$, C_2 consists of the proper transform of E_{11} , and C_3 consists of the proper transform of E_{19} .

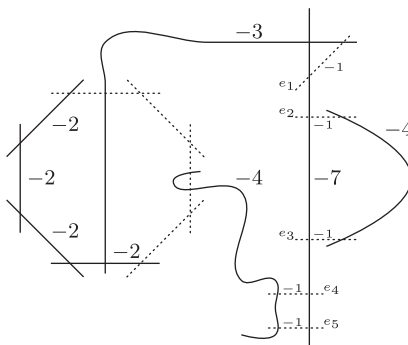


FIGURE 6. $Z = Y \# 5\overline{\mathbb{P}^2}$

We contract these three chains of \mathbb{P}^1 's from the surface Z so that it produces a normal projective surface X with three singularities of class T . In Section 4 we will show that the singular surface X has a global \mathbb{Q} -Gorenstein smoothing; see Theorem 4.1. Let X_t be a general fiber of a \mathbb{Q} -Gorenstein smoothing of the singular surface X .

Theorem 3.1. *A general fiber X_t is a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.*

Proof. Since X is a singular surface with $p_g = 0$ and $K^2 = 2$, by applying general results of complex surface theory and \mathbb{Q} -Gorenstein smoothing theory, one may conclude that a general fiber X_t is a complex surface with $p_g = 0$ and $K^2 = 2$.

We now prove the minimality of X_t . Let $f : Z \rightarrow X$ be the contraction map and let $h : Z \rightarrow Y$ be the blowing-up. Then we have

$$K_Z = f^*K_X - \left(\frac{5}{6}u_1 + \frac{5}{6}u_2 + \frac{4}{6}u_3 + \frac{3}{6}u_4 + \frac{2}{6}u_5 + \frac{1}{6}u_6 \right) - \frac{1}{2}u_7 - \frac{1}{2}u_8,$$

$$K_Z = h^*K_Y + e_1 + e_2 + e_3 + e_4 + e_5.$$

Since K_Y is numerically trivial, by combining these relations we get

$$\begin{aligned} f^*K_X \equiv & \frac{5}{6}u_1 + \frac{5}{6}u_2 + \frac{4}{6}u_3 + \frac{3}{6}u_4 + \frac{2}{6}u_5 + \frac{1}{6}u_6 + \frac{1}{2}u_7 + \frac{1}{2}u_8 \\ & + e_1 + e_2 + e_3 + e_4 + e_5, \end{aligned}$$

where \equiv denotes the numerical equivalence. Since all the coefficients are positive in the expression of f^*K_X , the \mathbb{Q} -divisor f^*K_X is nef if $f^*K_X \cdot e_i \geq 0$ for all $i = 1, \dots, 5$. In fact, we have $f^*K_X \cdot e_1 = \frac{2}{3}$, $f^*K_X \cdot e_2 = \frac{1}{3}$, $f^*K_X \cdot e_3 = \frac{1}{3}$, $f^*K_X \cdot e_4 = \frac{1}{3}$, $f^*K_X \cdot e_5 = \frac{1}{3}$. Therefore f^*K_X is nef; hence K_X is also nef. Let $\pi : \chi \rightarrow \Delta$ be a \mathbb{Q} -Gorenstein smoothing of X . Since the \mathbb{Q} -Cartier divisor $K_{\chi/\Delta}$ is π -big over Δ and π -nef at the point 0, the nefness of K_{X_t} is also obtained by shrinking Δ if it is necessary; cf. Nakayama [16]. Therefore a general fiber X_t is minimal.

Since K_{X_t} is nef, the Kodaira dimension of X_t is non-negative. By Enriques-Kodaira classification, all minimal surfaces with Kodaira dimension 0 or 1 have $K^2 = 0$. But $K_{X_t}^2 = 2$; hence X_t is of general type.

Finally we will prove that $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ in Theorem 5.1. \square

Proposition 3.2. *There is a \mathbb{Q} -Gorenstein smoothing of X such that the canonical divisor K_{X_t} of a general fiber X_t is not ample.*

Proof. The proper transform \tilde{E}_{10} of E_{10} is a (-2) -curve in Z ; cf. Figures 5 and 6. We now construct a \mathbb{Q} -Gorenstein smoothing of the singular surface X which can be extended to the deformation of the pair (X, \tilde{E}_{10}) .

We once more contract \tilde{E}_{10} from X so that we obtain a singular surface X' with four singular points of class T : Three of them are the original singular points on X , say, p_1, p_2, p_3 , and the new one is a rational double point, denoted by q . By applying a similar method as in the proof of Theorem 4.1, it is not difficult to show that the local-to-global obstruction space of the singular surface X' also vanishes. Therefore every local deformation of the four singularities can be globalized. Consider the local deformation of X' consisting of \mathbb{Q} -Gorenstein smoothings of the original singularities p_1, p_2, p_3 and the trivial deformation of the rational double point q . Then the local deformation is globalized and we obtain a family $\mathcal{X}' \rightarrow \Delta$ over a small disk Δ with the central fiber $X'_0 = X'$. Note that every fiber \mathcal{X}'_t has a rational double point as its unique singularity. By shrinking Δ if necessary, we simultaneously resolve the singularity of each fiber \mathcal{X}'_t . We then obtain a family $\mathcal{X}'' \rightarrow \Delta$ such that every fiber \mathcal{X}''_t has a (-2) -curve. Note that the central fiber of the family $\mathcal{X}'' \rightarrow \Delta$ is the singular surface X . Therefore we construct a \mathbb{Q} -Gorenstein smoothing of X that is extended to a deformation of the pair (X, \tilde{E}_{10}) .

By applying the same method in the proof of the above Theorem 3.1, we can show that a general fiber \mathcal{X}''_t is a minimal complex surface of general type with $p_g = 0$,

$K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$. But, since a general fiber X_t'' contains a (-2) -curve, the canonical divisor $K_{X_t''}$ is not ample. \square

Remark 3.3. We don't know whether the canonical divisor K_{X_t} of a general fiber X_t is *not* ample for any \mathbb{Q} -Gorenstein smoothing of X . In fact, we will construct an example with the ample canonical divisor in §6.1 by modifying the main example in the above Theorem 3.1.

4. EXISTENCE OF A GLOBAL \mathbb{Q} -GORENSTEIN SMOOTHING

This section is devoted to a proof of the following theorem.

Theorem 4.1. *The singular surface X has a global \mathbb{Q} -Gorenstein smoothing.*

The following proposition gives a sufficient condition for the existence of a global \mathbb{Q} -Gorenstein smoothing of X .

Proposition 4.2 (Y. Lee-J. Park [11]). *Let X be a normal projective surface with singularities of class T . Let $f : \tilde{X} \rightarrow X$ be the minimal resolution and let A be the reduced exceptional divisor. If $H^2(T_{\tilde{X}}(-\log A)) = 0$, then there is a global \mathbb{Q} -Gorenstein smoothing of X .*

Since the contraction map $f : Z \rightarrow X$ is the minimal resolution of the singular surface X , the existence of a global \mathbb{Q} -Gorenstein smoothing of X follows from the vanishing of the cohomology $H^2(T_Z(-\log A))$, where $A = u_1 + \dots + u_8$ is the divisor on Z consisting of the contracted rational curves. On one hand the vanishing $H^2(T_Z(-\log A)) = 0$ is preserved under blowing-downs.

Proposition 4.3 (Flenner-Zaidenberg [5]). *Let S be a non-singular surface and let A be a simple normal crossing divisor in S . Let $f : S' \rightarrow S$ be a blowing up of S at a point p of A . Set $A' = f^{-1}(A)_{red}$. Then $h^2(T_{S'}(-\log A')) = h^2(T_S(-\log A))$.*

Let

$$D = E_2 + E_3 + E_4 + E_5 + E_6 + E_{11} + E_{16} + E_{19} \in \text{Div}(Y)$$

be the divisor in the Enriques surface Y consisting of the rational curves whose proper transforms in the blown-up Enriques surface Z are contracted. Note that $f^{-1}(D)_{red} = A + e_1 + e_2 + e_3 + e_4 + e_5$. Therefore if $h^2(T_Y(-\log D)) = 0$, then $h^2(T_Z(-\log A)) = 0$ by Proposition 4.3. Thus Theorem 4.1 follows from Proposition 4.4.

