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GEOMETRY AND MARKOFF'S SPECTRUM FOR $\mathbb{Q}(i)$, I

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ABSTRACT. We develop a study of the relationship between geometry of geodesics and Markoff's spectrum for $\mathbb{Q}(i)$. There exists a particular immersed totally geodesic twice punctured torus in the Borromean rings complement, which is a double cover of the once punctured torus having Fricke coordinates $(2\sqrt{2}, 2\sqrt{2}, 4)$. The set of the simple closed geodesics on this once punctured torus is decomposed into two subsets. The discrete part of Markoff's spectrum for $\mathbb{Q}(i)$ (except for one) is given by the maximal Euclidean height of the lifts of the simple closed geodesics composing one of the subsets.

1. INTRODUCTION

Let $f(x,y) = ax^2 + bxy + cy^2$ be a binary indefinite quadratic form with real coefficients and with *discriminant* $D(f) = b^2 - 4ac$. We define

$$m(f) = \inf_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} |f(x,y)|.$$

The set

$$\mathcal{M} = \left\{ \sqrt{D(f)}/m(f) \,\middle|\, (a,b,c) \in \mathbb{R}^3, D(f) > 0 \right\}$$

is called the *Markoff spectrum* for the rational number field $\mathbb{Q}$.

Let $\mathbb{Q}(i)$ denote the imaginary quadratic number field whose ring of integers is the set of Gaussian integers $\mathbb{Z}[i]$. The Markoff spectrum for $\mathbb{Q}(i)$ can be defined in the same way: for $f(x,y) = ax^2 + bxy + cy^2$

$$\mathcal{M}_1 = \left\{ \sqrt{|D(f)|}/m_1(f) \,\middle|\, (a,b,c) \in \mathbb{C}^3, D(f) \neq 0 \right\},$$

where $m_1(f) = \inf_{(x,y)\in\mathbb{Z}[i]^2-\{(0,0)\}} |f(x,y)|$.

In this paper we develop a study of the relationship between these spectra and geometry of geodesics.

We begin by giving a summary of the study of the Markoff spectrum for $\mathbb{Q}$, in particular, its characterization by geodesics on a once punctured torus. This becomes a model of our study of the Markoff spectrum for $\mathbb{Q}(i)$.

Markoff triples are triples of integers (p,q,r) satisfying Markoff's equation $p^2 + q^2 + r^2 = 3pqr$. Here we suppose $1 \le p \le q \le r$. The set of all Markoff triples is obtained by building the infinite binary tree: starting with $(1,1,1)$, the two children of any node (p,q,r) is defined by $(p,r,3pr-q)$ and $(q,r,3qr-p)$ (exceptionally, $(1,1,1)$ and $(1,1,2)$ have only one child). The set of *Markoff numbers* $\mathcal{K} = \{1,2,5,13,29,\cdots\}$ which appear in these triples allows us to describe the discrete part of the Markoff spectrum for $\mathbb{Q}$: $\mathcal{M} \cap [0,3) = \{\sqrt{9-(4/k^2)} \mid k \in \mathcal{K}\}$. It

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is known that a minimal form attaining $\sqrt{9-(4/r^2)}$ is made from a Markoff triple (p,q,r). (See [27], [28], and also [17].)

Let $\mathbb{H}^2 = \{z = x+iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $ds^2 = (dx^2+dy^2)/y^2$. Recall that a geodesic in $\mathbb{H}^2$ is a semicircle or a ray perpendicular to the real axis. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ as fractional linear transformations. It also acts on its boundary $\mathbb{R} \cup \{\infty\}$. We always identify an element $g \in \mathrm{SL}(2,\mathbb{R})$ with the fractional linear transformation induced by g. If g fixes a unique point on the boundary of $\mathbb{H}^2$, it is called *parabolic* and if it fixes a pair of distinct points on the boundary, it is called *hyperbolic*. The geodesic in $\mathbb{H}^2$ fixed by a hyperbolic element g is called the *axis* of g.

Generally, if two elements g and h in $\mathrm{SL}(2,\mathbb{R})$ are hyperbolic, their axes intersect, and $h^{-1}g^{-1}hg$ is parabolic, then the quotient space of $\mathbb{H}^2$ by a Fuchsian group $\langle g,h\rangle$ is identified with a once punctured torus. Setting $X = \mathrm{tr}(g)$, $Y = \mathrm{tr}(h)$, and $Z = \mathrm{tr}(gh)$, we know g and h are hyperbolic, their axes intersect, and $h^{-1}g^{-1}hg$ is parabolic if and only if they satisfy $X^2+Y^2+Z^2 = XYZ$ and all of X,Y,Z are greater than 2. Hence, the triples (X,Y,Z) satisfying these conditions give coordinates of the Teichmüller space of the once punctured torus. This was first studied by R. Fricke in [20] (see also [23]); $X^2+Y^2+Z^2 = XYZ$ is called Fricke's moduli equation.

Let $\langle A_3, B_3\rangle$ be a free group generated by

$$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}. \tag{1.1}$$

It has Fricke coordinates $(3,3,3)$ and the quotient space $\mathbb{H}^2/\langle A_3,B_3\rangle$ is a once punctured torus denoted by $\mathbb{T}_{\langle A_3,B_3\rangle}$.

Observing the similarity between Fricke's moduli equation and Markoff's one (setting $X = 3x$, $Y = 3y$, and $Z = 3z$ in Fricke's, we get Markoff's), H. Cohn began geometric study of the Markoff spectrum for $\mathbb{Q}$. Recall that a matrix g is a *generator* of $\langle A_3,B_3\rangle$ if there is another matrix h which generates $\langle A_3,B_3\rangle$ together with g. Since generators of $\langle A_3,B_3\rangle$ have some special properties, defining a bijection between the set of Markoff numbers and a set of generators of $\langle A_3,B_3\rangle$, some theorems are obtained.

We define a form for a matrix $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ given by the fixed point equation of the action of N: $f_N(x,y) = cx^2+(d-a)xy-by^2$.

Theorem 1.1 ([11]). *For any μ in the discrete part of $\mathcal{M}$, there exists a generator N of $\langle A_3,B_3\rangle$ such that $\mu = \sqrt{D(f_N)}/m(f_N)$ and $m(f_N) = f_N(1,0)$.*

A geodesic on a quotient space of $\mathbb{H}^2$ is defined as the projection of a geodesic in $\mathbb{H}^2$. We say a geodesic is *simple* if it has no self-intersections.

Theorem 1.2 ([12]). *A geodesic γ in $\mathbb{H}^2$ is the axis of a generator of $\langle A_3,B_3\rangle$ if and only if γ projects to a simple closed geodesic on the once punctured torus $\mathbb{T}_{\langle A_3,B_3\rangle}$.*

The *Euclidean height* of a geodesic γ in $\mathbb{H}^2$ with endpoints η and ξ is defined as $|\eta-\xi|/2$ if η and ξ are finite or ∞ otherwise. Denote this by $h_E(\gamma)$. Let γ_{f_N} be a geodesic whose endpoints are the solutions of $f_N(x,1) = 0$, where f_N is a minimal form in Theorem 1.1.

Theorem 1.3 ([12]). *The Euclidean height of γ_{f_N} attains the maximum of the set $\{h_E(g(\gamma_{f_N})) \mid g \in \langle A_3, B_3\rangle\}$ and $\sqrt{D(f_N)}/m(f_N) = 2h_E(\gamma_{f_N})$.*

Finally, a theorem is deduced from these results:

Theorem 1.4 (Cohn; see also [22]). *The discrete part of the Markoff spectrum for $\mathbb{Q}$ is given by the twice maximal Euclidean height of the lifts of the simple closed geodesics on $\mathbb{T}_{\langle A_3,B_3\rangle}$.*

Since [11] such a geometric study of the Markoff spectrum for $\mathbb{Q}$ has been developed by Cohn himself and by several authors (see [12], [15], [33], [24], [35], [8], [21], [34], etc.). The generators of $\langle A_3, B_3\rangle$ which satisfy Theorems 1.1 and 1.2 are obtained by inductively building an infinite binary tree the nodes of which are triples of matrices in $\langle A_3, B_3\rangle$ (see [13]). How to select an initial triple of matrices is important, because it must correspond to a good homotopy basis of $\mathbb{T}_{\langle A_3,B_3\rangle}$.

There exist geometric studies of the other Markoff type spectra. For example, L.Ya. Vulakh proposed in [38] and [39], using the Klein model of the hyperbolic plane, a method of determining infinite binary trees of triples of matrices concerning the Markoff spectrum for a Fuchsian group.

The structure of the discrete part of the Markoff spectrum for $\mathbb{Q}(i)$ is investigated in [37] by Vulakh using binary complex quadratic forms. This is also studied by A. Schmidt from the point of view of Diophantine approximation of complex numbers based on the concept of the Farey tessellation ([31]). After these works, in this paper we call the discrete part of $\mathcal{M}_1$ the *Vulakh-Schmidt (VS) spectrum*. Our aim is to give a geometric interpretation of this spectrum analogous to Theorem 1.4.

Vulakh-Schmidt (VS) quadruples are quadruples of positive integers $(x_1, x_2; y_1, y_2)$ satisfying Vulakh's system of equations introduced in [37]:

$$\begin{cases} x_1 + x_2 &= 2y_1y_2, \\ 2x_1x_2 &= y_1^2 + y_2^2. \end{cases} \tag{1.2}$$

The set of all VS quadruples is obtained by building an infinite ternary tree (see §5.1). Let $\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, \cdots\}$ be the set of x_1 and x_2 occurring in the VS quadruples and let $\mathcal{N}(M) = \{1, 3, 11, 17, 41, \cdots\}$ be the set of y_1 and y_2 occurring in them. The VS spectrum is described in the following way (see [37] and [31]):

$$\left\{\sqrt{4 - \frac{1}{\lambda^2}} \;\middle|\; \lambda \in \mathcal{N}(\Lambda)\right\} \cup \left\{\sqrt{\frac{3}{5}\sqrt{41}}\right\}. \tag{1.3}$$

Let $\mathbb{H}^3 = \{z + jt \mid z = x + iy \in \mathbb{C}, t > 0\}$ be the upper half-space endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. Here we identify $(x, y, t) \in \mathbb{R}^3$ with the quaternion $x + iy + jt$. The group $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\}$ as fractional linear transformations and acts on $\mathbb{H}^3$ as their Poincaré extensions.

The Picard group $\mathrm{SL}(2, \mathbb{Z}[i])$ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ with finite covolume. There are various subgroups of the Picard group leading to quotient spaces of $\mathbb{H}^3$ related to links. Some of these are described in [40]. We focus our attention on the Borromean rings complement. It is obtained by gluing together two hyperbolic regular ideal octahedra. We define its geometric model in §3. It is represented as a quotient space $\mathbb{H}^3/\Gamma$, where Γ is a torsion-free normal subgroup of the Picard group with index 24.

We now introduce another once punctured torus. Let $\langle A_4, B_4\rangle$ be a free group generated by

$$A_4 = \begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B_4 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}. \tag{1.4}$$

This group has Fricke coordinates $(2\sqrt{2}, 2\sqrt{2}, 4)$ and the quotient space $\mathbb{H}^2/\langle A_4, B_4\rangle$ is a once punctured torus denoted by $\mathbb{T}_{\langle A_4,B_4\rangle}$. This is involved in representing a particular twice punctured torus in the Borromean rings complement.

Theorem 1.5. *There exists a hypersurface $\mathcal{HS}$ in the geometric model of the Borromean rings complement which is an immersed totally geodesic twice punctured torus conformally equivalent to a double cover of $\mathbb{T}_{\langle A_4,B_4\rangle}$.*

This is suggested by the equations and the coordinates $(2\sqrt{2}, 2\sqrt{2}, 4)$. Vulakh's system of equations is obtained from Fricke's: setting $X = 4x$, $Y = 2\sqrt{2}y_1$, and $Z = 2\sqrt{2}y_2$, we get $2x^2 - 4y_1y_2x + (y_1^2 + y_2^2) = 0$, which is equivalent to Vulakh's (see [38] and [39]). It is also obtained from the Nakanishi-Näätänen's equation (see [29]) which parametrizes the Teichmüller space of the twice punctured torus:

$$\frac{d}{ae} + \frac{c}{be} + \frac{e}{bc} + \frac{b}{ce} + \frac{a}{de} + \frac{e}{ad} = \alpha$$

by setting $e = 4x$, $a = c = 2\sqrt{2}y_1$, $b = d = 2\sqrt{2}y_2$, and $\alpha = 2$.

From the point of view of the minimum of quadratic forms, in [33] Schmidt generalized Theorem 1.4 to all of the once punctured tori with different complex structures and discussed the case of $\mathbb{T}_{\langle A_4,B_4\rangle}$ as an example. He used Vulakh's system of equations both in this example and in the study of the Markoff spectrum for $\mathbb{Q}(i)$ ([31]). This fact also suggests Theorem 1.5. A. Haas proved the same generalized result of Theorem 1.4 in [21] using geometric and topological arguments.

Theorem 1.5 is proved (see §2 and §4) conjugating $\langle A_4, B_4\rangle$ to $\Gamma_o = \langle A, B\rangle$, where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 5 & 7 \\ -1 & -1 \end{pmatrix} \text{ and } B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$$

and conjugating Γ_o to $\tilde{\Gamma}_o = \langle \tilde{A}, \tilde{B}\rangle$, where

$$\tilde{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 5-i & -3-3i \\ -2i & -1+i \end{pmatrix} \text{ and } \tilde{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 1-i \\ 2i & 3-i \end{pmatrix}.$$

Note that the group Γ_o is not included in the modular group and that the group $\tilde{\Gamma}_o$ is not included in the Picard group. However, two conformally equivalent surfaces of $\mathcal{HS}$ obtained by these conjugates are represented as quotient spaces by subgroups of the modular group and the Picard group (see §2 and §4).

In §5.2 we propose an algorithm which gives a way of building an infinite ternary tree, the nodes of which are quadruples of matrices in Γ_o (see Algorithm VS). We take an initial quadruple of matrices corresponding to a homotopy basis $\{B, BA\}$ of the once punctured torus $\mathbb{T}_o = \mathbb{H}^2/\Gamma_o$. We define two types of matrices Λ_λ and M_m related to $\lambda \in \mathcal{N}(\Lambda)$ and $m \in \mathcal{N}(M)$. The matrix Λ_λ is in $\mathrm{SL}(2,\mathbb{Z})$ and M_m is in $\mathrm{SL}(2,\mathbb{Z}/\sqrt{2})$, and they have special forms (see Definition 5.2). The quadruples of matrices in the tree consist of two Λ_λ's and two M_m's. These matrices are represented as words on $\{A, B\}$ and all of them are generators of Γ_o (see Proposition 6.1). Analogizing to Theorem 1.2, we can prove that the axes of Λ_λ and M_m project to simple closed geodesics on the once punctured torus $\mathbb{T}_o$; conversely, the simple closed geodesics on $\mathbb{T}_o$ must be the projection of the axes of Λ_λ, M_m and their equivalent matrices (see Theorem 6.4 and Corollary 6.8). By the form of matrices Λ_λ and M_m, we can directly compute the Euclidean height of their axes.

The matrices $\tilde{\Lambda}_\lambda$ and $\tilde{M}_m$ in $\tilde{\Gamma}_o$ are defined as the conjugates of Λ_λ and M_m (see §5.3). The matrix $\tilde{\Lambda}_\lambda$ is in $\mathrm{SL}(2,\mathbb{Z}[i])$ and $\tilde{M}_m$ is in $\mathrm{SL}(2,\mathbb{Z}[i]/\sqrt{2})$ (see Definition 5.10). They are represented as words on $\{\tilde{A},\tilde{B}\}$. All of them are generators of $\tilde{\Gamma}_o$ (see Proposition 6.9). The results on Λ_λ and M_m are translated to those on $\tilde{\Lambda}_\lambda$ and $\tilde{M}_m$. We are thus led to a theorem.

Theorem 1.6. *For any μ in $\left\{\sqrt{4-1/\lambda^2}\,\middle|\,\lambda\in\mathcal{N}(\Lambda)\right\}$, we constructively obtain a matrix $\tilde{\Lambda}_\lambda$ such that μ is equal to the twice Euclidean height of the axis of $\tilde{\Lambda}_\lambda$ and the axis projects to a simple closed geodesic on the hypersurface $\mathcal{HS}$ in the Borromean rings complement.*

As an analogue of Theorem 1.4, we have

Theorem 1.7. *Let $\mathcal{G}(\Lambda)$ be a set of simple closed geodesics on $\mathcal{HS}$ whose lifts are the axes of $\tilde{\Lambda}_\lambda$, $\lambda\in\mathcal{N}(\Lambda)$ and their equivalents. The VS spectrum (except for one) is given by the twice maximal Euclidean height of the lifts of the geodesics in $\mathcal{G}(\Lambda)$.*

We also obtain the following result as a by-product of building the ternary tree, the nodes of which are quadruples of matrices. The set $\lambda\in\mathcal{N}(\Lambda)$ gives a sequence of the Markoff spectrum $\mathcal{M}$; the Euclidean height of the axis of Λ_λ, $\lambda\in\mathcal{N}(\Lambda)$, corresponds to a value of $\mathcal{M}$.

Theorem 1.8. *For each $\mu=\sqrt{16-(4/\lambda^2)}$, $\lambda\in\mathcal{N}(\Lambda)$, there exists a matrix Λ_λ in the tree such that $\mu=\sqrt{D(f_{\Lambda_\lambda})}/m(f_{\Lambda_\lambda})$ and $m(f_{\Lambda_\lambda})=f_{\Lambda_\lambda}(1,0)$. Moreover, $\mu=2h_E(\gamma_{f_{\Lambda_\lambda}})$, where f_{Λ_λ} is defined by the fixed point equation of the action of Λ_λ and $\gamma_{f_{\Lambda_\lambda}}$ is a geodesic whose endpoints are its fixed points.*

Since the value of $\mathcal{M}$ in this theorem is larger than 3, the sequence is included in the non-discrete part of $\mathcal{M}$. We then conclude from Theorem 1.4 that the axis of Λ_λ cannot project to a simple closed geodesic on the once punctured torus $\mathbb{T}_{\langle A_3,B_3\rangle}$, while it projects to a simple closed one on $\mathbb{T}_o$.