Proposition 4.4. $H^2(T_Y(-\log D)) = H^0(\Omega_Y(\log D)(K_Y)) = 0$.

In order to prove Proposition 4.4, we follow a similar strategy as in J. Keum-Y. Lee-H. Park [9]. That is, we consider a lifting of the divisor D on the Enriques surface Y into the K3 surface V and then we use the push-forward map of the double covering $V \rightarrow Y$ to prove that $H^0(\Omega_Y(\log D)(K_Y)) = 0$.

Proof of Proposition 4.4. Let $\pi : V \rightarrow Y$ be the unramified double covering from the K3 surface V to the Enriques surface Y . The K3 surface V has two I_8 -singular fibers and two I_2 -singular fibers and four sections induced from the elliptic fibration structure of Y ; Figure 7. Note that $\pi(E_i^+) = \pi(E_i^-) = E_i$. We choose a divisor $\Delta \in \text{Div}(V)$ of the form

$$\Delta = E_2^+ + E_2^- + E_3^- + E_4^- + E_5^- + E_6^- + E_{11}^+ + E_{11}^- + E_{16}^+ + E_{19}^+$$

so that $\Delta \leq \pi^*D$ and $\pi_*\Delta = D$; Figure 8.

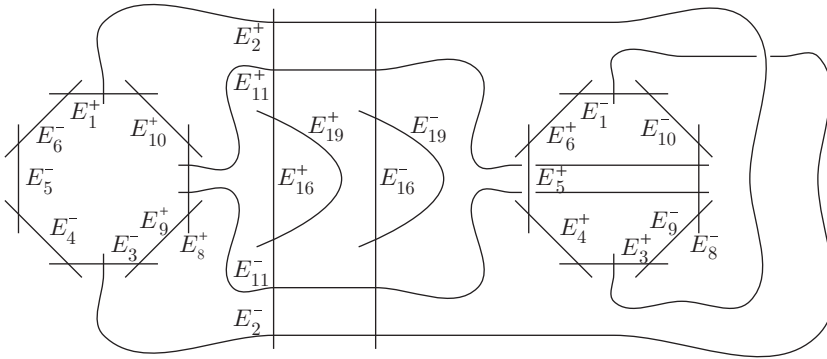


FIGURE 7. A K3 surface V

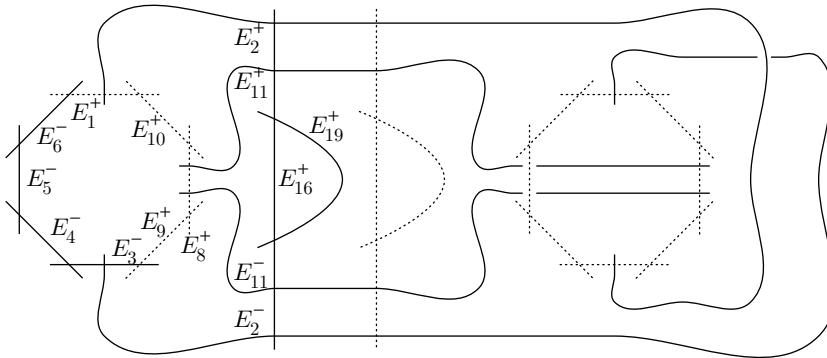


FIGURE 8. The divisor Δ

We have an exact sequence

$$0 \rightarrow \Omega_V \rightarrow \Omega_V(\log \Delta) \rightarrow \bigoplus_i \mathcal{O}_{\Delta_i} \rightarrow 0,$$

where the Δ_i 's are irreducible components of the divisor Δ . In the long exact sequence

$$0 \rightarrow H^0(\Omega_V) \rightarrow H^0(\Omega_V(\log \Delta)) \rightarrow \bigoplus_i H^0(\mathcal{O}_{\Delta_i}) \xrightarrow{\delta} H^1(\Omega_V^1) \rightarrow \dots,$$

the connecting homomorphism $\delta : \bigoplus_i H^0(\mathcal{O}_{\Delta_i}) \rightarrow H^1(\Omega_V^1)$ is the first Chern class map. Since the intersection matrix of the irreducible components of Δ is invertible, their images by the first Chern class map δ are linearly independent. Hence δ is injective. Furthermore, since V is a K3 surface, $H^0(\Omega_V) = 0$. Therefore we have

$$H^0(V, \Omega_V(\log \Delta)) = 0.$$

By the choice of Δ , we have $\Omega_Y(\log D) \subset \pi_*(\Omega_V(\log \Delta))$. On the other hand, since $0 = K_V = \pi^*K_Y$, it follows by the projection formula that

$$\pi_*(\Omega_V(\log \Delta)) = \pi_*(\Omega_V(\log \Delta)(K_V)) = \pi_*(\Omega_V(\log \Delta)) \otimes K_Y.$$

Therefore we have

$$\begin{aligned} H^0(Y, \Omega_Y(\log D)(K_Y)) &\subset H^0(Y, \pi_*(\Omega_V(\log \Delta))(K_Y)) \\ &= H^0(Y, \pi_*(\Omega_V(\log \Delta))) \\ &= H^0(V, \Omega_V(\log \Delta)) \\ &= 0. \end{aligned}$$

□

5. PROOF OF $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$

In this section we calculate the first homology group of a general fiber X_t of a \mathbb{Q} -Gorenstein smoothing of X , which is the key part of this article.

Theorem 5.1. *Let X_t be a general fiber of a \mathbb{Q} -Gorenstein smoothing of the singular surface X . Then $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.*

We denote by

$$C_1 : \begin{matrix} -7 \\ \circ \\ u_1 \end{matrix} - \begin{matrix} -3 \\ \circ \\ u_2 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_3 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_4 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_5 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_6 \end{matrix}, \quad C_2 : \begin{matrix} -4 \\ \circ \\ u_7 \end{matrix}, \quad C_3 : \begin{matrix} -4 \\ \circ \\ u_8 \end{matrix}$$

the plumbings of the linear chains of the rational curves. Let \bar{Z} be a smooth 4-manifold obtained from Z by replacing three plumbings C_1, C_2, C_3 with the corresponding Milnor fibers M_1, M_2, M_3 , respectively. Since a general fiber X_t is diffeomorphic to the rational blow-down 4-manifold \bar{Z} by Milnor fiber theory, we have $H_1(X_t; \mathbb{Z}) = H_1(\bar{Z}; \mathbb{Z})$. Hence it suffices to show that

$$H_1(\bar{Z}; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}.$$

We will prove it in Proposition 5.11. We start by decomposing the surface Z into

$$Z = Z_0 \cup (C_1 \cup C_2 \cup C_3).$$

Then the 4-manifold \bar{Z} can be decomposed into

$$\bar{Z} = Z_0 \cup (M_1 \cup M_2 \cup M_3).$$

In order to prove that $H_1(\bar{Z}; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$, we first calculate $H_1(Z_0; \mathbb{Z})$.

Proposition 5.2. $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.

We divide the proof of Proposition 5.2 into the following several lemmas. Proposition 5.2 will be proved in Lemma 5.10.

Let $\alpha_1, \alpha_2, \alpha_3$ be the normal circles of the disk bundles C_1, C_2, C_3 over u_1, u_7, u_8 , respectively; cf. Figure 6. We again denote by α_i the homology classes of α_i in $H_1(Z_0; \mathbb{Z})$ for convenience.

Lemma 5.3. $\alpha_1 = \alpha_2 = \alpha_3$ and $2\alpha_1 = 2\alpha_2 = 2\alpha_3 = 0$ in $H_1(Z_0; \mathbb{Z})$.

Proof. Refer to Figure 6. Since the exceptional curve e_2 intersects transversely the curves u_1 and u_8 , we have

$$\alpha_1 + \alpha_3 = 0$$

in $H_1(Z_0; \mathbb{Z})$. Since the exceptional curve e_4 intersects transversely the curves u_1 and u_7 , we have

$$\alpha_1 + \alpha_2 = 0$$

in $H_1(Z_0; \mathbb{Z})$. Since the curve u_7 intersects transversely the proper transform of E_8 at two different points, we have

$$2\alpha_2 = 0.$$

Combining these relations, we have $\alpha_1 = \alpha_2 = \alpha_3$ and $2\alpha_1 = 2\alpha_2 = 2\alpha_3 = 0$ in $H_1(Z_0; \mathbb{Z})$. □

From now on we set $\alpha = \alpha_1 = \alpha_2 = \alpha_3 \in H_1(Z_0; \mathbb{Z})$.

Lemma 5.4. $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ or $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. We first consider the Mayer-Vietoris sequence of a pair $(Z_0, C_1 \cup C_2 \cup C_3)$:

$$\begin{aligned} H_1(\partial Z_0; \mathbb{Z}) \xrightarrow{k_*} H_1(Z_0; \mathbb{Z}) \oplus H_1(C_1 \cup C_2 \cup C_3; \mathbb{Z}) &\rightarrow H_1(Z; \mathbb{Z}) \\ &\rightarrow \widetilde{H}_0(\partial Z_0; \mathbb{Z}) \xrightarrow{\xi} \widetilde{H}_0(Z_0; \mathbb{Z}) \oplus \widetilde{H}_0(C_1 \cup C_2 \cup C_3; \mathbb{Z}) \rightarrow 0. \end{aligned}$$

Since $\xi : \widetilde{H}_0(\partial Z_0; \mathbb{Z}) \rightarrow \widetilde{H}_0(C_1 \cup C_2 \cup C_3; \mathbb{Z})$ is isomorphic, we get the following short exact sequence:

$$(5.1) \quad 0 \rightarrow \text{Im}(k_*) \hookrightarrow H_1(Z_0; \mathbb{Z}) \rightarrow H_1(Z; \mathbb{Z}) \rightarrow 0.$$

We claim that $\text{Im}(k_*) = \langle \alpha \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then, since $\pi_1(Z) = \mathbb{Z}/2\mathbb{Z} = H_1(Z; \mathbb{Z})$, it follows that $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ or $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof of the claim. By Lemma 5.3 above, it suffices to show that $\text{Im}(k_*) \neq 0$ in $H_1(Z_0; \mathbb{Z})$. Assume on the contrary that $\text{Im}(k_*) = 0$, and we consider an exact sequence of a pair $(Z_0, \partial Z_0 = \partial C_1 \cup \partial C_2 \cup \partial C_3)$:

$$\cdots \rightarrow H_2(Z_0, \partial Z_0; \mathbb{Z}) \xrightarrow{\partial_*} H_1(\partial Z_0; \mathbb{Z}) \xrightarrow{k_*} H_1(Z_0; \mathbb{Z}) \rightarrow \cdots .$$

Then we conclude that the map $\partial_* : H_2(Z_0, \partial Z_0; \mathbb{Z}) \rightarrow H_1(\partial Z_0; \mathbb{Z})$ is surjective.

On the other hand, since a homomorphism

$$j_* : H_2(Z; \mathbb{Z}) \rightarrow H_2(Z, C_1 \cup C_2 \cup C_3; \mathbb{Z})$$

induced by an inclusion is surjective and

$$H_2(Z, C_1 \cup C_2 \cup C_3; \mathbb{Z}) \cong H_2(Z_0, \partial C_1 \cup \partial C_2 \cup \partial C_3; \mathbb{Z})$$

by the excision principle, the space $\text{Im}(\partial_*) = \text{Im}(\partial_* \circ j_*)$ is completely determined by the images of elements in $H_2(Z; \mathbb{Z})$. Furthermore, by choosing a suitable set \mathcal{B} of homology classes expanding $H_2(Z; \mathbb{Z})$ and by computing the images of all elements in \mathcal{B} under $\partial_* \circ j_*$, one can compute the space $\text{Im}(\partial_*)$.

For example, we can choose such a set

$$\mathcal{B} = \{u_1, \dots, u_8, e_1, \dots, e_5, \widetilde{E}_1, \dots, \widetilde{E}_{20}\} \cup \{\widetilde{F}_\alpha \mid \alpha \in \Lambda\},$$

where the u_i 's are rational curves in $C_1 \cup C_2 \cup C_3$, the e_i 's denote all exceptional curves, the \widetilde{E}_i 's denote the proper transforms on the blown-up space $Z = Y \# 5\overline{\mathbb{P}^2}$, and $\{\widetilde{F}_\alpha \mid \alpha \in \Lambda\}$ denotes the set of all half pencils of all possible elliptic fibration structures of Y . One can show that the images of \widetilde{F}_α under $\partial_* \circ j_*$ are generated by the images of $\{u_1, \dots, u_8, e_1, \dots, e_5, \widetilde{E}_1, \dots, \widetilde{E}_{20}\}$ under $\partial_* \circ j_*$. Hence, by the same argument used in the proof of Lemma 2.4 in the work of Y. Lee-J. Park [12], we conclude that the element $(0, 0, 1)$ in the space

$$H_1(\partial Z_0; \mathbb{Z}) = H_1(\partial C_1 \cup \partial C_2 \cup \partial C_3; \mathbb{Z}) \cong \mathbb{Z}/72\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

is not contained in the space $\text{Im}(\partial_* \circ j_*) = \text{Im}(\partial_*)$, which contradicts the surjectivity of ∂_* . □

5.1. **A special loop β in Z_0 .** We construct a loop β in $H_1(Z_0; \mathbb{Z})$ which lies on a certain curve of genus 2 in Z , which will be shown to be a generator of $H_1(Z_0; \mathbb{Z})$; see Lemma 5.10.

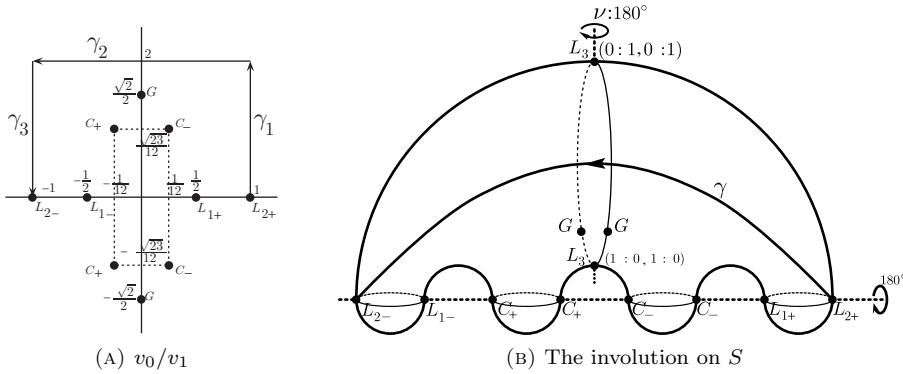


FIGURE 9. A rational curve S in $\mathbb{P}^1 \times \mathbb{P}^1$

Set

$$S = \{u_0v_1 = 2v_0u_1\} \subset \mathbb{P}^1 \times \mathbb{P}^1 = \{[u_0 : u_1, v_0 : v_1]\}.$$

The rational curve S intersects transversally the $(4, 4)$ -divisor $B = C_{\pm} + L_{1\pm} + L_{2\pm}$ at eight points. In Figure 9(A) we indicate by dots, \bullet , the v_0/v_1 -coordinates of the intersection points of S with the curves C_{\pm} , $L_{1\pm}$, $L_{2\pm}$ and G , where the horizontal axis is the real part of v_0/v_1 and the vertical axis is the imaginary part of v_0/v_1 .