The Markoff spectrum on sublattices of $\mathbb{Z}^2$, which is a subset of the classical Markoff spectrum $\mathcal{M}$, has been studied. For definition and results, we refer the reader to [26] (also to [39] and [32]). The sequence in Theorem 1.8 coincides with such a spectrum on two disjoint (even and odd) sublattices of $\mathbb{Z}^2$. This suggests that we need the twice punctured torus to geometrically characterize the VS spectrum (the two sublattices correspond to the two punctures; see §6). It is asked in [26] why the Markoff spectrum on two disjoint sublattices of $\mathbb{Z}^2$ coincides with the VS spectrum (except for one) multiplied by two. The fact reflects some geometric regularity: the matrices which give geodesics corresponding to these spectra also coincide with each other up to conjugacy.

The paper is organized in the following way. In §2 we introduce some models of the once and twice punctured tori we need and in §3 define our geometric model of the Borromean rings complement. §4 is devoted to a proof of Theorem 1.5. The infinite ternary trees, the nodes of which are VS quadruples, quadruples of Λ_λ's and M_m's, and quadruples of $\tilde{\Lambda}_\lambda$'s and $\tilde{M}_m$'s are defined in §5.1, §5.2, and §5.3, respectively. One of the most important parts of this paper is §7 in which we prove that these trees and the matrices are well defined. Using the properties of matrices Λ_λ and $\tilde{\Lambda}_\lambda$, Theorems 1.6 and 1.7 are proved in §6.

Before ending the introduction, we give some additional comments.

In this paper, after Cohn's idea (see the preliminary remarks of [14]), we use the designation "Markoff" to refer to A.A. Markoff (1856-1922) as the number theorist. His name is customarily spelled "Markov" as the probabilist.

We can regard a triple (X, Y, Z) satisfying Fricke's equation as a representation of the free group on two generators into $\mathrm{SL}(2,\mathbb{R})$ with the property that the commutator has trace equal to -2. Solving the equation over the complex numbers, we can consider representations into $\mathrm{SL}(2,\mathbb{C})$ having the same property. In this context, the equation is written by $x^2 + y^2 + z^2 = xyz$ and is called the "Markoff equation" (see [10] and [6]).

McShane's identity is an identity concerning the length of simple closed geodesics on a once punctured torus and is universal for all elements in the Teichmüller space of the once punctured torus. Together with the generalization of Theorem 1.4 by Haas and Schmidt, it is one of the most spectacular results about simple closed geodesics on a once punctured torus and its Teichmüller space.

What is interesting for us is Bowditch's alternative proof of McShane's identity using the tree of "Markoff triples". Since the traces of matrices give hyperbolic lengths of their axes, it can be interpreted as an identity concerning the traces of a representation. Hence, considering complex "Markoff triples", McShane's identity extends to that of quasifuchsian representations (see [9]). In this setting, we will discuss Vulakh's equation in a forthcoming paper.

2. Hecke groups, once and twice punctured tori

Let $\mathbb{H}^2 = \{z = x + iy \mid y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. A geodesic in $\mathbb{H}^2$ is a semicircle or a ray perpendicular to the real axis. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ as fractional linear transformations. It also acts on its boundary $\mathbb{R} \cup \{\infty\}$. We always identify an element $g \in \mathrm{SL}(2,\mathbb{R})$ with the fractional linear transformation induced by g. Recall that an element $g \in \mathrm{SL}(2,\mathbb{R})$ is *parabolic* if g has a unique fixed point on $\mathbb{R} \cup \{\infty\}$ and that g is *hyperbolic* if g has two fixed points on it. They are equivalent to $|\mathrm{tr}(g)| = 2$ and $|\mathrm{tr}(g)| > 2$, respectively.

Hecke groups are the groups generated by two elements

$$G_q = \left\langle \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

for integers $q \geq 3$. When $q = 3$, we simply have $\mathrm{SL}(2,\mathbb{Z})$. We use both G_3 and G_4 in our discussion.

Let $\langle A_3, B_3 \rangle$ be a free group generated by A_3 and B_3 of (1.1). These matrices are hyperbolic: the map A_3 sends ∞ to 1 and -1 to 0; the map B_3 sends ∞ to -1 and 1 to 0. Gluing the vertical ray $\mathrm{Re}(z) = -1$ to the semicircle sending 0 and 1 by A_3, and the vertical ray $\mathrm{Re}(z) = 1$ to the semicircle sending 0 and -1 by B_3, we obtain a fundamental domain of $\langle A_3, B_3 \rangle$. The quotient space $\mathbb{H}^2/\langle A_3, B_3 \rangle$ is thus a once punctured torus, denoted by $\mathbb{T}_{\langle A_3, B_3 \rangle}$. Using a well-known fundamental domain of the modular group, we can verify that the once punctured torus $\mathbb{T}_{\langle A_3, B_3 \rangle}$ is a six-fold cover of the modular surface. From the point of view of a group, $\langle A_3, B_3 \rangle$ is a torsion-free normal subgroup of the modular group with index 6.

Let $\langle A_4, B_4\rangle$ be a free group generated by A_4 and B_4 of (1.4). They are also hyperbolic: the map A_4 sends 0 to ∞ and $-1/\sqrt{2}$ to $-\sqrt{2}$; the map B_4 sends $-\sqrt{2}$ to ∞ and $-1/\sqrt{2}$ to 0. We obtain a fundamental domain of $\langle A_4, B_4\rangle$ by gluing the semicircle sending $-1/\sqrt{2}$ and 0 to the vertical ray $\mathrm{Re}(z) = -\sqrt{2}$ by A_4, and the semicircle sending $-\sqrt{2}$ and $-1\sqrt{2}$ to the vertical ray $\mathrm{Re}(z) = 0$ by B_4. Note that this domain is a four-fold cover of a typical fundamental domain F_4 of G_4:

$$F_4 = \left\{ z = x + iy \in \mathbb{H}^2 \,\middle|\, |z| \geq 1,\ -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \right\}.$$

The quotient space $\mathbb{H}^2/\langle A_4, B_4\rangle$ is also a once punctured torus, denoted by $\mathbb{T}_{\langle A_4,B_4\rangle}$. In this case, $\langle A_4, B_4\rangle$ is a torsion-free normal subgroup of the Hecke group G_4 with index 4.

Consider a free group $\langle g, h\rangle$ generated by $g, h \in \mathrm{SL}(2,\mathbb{R})$, assuming that g and h are hyperbolic, their axes intersect, and the commutator $[h, g]$ is parabolic. In general, the quotient space $\mathbb{H}^2/\langle g, h\rangle$ is a once punctured torus. Set $X = \mathrm{tr}(g)$, $Y = \mathrm{tr}(h)$, and $Z = \mathrm{tr}(gh)$. The condition that g and h are hyperbolic, their axes intersect, and $[h, g]$ is parabolic is equivalent to the condition that they satisfy $X^2 + Y^2 + Z^2 = XYZ$ and all of X, Y, Z are greater than 2. Such triples (X, Y, Z), therefore, provide coordinates of the Teichmüller space of the once punctured torus ([20], [23]). Note that $\langle A_3, B_3\rangle$ is represented by $(3, 3, 3)$ and $\langle A_4, B_4\rangle$ by $(2\sqrt{2}, 2\sqrt{2}, 4)$.

If we abelianize such a group $\langle g, h\rangle$, then the commutator $[h, g]$ becomes the identity (geometrically, the cusp of the quotient space disappears). We thus have the closed torus corresponding to a once punctured torus. Let 1, $\rho = e^{\frac{2}{3}\pi i}$, and i denote, respectively, the following three translations on $\mathbb{C}$: $z \mapsto z+1$, $z \mapsto z+\rho$, and $z \mapsto z+i$. The groups $\langle 1, \rho\rangle$ and $\langle 1, i\rangle$ are abelian; the quotient spaces $\mathbb{C}/\langle 1, \rho\rangle$ and $\mathbb{C}/\langle 1, i\rangle$ are closed flat tori. The former is a torus consisting of two regular triangles and the latter is by a square. We can define an explicit conformal mapping φ_3 from $\mathbb{T}_{\langle A_3,B_3\rangle}$ to $\mathbb{C}/\langle 1, \rho\rangle$ so that it gives identification of A_3 with ρ and of B_3 with 1. We can also define an explicit conformal mapping φ_4 between $\mathbb{T}_{\langle A_4,B_4\rangle}$ and $\mathbb{C}/\langle 1, i\rangle$, identifying A_4 with i and B_4 with 1. (See [11], [16], and [1].) The construction of these conformal mappings means the abelianization of the groups $\langle A_3, B_3\rangle$ and $\langle A_4, B_4\rangle$. The identification of generators given by φ_3 and φ_4 allows us to conclude that the once punctured tori $\mathbb{T}_{\langle A_3,B_3\rangle}$ and $\mathbb{T}_{\langle A_4,B_4\rangle}$ possess the highest and the second highest degree of symmetry in the Teichmüller space of the once punctured torus.

Let $\langle P_4, Q_4, R_4\rangle$ be a group generated by

$$P_4 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix},\quad Q_4 = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{pmatrix},\quad R_4 = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix}.$$

These matrices are hyperbolic and satisfy the following relations:

$$P_4 = B_4A_4^{-1},\ \ Q_4 = A_4^{-1}B_4^{-1},\ \ R_4 = B_4^2,\ \ Q_4P_4 = A_4^{-2}. \tag{2.1}$$

We show a fundamental domain of $\langle P_4, Q_4, R_4\rangle$. In Figure 1 we set

$$\begin{gathered} p_0 = \infty,\ p_1 = -2\sqrt{2},\ p_2 = -\sqrt{2},\ p_3 = -\frac{1}{\sqrt{2}}, \\ p_4 = 0,\ p_5 = \frac{1}{\sqrt{2}},\ p_6 = \sqrt{2},\ p_7 = 2\sqrt{2}. \end{gathered} \tag{2.2}$$

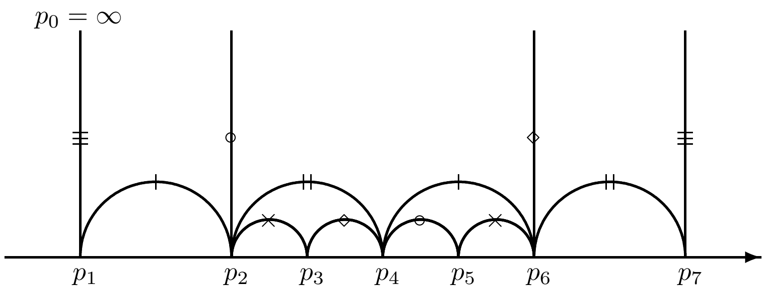

FIGURE 1. Two fundamental domains of $\langle P_4, Q_4, R_4\rangle$

Observe how P_4, Q_4, and R_4 send points:

$$P_4 : p_0 \mapsto p_5,\ p_1 \mapsto p_6,\ p_2 \mapsto p_4,\quad Q_4 : p_0 \mapsto p_3,\ p_6 \mapsto p_4,\ p_7 \mapsto p_2,\\ R_4 : p_2 \mapsto p_6,\ p_3 \mapsto p_5. \tag{2.3}$$

We can glue the vertical ray $(p_2 \to p_0)$ to the semicircle $(p_4 \to p_5)$ by P_4, the vertical ray $(p_6 \to p_0)$ to the semicircle $(p_4 \to p_3)$ by Q_4, and the semicircle $(p_2 \to p_3)$ to the one $(p_6 \to p_5)$ by R_4. A fundamental domain of $\langle P_4, Q_4, R_4\rangle$ is thus obtained (see Figure 1). The quotient space $\mathbb{H}^2/\langle P_4, Q_4, R_4\rangle$ is then a twice punctured torus, denoted by $\mathbb{T}_{\langle P_4,Q_4,R_4\rangle}$. Note that in this setting the fundamental domain of $\langle A_4, B_4\rangle$ introduced above is obtained by gluing the vertical ray $(p_2 \to p_0)$ to the semicircle $(p_3 \to p_4)$ by B_4, and the vertical ray $(p_4 \to p_0)$ with the semicircle $(p_3 \to p_2)$ by A_4.

Proposition 2.1. *The twice punctured torus $\mathbb{T}_{\langle P_4,Q_4,R_4\rangle}$ is a two-fold cover of the once punctured torus $\mathbb{T}_{\langle A_4,B_4\rangle}$. In other words, the group $\langle P_4, Q_4, R_4\rangle$ is a subgroup of $\langle A_4, B_4\rangle$ with index* 2. *The group $\langle P_4, Q_4, R_4\rangle$ is, therefore, a subgroup of G_4 with index* 8.

Proof. These are deduced from (2.3) and (2.1). □

Let us show another fundamental domain of $\langle P_4, Q_4, R_4\rangle$. It follows from (2.3) that the geodesic going from p_1 to p_2 is glued to the one from p_6 to p_4 by P_4 and that the geodesic going from p_6 to p_7 is glued to the one from p_4 to p_2 by Q_4. Moreover, the vertical ray $\mathrm{Re}(z) = p_1$ is glued to the one $\mathrm{Re}(z) = p_7$ by the map

$$Q_4^{-1} R_4^{-1} P_4 = B_4 A_4 B_4^{-1} A_4^{-1} = \begin{pmatrix} -1 & -4\sqrt{2} \\ 0 & -1 \end{pmatrix}.$$

We thus obtain another fundamental domain of $\langle P_4, Q_4, R_4\rangle$ (see Figure 1).

We now conjugate $\langle A_4, B_4\rangle$ and $\langle P_4, Q_4, R_4\rangle$ by a matrix $U = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{pmatrix}$. The conjugate of the former is a group $\Gamma_o = \langle A, B\rangle$ generated by

$$A = UA_4U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 5 & 7 \\ -1 & -1 \end{pmatrix} \quad\text{and}\quad B = UB_4U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}; \tag{2.4}$$

that of the latter is a group $\Gamma_t = \langle P, Q, R\rangle$ generated by

$$P = UP_4U^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}, \quad Q = UQ_4U^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix},$$

$$R = UR_4U^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Note that U was taken so that Γ_t becomes a subgroup of the modular group (see Proposition 2.2).

Since the translation by U is conformal, the quotient space $\mathbb{T}_o = \mathbb{H}^2/\Gamma_o$ is conformally equivalent to $\mathbb{T}_{\langle A_4,B_4\rangle}$, and $\mathbb{T}_t = \mathbb{H}^2/\Gamma_t$ is to $\mathbb{T}_{\langle P_4,Q_4,R_4\rangle}$. In Figure 1 we now set

$$p_0 = \infty,\ p_1 = -5,\ p_2 = -3,\ p_3 = -2,\ p_4 = -1,\ p_5 = 0,\ p_6 = 1,\ p_7 = 3.$$

These are the images by U of the points in (2.2). Using the same gluing patterns as before, we obtain representation of the quotient spaces $\mathbb{T}_o$ and $\mathbb{T}_t$.

Proposition 2.2. *The group Γ_t is a subgroup of the modular group with index* 12.

Proof. We can directly verify that a fundamental domain of Γ_t is a 12-fold cover of the well-known fundamental domain of the modular group. □

3. Model of the Borromean rings complement

Let $\mathbb{H}^3 = \{z + jt \mid z = x + iy \in \mathbb{C}, t > 0\}$ be the upper half-space endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. Here we identify $(x, y, t) \in \mathbb{R}^3$ with the quaternion $x + iy + jt$ (for this expression, we refer the reader to [7]). A geodesic in $\mathbb{H}^3$ is a semicircle or a ray perpendicular to the complex plane $\mathbb{C}$. The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}\cup\{\infty\}$ as fractional linear transformations and acts on $\mathbb{H}^3$ as their Poincaré extensions. Note that $\mathbb{C}\cup\{\infty\}$ is the boundary of $\mathbb{H}^3$. We always identify an element $g \in \mathrm{SL}(2,\mathbb{C})$ with the fractional linear transformation and its Poincaré extension induced by g. Recall that an element $g \in \mathrm{SL}(2,\mathbb{C})$ is *parabolic* if g has a unique fixed point on the boundary of $\mathbb{H}^3$ and g is *loxodromic* if g has two fixed points on it. The former is equivalent to $\mathrm{tr}^2(g) = 4$. Loxodromic elements split into two cases: g is *hyperbolic* if $\mathrm{tr}^2(g) \in (4, +\infty)$; g is *strictly loxodromic* if $\mathrm{tr}^2(g) \notin [0, +\infty)$ ([7]).

The group defined as

$$\mathrm{SL}(2,\mathbb{Z}[i]) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \alpha,\beta,\gamma,\delta \in \mathbb{Z}[i],\ \alpha\delta - \beta\gamma = 1 \right\}$$

is called the *Picard group*, where $\mathbb{Z}[i]$ is the set of Gaussian integers. The Picard group is a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ with finite covolume. A fundamental region of $\mathrm{SL}(2,\mathbb{Z}[i])$ is described as follows:

$$\left\{ x + iy + jt \in \mathbb{H}^3 \middle| x^2 + y^2 + t^2 \geq 1,\ |x| \leq \frac{1}{2},\ 0 \leq y \leq \frac{1}{2} \right\}.$$

It has a single parabolic vertex at ∞. It is also known that the Picard group is generated by the four elements:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(see [18]). Note that S and T generate $\mathrm{SL}(2,\mathbb{Z})$.

Analogous to the Farey tessellation (i.e., a tiling of the upper half-plane by ideal regular hyperbolic triangles), the upper half-space $\mathbb{H}^3$ is tiled by the images of a hyperbolic regular ideal octahedron under the action of the Picard group.

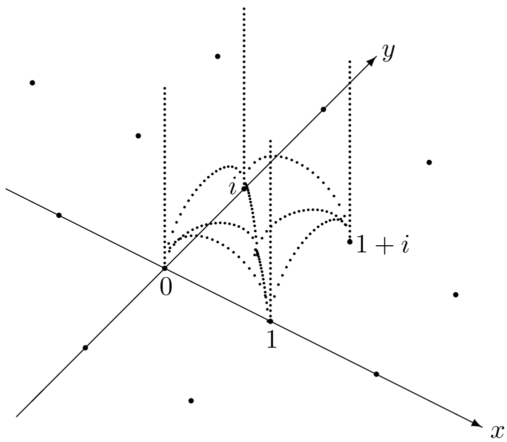

FIGURE 2. Hyperoblic regular ideal octahedron Δ in $\mathbb{H}^3$

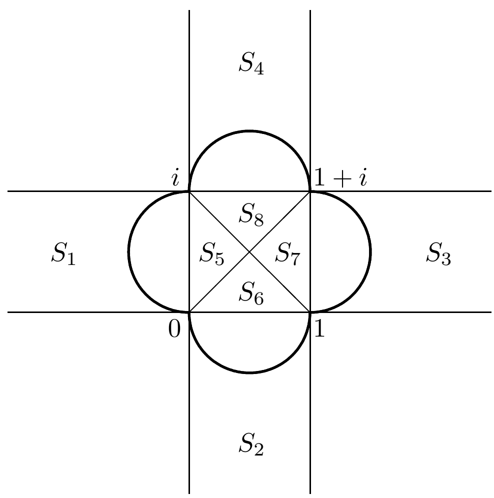

FIGURE 3. Illustration of the hyperbolic regular ideal octahedron Δ. The upper faces are hyperbolic triangles in the vertical planes $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The lower faces are isosceles triangles on the hemispheres of the radius $1/2$ with the centers $1/2$, $i/2$, $U(1/2)$, and $T(i/2)$.

We introduce some parts of planes and hemispheres in $\mathbb{H}^3$:

$$S_1 = \left\{x + iy + jt \in \mathbb{H}^3 \,\middle|\, x^2 + \left(y - \tfrac{1}{2}\right)^2 + t^2 \geq \tfrac{1}{4}, x = 0, 0 \leq y \leq 1\right\},$$

$$S_2 = \left\{x + iy + jt \in \mathbb{H}^3 \,\middle|\, \left(x - \tfrac{1}{2}\right)^2 + y^2 + t^2 \geq \tfrac{1}{4}, y = 0, 0 \leq x \leq 1\right\},$$

$$S_5 = \left\{x + iy + jt \in \mathbb{H}^3 \,\middle|\, x^2 + \left(y - \tfrac{1}{2}\right)^2 + t^2 = \tfrac{1}{4}, x + y \leq 1, x - y \leq 0, x \geq 0\right\},$$

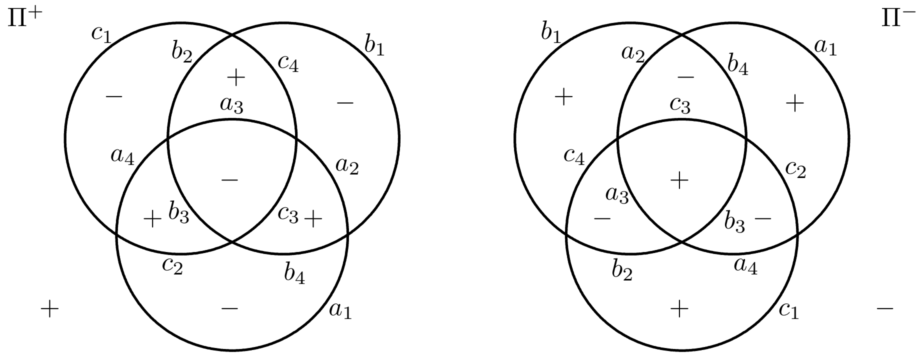

FIGURE 4. Two planar Borromean rings topologically regarded as two octahedra

$$S_6 = \left\{ x + iy + jt \in \mathbb{H}^3 \,\middle|\, \left(x - \tfrac{1}{2}\right)^2 + y^2 + t^2 = \tfrac{1}{4}, x + y \leq 1, x - y \geq 0, y \geq 0 \right\},$$

$$S_3 = TS_1, \quad S_4 = US_2, \quad S_7 = TULS_5, \quad S_8 = TULS_6.$$

Let Δ denote the hyperbolic regular ideal octahedron determined by the eight faces S_k, $k = 1$ to 8 (see a sketch in Figure 2). We call S_r, $r = 1$ to 4, *upper faces* of Δ and S_r, $r = 5$ to 8, *lower faces* of Δ. We sometimes illustrate Δ as in Figure 3. The tessellation of $\mathbb{H}^3$ by the hyperbolic regular ideal octahedra is defined as $\mathcal{F}^3 = \{g\Delta \mid g \in \mathrm{SL}(2, \mathbb{Z}[i])\}$.

Some examples of hyperbolic 3-manifolds of finite volume are obtained by gluing together a finite number of regular ideal polyhedra in $\mathbb{H}^3$ along their sides. In many cases, such an example is homeomorphic to the complement of a knot or link in the 3-ball. (See [36] and [30].) Here we recall a gluing pattern for the Borromean rings complement after [36], [3], [4], and [5].

The three rings on the left side of Figure 4 are a planar representation of the Borromean rings. Since they decompose the plane into eight regions, the picture is topologically regarded as an octahedron, denoted by Π^+. If we associate a $+$ sign to white and a $-$ sign to black in the figure, then Π^+ is 2-colored in a checkerboard fashion. Take an identical copy of Π^+, reverse all signs (and colors), and denote the resulting octahedron by Π^- (see the right side of Figure 4).

Taking the truncation of $\Pi^\pm$, we get the truncated octahedra bounded by six squares and eight hexagons, denoted by $\Pi_t^\pm$. They are illustrated in Figure 5. The six squares of $\Pi_t^\pm$ correspond to cuts of the vertices of $\Pi^\pm$. We attach labels a_l, b_l, and c_l ($l = 1$ to 4) to the four parts of each rings of Π^+, and to those of Π^- by rotating the three rings by $2\pi/3$. The same labels are attached to the corresponding edges of Π_t^+ and Π_t^-. (See Figures 4 and 5.) We also attach labels $\phi_r^\pm$ ($r = 1$ to 8) to the eight faces of $\Pi^\pm$ and $\Pi_t^\pm$. Each face $\phi_r^\pm$ in $\Pi^\pm$ is a triangle with sign allocation $\sigma_r^\pm$ (black or white). The signs of the faces are depicted in Figure 4 and the names of the faces $\phi_r^\pm$ are depicted in Figure 5. The correspondence between the faces of $\Pi^\pm$ and $\Pi_t^\pm$ are clearly obtained from the labels a_l, b_l, c_l of edges. The information for $\Pi^\pm$ and $\Pi_t^\pm$ are summarized in Table 1. Note that σ_r^+ and σ_r^- have opposite signs.

TABLE 1. The information for $\Pi^{\pm}$ and $\Pi_t^{\pm}$

r	σ_r^+	labels for the sides of ϕ_r^+	σ_r^-	labels for the sides of ϕ_r^-
1	$-$	$a_4b_2c_1$	$+$	$a_2b_1c_4$
2	$+$	$a_1b_1c_1$	$-$	$a_1b_1c_1$
3	$-$	$a_1b_4c_2$	$+$	$a_4b_2c_1$
4	$+$	$a_4b_3c_2$	$-$	$a_3b_2c_4$
5	$+$	$a_3b_2c_4$	$-$	$a_2b_4c_3$
6	$-$	$a_2b_1c_4$	$+$	$a_1b_4c_2$
7	$+$	$a_2b_4c_3$	$-$	$a_4b_3c_2$
8	$-$	$a_3b_3c_3$	$+$	$a_3b_3c_3$

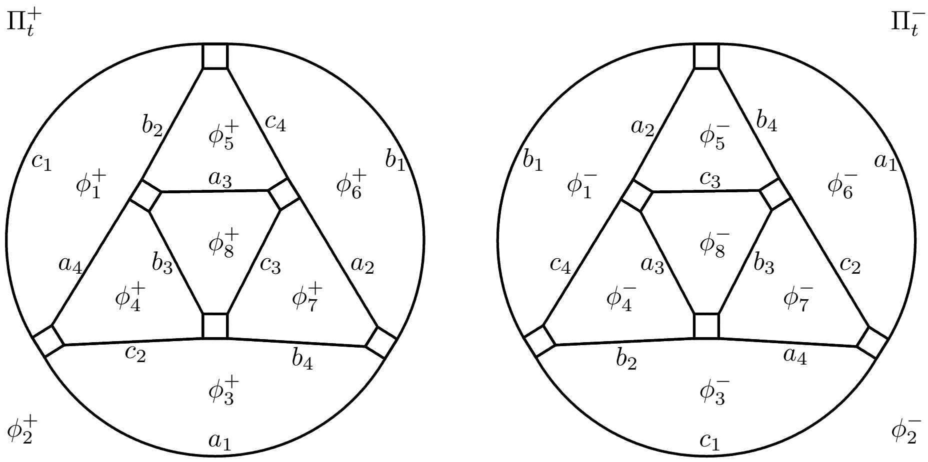

FIGURE 5. Two truncated octahedra corresponding to Π^+ and Π^-

We glue ϕ_r^+ with the corresponding face ϕ_r^- by a rotation of $2\pi\sigma_r^+/3$, where the sign ($+$ or $-$) means the clockwise or counterclockwise direction. The three edges of each triangle always have labels a_{*_1}, b_{*_2}, and c_{*_3}, where $*_k$, $k = 1, 2, 3$, stand for some integers between 1 and 4 (see Table 1). The gluing of ϕ_r^+ with ϕ_r^- by rotation is equivalent to gluing them together by identifying their edges with labels a_{*_1}, b_{*_2}, and c_{*_3}. For example, ϕ_1^+ and ϕ_1^- are glued by identifying a_4, b_2, and c_1 with a_2, b_1, and c_4, respectively.

We denote the resulting topological space by $\overline{M}_{BR}$. Denote also by M_{BR} the topological space deleting vertices from $\overline{M}_{BR}$. The space M_{BR} is equal to a topological space obtained by the same gluing of the truncated octahedra $\Pi_t^{\pm}$. Then it is known that M_{BR} is canonically homeomorphic to $S^3 - \mathcal{L}$, where $\mathcal{L}$ denotes the Borromean rings. The space M_{BR} is called the *Borromean rings complement*.

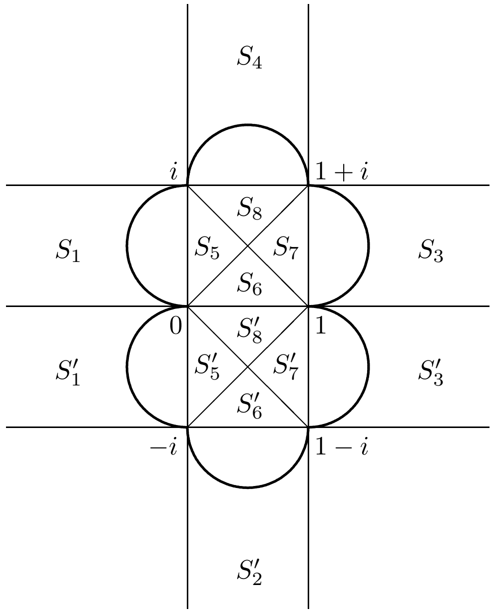

FIGURE 6. Two adjacent hyperbolic regular ideal octahedra Δ and $U^{-1}\Delta$. Here we use $S'_r = U^{-1}S_r$ ($r = 1$ to 8).

We now make a geometric model of the Borromean rings complement from the two adjacent hyperbolic regular ideal octahedra Δ and $U^{-1}\Delta$. Denoting $U^{-1}S_r = S'_r$ ($r = 1$ to 8), we illustrate them as in Figure 6. Looking at the truncated octahedra $\Pi_t^{\pm}$ depicted in Figure 5, we take correspondence between each face S_r of Δ and each ϕ_r^- of Π_t^-. Then, the ideal vertices of Δ (∞, 0, 1, i, $1+i$, and $(1+i)/2$) correspond to six squares of Π_t^-. Comparing Figures 5 and 6, we get the correspondence illustrated in the upper part of Figure 7. From the gluing pattern for making M_{BR}, identifying ϕ_2^+ with ϕ_2^-, we obtain the following correspondence between the faces S'_r of $U^{-1}\Delta$ and ϕ_r^+ of Π_t^+:

$$\phi_1^+ \leftrightarrow S'_3,\ \ \phi_2^+ \leftrightarrow S_2 = S'_4,\ \ \phi_3^+ \leftrightarrow S'_8,\ \ \phi_4^+ \leftrightarrow S'_7,$$
$$\phi_5^+ \leftrightarrow S'_2,\ \ \phi_6^+ \leftrightarrow S'_1,\ \ \phi_7^+ \leftrightarrow S'_5,\ \ \phi_8^+ \leftrightarrow S'_6.$$

It is illustrated in the lower part of Figure 7. We now get the gluing pattern for Δ and $U^{-1}\Delta$:

$$S_1 \leftrightarrow S'_3,\ \ S_3 \leftrightarrow S'_8,\ \ S_4 \leftrightarrow S'_7,\ \ S_5 \leftrightarrow S'_2,$$
$$S_6 \leftrightarrow S'_1,\ \ S_7 \leftrightarrow S'_5,\ \ S_8 \leftrightarrow S'_6 \tag{3.1}$$

(see Figure 6). Note that we have obtained not only the correspondence of the faces between Δ and $U^{-1}\Delta$ but also that of the vertices and the edges of each face. For example, the ideal vertices 0, i, and ∞ of S_1 are identified with $1-i$, 1, and ∞ of S'_3, respectively (see Figure 7).

Remark 3.1. Consider the set of the images of Δ by translations of the group generated by T and U. Applying the preceding argument to any two adjacent hyperbolic regular ideal octahedra in this set, we obtain the gluing pattern for them. We illustrate in Figure 8 a part of the gluing pattern of the images of the lower faces

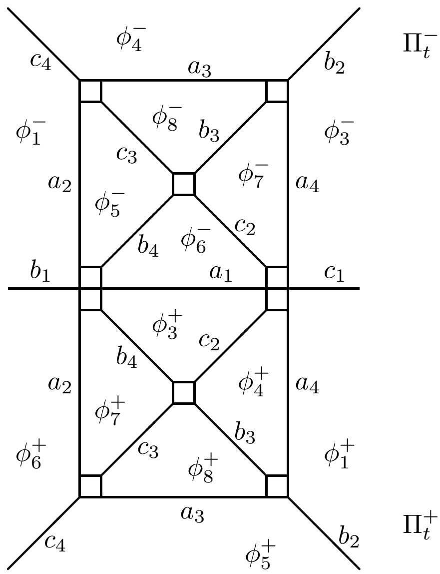

FIGURE 7. Gluing pattern for the two truncated octahedra corresponding to Δ and $U^{-1}\Delta$

of Δ. The gluing pattern of the images of the upper faces is obtained from Figures 8 and 9. Gluing ϕ_r^+ and ϕ_r^- of any two adjacent hyperbolic regular ideal octahedra, we can construct a geometric model of the Borromean rings complement.

In Figure 8, we can find a combinatorial structure of the cusp ∞ of the Borromean rings complement: the link of the cusp ∞ (i.e., the surface around ∞ obtained by gluing together the two hyperbolic regular ideal octahedra Δ and $U^{-1}\Delta$) is a torus consisting of four unit squares. For example, the four unit squares $\{x + iy \mid 0 \leq x \leq 4 \text{ and } 1 \leq y \leq 2\}$ make a periodic pattern in Figure 8.

By the definition of the tessellation $\mathcal{F}^3$ of $\mathbb{H}^3$, the faces of Δ and $U^{-1}\Delta$ must be glued by some actions of the Picard group.

Proposition 3.2. *The gluing pattern* (3.1) *of the faces of* Δ *and* $U^{-1}\Delta$ *is represented by*

$$P_\infty = \begin{pmatrix} 1 & 1-i \\ 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ -1+i & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} i & 1-i \\ -1+i & 2-i \end{pmatrix}$$

in the following way:

$$\begin{gathered} P_\infty : S_1 \to S_3', \quad P_0 : S_6 \to S_1', \quad P_1 : S_3 \to S_8', \\ P_2 = P_1 P_\infty : S_4 \to S_7', \quad P_3 = P_\infty P_0 : S_5 \to S_2', \\ P_4 = P_0 P_1 : S_7 \to S_5', \\ P_5 = P_\infty P_0 (P_\infty^{-1} P_1 P_\infty) = P_1 P_\infty (P_1^{-1} P_0 P_1) : S_8 \to S_6'. \end{gathered}$$

Proof. These are verified by direct calculations.

For example, P_∞ sends 0, i, and ∞ of S_1 to $1-i$, 1, and ∞ of S_3', respectively. □

Remark 3.3. The actions P_∞, P_0, and P_1 are parabolic, P_2, P_3, and P_4 are strictly loxodromic, and P_5 is hyperbolic.

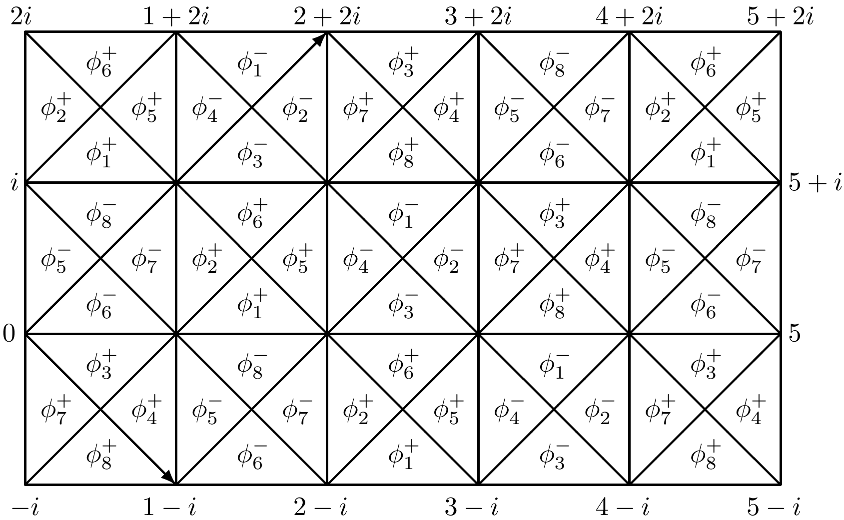

FIGURE 8. A part of the gluing pattern of the images of the lower faces of Δ. Each unit square, the vertices of which are points of Gaussian integers, is an image of the lower faces of Δ. We can find the lower part of Figure 7 in the left-bottom corner.