Since S does not pass through any singular points of the $(4, 4)$ -divisor B , the proper transform of S in $R = (\mathbb{P}^1 \times \mathbb{P}^1) \# 16\overline{\mathbb{P}^2}$ intersects the branch divisor $B' = C'_{\pm} + L'_{1\pm} + L'_{2\pm} + H'_1 + H'_2$ again in eight points. We denote the proper transform of S again by S . Therefore the inverse image T of the rational curve S by the branched double covering $\phi : V \rightarrow R$ is a curve of genus 3 in the K3 surface V ; Figure 10. The dots, \bullet , indicate the intersection points with the rational curves $E_{i\pm}$.

We denote the restriction of the double covering $\phi : V \rightarrow R$ on T again by ϕ . Hence we have a branched double covering $\phi : T \rightarrow S$. Recall that the involution $\sigma : V \rightarrow V$ that induces the Enriques surface $Y = V/\sigma$ is the composition of the extension of the involution $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defined by $(u_0 : u_1, v_0 : v_1) \mapsto (-u_0 : u_1, -v_0 : v_1)$ and the covering involution $\tau : V \rightarrow V$. Since the rational curve S is invariant under the involution ν on $\mathbb{P}^1 \times \mathbb{P}^1$, the involution $\sigma : V \rightarrow V$ induces an involution on T without fixed points. We denote the involution of T again by $\sigma : T \rightarrow T$. We visualize the involutions on T in Figure 10. Topologically the involution ν (or τ) is the rotation around the vertical (resp. horizontal) line through the middle point marked by \otimes by an angle 180° . Therefore the involution $\sigma : T \rightarrow T$ is the rotation around the axis through the paper centered at the marked point \otimes by an angle 180° . Hence the quotient T/σ is a smooth curve of genus 2; see Figure 11. Let $\pi : T \rightarrow T/\sigma$ denote the quotient map.

Note that $S \cap L'_{2+}$ and $S \cap L'_{2-}$ are two of the branch points of the covering $\phi : T \rightarrow S$. Let γ be a path lying on S which connects $S \cap L'_{2+}$ and $S \cap L'_{2-}$ and does not pass through any intersection points with C_{\pm} , $L_{1\pm}$, $L_{2\pm}$, L_1 , L_2 , L_3 , $F_{1\pm}$, $F_{2\pm}$;

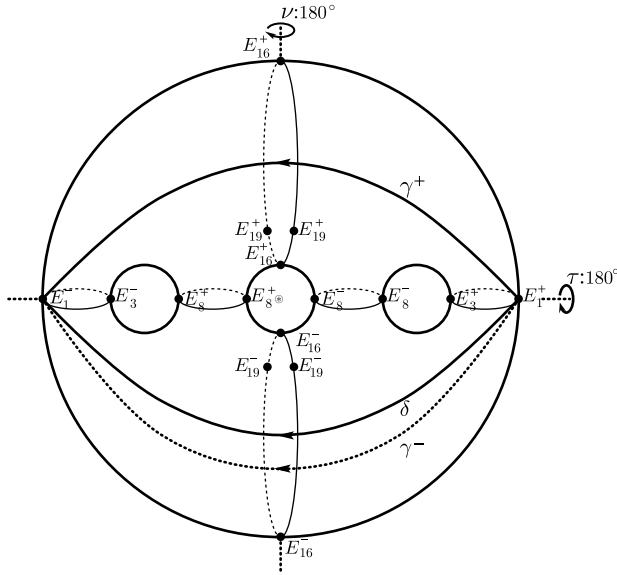


FIGURE 10. A curve T of genus 3 in K3 surface V

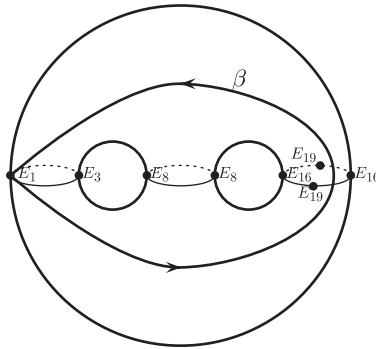


FIGURE 11. A curve T/σ of genus 2 in Enriques surface Y

cf. Figure 9(A). More precisely, we define the paths $\gamma_i \subset \mathbb{P}^1 \times \mathbb{P}^1 = \{[u_0 : u_1, v_0 : v_1]\}$ on $S = \{u_0 v_1 = 2v_0 u_1\}$ by

$$\gamma_1(s) = (2 + 2s\sqrt{-1} : 1, 1 + s\sqrt{-1} : 1), \quad 0 \leq s \leq 2,$$

$$\gamma_2(s) = (-2s + 4\sqrt{-1} : 1, -s + 2\sqrt{-1} : 1), \quad -1 \leq s \leq 1,$$

$$\gamma_3(s) = (-2 + 2(2-s)\sqrt{-1} : 1, -1 + (2-s)\sqrt{-1} : 1), \quad 0 \leq s \leq 2$$

(cf. Figure 9(A)), and the path γ is defined by their union, that is,

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Note that the inverse image $\phi^{-1}(\gamma)$ on the curve T in K3 surface V consists of two paths $\gamma^+ \cup \gamma^-$ as in Figure 10. We define a loop β in the curve T/σ by the image of γ^+ under the quotient map $\pi : T \rightarrow T/\sigma$, that is,

$$\beta = \pi(\gamma^+) \subset T/\sigma;$$

cf. Figure 11. Since the loop β does not pass through any blowing-up points of the blowing-up $h : Z = Y \# 5\overline{\mathbb{P}^2} \rightarrow Y$, we may regard β as a loop in Z .

Lemma 5.5. *The loop β is a generator of $H_1(Z; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. The path γ^+ is a lifting of $\beta \subset Y$ to K3 surface V , but γ^+ is not a loop. Therefore $\beta \neq 0$ in $\pi_1(Y) = \mathbb{Z}/2\mathbb{Z}$. Since $\pi_1(Y) \cong \pi_1(Z)$, we have $\beta \neq 0$ in $H_1(Z; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$; hence it generates $H_1(Z; \mathbb{Z})$. \square

Note that the loop β does not pass through any intersection points of T/σ with the rational curves consisting of the configurations C_1, C_2 and C_3 by a choice of the path γ . Since Z_0 is obtained by removing C_1, C_2 and C_3 from Z , it follows that the loop β may be considered as a loop in Z_0 , that is, $\beta \in H_1(Z_0, \mathbb{Z})$.

Lemma 5.6. $H_1(Z_0; \mathbb{Z}) = \langle \alpha, \beta \rangle$.

Proof. In the proof of Lemma 5.4, we show that $\text{Im}(k_*)$ is generated by α , and in the above lemma we show that $H_1(Z; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is generated by β . Therefore the result follows from the exact sequence (5.1). \square

5.2. The loop β generates $H_1(Z_0; \mathbb{Z})$. We will show that $H_1(Z_0; \mathbb{Z})$ is generated only by β in Lemma 5.10, which completes the proof of $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.