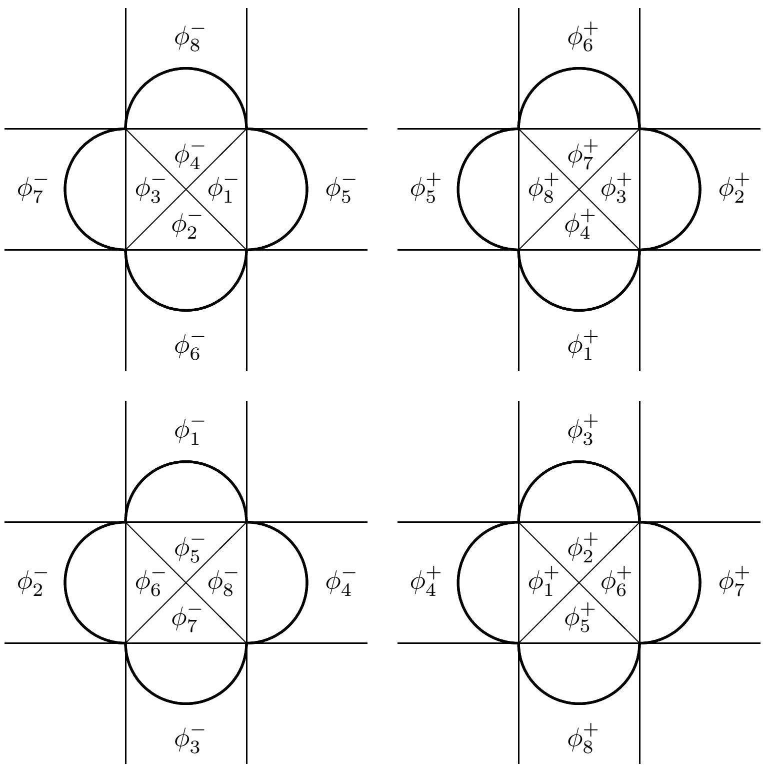

FIGURE 9. Four patterns of labeling of a hyperbolic regular ideal octahedron

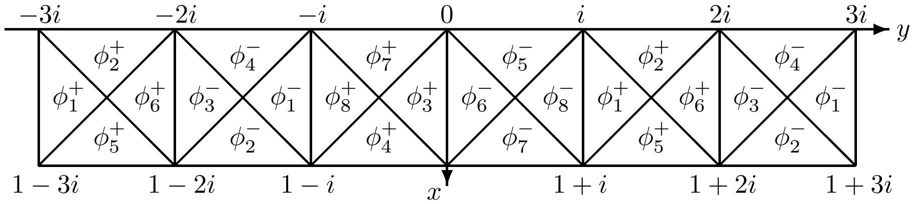

FIGURE 10. Gluing pattern of the lower faces of $U^l\Delta$, $l = -3$ to 2

Proposition 3.4. *The actions $P_1^{-1}P_0P_1P_0^{-1}$, $P_\infty^{-1}P_1P_\infty P_1^{-1}$, and $P_0^{-1}P_\infty P_0 P_\infty^{-1}$ are parabolic. Moreover, they have the same fixed points ∞, 0, and 1 as P_∞, P_0, and P_1, respectively.*

Proof. These are also verified by direct calculations.

For example, $P_1^{-1}P_0P_1P_0^{-1} = \begin{pmatrix} -1 & -2-2i \\ 0 & -1 \end{pmatrix}$ is parabolic and has the fixed point ∞ which is the same as P_∞. □

Hence, the geometric model of the Borromean rings complement by Δ and $U^{-1}\Delta$ has the three cusps ∞, 0 and 1; the links of them are tori. Moreover, P_∞ and $P_1^{-1}P_0P_1P_0^{-1}$ (P_0 and $P_\infty^{-1}P_1P_\infty P_1^{-1}$, P_1 and $P_0^{-1}P_\infty P_0P_\infty^{-1}$) are transformations corresponding to a meridian and a longitude of the torus. The translations P_∞ and $P_1^{-1}P_0P_1P_0^{-1}$ are indicated in Figure 8 by two arrows. They span a rectangle which also makes a periodic pattern in the figure (compare Remark 3.1).

We finally deduce a conclusion from the preceding discussion:

Theorem 3.5. *The quotient space $\mathcal{BRC} = \mathbb{H}^3/\Gamma_{BR}$ is a geometric model of the Borromean rings complement, where*

$$\Gamma_{BR} = \left\langle P_\infty, P_0, P_1 \;\middle|\; \begin{array}{l} P_\infty(P_1^{-1}P_0P_1P_0^{-1}) = (P_1^{-1}P_0P_1P_0^{-1})P_\infty \\ P_0(P_\infty^{-1}P_1P_\infty P_1^{-1}) = (P_\infty^{-1}P_1P_\infty P_1^{-1})P_0 \\ P_1(P_0^{-1}P_\infty P_0P_\infty^{-1}) = (P_0^{-1}P_\infty P_0P_\infty^{-1})P_1 \end{array} \right\rangle.$$

Remark 3.6. N. Wielenberg investigated in [40], using an HNN extension, a subgroup of the Picard group, the corresponding link of which is the Borromean rings. His result is a little different from ours.

4. HYPERSURFACE IN $\mathcal{BRC}$

We continue a discussion on the geometric model of the Borromean rings complement. We take the following planes in $\mathbb{H}^3$ perpendicular to $\mathbb{C}$:

$$W_1 = \left\{ \frac{1}{2} + yi + tj \;\middle|\; (y,t) \in \mathbb{R}^2, t > 0 \right\}, \quad W_2 = \left\{ x + \frac{1}{2}i + tj \;\middle|\; (x,t) \in \mathbb{R}^2, t > 0 \right\},$$

and $W_3 = U^{-1}W_2$, where we use the same notation as in §3. We define $\mathcal{HS}$ as the intersection of $W_1 \cup W_2 \cup W_3$ with $\Delta \cup U^{-1}\Delta$. The aim of this section is to prove that $\mathcal{HS}$ can be regarded as an immersed totally geodesic twice punctured torus in $\mathcal{BRC}$ and that it is conformally equivalent to the twice punctured torus $\mathbb{T}_t$ studied in §2.

Consider four aligning hyperbolic regular ideal octahedra $U^{-2}\Delta$, $U^{-1}\Delta$, U, and $U\Delta$. As we noted in Remark 3.1, they have a gluing pattern by which a Borromean

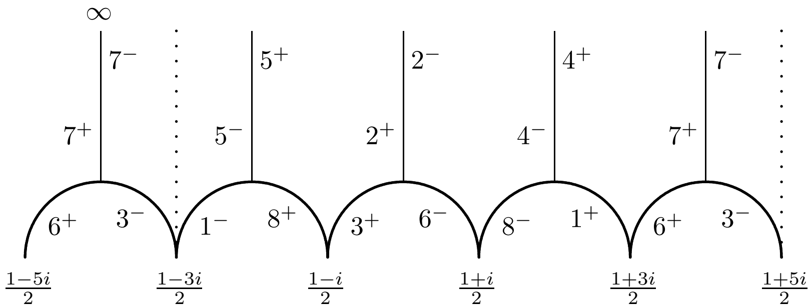

FIGURE 11. Labeling of the intersection of W_1 with $U^{-2}\Delta \cup U^{-1}\Delta \cup \Delta \cup U\Delta$. The edge with $r^{\pm}$ ($r = 1$ to 8) stands for the intersection of W_1 with a face with $\phi_r^{\pm}$.

rings complement is obtained from any two adjacent octahedra. The gluing pattern of the lower faces of them is depicted in Figure 10. Using the diagrams in Figure 9, we can get the gluing pattern of the upper faces.

We now consider the intersection of W_1 with $U^{-2}\Delta \cup U^{-1}\Delta \cup \Delta \cup U\Delta$. Making use of the patterns in Figures 10 and 9, we can illustrate it by a diagram in Figure 11. Here and in the following, we use the notation: an edge with $r^{\pm}$ stands for the intersection of the plane W_1 (W_2 or W_3) with a face with label $\phi_r^{\pm}$ ($r = 1$ to 8).

Lemma 4.1. *The quadrilateral $W_1 \cap U\Delta$ is identified with $W_3 \cap U^{-1}\Delta$.*

Proof. Using Figure 10 and the two right diagrams of Figure 9, we know both $W_1 \cap U\Delta$ and $W_3 \cap U^{-1}\Delta$ are a quadrilateral composed by edges with $1^+, 6^+, 7^+$, and 4^+. These edges are identified by the mapping P_2 defined in Proposition 3.2. ☐

Lemma 4.2. *The quadrilateral $W_1 \cap U^{-2}\Delta$ is identified with $W_2 \cap \Delta$.*

Proof. Using Figure 10 and the two left diagrams of Figure 9, we know both $W_1 \cap U^{-2}\Delta$ and $W_2 \cap \Delta$ are a quadrilateral composed by edges with $3^-, 1^-, 5^-$, and 7^-. These edges are identified by P_3 of Proposition 3.2. ☐

Hence, in order to prove that $\mathcal{HS}$ is a twice punctured torus, we have only to show that $W_1 \cap (U^{-2}\Delta \cup U^{-1}\Delta \cup \Delta \cup U\Delta)$ is a twice punctured torus.

Since the model of the Borromean rings complement $\mathcal{BRC}$ is obtained by gluing faces with ϕ_r^+ and ϕ_r^-, the edge with 7^- in $W_1 \cap U^{-1}\Delta$ (the left edge with 7^- in Figure 11) and the one with 7^+ in $W_1 \cap U^2\Delta$ (the right edge with 7^+ in Figure 11) must be identified. We then get a diagram in Figure 12. This gives a gluing pattern: the upper side and the lower side, and the left side and the right side of the biggest square in Figure 12 are glued together. We thus have a torus. Since $(1-3i)/2, (1-i)/2, (1+i)/2, (1+3i)/2$, and ∞ are ideal vertices, a point obtained by identifying the first four vertices is a cusp of the torus and the center of the square (∞) is another cusp. Hence, we have proved that $W_1 \cap (U^{-2}\Delta \cup U^{-1}\Delta \cup \Delta \cup U\Delta)$ is a twice punctured torus.

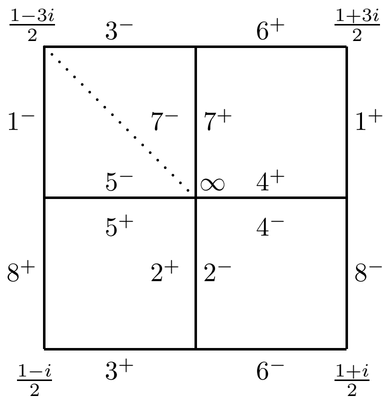

FIGURE 12. Gluing pattern for the twice punctured torus

We now determine the mappings giving the gluing of the sides of $W_1 \cap (U^{-2}\Delta \cup U^{-1}\Delta \cup \Delta \cup U\Delta)$. We define three matrices $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$ as follows:

$$\tilde{P} = P_0^{-1}P_1P_\infty = \begin{pmatrix} i & 2 \\ 2i & 4-i \end{pmatrix}, \quad \tilde{Q} = P_5^{-1} = \begin{pmatrix} 2-i & 2i \\ -2i & 2+i \end{pmatrix},$$
$$\tilde{R} = P_1P_\infty^{-1}P_1P_\infty = \begin{pmatrix} 1+2i & 2-2i \\ 4i & 5-2i \end{pmatrix}.$$

Lemma 4.3. *The plane W_1 is invariant under the actions of $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$.*

Proof. This is verified by direct calculation. □

The action of $\tilde{Q}^{-1}$ maps the semicircle with labels 8^- and 1^+ to the one with 8^+ and 1^- in Figure 11. Indeed, $\tilde{Q}^{-1}$ sends $(1+i)/2$ to $(1-i)/2$ and $(1+3i)/2$ to $(1-3i)/2$. Next, we take

$$\tilde{Q}^{-1}\tilde{R}^{-1}\tilde{P} = \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}.$$

This is a translation on W_1 and glues the edge with 7^- in $W_1 \cap U^{-1}\Delta$ to the one with 7^+ in $W_1 \cap U^2\Delta$. Finally, since $\tilde{P}$ sends $(1+3i)/2$ to $(1+i)/2$ and $(1+5i)/2$ to $(1-i)/2$, it maps the semicircle with 6^+ and 3^- to the one with 6^- and 3^+. We thus get the desired mappings.

Next, we conjugate actions on W_1 to those on the upper half-plane. By the definition of W_1, we require a translation by $-1/2$ and a rotation by $\pi/2$:

$$V_t = \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V_r = \begin{pmatrix} 1+i & 0 \\ 0 & (1-i)/2 \end{pmatrix}.$$

Define V as V_rV_t:

$$V = V_rV_t = \begin{pmatrix} 1+i & -(1+i)/2 \\ 0 & (1-i)/2 \end{pmatrix}. \tag{4.1}$$

We then get

$$P = V\tilde{P}V^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}, \quad Q = V\tilde{Q}V^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix},$$
$$R = V\tilde{R}V^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

All these are matrices used in §2. Let us define the group $\tilde{\Gamma}_t$ generated by $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$. The group $\tilde{\Gamma}_t$ is conjugate to Γ_t, that is, the twice punctured torus $\mathbb{T}_t$ is mapped to the quotient space $\tilde{\mathbb{T}}_t = W_1/\tilde{\Gamma}_t$. We have already seen that in the model of the Borromean rings complement $\mathcal{BRC}$, $\tilde{\mathbb{T}}_t$ is identified with $\mathcal{HS}$. Thus, there is an immersion of $\mathbb{T}_t$ into the model of the Borromean rings complement $\mathcal{BRC}$. We deduce a conclusion from the discussion:

Theorem 4.4. *The plane $\mathcal{HS}$ is an immersed totally geodesic twice punctured torus which is conformally equivalent to $\mathbb{T}_t$.*

Inversely conjugating the action of A and B on $\mathbb{H}^2$ to W_1, we get

$$\begin{aligned}\tilde{A} &= V^{-1}AV = \frac{1}{\sqrt{2}}\begin{pmatrix} 5-i & -3-3i \\ -2i & -1+i \end{pmatrix},\\ \tilde{B} &= V^{-1}BV = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 1-i \\ 2i & 3-i \end{pmatrix}.\end{aligned} \tag{4.2}$$

The plane W_1 is invariant under these actions. Let $\tilde{\Gamma}_o$ be a group generated by $\tilde{A}$ and $\tilde{B}$. Then the quotient space $\tilde{\mathbb{T}}_o = W_1/\tilde{\Gamma}_o$ can be regarded as an immersed totally geodesic once punctured torus in the Borromean rings complement $\mathcal{BRC}$.

5. Vulakh-Schmidt tree

5.1. **Tree of quadruples of integers.** In order to investigate the discrete part of the Markoff spectrum for the imaginary quadratic number field $\mathbb{Q}(i)$, L.Ya. Vulakh introduced the system of equations (1.2), which is an analogue of Markoff's equation. Let $(x_1, x_2; y_1, y_2)$ denote a solution of (1.2) such that

$$(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4, \ \ 1 \le x_1 \le x_2, \ \ 1 \le y_1 \le y_2.$$

We call such a solution a *Vulakh-Schmidt quadruple* (or, for brevity, a *VS quadruple*) after the works of L.Ya. Vulakh [37] and A. Schmidt [31].

We can simply verify that $(1,1;1,1)$ is the unique solution satisfying $x_1 = x_2$ or $y_1 = y_2$. Setting $x_1 = 1$ and $y_1 = 1$, we get $(1,5;1,3)$, which is the unique solution derived from $(1,1;1,1)$. Suppose that $q = (\bar{x}_1, \bar{x}_2; \bar{y}_1, \bar{y}_2)$ is a VS quadruple different from $(1,1;1,1)$. Then, setting $(x_1, y_1) = (\bar{x}_1, \bar{y}_1)$, $(\bar{x}_1, \bar{y}_2)$, $(\bar{x}_2, \bar{y}_1)$, and $(\bar{x}_2, \bar{y}_2)$ in (1.2), we get from q four quadratic equations. Solving them, four other quadruples are obtained. Since one of those derived from q is the quadruple from which q itself is obtained, three of them are new. Hence, a VS quadruple different from $(1,1;1,1)$ always has three children which are explicitly written. Indeed, an infinite ternary tree, the nodes of which are VS quadruples, is built in the following way: the root of the tree is $(1,5;1,3)$ and, a node $(x_1, x_2; y_1, y_2)$ being given, its three children are

$$\begin{array}{ll} (x_1, 2y_2(4x_1y_2 - y_1) - x_1; y_2, 4x_1y_2 - y_1) & \text{for its left child,} \\ (x_2, 2y_2(4x_2y_2 - y_1) - x_2; y_2, 4x_2y_2 - y_1) & \text{for its center child,} \\ (x_2, 2y_1(4x_2y_1 - y_2) - x_2; y_1, 4x_2y_1 - y_2) & \text{for its right child.} \end{array}$$

(See §5.2 in [31].) We call this a *Vulakh-Schmidt tree* (or, for brevity, a *VS tree*).

In order to illustrate the VS tree, we define labels for its nodes. We label the root $\varnothing$. Its left child is labeled by a, its center child by b and its right child by c. If a node has L as a label, we label its left child La, its center child Lb and its right child Lc. (See Figure 13.) In the following, we frequently use the expression

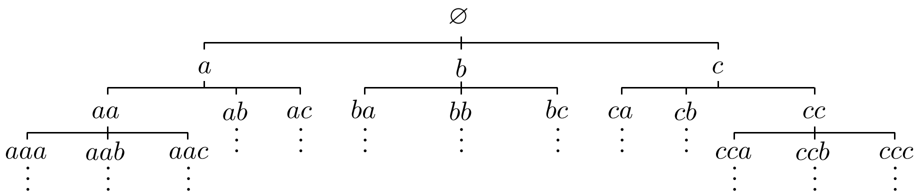

FIGURE 13. Labeling of the Vulakh-Schmidt tree

TABLE 2. VS quadruples

label	VS quadruple
$\varnothing$	(1, 5; 1, 3)
a	(1,65; 3,11)
b	(5, 349; 3, 59)
c	(5, 29; 1, 17)
aa	(1, 901; 11, 41)
ab	(65, 62789; 11, 2857)
ac	(65, 4549; 3, 769)
ba	(5, 138881; 59, 1177)
bb	(349, 9718249; 59,82361)
bc	(349, 24425; 3, 4129)
ca	(5, 11521; 17, 339)
cb	(29, 66985; 17, 1971)
cc	(29, 169; 1, 99)

"the node L" instead of "the node with a label L". Table 2 gives correspondence between nodes of VS quadruples and labels with length less than 2. The beginning of the Vulakh-Schmidt tree can be built by Figure 13 and Table 2.

Let $\mathcal{N}(\Lambda)$ denote the set of members x_1 and x_2 in Vulakh-Schmidt quadruples, and $\mathcal{N}(M)$ the set of members y_1 and y_2:

$$\mathcal{N}(\Lambda) = \{\lambda\} = \{1, 5, 29, 65, 169, 349, 901, 985, 4549, 5741, \cdots\},$$
$$\mathcal{N}(M) = \{m\} = \{1, 3, 11, 17, 41, 59, 99, 153, 339, 571, 577, \cdots\}.$$

The set $\mathcal{N}(\Lambda)$ is used to represent the discrete part (1.3) of the Markoff spectrum for $\mathbb{Q}(i)$ (see [37] and [31]).

Remark 5.1. We use the prime $'$ to represent a relationship between a parent and its child in the VS tree, that is, $(x_1', x_2'; y_1', y_2')$ denotes a child of $(x_1, x_2; y_1, y_2)$. The quadruple $(x_1', x_2'; y_1', y_2')$ is uniquely determined if we know which child it is.

For example, if $(x_1', x_2'; y_1', y_2')$ is the left child of $(x_1, x_2; y_1, y_2)$, then $(x_1', x_2'; y_1', y_2') = (x_1, 2y_2(4x_1y_2 - y_1) - x_1; y_2, 4x_1y_2 - y_1)$.

5.2. **Tree of quadruples of matrices.** We now propose an algorithm building a ternary tree, the nodes of which are quadruples of matrices. The matrices are the following special ones corresponding to the elements of $\mathcal{N}(\Lambda)$ and $\mathcal{N}(M)$. In the definition (also in the sequel) we use two notation: for any matrix $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\kappa(N)$ denotes the $(2,1)$-entry of N, namely, $\kappa(N) = c$; $\mathbb{Z}/\sqrt{2}$ stands for the set of numbers of the form $u/\sqrt{2}$ with $u \in \mathbb{Z}$.

Definition 5.2. (i) For each $\lambda \in \mathcal{N}(\Lambda)$, if a matrix Λ_λ belongs to $\mathrm{SL}(2,\mathbb{Z})$, and satisfies $\kappa(\Lambda_\lambda) = \lambda$ and $\mathrm{tr}(\Lambda_\lambda) = 4\lambda$, then we call it *$\Lambda$-matrix associated with λ*.

(ii) For each $m \in \mathcal{N}(M)$, if a matrix M_m belongs to $\mathrm{SL}(2,\mathbb{Z}/\sqrt{2})$, and satisfies $\kappa(M_m) = m/\sqrt{2}$ and $\mathrm{tr}(M_m) = 2\sqrt{2}\cdot m$, then we call it *$M$-matrix associated with m*.

A quadruple of matrices is written in the form $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, where the quadruple of the subscripts $(x_1, x_2; y_1, y_2)$ is a VS quadruple. Corresponding to the root of the VS tree $(1,5;1,3)$, we define a quadruple of matrices of the root by $(\Lambda_1, \Lambda_5; M_1, M_3)$, where

$$\Lambda_1 = Q^{-1} = BA = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad M_1 = B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$$

(5.1)

$$\Lambda_5 = M_3M_1 = \Lambda_1 M_1^2 = \begin{pmatrix} 8 & 19 \\ 5 & 12 \end{pmatrix}, \quad M_3 = \Lambda_1 M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 5 & 11 \\ 3 & 7 \end{pmatrix}.$$

Note that A, B, and Q are matrices used in §2 and that Λ_1 and Λ_5 are Λ-matrices associated with 1 and 5; M_1 and M_3 are M-matrices associated with 1 and 3.

Any node of the tree is either of type *I* or of type *II*. The root is assumed to be of type *I*. The children of a node $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ are defined in the following way.

Algorithm VS (see Figure 14).

- If $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is of type *I*, then
  - its left child is $(\Lambda_{x_1}, \Lambda_{x_1}M_{y_2}^2; M_{y_2}, \Lambda_{x_1}M_{y_2})$ and is of type *I*;
  - its center child is $(\Lambda_{x_2}, M_{y_2}^2\Lambda_{x_2}; M_{y_2}, M_{y_2}\Lambda_{x_2})$ and is of type *II*;
  - its right child is $(\Lambda_{x_2}, \Lambda_{x_2}M_{y_1}^2; M_{y_1}, \Lambda_{x_2}M_{y_1})$ and is of type *I*.
- If $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is of type *II*, then
  - its left child is $(\Lambda_{x_1}, M_{y_2}^2\Lambda_{x_1}; M_{y_2}, M_{y_2}\Lambda_{x_1})$ and is of type *II*;
  - its center child is $(\Lambda_{x_2}, \Lambda_{x_2}M_{y_2}^2; M_{y_2}, \Lambda_{x_2}M_{y_2})$ and is of type *I*;
  - its right child is $(\Lambda_{x_2}, M_{y_1}^2\Lambda_{x_2}; M_{y_1}, M_{y_1}\Lambda_{x_2})$ and is of type *II*.

We call the tree built by this algorithm, starting with the matrices of (5.1), a *Vulakh-Schmidt matrix tree* (or, for brevity, a *VS matrix tree*). The beginning of the Vulakh-Schmidt matrix tree can be built by Figure 13 and Table 3. In Table 4 we show some Λ- and M-matrices in the VS matrix tree.

Remark 5.3. By $|L|_b$ we write the number of occurrences of the letter b in a label L. It is easily verified that the node L is of type *I* if and only if $|L|_b$ is even; the node L is of type *II* if and only if $|L|_b$ is odd.

$$(\Lambda_{x_1}, \Lambda_{x_2}\ ;\ M_{y_1}, M_{y_2}), I$$

$$(\Lambda_{x_1}, \Lambda_{x_1} M_{y_2}^2\ ;\ M_{y_2}, \Lambda_{x_1} M_{y_2}), I \qquad (\Lambda_{x_2}, M_{y_2}^2 \Lambda_{x_2}\ ;\ M_{y_2}, M_{y_2}\Lambda_{x_2}), II \qquad (\Lambda_{x_2}, \Lambda_{x_2} M_{y_1}^2\ ;\ M_{y_1}, \Lambda_{x_2} M_{y_1}), I$$

$$(\Lambda_{x_1}, \Lambda_{x_2}\ ;\ M_{y_1}, M_{y_2}), II$$

$$(\Lambda_{x_1}, M_{y_2}^2 \Lambda_{x_1}\ ;\ M_{y_2}, M_{y_2}\Lambda_{x_1}), II \qquad (\Lambda_{x_2}, \Lambda_{x_2} M_{y_2}^2\ ;\ M_{y_2}, \Lambda_{x_2} M_{y_2}), I \qquad (\Lambda_{x_2}, M_{y_1}^2 \Lambda_{x_2}\ ;\ M_{y_1}, M_{y_1}\Lambda_{x_2}), II$$

FIGURE 14. Illustration of Algorithm VS. The first is the case in which the parent is of type I. The second is the case in which the parent is of type II.

TABLE 3. Quadruples of matrices

label	quadruple of matrices	type
$\varnothing$	$(\Lambda_1,\ \Lambda_5 = M_3 M_1;\ M_1,\ M_3 = \Lambda_1 M_1)$	I
a	$(\Lambda_1,\ \Lambda_{65} = M_{11} M_3;\ M_3,\ M_{11} = \Lambda_1 M_3)$	I
b	$(\Lambda_5,\ \Lambda_{349} = M_3 M_{59};\ M_3,\ M_{59} = M_3 \Lambda_5)$	II
c	$(\Lambda_5,\ \Lambda_{29} = M_{17} M_1;\ M_1,\ M_{17} = \Lambda_5 M_1)$	I
aa	$(\Lambda_1,\ \Lambda_{901} = M_{41} M_{11};\ M_{11},\ M_{41} = \Lambda_1 M_{11})$	I
ab	$(\Lambda_{65},\ \Lambda_{62789} = M_{11} M_{2857};\ M_{11},\ M_{2857} = M_{11} \Lambda_{65})$	II
ac	$(\Lambda_{65},\ \Lambda_{4549} = M_{769} M_3;\ M_3,\ M_{769} = \Lambda_{65} M_3)$	I
ba	$(\Lambda_5,\ \Lambda_{138881} = M_{59} M_{1177};\ M_{59},\ M_{1177} = M_{59} \Lambda_5)$	II
bb	$(\Lambda_{349},\ \Lambda_{9718249} = M_{82361} M_{59};\ M_{59},\ M_{82361} = \Lambda_{349} M_{59})$	I
bc	$(\Lambda_{349},\ \Lambda_{24425} = M_3 M_{4129};\ M_3,\ M_{4129} = M_3 \Lambda_{349})$	II
ca	$(\Lambda_5,\ \Lambda_{11521} = M_{339} M_{17};\ M_{17},\ M_{339} = \Lambda_5 M_{17})$	I
cb	$(\Lambda_{29},\ \Lambda_{66985} = M_{17} M_{1971};\ M_{17},\ M_{1971} = M_{17} \Lambda_{29})$	II
cc	$(\Lambda_{29},\ \Lambda_{169} = M_{99} M_1;\ M_1,\ M_{99} = \Lambda_{29} M_1)$	I

Remark 5.4. We also use the prime $'$ to represent a relationship between a parent and its child in the VS matrix tree, that is, $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'})$ denotes a child of $(\Lambda_{x_1}, \Lambda_{x_2};\ M_{y_1}, M_{y_2})$. The quadruple of matrices is uniquely determined if we know the type of the parent and which child it is. For example, if it is the left child of $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ of type I, then $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'}) = (\Lambda_{x_1}, \Lambda_{x_1} M_{y_2}^2; M_{y_2}, \Lambda_{x_1} M_{y_2})$.

With this notation, we can write Algorithm VS more simply. If a child $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'})$ is defined as type I, then $M_{y_2'} = \Lambda_{x_1'} M_{y_1'}$ and $\Lambda_{x_2'} = M_{y_2'} M_{y_1'}$; if it is defined as type II, then $M_{y_2'} = M_{y_1'} \Lambda_{x_1'}$ and $\Lambda_{x_2'} = M_{y_1'} M_{y_2'}$.

Definition 5.5. Let $(x_1, x_2; y_1, y_2)$ be a Vulakh-Schmidt quadruple. If both Λ_{x_1} and Λ_{x_2} are Λ-matrices associated with x_1 and x_2 and if both M_{y_1} and M_{y_2} are M-matrices associated with y_1 and y_2, then we say that $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is *associated with* $(x_1, x_2; y_1, y_2)$.

We have to verify that the VS matrix tree and the quadruples of matrices in the nodes are well defined. With the notation in Remarks 5.1 and 5.4, a theorem to be proved is written in the following way:

Theorem 5.6. *If a node* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ *in the VS matrix tree is associated with a VS quadruple* $(x_1, x_2; y_1, y_2)$*, then each child* $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'})$ *defined by Algorithm VS is associated with* $(x_1', x_2'; y_1', y_2')$.

Note that $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'})$ has six possibilities by two types and three children. This will be proved in §7. The theorem ensures that the VS tree and the VS matrix tree correspond to each other in a natural way. We now have

Corollary 5.7. *For each* $\lambda \in \mathcal{N}(\Lambda)$*, the matrix* Λ_λ *in the VS matrix tree is associated with* λ*. For each* $m \in \mathcal{N}(M)$*, the matrix* M_m *in the VS matrix tree is associated with* m.

We deduce the following proposition from Definition 5.2 and the construction of the VS matrix tree.

Proposition 5.8. *All* Λ*-matrices and* M*-matrices in the VS matrix tree are hyperbolic and are represented as words on the alphabet* $\{A, B\}$*, where* A *and* B *are the matrices in* (2.4).

Theorem 5.6 is proved by inductive method (see §7). We can directly check that $(\Lambda_1, \Lambda_5; M_1, M_3)$, $(\Lambda_1, \Lambda_{65}; M_3, M_{11})$, $(\Lambda_5, \Lambda_{349}; M_3, M_{59})$, and $(\Lambda_5, \Lambda_{29}; M_1, M_{17})$ are associated with $(1, 5; 1, 3)$, $(1, 65; 3, 11)$, $(5, 349; 3, 59)$, and $(5, 29; 1, 17)$, respectively (see Table 4). Assuming that $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is associated with $(x_1, x_2; y_1, y_2)$, we prove that, in each child, $(\Lambda_{x_1'}, \Lambda_{x_2'}; M_{y_1'}, M_{y_2'})$ defined by Algorithm VS is associated with $(x_1', x_2'; y_1', y_2')$.

Remark 5.9. Note that the Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and the M-matrix associated with $m \in \mathcal{N}(M)$ are not unique. For example, we can construct another VS matrix tree by Algorithm VS starting with

$$\Lambda_1 = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then we can prove that if a quadruple of matrices $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ in the tree is associated with a VS quadruple $(x_1, x_2; y_1, y_2)$, then its three children are also associated with the children of $(x_1, x_2; y_1, y_2)$.

Fortunately, this ambiguity will not cause any trouble, because in the VS matrix tree built by Algorithm VS starting with (5.1), a matrix Λ_λ for $\lambda \in \mathcal{N}(\Lambda)$ and a matrix M_m for $m \in \mathcal{N}(M)$ are uniquely determined (see [2]).

TABLE 4. Examples of Λ-matrix and M-matrix

label	the second matrix	the fourth matrix
∅	$\Lambda_5 = \begin{pmatrix} 8 & 19 \\ 5 & 12 \end{pmatrix}$	$M_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 5 & 11 \\ 3 & 7 \end{pmatrix}$
a	$\Lambda_{65} = \begin{pmatrix} 112 & 255 \\ 65 & 148 \end{pmatrix}$	$M_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 19 & 43 \\ 11 & 25 \end{pmatrix}$
b	$\Lambda_{349} = \begin{pmatrix} 562 & 1343 \\ 349 & 834 \end{pmatrix}$	$M_{59} = \frac{1}{\sqrt{2}} \begin{pmatrix} 95 & 227 \\ 59 & 141 \end{pmatrix}$
c	$\Lambda_{29} = \begin{pmatrix} 46 & 111 \\ 29 & 70 \end{pmatrix}$	$M_{17} = \frac{1}{\sqrt{2}} \begin{pmatrix} 27 & 65 \\ 17 & 41 \end{pmatrix}$

5.3. **Tree of quadruples of complex matrices.** We introduce complex matrices which are analogues of Λ- and M-matrices in §5.2.

Definition 5.10. (i) For each $\lambda \in \mathcal{N}(\Lambda)$, if a matrix $\tilde{\Lambda}_\lambda$ belongs to $\mathrm{SL}(2, \mathbb{Z}[i])$, has the form

$$\tilde{\Lambda}_\lambda = \begin{pmatrix} a + \lambda i & b + ci \\ 2\lambda i & d - \lambda i \end{pmatrix}, \tag{5.2}$$

and satisfies $\mathrm{tr}(\tilde{\Lambda}_\lambda) = 4\lambda$, then we call it a $\tilde{\Lambda}$-*matrix associated with* λ.

(ii) For each $m \in \mathcal{N}(M)$, if a matrix $\tilde{M}_m$ belongs to $\mathrm{SL}(2, \mathbb{Z}[i]/\sqrt{2})$, has the form

$$\tilde{M}_m = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + mi & \beta + \gamma i \\ 2mi & \delta - mi \end{pmatrix}, \tag{5.3}$$

and satisfies $\mathrm{tr}(\tilde{M}_m) = 2\sqrt{2} \cdot m$, then we call it an $\tilde{M}$-*matrix associated with* m.

Here $\mathbb{Z}[i]/\sqrt{2}$ stands for the set of complex numbers of the form $u/\sqrt{2}$ with $u \in \mathbb{Z}[i]$; $a, b, c, d, \alpha, \beta, \gamma$, and δ in the entries of the matrices are some integers.

The aim of this subsection is to construct a ternary tree, the nodes of which are quadruples of these complex matrices associated with the elements of $\mathcal{N}(\Lambda)$ and $\mathcal{N}(M)$. The tree is also defined inductively. A quadruple of complex matrices is written in the form $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_2})$, where the quadruple of the subscripts $(x_1, x_2; y_1, y_2)$ is a VS quadruple. Each node of the tree is either of type *I* or of type *II*.

Corresponding to the root of the VS tree $(1, 5; 1, 3)$, we define a quadruple of complex matrices of the root by $(\tilde{\Lambda}_1, \tilde{\Lambda}_5 = \tilde{M}_3\tilde{M}_1; \tilde{M}_1, \tilde{M}_3 = \tilde{\Lambda}_1\tilde{M}_1)$, where $\tilde{\Lambda}_1$ and $\tilde{M}_1$ are the conjugate of Λ_1 and M_1 by V (see (5.1) and (4.1)):

$$\tilde{\Lambda}_1 = V^{-1}\Lambda_1 V = \begin{pmatrix} 2 + i & -2i \\ 2i & 2 - i \end{pmatrix},$$
$$\tilde{M}_1 = V^{-1}M_1 V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i & 1 - i \\ 2i & 3 - i \end{pmatrix}. \tag{5.4}$$

(Note that $\tilde{\Lambda}_1$ is used in [19].) The type of the root is assumed to be of type *I*. For a node $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_2})$, we define its children by Algorithm VS; to be precise,

TABLE 5. Examples of $\tilde{\Lambda}$-matrix and $\tilde{M}$-matrix

label	the second matrix
$\varnothing$	$\tilde{\Lambda}_5 = \begin{pmatrix} 8+5i & 2-12i \\ 10i & 12-5i \end{pmatrix}$
a	$\tilde{\Lambda}_{65} = \begin{pmatrix} 112+65i & 18-160i \\ 130i & 148-65i \end{pmatrix}$
b	$\tilde{\Lambda}_{349} = \begin{pmatrix} 562+349i & 136-846i \\ 698i & 834-349i \end{pmatrix}$
c	$\tilde{\Lambda}_{29} = \begin{pmatrix} 46+29i & 12-70i \\ 58i & 70-29i \end{pmatrix}$

label	the fourth matrix
$\varnothing$	$\tilde{M}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 5+3i & 1-7i \\ 6i & 7-3i \end{pmatrix}$
a	$\tilde{M}_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 19+11i & 3-27i \\ 22i & 25-11i \end{pmatrix}$
b	$\tilde{M}_{59} = \frac{1}{\sqrt{2}} \begin{pmatrix} 95+59i & 23-143i \\ 118i & 141-59i \end{pmatrix}$
c	$\tilde{M}_{17} = \frac{1}{\sqrt{2}} \begin{pmatrix} 27+17i & 7-41i \\ 34i & 41-17i \end{pmatrix}$

Λ_{x_k} and M_{y_k} $(k = 1, 2)$ in the algorithm must be replaced by $\tilde{\Lambda}_{x_k}$ and $\tilde{M}_{y_k}$. The tree built in this way is called a *Vulakh-Schmidt* (for brevity, a *VS*) *complex matrix tree*. Some $\tilde{\Lambda}$- and $\tilde{M}$-matrices in this tree are shown in Table 5, where we use the same labeling as the tree defined in §5.2.