We blow up the K3 surface V at the corresponding two points when we blow up the Enriques surface Y at each marked point in order to construct the surface Z ; cf. Figure 6. Then we get an unramified double covering $W = V \# 10\overline{\mathbb{P}^2} \rightarrow Z = Y \# 5\overline{\mathbb{P}^2}$ that induces the double covering $\pi : T \rightarrow T/\sigma$, where T and T/σ are considered as the proper transforms of T and T/σ , respectively. We denote the covering $W \rightarrow Z$ again by $\pi : W \rightarrow Z$. Let us denote by \tilde{E}_i^\pm the proper transform of E_i^\pm . Note that the covering W has six configurations of linear chains of \mathbb{P}^1 's:

$$\begin{aligned} C_1^+ &: \begin{array}{cccccc} \overset{-7}{\circ} & \overset{-3}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} \\ \tilde{E}_{16}^+ & \tilde{E}_2^- & \tilde{E}_3^- & \tilde{E}_4^- & \tilde{E}_5^- & \tilde{E}_6^- \end{array}, \\ C_1^- &: \begin{array}{cccccc} \overset{-7}{\circ} & \overset{-3}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} & \overset{-2}{\circ} \\ \tilde{E}_{16}^- & \tilde{E}_2^+ & \tilde{E}_3^+ & \tilde{E}_4^+ & \tilde{E}_5^+ & \tilde{E}_6^+ \end{array}, \\ C_2^+ &: \begin{array}{c} \overset{-4}{\circ} \\ \tilde{E}_{11}^+ \end{array}, & C_2^- &: \begin{array}{c} \overset{-4}{\circ} \\ \tilde{E}_{11}^- \end{array}, \\ C_3^+ &: \begin{array}{c} \overset{-4}{\circ} \\ \tilde{E}_{19}^+ \end{array}, & C_3^- &: \begin{array}{c} \overset{-4}{\circ} \\ \tilde{E}_{19}^- \end{array}, \end{aligned}$$

which are the inverse images of the configurations C_1, C_2, C_3 in Z , respectively; see Figure 12. We decompose

$$W = W_0 \cup \{C_1^+ \cup C_1^- \cup C_2^+ \cup C_2^- \cup C_3^+ \cup C_3^-\}.$$

Then the restriction $W_0 \rightarrow Z_0$ of the covering $\pi : W \rightarrow Z$ is also an unramified double covering. We also denote the covering $W_0 \rightarrow Z_0$ by $\pi : W_0 \rightarrow Z_0$ for convenience.

Lemma 5.7. *Let $\tilde{2}\beta$ be the lifting of $2\beta \subset Z_0$ to W_0 by the covering map $\pi : W_0 \rightarrow Z_0$. Set $\gamma^* = \gamma^+ - \gamma^-$ and let $\bar{\alpha}$ be a normal circle of \tilde{E}_{16}^- lying in W_0 . Then $\tilde{2}\beta$ is homologous to $\gamma^* + \bar{\alpha}$ in W_0 .*

Proof. In Figure 10 it follows that $\tilde{2}\beta$ is homologous to $\gamma^+ - \delta$. On the other hand, the curve E_{16}^- intersects T at two points and one of them lies in the south pole; cf.

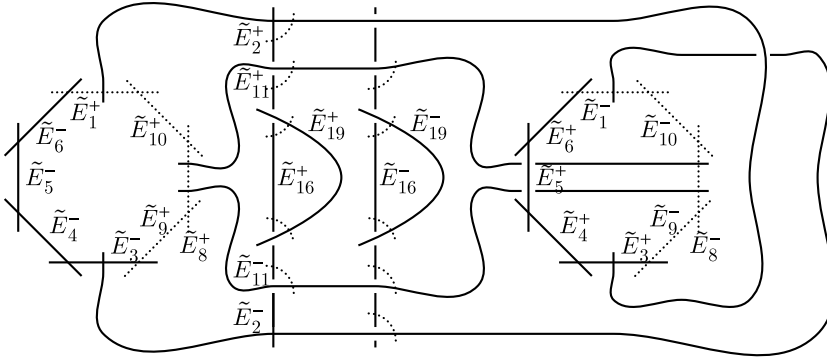


FIGURE 12. $W = V_{\sharp}10\overline{\mathbb{P}^2}$

Figure 10. The proper transform \tilde{E}_{16}^- is contained in C_1^- , and the configuration C_1^- will be removed from W to obtain W_0 . Since the homology class of a normal circle of the disk bundle over \tilde{E}_{16}^- is $\bar{\alpha}$, we have the following picture in W_0 : Figure 13. Hence $\delta - \gamma^- + \bar{\alpha} = 0$ in $H_1(W_0; \mathbb{Z})$. Therefore $\tilde{2}\beta$ is homologous to $\gamma^+ - \gamma^- + \bar{\alpha}$ in W_0 . \square

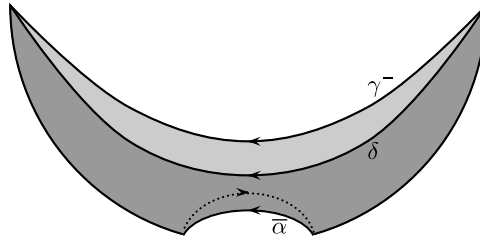


FIGURE 13. A lifting $\tilde{2}\beta$

Since the blown-up K3 surface $W = V_{\sharp}10\overline{\mathbb{P}^2}$ is simply connected, we can apply the same argument used in the proof of Lemma 2.4 in the work of Y. Lee-J. Park [12] so that we can prove:

Lemma 5.8. $H_1(W_0; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cong \langle \bar{\alpha} \rangle$.

The loop γ^* in Lemma 5.7 vanishes in $H_1(W_0; \mathbb{Z})$:

Lemma 5.9. $\gamma^* = 0$ in $H_1(W_0; \mathbb{Z})$.

Proof. We construct a real 2-dimensional surface U in $\mathbb{P}^1 \times \mathbb{P}^1$ such that the path γ is a boundary component of U . Define a path γ' lying on F_{1-} which connects the two points $F_{1-} \cap L_{2+} = \{(-1/\sqrt{-3} : 1, 1 : 1)\}$ and $F_{1-} \cap L_{2-} = \{(-1/\sqrt{-3} : 1, -1 : 1)\}$ as follows: Let us define paths γ'_i on F_{1-} by

$$\begin{aligned} \gamma'_1(s) &= (-1/\sqrt{-3} : 1, 1 + s\sqrt{-1} : 1), 0 \leq s \leq 2, \\ \gamma'_2(s) &= (-1/\sqrt{-3} : 1, -s + 2\sqrt{-1} : 1), -1 \leq s \leq 1, \\ \gamma'_3(s) &= (-1/\sqrt{-3} : 1, -1 + (2-s)\sqrt{-1} : 1), 0 \leq s \leq 2, \end{aligned}$$

and the path γ' is defined by their union, that is,

$$\gamma' = \gamma'_1 \cup \gamma'_2 \cup \gamma'_3.$$

Then the surface U is defined by connecting the paths γ and γ' by line segments. More precisely,

$$\begin{aligned} U &= (t\gamma_1(s) + (1-t)\gamma'_1(s)) \cup (t\gamma_2(s) + (1-t)\gamma'_2(s)) \cup (t\gamma_2(s) + (1-t)\gamma'_2(s)) \\ &= \{(t(2 + 2s\sqrt{-1}) + (1-t)(-1/\sqrt{-3}) : 1, 1 + s\sqrt{-1} : 1) \mid 0 \leq s \leq 2\} \\ &\cup \{(t(-2s + 4\sqrt{-1}) + (1-t)(-1/\sqrt{-3}) : 1, -s + 2\sqrt{-1} : 1) \mid -1 \leq s \leq 1\} \\ &\cup \{(t(-2 + 2(2-s)\sqrt{-1}) + (1-t)(-1/\sqrt{-3}) : 1, -1 + (2-s)\sqrt{-1} : 1) \mid 0 \leq s \leq 2\}, \end{aligned}$$

where $0 \leq t \leq 1$; see Figure 14(A).