Recall the Vulakh-Schmidt matrix tree, built in §5.2, the nodes of which are quadruples of real matrices associated with VS quadruples. Let Λ_λ be a Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and let M_m be an M-matrix associated with $m \in \mathcal{N}(M)$ in the VS matrix tree (see Corollary 5.7). We define $\tilde{\Lambda}_\lambda$ and $\tilde{M}_m$ as the conjugate of Λ_λ and M_m by V, namely,

$$\tilde{\Lambda}_\lambda = V^{-1}\Lambda_\lambda V \quad \text{and} \quad \tilde{M}_m = V^{-1}M_m V. \tag{5.5}$$

By construction, we get a proposition:

Proposition 5.11. *All matrices $\tilde{\Lambda}_\lambda$, $\lambda \in \mathcal{N}(\Lambda)$, and $\tilde{M}_m$, $m \in \mathcal{N}(M)$, in the VS complex matrix tree are equal to the matrices obtained by* (5.5) *from Λ_λ and M_m in the VS matrix tree.*

The following theorem analogous to Corollary 5.7 is proved at the end of §7.

Theorem 5.12. *For each $\lambda \in \mathcal{N}(\Lambda)$, the matrix $\tilde{\Lambda}_\lambda$ in the VS complex matrix tree is associated with λ. For each $m \in \mathcal{N}(M)$, the matrix $\tilde{M}_m$ in the VS complex matrix tree is associated with m.*

Thanks to this theorem, each quadruple of matrices $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_1})$ in the VS complex matrix tree is associated with $(x_1, x_2; y_1, y_2)$ in the following sense:

Definition 5.13. Let $(x_1, x_2; y_1, y_2)$ be a VS quadruple. If both $\tilde{\Lambda}_{x_1}$ and $\tilde{\Lambda}_{x_2}$ are $\tilde{\Lambda}$-matrices associated with x_1 and x_2 and if both $\tilde{M}_{y_1}$ and $\tilde{M}_{y_2}$ are $\tilde{M}$-matrices associated with y_1 and y_2, then we say that $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_2})$ is *associated with* $(x_1, x_2; y_1, y_2)$.

By virtue of (5.2), (5.3), and the construction of the VS complex matrix tree, we have a similar result to Proposition 5.8:

Proposition 5.14. *All $\tilde{\Lambda}$-matrices and $\tilde{M}$-matrices in the VS complex matrix tree are hyperbolic and are represented as words on the alphabet $\{\tilde{A}, \tilde{B}\}$, where $\tilde{A}$ and $\tilde{B}$ are the matrices in* (4.2).

Remark 5.15. Note that the $\tilde{\Lambda}$-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and the $\tilde{M}$-matrix associated with $m \in \mathcal{N}(M)$ are not unique, but they are unique in the VS complex matrix tree built by Algorithm VS starting with (5.4) (see Remark 5.9).

6. Simple closed geodesics

Let us begin by recalling a basic fact of the two generator free group. Let $\langle g, h\rangle$ be a free group generated by g and h. An element g' is called a *generator* of $\langle g, h\rangle$ if there exists an h' such that g' and h' generate $\langle g, h\rangle$. We then also say that g' is represented as a *primitive word* in the two generator free group $\langle g, h\rangle$. It is well known (see, for example, [25]) that the outer automorphism group of $\langle g, h\rangle$ is generated by the following three operations: exchanging g and h; replacing g by g^{-1}; replacing g by gh. Combining this fact with the definition of matrices in the VS matrix tree (Algorithm VS), we get

Proposition 6.1. *Every Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ in the VS matrix tree is a generator of Γ_o. Every M-matrix associated with $m \in \mathcal{N}(M)$ in the VS matrix tree is also a generator of Γ_o.*

Any pair of Λ-matrices cannot generate Γ_o, because the group generated by two Λ-matrices must be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, but Γ_o cannot be included in $\mathrm{SL}(2,\mathbb{Z})$. Contrasting with this fact, we can prove the following:

Proposition 6.2. *Let $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ be the quadruple of matrices associated with a VS quadruple $(x_1, x_2; y_1, y_2)$ in the VS matrix tree. Then M_{y_1} and M_{y_2} generate Γ_o.*

Proof. By Definition 5.2, we have $\mathrm{tr}(\Lambda_{x_2}) = 4x_2$, $\mathrm{tr}(M_{y_1}) = 2\sqrt{2}y_1$, and $\mathrm{tr}(M_{y_2}) = 2\sqrt{2}y_2$. From Algorithm VS, Λ_{x_2} is defined as $M_{y_1}M_{y_2}$ or $M_{y_2}M_{y_1}$. Consider the triple $(\mathrm{tr}(M_{y_1}), \mathrm{tr}(M_{y_2}), \mathrm{tr}(M_{y_1}M_{y_2}))$. Since $(x_1, x_2; y_1, y_2)$ satisfies (1.2), this triple satisfies Fricke's equation. Hence, $\Gamma_o = \langle M_{y_1}, M_{y_2}\rangle$ is proved. □

The following proposition is very useful (see [22]):

Proposition 6.3 (Nielsen)**.** *Let $\langle g, h\rangle$ be a group, the quotient space of which is a once punctured torus. Then, a geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$ is the axis of a generator of $\langle g, h\rangle$ if and only if $\tilde{\gamma}$ projects to a simple closed geodesic on the once punctured torus $\mathbb{H}^2/\langle g, h\rangle$.*

Recall that the geodesic in $\mathbb{H}^2$ fixed by a hyperbolic element g is called the *axis* of g and that a *simple* geodesic is a geodesic having no self-intersections.

We now deduce a theorem from Propositions 6.1 and 6.3:

Theorem 6.4. *The axes of each Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and of each M-matrix associated with $m \in \mathcal{N}(M)$ in the VS matrix tree project to simple closed geodesics on $\mathbb{T}_o$.*

Since the twice punctured torus $\mathbb{T}_t$ is a 2-fold cover of $\mathbb{T}_o$ (see §2), we immediately get a corollary:

Corollary 6.5. *The axes in Theorem* 6.4 *also project to simple closed geodesics on $\mathbb{T}_t$.*

The *Euclidean height* of a geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$ is defined as $|\eta - \xi|/2$ if η and ξ are finite or ∞ otherwise, where η and ξ are the two endpoints of $\tilde{\gamma}$.

Proposition 6.6. *The Euclidean height of the axis of the Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and that of the axis of the M-matrix associated with $m \in \mathcal{N}(M)$ are*

$$\sqrt{4 - \frac{1}{\lambda^2}} \quad \textit{and} \quad \sqrt{4 - \frac{2}{m^2}}.$$

Proof. These are directly computed by Definition 5.2. ☐

Here we explain that the simple closed geodesics on $\mathbb{T}_o$ are essentially obtained from the axes of Λ- and M-matrices in the VS matrix tree.

The conformal mapping φ_4 between $\mathbb{T}_{\langle A_4, B_4 \rangle}$ and $\mathbb{C}/\langle 1, i \rangle$ mentioned in §2 gives the abelian image of $\mathbb{T}_{\langle P_4, Q_4, R_4 \rangle}$. Figure 15 depicts a tessellation of the complex plane $\mathbb{C}$ by the abelian images of $\mathbb{T}_{\langle P_4, Q_4, R_4 \rangle}$. Since $\mathbb{T}_t$ and $\mathbb{T}_{\langle P_4, Q_4, R_4 \rangle}$ are conformally equivalent, this can be regarded as a tessellation of $\mathbb{C}$ by the abelian images of $\mathbb{T}_t$. Here we take $\{A, B\}$ corresponding to $\{i, 1\}$ (as an ordered pair). There are the other three equivalent possibilities (see [16]): $\{A, B\}$ corresponding to $\{i, -1\}$, $\{1, i\}$, or $\{-1, i\}$.

Identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. Let $\Omega = \{(k, l) \in \mathbb{Z}^2\}$, namely, Ω is the set of the images of $(0, 0) \in \mathbb{R}^2$ by the action of the group $\langle 1, i \rangle$. The lattice Ω is decomposed into the following two disjoint sublattices, each of which corresponds to one of the two cusps of $\mathbb{T}_t$:

$$\Omega_w = \{(k, l) \in \Omega \mid (k, l) \equiv (0, 0) \text{ or } (1, 1) \pmod{2}\},$$

$$\Omega_b = \{(k, l) \in \Omega \mid (k, l) \equiv (1, 0) \text{ or } (0, 1) \pmod{2}\}.$$

The white points in Figure 15 stand for the elements of Ω_w and the black points stand for those of Ω_b. (The origin, $(0, 0)$, is a white point.)

Recall that each matrix N in the VS matrix tree is represented as a word on the alphabet $\{A, B\}$ (Proposition 5.8). We define the *Euclidean pair* (for brevity, the *E-pair*) of N as $(|N|_B, |N|_A)$, where $|N|_B$ stands for the number of B occurring in the word of N and $|N|_A$ for the number of A.

Let Λ_λ be a Λ-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ and M_m be an M-matrix associated with $m \in \mathcal{N}(M)$ in the VS matrix tree. We write (λ_1, λ_2) for the E-pair of Λ_λ and (m_1, m_2) for the E-pair of M_m. A quadruple of matrices induces a quadruple of E-pairs; for example, for the root $(\Lambda_1, \Lambda_5; M_1, M_3)$ of the VS matrix tree, the quadruple of E-pairs is $((1, 1), (3, 1); (1, 0), (2, 1))$.

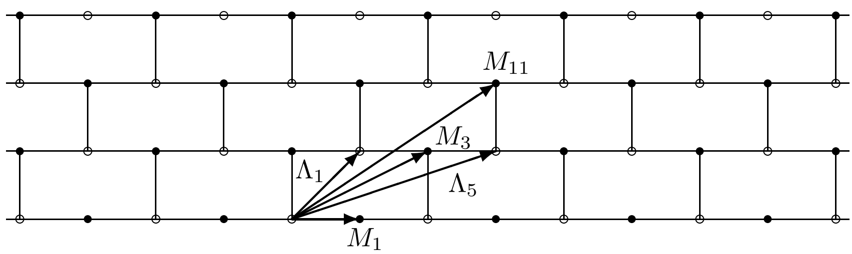

FIGURE 15. Two colored lattice in the complex plane and representation of some E-pairs in it. The beginning point of the vectors is the origin $(0, 0)$.

Let $((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2); (m_1, m_2), (m'_1, m'_2))$ be the quadruple of E-pairs for a quadruple of matrices $(\Lambda_\lambda, \Lambda_{\lambda'}; M_m, M_{m'})$ in the VS matrix tree. The following relations are deduced from Algorithm VS and the definition of the E-pair:

$$(m'_1, m'_2) = (\lambda_1, \lambda_2) + (m_1, m_2),$$
$$(\lambda'_1, \lambda'_2) = (m_1, m_2) + (m'_1, m'_2) = (\lambda_1, \lambda_2) + 2(m_1, m_2).$$

Since the E-pair of Λ_1 is $(1, 1)$ and that of M_1 is $(1, 0)$, using these relations, we easily verify that the E-pair (λ_1, λ_2) of any matrix Λ_λ satisfies $(\lambda_1, \lambda_2) \equiv (1, 1)$ (mod 2) and the E-pair (m_1, m_2) of any matrix M_m satisfies $(m_1, m_2) \equiv (1, 0)$ or $(0, 1)$ (mod 2) (see Figure 15).

Moreover, we can prove the following theorem (see [2]).

Theorem 6.7. (i) *There exists a bijection between the set of the E-pair of Λ_λ-matrices and the set $\Omega_w \cap \{(k, l) \mid (k, l)$ are mutually prime and $0 < l \leq k\}$.*

(ii) *There exists a bijection between the set of the E-pair of M_m-matrices and the set $\Omega_b \cap \{(k, l) \mid (k, l)$ are mutually prime and $0 \leq l < k\}$.*

Considering the other three equivalent correspondences of $\{A, B\}$ to $\{i, -1\}$, $\{1, i\}$, and $\{-1, i\}$, the set of the mutually prime pairs in Ω is covered by the set in Theorem 6.7 and its equivalent sets. We thus have

Corollary 6.8. *The set of the simple closed geodesics on the once punctured torus $\mathbb{T}_o$ is decomposed into two sets: a set of the projection of the axis of the Λ-matrix and its equivalent and a set of the projection of the axis of the M-matrix and its equivalent.*

In §4 we showed the conjugate between the generators of the groups Γ_o and $\tilde{\Gamma}_o$, and between those of Γ_t and $\tilde{\Gamma}_t$. Hence, Propositions 6.1 and 6.2, and Theorem 6.4 are translated into the context of complex matrices.

Proposition 6.9. *Every $\tilde{\Lambda}$-matrix associated with $\lambda \in \mathcal{N}(\Lambda)$ in the VS complex matrix tree is a generator of $\tilde{\Gamma}_o$. Every $\tilde{M}$-matrix associated with $m \in \mathcal{N}(M)$ in the VS complex matrix tree is also a generator of $\tilde{\Gamma}_o$.*

Proposition 6.10. *Let $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_2})$ be the quadruple of matrices associated with a VS quadruple $(x_1, x_2; y_1, y_2)$ in the VS complex matrix tree. Then $\tilde{M}_{y_1}$ and $\tilde{M}_{y_2}$ generate $\tilde{\Gamma}_o$.*

Theorem 6.11. *The axes of each* $\tilde{\Lambda}$*-matrix associated with* $\lambda \in \mathcal{N}(\Lambda)$ *and of each* $\tilde{M}$*-matrix associated with* $m \in \mathcal{N}(M)$ *in the VS matrix tree project to simple closed geodesics on* $\tilde{\mathbb{T}}_o$.

Corollary 6.12. *The axes in Theorem* 6.11 *also project to simple closed geodesics on* $\tilde{\mathbb{T}}_t$.

The *Euclidean height* of a geodesic $\tilde{\gamma}$ in $\mathbb{H}^3$ is defined in the same way as that of a geodesic in $\mathbb{H}^2$: let η and ξ be the two endpoints of $\tilde{\gamma}$; then it is defined as $|\eta - \xi|/2$ if η and ξ are finite or ∞ otherwise.

Proposition 6.13. *The Euclidean height of the axis of the* $\tilde{\Lambda}$*-matrix associated with* $\lambda \in \mathcal{N}(\Lambda)$ *and that of the axis of the* M*-matrix associated with* $m \in \mathcal{N}(M)$ *are*

$$\frac{1}{2}\sqrt{4 - \frac{1}{\lambda^2}} \ \ \textit{and} \ \ \frac{1}{2}\sqrt{4 - \frac{2}{m^2}}.$$

Proof. These are directly checked by (5.2) and (5.3). □

Theorem 1.6 follows from Theorem 4.4, Corollary 6.12, and Proposition 6.13.

Let us discuss the maximality of the Euclidean height of the axis of $\tilde{\Lambda}$.

Two complex binary indefinite quadratic forms f and g are *equivalent* if either there exists a non-zero $\alpha \in \mathbb{C}$ with $g = \alpha f$ or there exists an element $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{Z}[i])$ such that $g(x, y) = (f \circ L)(x, y) = f(ax + by, cx + dy)$. It is easily checked that $\sqrt{|D(f)|}/m_1(f) = \sqrt{|D(g)|}/m_1(g)$ if f and g are equivalent.

Let γ_f be a geodesic whose endpoints are the roots of the equation $f(x, 1) = 0$ for a complex binary quadratic form f. Define $H(\gamma_f) = \sup\{h_E(g(\gamma_f)) \mid g \in \mathrm{SL}(2, \mathbb{Z}[i])\}$, where $h_E(\gamma)$ stands for the Euclidean height of a geodesic γ.

Proposition 6.14. $2H(\gamma_f) = \sqrt{|D(f)|}/m_1(f)$.

Proof. We take a complex binary quadratic form $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$. There is no loss of generality if we suppose $D(f) = 1$.

If $\alpha \neq 0$, computing the root of $f(x, 1) = 0$, we get $h_E(\gamma_f) = 1/(2|\alpha|)$.

Let $\{(x_i, y_i)\}$ be a sequence of points in $\mathbb{Z}[i]$ with $\lim_{i\to\infty} |f(x_i, y_i)| = m_1(f)$. Choose $A_i \in \mathrm{SL}(2, \mathbb{Z}[i])$ with $A_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and define $(f \circ A_i)(x, y) = g_i(x, y) = a_i x^2 + b_i xy + c_i y^2$. We then have

$$\lim_{i\to\infty} |f(x_i, y_i)| = \lim_{i\to\infty} |(f \circ A_i)(1, 0)| = \lim_{i\to\infty} |g_i(1, 0)| = \lim_{i\to\infty} |a_i| = \lim_{i\to\infty} \frac{1}{2h_E(\gamma_i)},$$

where γ_i is a geodesic whose endpoints are the roots of $g_i(x, 1) = 0$. Since $H(\gamma_f) \geq h_E(\gamma_i) = 1/(2|a_i|)$, we get $2H(\gamma_f) \geq 1/m_1(f)$.

If $2H(\gamma_f) > 1/m_1(f)$, then there exists a form $g(x, y) = ax^2 + bxy + cy^2$ with $D(g) = 1$ which is equivalent to f and satisfies $2h_E(\gamma_g) > 1/m_1(f)$. Using $h_E(\gamma_g) = 1/(2|a|)$, we get $1/(2|a|) > 1/(2m_1(f))$. It follows that $m_1(g) = m_1(f) > |a| = |g(1, 0)|$, contrary to the definition of $m_1(g)$. $2H(\gamma_f) = 1/m_1(f)$ has been proved.

If $\alpha = 0$, then the equation is obtained from $H(\gamma_f) = \infty$ and $m_1(f) = 0$. □

For a matrix $\tilde{\Lambda}_\lambda$ in the complex VS matrix tree, associated with $\lambda \in \mathcal{N}(\Lambda)$, we define the form $f_{\tilde{\Lambda}_\lambda}$ by the fixed point equation of the action of $\tilde{\Lambda}_\lambda$. Let $\gamma_{f_{\tilde{\Lambda}_\lambda}}$ be a geodesic whose endpoints are its fixed points. It follows from Proposition 6.13 that $h_E(\gamma_{f_{\tilde{\Lambda}_\lambda}}) = (1/2)\sqrt{4-(1/\lambda^2)}$. Its double is the value of the VS spectrum for λ. Moreover, we can prove $m_1(f_{\tilde{\Lambda}_\lambda}) = f_{\tilde{\Lambda}_\lambda}(1,0) = \lambda$. A proposition is deduced from these facts and Proposition 6.14:

Proposition 6.15. *$H(\gamma_{f_{\tilde{\Lambda}_\lambda}})$ is attained by $h_E(\gamma_{f_{\tilde{\Lambda}_\lambda}})$ and $\sqrt{|D(f_{\tilde{\Lambda}_\lambda})|}/m_1(f_{\tilde{\Lambda}_\lambda}) = 2h_E(\gamma_{f_{\tilde{\Lambda}_\lambda}})$.*

Theorem 1.7 follows from Corollary 6.8 and Proposition 6.15.

7. Proofs

In this section we prove Theorems 5.6 and 5.12. Most of the section is a proof of the first theorem.

Let $(x_1,x_2;y_1,y_2)$ be a VS quadruple. Suppose that $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$ is a quadruple of matrices associated with $(x_1,x_2;y_1,y_2)$. In this section we always describe these matrices in the following way:

$$\Lambda_{x_k} = \begin{pmatrix} a_k & b_k \\ x_k & d_k \end{pmatrix} \text{ and } M_{y_k} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_k & \beta_k \\ y_k & \delta_k \end{pmatrix} \quad \text{for } k=1,2.$$

By Definition 5.2, $a_k, b_k, d_k, \alpha_k, \beta_k$, and δ_k are integers and satisfy

$$a_k + d_k = 4x_k,\ \ a_kd_k - b_kx_k = 1,\ \ \alpha_k+\delta_k = 4y_k,\ \ \alpha_k\delta_k - \beta_k y_k = 2. \tag{7.1}$$

Let $(x_1',x_2';y_1',y_2')$ be a child of $(x_1,x_2;y_1,y_2)$ and let $(\Lambda_{x_1'},\Lambda_{x_2'};M_{y_1'},M_{y_2'})$ be a quadruple of matrices defined by Algorithm VS from $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$. We give a very rough sketch of proofs: introducing some relations between a pair of matrices in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$, we prove that $M_{y_2'}$ is associated with y_2' and $\Lambda_{x_2'}$ is associated with x_2' provided that pairs of matrices satisfy these relations; then we show that pairs of matrices in $(\Lambda_{x_1'},\Lambda_{x_2'};M_{y_1'},M_{y_2'})$ satisfy the relations.

7.1. ΛM-pair and $M_{y_2'}$.

Recall that $M_{y_2'}$ is defined by $\Lambda_{x_1'}M_{y_1'}$ or $M_{y_1'}\Lambda_{x_1'}$ (see Remark 5.4), where both $\Lambda_{x_1'}$ and $M_{y_1'}$ are matrices inherited from the quadruple associated with $(x_1,x_2;y_1,y_2)$. To be precise, there are six ways of defining $M_{y_2'}$. We introduce six relations corresponding to them.

Definition 7.1. (i) If the entries of Λ_{x_1}, M_{y_1}, and M_{y_2} satisfy $a_1y_2 = x_1\alpha_2 + y_1$, then we say that (Λ_{x_1}, M_{y_2}) is *a ΛM-pair of type I_a* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

(ii) If the entries of Λ_{x_2}, M_{y_1}, and M_{y_2} satisfy $d_2y_2 = x_2\delta_2 + y_1$, then we say that (Λ_{x_2}, M_{y_2}) is *a ΛM-pair of type I_b* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

(iii) If the entries of Λ_{x_2}, M_{y_1}, and M_{y_2} satisfy

$$a_2y_1 = x_2\alpha_1 + y_2, \tag{7.2}$$

then we say that (Λ_{x_2}, M_{y_1}) is *a ΛM-pair of type I_c* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

(iv) If the entries of Λ_{x_1}, M_{y_1}, and M_{y_2} satisfy $d_1y_2 = x_1\delta_2 + y_1$, then we say that (Λ_{x_1}, M_{y_2}) is *a ΛM-pair of type II_a* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

(v) If the entries of Λ_{x_2}, M_{y_1}, and M_{y_2} satisfy $a_2y_2 = x_2\alpha_2 + y_1$, then we say that (Λ_{x_2}, M_{y_2}) is *a ΛM-pair of type II_b* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

(vi) If the entries of Λ_{x_2}, M_{y_1}, and M_{y_2} satisfy $d_2y_1 = x_2\delta_1 + y_2$, then we say that (Λ_{x_2}, M_{y_1}) is *a ΛM-pair of type II_c* in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$.

Example 7.2. Using (5.1) and Table 4, we can directly check that (Λ_1, M_3) (resp. (Λ_5, M_3), (Λ_5, M_1)) is a ΛM-pair of type I_a (resp. I_b, I_c) in $(\Lambda_1, \Lambda_5; M_1, M_3)$ and that (Λ_5, M_{59}) (resp. (Λ_{349}, M_{59}), (Λ_{349}, M_3)) is a ΛM-pair of type II_a (resp. II_b, II_c) in $(\Lambda_5, \Lambda_{349}; M_3, M_{59})$.

Lemma 7.3. *Suppose that $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is associated with $(x_1, x_2; y_1, y_2)$.*

(i) *If (Λ_{x_1}, M_{y_2}) is a ΛM-pair of type I_a (resp. of type II_a) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $\Lambda_{x_1} M_{y_2}$ (resp. $M_{y_2}\Lambda_{x_1}$) is an M-matrix associated with $4x_1y_2 - y_1$.*
(ii) *If (Λ_{x_2}, M_{y_2}) is a ΛM-pair of type I_b (resp. of type II_b) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $M_{y_2}\Lambda_{x_2}$ (resp. $\Lambda_{x_2} M_{y_2}$) is an M-matrix associated with $4x_2y_2 - y_1$.*
(iii) *If (Λ_{x_2}, M_{y_1}) is a ΛM-pair of type I_c (resp. of type II_c) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $\Lambda_{x_2} M_{y_1}$ (resp. $M_{y_1}\Lambda_{x_2}$) is an M-matrix associated with $4x_2y_1 - y_2$.*

Note that $4x_1y_2 - y_1$, $4x_2y_2 - y_1$, and $4x_2y_1 - y_2$ are equal to y_2' in the left, center, and right child of $(x_1, x_2; y_1, y_2)$, respectively.

Proof. These are proved by direct calculations. Here we prove (iii) for the case where (Λ_{x_2}, M_{y_1}) is a ΛM-pair of type I_c in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$.

Since

$$\Lambda_{x_2} M_{y_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_2\alpha_1 + b_2y_1 & a_2\beta_1 + b_2\delta_1 \\ x_2\alpha_1 + d_2y_1 & x_2\beta_1 + d_2\delta_1 \end{pmatrix}, \tag{7.3}$$

we have to verify

$$x_2\alpha_1 + d_2y_1 = 4x_2y_1 - y_2 \quad \text{and}$$
$$\mathrm{tr}(\Lambda_{x_2} M_{y_1}) = \frac{1}{\sqrt{2}}(a_2\alpha_1 + b_2y_1 + x_2\beta_1 + d_2\delta_1) = 2\sqrt{2}(4x_2y_1 - y_2).$$

The first equation follows from (7.2) and $\mathrm{tr}(\Lambda_{x_2}) = a_2 + d_2 = 4x_2$.

We modify some equations of (7.1) in the following forms:

$$b_2 = \frac{a_2d_2 - 1}{x_2},\ \beta_1 = \frac{\alpha_1\delta_1 - 2}{y_1},\ d_2 = 4x_2 - a_2,\ \text{and}\ \delta_1 = 4y_1 - \alpha_1.$$

Substituting these in the second term of the second equation, by (7.1), we get

$$\mathrm{tr}(\Lambda_{x_2} M_{y_1}) = \frac{1}{\sqrt{2}} \cdot \frac{1}{x_2y_1} \left(2a_2\alpha_1x_2y_1 - (a_2^2 + 1)y_1^2 - (\alpha_1^2 + 2)x_2^2 + 16x_2^2y_1^2\right).$$

By virtue of (1.2) and (7.2), it is changed into

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{x_2y_1}(16x_2^2y_1^2 - 2x_2^2 - (y_1^2 + y_2^2)) = \frac{4}{\sqrt{2}}(4x_2y_1 - y_2).$$

The other cases are proved in the same way. □

7.2. MM**-pair and** Λ_{x_2}**.** Let $(x_1, x_2; y_1, y_2)$ be a VS quadruple. Suppose that Λ_{x_1}, M_{y_1}, and M_{y_2} are associated with x_1, y_1, and y_2, respectively. We discuss a sufficient condition so that $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is associated with $(x_1, x_2; y_1, y_2)$.

Recall that in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ the matrix Λ_{x_2} is defined as either $M_{y_1}M_{y_2} = M_{y_1}^2\Lambda_{x_1}$ or $M_{y_2}M_{y_1} = \Lambda_{x_1}M_{y_1}^2$. We prove a lemma.

Lemma 7.4. *The matrix Λ_{x_2} is in the group* $\mathrm{SL}(2, \mathbb{Z})$.

Proof. Using $\beta_1y_1 = \alpha_1\delta_1 - 2$ and $\alpha_1 + \delta_1 = 4y_1$, we get

$$M_{y_1}^2 = \frac{1}{2} \begin{pmatrix} \alpha_1^2 + \beta_1y_1 & \beta_1(\alpha_1 + \delta_1) \\ y_1(\alpha_1 + \delta_1) & \beta_1y_1 + \delta_1^2 \end{pmatrix} = \begin{pmatrix} 2y_1\alpha_1 - 1 & 2y_1\beta_1 \\ 2y_1^2 & 2y_1\delta_1 - 1 \end{pmatrix}. \tag{7.4}$$

Hence, $M_{y_1}^2$ is an element of $\mathrm{SL}(2,\mathbb{Z})$. Since Λ_{x_2} is defined as $\Lambda_{x_1}M_{y_1}^2$ or $M_{y_1}^2\Lambda_{x_1}$, and $\Lambda_{x_1}\in\mathrm{SL}(2,\mathbb{Z})$, we conclude that Λ_{x_2} is in $\mathrm{SL}(2,\mathbb{Z})$. □

Definition 7.5. (i) If $x_2=\kappa(M_{y_2}M_{y_1})$, then we say that (M_{y_1},M_{y_2}) is *an MM-pair of type I* for $(x_1,x_2;y_1,y_2)$.

(ii) If $x_2=\kappa(M_{y_1}M_{y_2})$, then we say that (M_{y_1},M_{y_2}) is *an MM-pair of type II* for $(x_1,x_2;y_1,y_2)$.

Example 7.6. By virtue of (5.1) and Table 4, we can verify that (M_1,M_3) is an MM-pair of type I for $(1,5;1,3)$, (M_3,M_{11}) is an MM-pair of type I for $(1,65;3,11)$, (M_1,M_{17}) is an MM-pair of type I for $(5,29;1,17)$, and (M_3,M_{59}) is an MM-pair of type II for $(5,349;3,59)$.

Lemma 7.7. *Let $(x_1,x_2;y_1,y_2)$ be a VS quadruple. Suppose that Λ_{x_1}, M_{y_1}, and M_{y_2} are associated with x_1, y_1, and y_2, respectively.*

(i) *If (M_{y_1},M_{y_2}) is an MM-pair of type I for $(x_1,x_2;y_1,y_2)$, then $M_{y_2}M_{y_1}$ is a Λ-matrix associated with x_2.*

(ii) *If (M_{y_1},M_{y_2}) is an MM-pair of type II for $(x_1,x_2;y_1,y_2)$, then $M_{y_1}M_{y_2}$ is a Λ-matrix associated with x_2.*

Proof. Here we prove (i). Calculate $M_{y_2}M_{y_1}$:

$$M_{y_2}M_{y_1}=\frac{1}{2}\begin{pmatrix}\alpha_1\alpha_2+y_1\beta_2 & \beta_1\alpha_2+\delta_1\beta_2\\ \alpha_1y_2+y_1\delta_2 & \beta_1y_2+\delta_1\delta_2\end{pmatrix}.$$

Thanks to Lemma 7.4, we only have to check

$$x_2=\frac{1}{2}(\alpha_1y_2+y_1\delta_2)\ \text{ and }\ \mathrm{tr}(M_{y_2}M_{y_1})=\frac{1}{2}(\alpha_1\alpha_2+y_1\beta_2+\beta_1y_2+\delta_1\delta_2)=4x_2.$$

The former is just equal to $x_2=\kappa(M_{y_2}M_{y_1})$. This is changed into $2x_1=\alpha_2y_1-\alpha_1y_2$ by (1.2) and $\delta_2=4y_2-\alpha_2$.

The latter is verified by a similar computation to the proof of Lemma 7.3. Substituting β_1,β_2,δ_1, and δ_2, we get

$$\mathrm{tr}(M_{y_2}M_{y_1})=\frac{1}{2y_1y_2}\left(16y_1^2y_2^2-2(y_1^2+y_2^2)-(\alpha_2y_1-\alpha_1y_2)^2\right).$$

By virtue of (1.2) and $2x_1=\alpha_2y_1-\alpha_1y_2$, this is equal to $4x_2$.

We can prove (ii) in the same way. □

7.3. **ΛM-pair and $\Lambda_{x_2'}$.** Let $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$ be a quadruple of matrices associated with a VS quadruple $(x_1,x_2;y_1,y_2)$. We examine MM-pairs for $(x_1',x_2';y_1',y_2')$.

Lemma 7.8. (i) *If (Λ_{x_1},M_{y_2}) is a ΛM-pair of type I_a (resp. of type II_a) in $(\Lambda_{x_1},\Lambda_{x_2};\ M_{y_1},M_{y_2})$, then $(M_{y_1'},M_{y_2'})=(M_{y_2},\Lambda_{x_1}M_{y_2})$ (resp. $(M_{y_2},M_{y_2}\Lambda_{x_1})$) is an MM-pair of type I (resp. of type II) for the left child of $(x_1,x_2;y_1,y_2)$.*

(ii) *If (Λ_{x_2},M_{y_2}) is a ΛM-pair of type I_b (resp. of type II_b) in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$, then $(M_{y_1'},M_{y_2'})=(M_{y_2},M_{y_2}\Lambda_{x_2})$ (resp. $(M_{y_2},\Lambda_{x_2}M_{y_2})$) is an MM-pair of type II (resp. of type I) for the center child of $(x_1,x_2;y_1,y_2)$.*

(iii) *If (Λ_{x_2},M_{y_1}) is a ΛM-pair of type I_c (resp. of type II_c) in $(\Lambda_{x_1},\Lambda_{x_2};M_{y_1},M_{y_2})$, then $(M_{y_1'},M_{y_2'})=(M_{y_1},\Lambda_{x_2}M_{y_1})$ (resp. $(M_{y_1},M_{y_1}\Lambda_{x_2})$) is an MM-pair of type I (resp. of type II) for the right child of $(x_1,x_2;y_1,y_2)$.*

Proof. Making use of the proof of Lemma 7.3, here we give a proof of (iii) for the case where (Λ_{x_2}, M_{y_1}) is a ΛM-pair of type I_c in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$. The other cases are proved in the same way.

Using

$$M_{y'_k} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha'_k & \beta'_k \\ y'_k & \delta'_k \end{pmatrix} \quad \text{for } k = 1, 2, \tag{7.5}$$

we verify $2x'_2 = \alpha'_1 y'_2 + y'_1 \delta'_2$. For the right child of $(x_1, x_2; y_1, y_2)$, we have $x'_2 = 2y_1(4x_2y_1 - y_2) - x_2$, $y'_1 = y_1$, and $y'_2 = 4x_2y_1 - y_2$. By $(M_{y'_1}, M_{y'_2}) = (M_{y_1}, \Lambda_{x_2} M_{y_1})$ and (7.3), we get $\alpha'_1 = \alpha_1$ and $\delta'_2 = x_2\beta_1 + d_2\delta_1$. The relation is now written in the following way:

$$2(2y_1(4x_2y_1 - y_2) - x_2) = \alpha_1(4x_2y_1 - y_2) + y_1(x_2\beta_1 + d_2\delta_1).$$

This is directly checked by (7.2), $\alpha_1\delta_1 - y_1\beta_1 = 2$, $\alpha_1 + \delta_1 = 4y_1$, and $a_2 + d_2 = 4x_2$. □

Combining Lemmas 7.3, 7.7 and 7.8, we obtain the following proposition.

Proposition 7.9. (i) *If (Λ_{x_1}, M_{y_2}) is a ΛM-pair of type I_a (resp. type II_a) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $\Lambda_{x_1} M_{y_2}^2$ (resp. $M_{y_2}^2 \Lambda_{x_1}$) is a Λ-matrix associated with $2y_2(4x_1y_2 - y_1) - x_1$.*

(ii) *If (Λ_{x_2}, M_{y_2}) is a ΛM-pair of type I_b (resp. type II_b) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $M_{y_2}^2 \Lambda_{x_2}$ (resp. $\Lambda_{x_2} M_{y_2}^2$) is a Λ-matrix associated with $2y_2(4x_2y_2 - y_1) - x_2$.*

(iii) *If (Λ_{x_2}, M_{y_1}) is a ΛM-pair of type I_c (resp. type II_c) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, then $\Lambda_{x_2} M_{y_1}^2$ (resp. $M_{y_1}^2 \Lambda_{x_2}$) is a Λ-matrix associated with $2y_1(4x_2y_1 - y_2) - x_2$.*

Note that (i) corresponds to the case of the left child of $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$, that is, $\Lambda_{x'_2}$ is defined as either $\Lambda_{x_1} M_{y_2}^2$ or $M_{y_2}^2 \Lambda_{x_1}$ and $x'_2 = 2y_2(4x_1y_2 - y_1) - x_1$. Likewise, (ii) and (iii) correspond to the case of the center and right child of $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$.