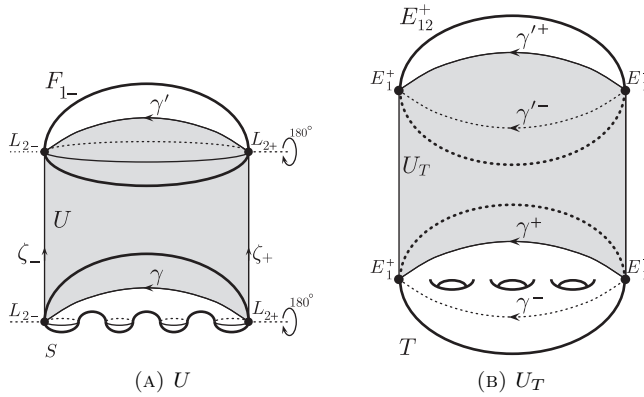


FIGURE 14. The real 2-dimensional surfaces U and U_T

The surface U does not pass through the blowing-up points of $R = (\mathbb{P}^1 \times \mathbb{P}^1) \# 16\overline{\mathbb{P}^2} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Hence one may assume that U is contained in R . The boundary ∂U of U consists of four components γ , γ' , ζ_- , ζ_+ , where ζ_{\pm} is the line segment connecting two points $S \cap L_{2\pm}$ and $F_{1-} \cap L_{2\pm}$, respectively; that is,

$$\zeta_{\pm} = \{(2t + (1-t)(-1/\sqrt{-3}) : 1, \pm 1 : 1) \mid 0 \leq t \leq 1\}.$$

Note that the curves $L'_{2\pm}$ are components of the branch divisors of the double covering $\phi : V \rightarrow R$. But the two boundary components ζ_- and ζ_+ of U are contained in $L'_{2\pm}$. Therefore the inverse image $U_T = \phi^{-1}(U)$ has only two boundaries:

$$\partial U_T = \phi^{-1}(\gamma) \cup \phi^{-1}(\gamma').$$

Moreover, since the inverse image of the sphere F_{1-} of the double covering ϕ is again the sphere E_{12}^+ , it follows that the loop $\phi^{-1}(\gamma') = \gamma'^+ - \gamma'^-$ lies in E_{12}^+ ; see Figure 14(B). In fact, by a simple calculation, one can show that the interior of U does not intersect with the $(4, 4)$ -divisor $B = C_{\pm} + L_{1\pm} + L_{2\pm}$. Therefore U_T is topologically a cylinder with two boundaries $\phi^{-1}(\gamma)$ and $\phi^{-1}(\gamma')$.

Set $\gamma'^* = \gamma'^+ - \gamma'^-$. It follows by a simple calculation that $U \cap (L_{1-} \cup L_{1+} \cup G) = \emptyset$ and $U \cap L_3 = \{(2\sqrt{-1} : 1, 2\sqrt{-1} : 1)\}$. Therefore $U_T \cap (E_3^+ \cup E_3^-) = U_T \cap (E_{19}^+ \cup E_{19}^-) = \emptyset$ and $U_T \cap (E_{16}^+ \cup E_{16}^-)$ consists of two points. Since U_T does not pass through any blowing-up points of $W = V\sharp 10\mathbb{P}^2$, U_T does not intersect $\tilde{E}_2^\pm, \tilde{E}_4^\pm, \tilde{E}_5^\pm, \tilde{E}_6^\pm, \tilde{E}_{11}^\pm$. Therefore $U'_T = U_T \cap W_0$ has four boundaries as in Figure 15. Since $2\bar{\alpha} = 0$, we have

$$(5.2) \quad \gamma^* + \gamma'^* = 0.$$

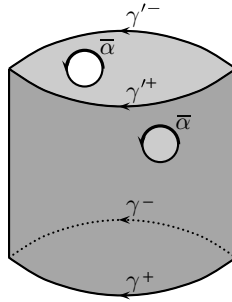


FIGURE 15. Four boundaries of $U'_T = U_T \cap W_0$

On the other hand, the sphere F'_{1-} intersects transversely with H'_3 at $(-1/\sqrt{-3} : 1, -1/\sqrt{-3} : 1)$ and with G' at $(-1/\sqrt{-3} : 1, \sqrt{-3} : 1)$; see Figure 16(A). Therefore the sphere E'_{12} intersects transversely with E'_{11} and E'_{19} on the upper half side of E'_{12} and with E'_{11} and E'_{19} on the lower half side of E'_{12} ; see Figure 17(A). Therefore the loop γ'^* encloses the two intersection points $E'_{12} \cap E'_{11}$ and $E'_{12} \cap E'_{19}$. The normal circles over the sphere bundles C_2^+ and C_3^+ over E'_{11} and E'_{19} are $\bar{\alpha}$. Therefore in W_0 the lower half sphere of \tilde{E}'_{12} has three boundaries: γ'^* and two $\bar{\alpha}$'s on the two intersection points $E'_{12} \cap E'_{11}$ and $E'_{12} \cap E'_{19}$; see Figure 17(B). Since $2\bar{\alpha} = 0$, we have $\gamma'^* = 0$. Therefore the relation (5.2) implies that $\gamma^* = 0$. \square

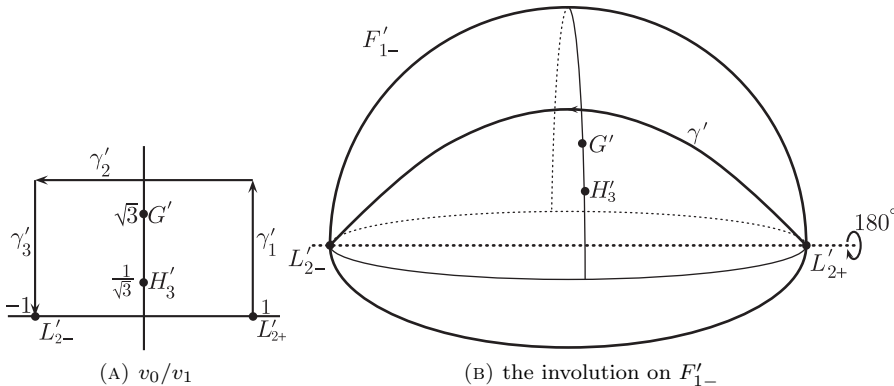


FIGURE 16. F'_{1-}

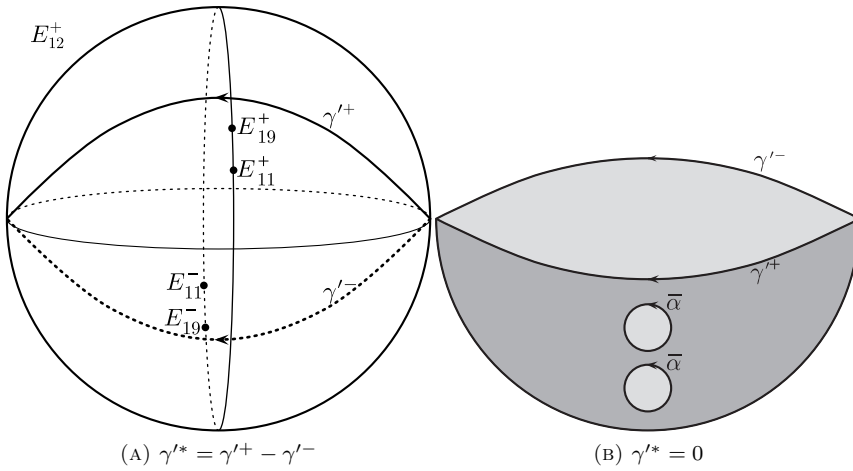


FIGURE 17. E_{12}^+

Lemma 5.10. *The loop β is a generator of $H_1(Z_0; \mathbb{Z})$; hence, $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.*

Proof. By the previous lemmas, we have $\widetilde{2\beta} = \bar{\alpha} \neq 0$ in $H_1(W_0; \mathbb{Z})$. Hence we get $2\beta = \alpha \neq 0$ in $H_1(Z_0; \mathbb{Z})$. But we have $H_1(Z_0; \mathbb{Z}) = \langle \alpha, \beta \rangle$ by Lemma 5.6. Therefore $H_1(Z_0; \mathbb{Z})$ is generated by the single element β . Since $H_1(Z_0; \mathbb{Z})$ is of order 4 by Lemma 5.4, we have $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$. \square

We finally prove the main result of this section.

Proposition 5.11. $H_1(\bar{Z}; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$.

Proof. We consider the Mayer-Vietoris sequence of a pair $(Z_0, M_1 \cup M_2 \cup M_3)$:

$$H_1(\partial M_1 \cup \partial M_2 \cup \partial M_3; \mathbb{Z}) \xrightarrow{k_* \oplus j_*} H_1(Z_0; \mathbb{Z}) \oplus H_1(M_1 \cup M_2 \cup M_3; \mathbb{Z}) \rightarrow H_1(\bar{Z}; \mathbb{Z}) \rightarrow 0.$$

Let $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ be generators of

$$\begin{aligned} H_1(\partial M_1 \cup \partial M_2 \cup \partial M_3; \mathbb{Z}) &\cong H_1(\partial M_1; \mathbb{Z}) \oplus H_1(\partial M_2; \mathbb{Z}) \oplus H_1(\partial M_3; \mathbb{Z}) \\ &\cong \mathbb{Z}/72\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

Then the images of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ under the map $k_* \oplus j_*$ are $(\alpha, 1, 0, 0)$, $(\alpha, 0, 1, 0)$ and $(\alpha, 0, 0, 1)$, respectively, in

$$\begin{aligned} H_1(Z_0; \mathbb{Z}) \oplus H_1(M_1 \cup M_2 \cup M_3; \mathbb{Z}) \\ &\cong H_1(Z_0; \mathbb{Z}) \oplus H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}) \oplus H_1(M_3; \mathbb{Z}) \\ &\cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Since $\alpha = 2\beta$ and β is a generator of $H_1(Z_0; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, it follows that

$$\begin{aligned} H_1(\bar{Z}; \mathbb{Z}) &\cong H_1(Z_0; \mathbb{Z}) \oplus H_1(M_1 \cup M_2 \cup M_3; \mathbb{Z}) / \text{Im}(k_* \oplus j_*) \\ &\cong (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) / \langle (2, 1, 0, 0), (2, 0, 1, 0), (2, 0, 0, 1) \rangle \\ &\cong \langle (1, 0, 0, 0) \rangle \cong \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

Therefore we finally get $H_1(\bar{Z}; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$. \square

6. MORE EXAMPLES

In this section we construct three more examples.

6.1. An example with an ample canonical divisor. We construct an example with an ample canonical divisor by modifying the configuration in the main construction, while the main example in Section 3 may have a non-ample canonical divisor as stated in Proposition 3.2.

In order to construct a singular surface X with three singularities of class T in Section 3, we contracted three linear chains C_1, C_2, C_3 of rational curves from the blown-up surface Z so that we obtained a singular surface X with three singularities of class T ; cf. Figure 6. Together with the C_i 's, contract one more rational curve, the proper transform of E_{10} . Since the proper transform of E_{10} is a (-2) -curve, we obtain a projective singular surface X' with four singularities of class T . Let $f : Z \rightarrow X'$ be the contraction. By similar methods in Section 4 and Section 5, one can show that there is a global \mathbb{Q} -Gorenstein smoothing of X' and that its general fiber X'_t is a minimal surface of general type with $p_g = 0, K^2 = 2,$ and $H_1 = \mathbb{Z}/4\mathbb{Z}.$

Proposition 6.1. *A general fiber X'_t of a \mathbb{Q} -Gorenstein smoothing of X' has an ample canonical divisor.*

Proof. We use a similar method found in the author's paper [18] and J. Keum-Y. Lee-H. Park [9]. We denote by

$$C_1 : \begin{matrix} -7 \\ \circ \\ u_1 \end{matrix} - \begin{matrix} -3 \\ \circ \\ u_2 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_3 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_4 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_5 \end{matrix} - \begin{matrix} -2 \\ \circ \\ u_6 \end{matrix}, C_2 : \begin{matrix} -4 \\ \circ \\ u_7 \end{matrix}, C_3 : \begin{matrix} -4 \\ \circ \\ u_8 \end{matrix}, C_4 : \begin{matrix} -2 \\ \circ \\ u_9 \end{matrix}$$

the linear chains of rational curves, where $u_1 = \tilde{E}_{16}, u_2 = \tilde{E}_2, u_3 = \tilde{E}_3, u_4 = \tilde{E}_4, u_5 = \tilde{E}_5, u_6 = \tilde{E}_6, u_7 = \tilde{E}_{11}, u_8 = \tilde{E}_{19}, u_9 = \tilde{E}_{10}$ denote the proper transforms of E_i on the blown-up space Z . By a similar calculation in the proof of Theorem 3.1, it follows that

$$(6.1) \quad \begin{aligned} f^*K_{X'} \equiv & \frac{5}{6}u_1 + \frac{5}{6}u_2 + \frac{4}{6}u_3 + \frac{3}{6}u_4 + \frac{2}{6}u_5 + \frac{1}{6}u_6 + \frac{1}{2}u_7 + \frac{1}{2}u_8 \\ & + e_1 + e_2 + e_3 + e_4 + e_5 \end{aligned}$$

and that $f^*K_{X'}$ is nef and, furthermore, $f^*K_{X'} \cdot e_j > 0$ for all $j = 1, \dots, 5$.

Since $K_{X'}^2 = 2 > 0,$ in order to show the ampleness of $K_{X'}$, it is enough to show that $K_{X'} \cdot C > 0$ for any irreducible curve $C \subset X'$. Let $\bar{C} \subset Z$ be the proper transform of C . Note that

$$K_{X'} \cdot C = f^*K_{X'} \cdot f^*C = f^*K_{X'} \cdot \bar{C}.$$

Hence it is enough to show that $f^*K_{X'} \cdot \bar{C} > 0$. But we already know that $f^*K_{X'} \cdot e_j > 0$ for all $j = 1, \dots, 5$. Thus we may assume that $\bar{C} \neq e_j$ for any j . Since the coefficients of $f^*K_{X'}$ in (6.1) are positive and $\bar{C} \neq u_i$ for any $i = 1, \dots, 9$ and $\bar{C} \neq e_j$ for any $j = 1, \dots, 5,$ if $\bar{C} \cdot u_i > 0$ for some i or $\bar{C} \cdot e_j > 0$ for some $j,$ then $f^*K_{X'} \cdot \bar{C} > 0$.

Let $h : Z \rightarrow Y$ be the blowing-up that produces the exceptional curve e_j . We denote $h(\bar{C})$ by \hat{C} . If $p_a(\bar{C}) \geq 2 = p_a(\hat{C}) \geq 2,$ then $\hat{C} \cdot (E_{16} + E_{19}) > 0$. Therefore $\bar{C} \cdot h^*(E_{16} + E_{19}) > 0$. If $p_a(\bar{C}) = p_a(\hat{C}) = 1,$ then \hat{C} is a fiber or a half pencil of an elliptic pencil. If \hat{C} is numerically equivalent to $|E_{16} + E_{19}|$ or $(1/2)(E_{16} + E_{19}),$ then we have $\hat{C} \cdot E_2 > 0$ because E_2 is a bisection; hence, $\bar{C} \cdot h^*E_2 > 0$. If not, then $\hat{C} \cdot (E_{16} + E_{19}) > 0$. Finally, if $p_a(\bar{C}) = 0,$ then \bar{C} is a rational curve; hence

$\overline{C} = \widetilde{E}_k$ for some $1 \leq k \leq 20$. However, according to the dual graph (Figure 4) of (all) rational curves lying on the Enriques surface Y , every rational curve lying on Z which is not contracted to a singular point of X' intersects with some u_i 's. In any case we show that $\overline{C} \cdot u_i > 0$ for some i or $\overline{C} \cdot e_j > 0$ for some j , which completes the proof. \square

6.2. An example obtained from a different configuration. We construct two more examples of a minimal surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$ using different configurations of rational curves coming from the same Enriques surface Y . We first construct an example with a non-ample canonical divisor; see Remark 6.2. We then construct another example by modifying the configuration. But we don't know whether the canonical divisor of the latter example is ample or not; see Remark 6.3. Since all proofs are basically the same as the case of the main example constructed in Section 3, we only explain how to construct the blown-up surface Z from the Enriques surface Y containing linear chains of rational curves.

On the Enriques surface Y used in Section 3 we blow up 5 times at the marked points \bullet on E_{16} and we blow up again successively 5 times at the marked point \odot on E_{16} ; see Figure 18(A). The blown-up surface $Z = Y \# 10\overline{\mathbb{P}^2}$ contains the following four disjoint linear chains of rational curves as in Figure 18(B):

$$\begin{aligned}
 C_1 = C_{6,1} &: \begin{matrix} -8 & -2 & -2 & -2 & -2 \\ \circ & -\circ & -\circ & -\circ & -\circ \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix}, \\
 C_2 = C_{6,1} &: \begin{matrix} -8 & -2 & -2 & -2 & -2 \\ \circ & -\circ & -\circ & -\circ & -\circ \\ u_6 & u_7 & u_8 & u_9 & u_{10} \end{matrix}, \\
 C_3 = C_{2,1} &: \begin{matrix} -4 \\ \circ \\ u_{11} \end{matrix}, \quad C_4 = C_{2,1} : \begin{matrix} -4 \\ \circ \\ u_{12} \end{matrix},
 \end{aligned}$$

where C_1 consists of the proper transforms of E_2, E_3, E_4, E_5, E_6 , and C_2, C_3, C_4 contain the proper transforms of E_{16}, E_{11}, E_{19} , respectively.

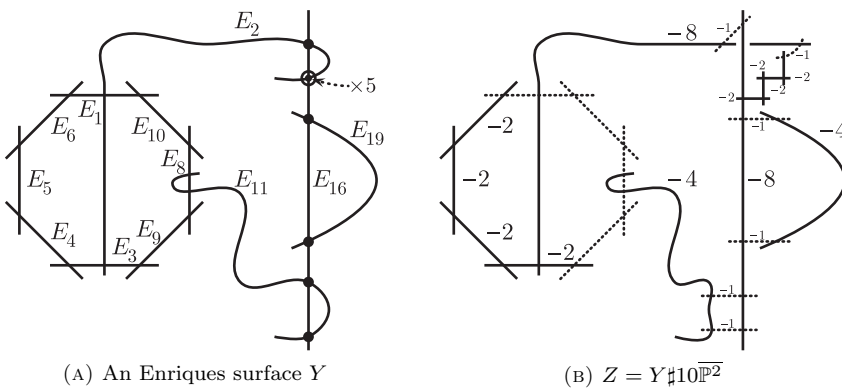


FIGURE 18. Another example with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$

We contract the four linear chains C_1, C_2, C_3, C_4 from Z . Then we obtain a projective singular surface X with four singularities of class T . By applying the same method to the singular surface X as in the previous sections, one can show

that a general fiber X_t of a \mathbb{Q} -Gorenstein smoothing of X is a minimal surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$.

Remark 6.2. By applying the same method used in the proof of Proposition 3.2, we can show that there is a \mathbb{Q} -Gorenstein smoothing of the singular surface X such that the canonical divisor K_{X_t} of a general fiber X_t of the \mathbb{Q} -Gorenstein smoothing of X is *not* ample.

Remark 6.3. As in the previous subsection, let us contract one more (-2) -curve \widetilde{E}_{10} as well as four linear chains C_1, C_2, C_3, C_4 from Z . Let $f : Z \rightarrow X'$ be the contraction. One can show that there is a global \mathbb{Q} -Gorenstein smoothing of X' and that its general fiber X'_t is a minimal surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/4\mathbb{Z}$. However, we cannot conclude that $K_{X'}$ is ample because, letting e be the exceptional divisor connecting u_1 and u_6 , we have $f^*K_{X'} \cdot e = 0$. We don't know whether or not the canonical divisor $K_{X'_t}$ of a general fiber X'_t is ample. We leave this problem for future research.

7. APPENDIX: AN EXAMPLE WITH $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

As mentioned in the introduction, minimal complex surfaces with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ have been constructed in works of Inoue [7] and Keum [8] by the classical method: Quotient by group action. In this appendix we construct such a surface by a \mathbb{Q} -Gorenstein smoothing method. We briefly explain how to construct such an example, and we sketch how to prove $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ because almost all proofs are basically the same as the case of $H_1 = \mathbb{Z}/4\mathbb{Z}$.

Construction. We begin with the same Enriques surface Y used in Section 3 for constructing an example with $H_1 = \mathbb{Z}/4\mathbb{Z}$. However, we use another bisection E_7 instead of the bisection E_{11} ; see Figure 19(A). We blow up the four marked points \bullet and we blow up again 3 times at the marked point \odot . The blown-up surface $Z = Y \# 7\overline{\mathbb{P}^2}$ has four disjoint linear chains of rational curves as in Figure 19(B):

$$C_1 : \overset{-5}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}, \quad C_2 : \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}, \quad C_3 : \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ},$$

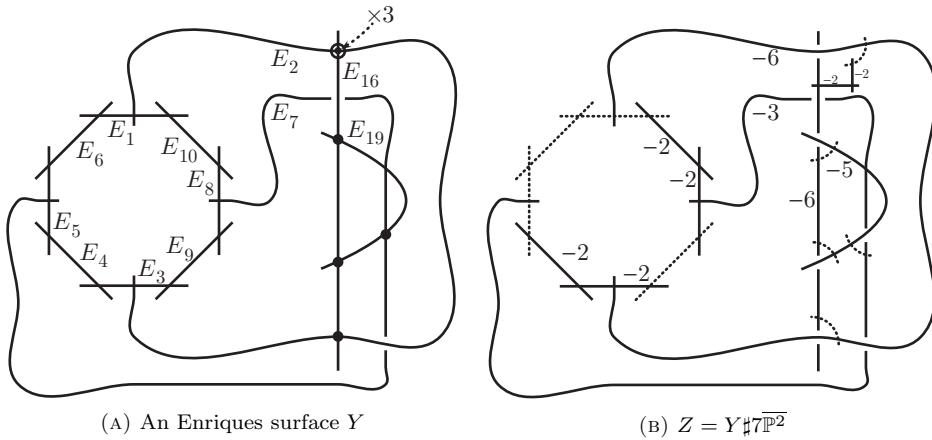
where C_1 consists of the proper transforms of E_{19}, E_7, E_8, E_{10} and C_2 consists of the proper transforms of E_2, E_3, E_4 . The chain C_3 contains the proper transforms of E_{16} .

By contracting these three disjoint chains from Z , we get a projective surface X with three singularities of class T . Then, by the same argument used in Section 3 and Section 4, we see that the surface X has a \mathbb{Q} -Gorenstein smoothing and that a general fiber X_t of the \mathbb{Q} -Gorenstein smoothing for X is a minimal complex surface of general type with $p_g = 0$ and $K^2 = 2$.

Proposition 7.1. $H_1(X_t, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. Let \overline{Z} be the rational blow-down 4-manifold obtained from Z by replacing the C_i 's with the corresponding Milnor fibers, respectively. As in the case of the proof of $H_1 = \mathbb{Z}/4\mathbb{Z}$, it is enough to show that $H_1(\overline{Z}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We decompose $Z = Z_0 \cup \{C_1 \cup C_2 \cup C_3\}$. Applying the same proof of Proposition 5.11 to this case, one can show that $H_1(\overline{Z}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We consider the loop β on the curve T/σ constructed in §5.1. In Lemma 5.10 we proved that if $2\widetilde{\beta} \neq 0$, then $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$. By a similar argument, one can show that if $2\widetilde{\beta} = 0$, then $H_1(Z_0; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.



(A) An Enriques surface Y (B) $Z = Y \# 7\overline{\mathbb{P}^2}$

FIGURE 19. An example with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In order to prove $\widetilde{2\beta} = 0$, we consider the real 2-dimensional surface U'_T constructed in Lemma 5.9. Since the proper transform of E_{16} is removed to obtain Z_0 , one may apply the same proof of equation (5.2) to our case so that we get

$$\gamma^* + \gamma'^* = 0.$$

On the other hand, we use E_7 instead of E_{11} to construct the linear chains C_i . Since E_7 does not intersect with E_{12} , we have only one α and γ'^* as a boundary in Figure 17(B). Hence we have

$$\gamma'^* + \bar{\alpha} = 0.$$

Since $\widetilde{2\beta} = \gamma^* + \bar{\alpha}$ by Lemma 5.7, combining these relations we have $\widetilde{2\beta} = 0$. \square

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