7.4. ΛM**-pairs.** We have already proved that the matrices $\Lambda_{x'_2}$ and $M_{y'_2}$ defined by Algorithm VS from a quadruple $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ are associated with x'_2 and y'_2, respectively, provided that the pairs of matrices in the quadruple satisfy the hypotheses of Lemma 7.3. In the following, we prove that pairs of $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ satisfy these hypotheses.

Corresponding to the proof of Lemmas 7.3 and 7.8, here we discuss the case of the right child of $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$.

Proposition 7.10. *Let $(x_1, x_2; y_1, y_2)$ be a VS quadruple and let $(x'_1, x'_2; y'_1, y'_2)$ be its right child, that is, $(x'_1, x'_2; y'_1, y'_2) = (x_2, 2y_1(4x_2y_1 - y_2) - x_2; y_1, 4x_2y_1 - y_2)$. Suppose that (Λ_{x_2}, M_{y_1}) is a ΛM-pair of type I_c (resp. type II_c) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ associated with $(x_1, x_2; y_1, y_2)$. Then we have*

(i) *$(\Lambda_{x'_1}, M_{y'_2}) = (\Lambda_{x_2}, \Lambda_{x_2} M_{y_1})$ (resp. $(\Lambda_{x_2}, M_{y_1} \Lambda_{x_2})$) is a ΛM-pair of type I_a (resp. type II_a) in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$;*

(ii) *$(\Lambda_{x'_2}, M_{y'_2}) = (\Lambda_{x_2} M_{y_1}^2, \Lambda_{x_2} M_{y_1})$ (resp. $(M_{y_1}^2 \Lambda_{x_2}, M_{y_1} \Lambda_{x_2})$) is a ΛM-pair of type I_b (resp. type II_b) in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$;*

(iii) *$(\Lambda_{x'_2}, M_{y'_1}) = (\Lambda_{x_2} M_{y_1}^2, M_{y_1})$ (resp. $(M_{y_1}^2 \Lambda_{x_2}, M_{y_1})$) is a ΛM-pair of type I_c (resp. type II_c) in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$.*

Proof. Here we use (7.5) and

$$\Lambda_{x'_k} = \begin{pmatrix} a'_k & b'_k \\ x'_k & d'_k \end{pmatrix} \quad \text{for} \quad k = 1, 2.$$

We prove (i) for $(\Lambda_{x'_1}, M_{y'_2}) = (\Lambda_{x_2}, \Lambda_{x_2} M_{y_1})$. We have to check $a'_1 y'_2 = x'_1 \alpha'_2 + y'_1$. Since $\alpha'_2 = a_2\alpha_1 + b_2 y_1$ by (7.3), it is written as follows:

$$a_2(4x_2y_1 - y_2) = x_2(a_2\alpha_1 + b_2y_1) + y_1.$$

This is verified by (7.2), $a_2 + d_2 = 4x_2$, and $a_2d_2 - b_2x_2 = 1$.

Likewise, we can prove the other case of (i).

We prove (iii) for $(\Lambda_{x'_2}, M_{y'_1}) = (\Lambda_{x_2} M^2_{y_1}, M_{y_1})$. We have to check $a'_2 y'_1 = x'_2 \alpha'_1 + y'_2$. Using (7.4), we compute $\Lambda_{x_2} M^2_{y_1}$ and get $a'_2 = a_2(2y_1\alpha_1 - 1) + 2b_2y_1^2$. The relation is then written in the following way:

$$(a_2(2y_1\alpha_1 - 1) + 2b_2y_1^2)y_1 = (2y_1(4x_2y_1 - y_2) - x_2)\alpha_1 + (4x_2y_1 - y_2).$$

Using (1.2), (7.2) and $a_2 + d_2 = x_2$, it is changed into the form

$$a_2(2y_1\alpha_1 - 1) + 2b_2y_1^2 = \frac{1}{x_2}(2a_2y_1(4x_2y_1 - y_2) - a_2x_2 - 2y_1^2).$$

This is verified by (7.2), $a_2 + d_2 = 4x_2$, and $a_2d_2 - b_2x_2 = 1$.

Likewise, we can prove the other case of (iii).

We prove (ii) for $(\Lambda_{x'_2}, M_{y'_2}) = (\Lambda_{x_2} M^2_{y_1}, \Lambda_{x_2} M_{y_1})$. We have to check $d'_2 y'_2 = x'_2 \delta'_2 + y'_1$. In the same way as before, computing $\Lambda_{x_2} M^2_{y_1}$, we get $d'_2 = 2x_2y_1\beta_1 + d_2(2y_1\delta_1 - 1)$, and by (7.3) we have $\delta'_2 = x_2\beta_1 + d_2\delta_1$. The relation is then written in the following way:

$$\begin{aligned} &(2x_2y_1\beta_1 + d_2(2y_1\delta_1 - 1))(4x_2y_1 - y_2) \\ &= (2y_1(4x_2y_1 - y_2) - x_2)(x_2\beta_1 + d_2\delta_1) + y_1. \end{aligned} \tag{7.6}$$

Using (7.2), $a_2 + d_2 = 4x_2$, and $\alpha_1\delta_1 - \beta_1 y_1 = 2$, we get

$$2x_2y_1\beta_1 + d_2(2y_1\delta_1 - 1) = 2(4x_2y_1 - y_2)\delta_1 - 4x_2 - d_2,$$

$$x_2\beta_1 + d_2\delta_1 = \frac{1}{y_1}((4x_2y_1 - y_2)\delta_1 - 2x_2).$$

Substituting these in (7.6) and using (1.2), (7.2), $a_2 + d_2 = 4x_2$, and $\alpha_1 + \delta_1 = y_1$, we know that relation (7.6) is equivalent to the following:

$$(x_2\delta_1 - d_2y_1)(4x_2y_1 - y_2) = (a_2y_1 - x_2\alpha_1)(4x_2y_1 - y_2) = 2x_2^2 + y_1^2.$$

Likewise, the other case of (ii) is proved. □

In the same way, we can discuss the case of the left and center child of $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ and prove the following.

Proposition 7.11. *Let $(x_1, x_2; y_1, y_2)$ be a VS quadruple and let $(x'_1, x'_2; y'_1, y'_2)$ be its left child, that is, $(x'_1, x'_2; y'_1, y'_2) = (x_1, 2y_2(4x_1y_2 - y_1) - x_1; y_2, 4x_1y_2 - y_1)$. Suppose that (Λ_{x_1}, M_{y_2}) is a ΛM-pair of type I_a (resp. type II_a) in $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ associated with $(x_1, x_2; y_1, y_2)$. Then we have*

- (i) *$(\Lambda_{x'_1}, M_{y'_2}) = (\Lambda_{x_1}, \Lambda_{x_1} M_{y_2})$ (resp. $(\Lambda_{x_1}, M_{y_2}\Lambda_{x_1})$) is a ΛM-pair of type I_a (resp. type II_a);*
- (ii) *$(\Lambda_{x'_2}, M_{y'_2}) = (\Lambda_{x_1} M^2_{y_2}, \Lambda_{x_1} M_{y_2})$ (resp. $(M^2_{y_2}\Lambda_{x_1}, M_{y_2}\Lambda_{x_1})$) is a ΛM-pair of type I_b (resp. type II_b);*

(iii) $(\Lambda_{x'_2}, M_{y'_1}) = (\Lambda_{x_1} M_{y_2}^2, M_{y_2})$ *(resp.* $(M_{y_2}^2 \Lambda_{x_1}, M_{y_2})$*) is a* ΛM*-pair of type* I_c *(resp. type* II_c*).*

Proposition 7.12. *Let* $(x_1, x_2; y_1, y_2)$ *be a VS quadruple and let* $(x'_1, x'_2; y'_1, y'_2)$ *be its center child, that is,* $(x'_1, x'_2; y'_1, y'_2) = (x_2, 2y_2(4x_2y_2 - y_1) - x_2; y_2, 4x_2y_2 - y_1)$*. Suppose that* (Λ_{x_2}, M_{y_2}) *is a* ΛM*-pair of type* I_b *(resp. type* II_b*) in* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ *associated with* $(x_1, x_2; y_1, y_2)$*. Then we have*

(i) $(\Lambda_{x'_1}, M_{y'_2}) = (\Lambda_{x_2}, M_{y_2} \Lambda_{x_2})$ *(resp.* $(\Lambda_{x_2}, \Lambda_{x_2} M_{y_2})$*) is a* ΛM*-pair of type* II_a *(resp. type* I_a*);*

(ii) $(\Lambda_{x'_2}, M_{y'_2}) = (M_{y_2}^2 \Lambda_{x_2}, M_{y_2} \Lambda_{x_2})$ *(resp.* $(\Lambda_{x_2} M_{y_2}^2, \Lambda_{x_2} M_{y_2})$*) is a* ΛM*-pair of type* II_b *(resp. type* I_b*);*

(iii) $(\Lambda_{x'_2}, M_{y'_1}) = (M_{y_2}^2 \Lambda_{x_2}, M_{y_2})$ *(resp.* $(\Lambda_{x_2} M_{y_2}^2, M_{y_2})$*) is a* ΛM*-pair of type* II_c *(resp. type* I_c*).*

7.5. **Proof of Theorem 5.6.** Collecting the preceding results, we complete the proof of Theorem 5.6.

Lemma 7.13. *Let* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ *be a quadruple of matrices associated with a VS quadruple* $(x_1, x_2; y_1, y_2)$*.*

(i) *If* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ *is of type I, then* (Λ_{x_1}, M_{y_2})*,* (Λ_{x_2}, M_{y_2})*, and* (Λ_{x_2}, M_{y_1}) *are* ΛM*-pairs of type* I_a*,* I_b*, and* I_c*, respectively, in* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$*, and* (M_{y_1}, M_{y_2}) *is an* MM*-pair of type I for* $(x_1, x_2; y_1, y_2)$*.*

(ii) *If* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ *is of type II, then* (Λ_{x_1}, M_{y_2})*,* (Λ_{x_2}, M_{y_2})*, and* (Λ_{x_2}, M_{y_1}) *are* ΛM*-pairs of type* II_a*,* II_b*, and* II_c*, respectively, in* $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$*, and* (M_{y_1}, M_{y_2}) *is an* MM*-pair of type II for* $(x_1, x_2; y_1, y_2)$*.*

Proof. We prove this inductively.

For the node $\varnothing$, a, b, c in the VS matrix tree, namely, for $(\Lambda_1, \Lambda_5; M_1, M_3)$, $(\Lambda_1, \Lambda_{65}; M_3, M_{11})$, $(\Lambda_5, \Lambda_{349}; M_3, M_{59})$, and $(\Lambda_5, \Lambda_{29}; M_1, M_{17})$, we can directly verify the claim using (5.1) and Table 4 (see Examples 7.2 and 7.6).

Suppose that $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is of type I and claim (i) is true. Thanks to Lemma 7.8 and Propositions 7.10, 7.11, and 7.12, we obtain

- its left and right child $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ is of type I, and $(\Lambda_{x'_1}, M_{y'_2})$, $(\Lambda_{x'_2}, M_{y'_2})$, and $(\Lambda_{x'_2}, M_{y'_1})$ are ΛM-pairs of type I_a, I_b, and I_c, respectively, in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$, and $(M_{y'_1}, M_{y'_2})$ is an MM-pair of type I for $(x'_1, x'_2; y'_1, y'_2)$,
- its center child $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ is of type II, and $(\Lambda_{x'_1}, M_{y'_2})$, $(\Lambda_{x'_2}, M_{y'_2})$, and $(\Lambda_{x'_2}, M_{y'_1})$ are ΛM-pairs of type II_a, II_b, and II_c, respectively, in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$, and $(M_{y'_1}, M_{y'_2})$ is an MM-pair of type II for $(x'_1, x'_2; y'_1, y'_2)$.

In the same way, if $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is of type II and claim (ii) is true, we obtain

- its left and right child $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ is of type II, and $(\Lambda_{x'_1}, M_{y'_2})$, $(\Lambda_{x'_2}, M_{y'_2})$, and $(\Lambda_{x'_2}, M_{y'_1})$ are ΛM-pairs of type II_a, II_b, and II_c, respectively, in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$, and $(M_{y'_1}, M_{y'_2})$ is an MM-pair of type II for $(x'_1, x'_2; y'_1, y'_2)$,
- its center child $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ is of type I, and $(\Lambda_{x'_1}, M_{y'_2})$, $(\Lambda_{x'_2}, M_{y'_2})$, and $(\Lambda_{x'_2}, M_{y'_1})$ are ΛM-pairs of type I_a, I_b, and I_c, respectively, in $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$, and $(M_{y'_1}, M_{y'_2})$ is an MM-pair of type I for $(x'_1, x'_2; y'_1, y'_2)$.

The lemma is thus proved. □

By virtue of this lemma, we can apply Lemma 7.7 and Proposition 7.9 to each quadruple of matrices in the VS matrix tree. Finally, we conclude that, if $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is associated with $(x_1, x_2; y_1, y_2)$, then $(\Lambda_{x'_1}, \Lambda_{x'_2}; M_{y'_1}, M_{y'_2})$ is also associated with $(x'_1, x'_2; y'_1, y'_2)$.

Hence, Theorem 5.6 is proved.

7.6. **Proof of Theorem 5.12.** Let Λ_λ and M_m be matrices in the VS matrix tree. Thanks to Corollary 5.7, we can suppose that Λ_λ is associated with $\lambda \in \mathcal{N}(\Lambda)$ and M_m is associated with $m \in \mathcal{N}(M)$. Recall basic facts: trace and determinant are invariant on conjugacy classes, that is,

$$\mathrm{tr}(D) = \mathrm{tr}(C^{-1}DC) \text{ and } \det(D) = \det(C^{-1}DC) \text{ for } C, D \in \mathrm{SL}(2, \mathbb{C}).$$

It follows from (5.5) and these facts that $\mathrm{tr}(\Lambda_\lambda) = \mathrm{tr}(\tilde{\Lambda}_\lambda) = 4\lambda$, $\mathrm{tr}(M_m) = \mathrm{tr}(\tilde{M}_m) = 2\sqrt{2}m$, $\det(\Lambda_\lambda) = \det(\tilde{\Lambda}_\lambda) = 1$, and $\det(M_m) = \det(\tilde{M}_m) = 1$. Hence, to prove Theorem 5.12, we will check the forms of matrices $\tilde{\Lambda}_\lambda$ and $\tilde{M}_m$.

We can directly verify that $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_5$ are $\tilde{\Lambda}$-matrices associated with 1 and 5 and that $\tilde{M}_1$ and $\tilde{M}_3$ are $\tilde{M}$-matrices associated with 1 and 3 (see (5.4) and Table 5). We check the forms of matrices by induction. Let $(x_1, x_2; y_1, y_2)$ be a VS quadruple and let $(x'_1, x'_2; y'_1, y'_2)$ be a child of $(x_1, x_2; y_1, y_2)$. Recall that the parent has three children. For a quadruple of complex matrices $(\tilde{\Lambda}_{x_1}, \tilde{\Lambda}_{x_2}; \tilde{M}_{y_1}, \tilde{M}_{y_2})$, we suppose that $\tilde{\Lambda}_{x_1}$ is associated with x_1, $\tilde{\Lambda}_{x_2}$ with x_2, $\tilde{M}_{y_1}$ with y_1, and $\tilde{M}_{y_2}$ with y_2. Equivalently, in the VS matrix tree there exists a quadruple of real matrices $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ associated with $(x_1, x_2; y_1, y_2)$.

Recall that $\tilde{\Lambda}_{x'_2}$ is defined as either $\tilde{\Lambda}_{x'_1}\tilde{M}^2_{y'_1}$ or $\tilde{M}^2_{y'_1}\tilde{\Lambda}_{x'_1}$ (see Remark 5.4). Here we describe $\tilde{M}_{y'_1}$ by the following form:

$$\tilde{M}_{y'_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha'_1 + y'_1 i & \beta'_1 + \gamma'_1 i \\ 2y'_1 i & \delta'_1 - y'_1 i \end{pmatrix}, \tag{7.7}$$

where α'_1, β'_1, γ'_1, and δ'_1 are some integers. Since $y'_1 = y_1$ or y_2, by inductive hypothesis, $\tilde{M}_{y'_1}$ is associated with y'_1. Using $\mathrm{tr}(\tilde{M}_{y'_1}) = 2\sqrt{2}y'_1$ and $\det(\tilde{M}_{y'_1}) = 1$, we get

$$\tilde{M}^2_{y'_1} = \begin{pmatrix} 2y'_1\alpha'_1 - 1 + y'_1(\alpha'_1 + \beta'_1)i & 2y'_1(\beta'_1 + \gamma'_1 i) \\ 4(y'_1)^2 i & 2y'_1\delta'_1 - 1 + y'_1(\beta'_1 - \delta'_1)i \end{pmatrix}.$$

We thus know $\tilde{M}^2_{y'_1}$ is an element of $\mathrm{SL}(2, \mathbb{Z}[i])$. By virtue of $\tilde{\Lambda}_{x'_1} \in \mathrm{SL}(2, \mathbb{Z}[i])$, the matrix $\tilde{\Lambda}_{x'_2}$ must be in $\mathrm{SL}(2, \mathbb{Z}[i])$.

By inductive hypothesis and Corollary 5.7, $\Lambda_{x'_2}$ is associated with x'_2 and $M_{y'_2}$ is associated with y'_2. These matrices can then be described in the following way:

$$\Lambda_{x'_2} = \begin{pmatrix} a'_2 & b'_2 \\ x'_2 & d'_2 \end{pmatrix} \text{ and } \mathrm{tr}(\Lambda_{x'_2}) = 4x'_2,$$

$$M_{y'_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha'_2 & \beta'_2 \\ y'_2 & \delta'_2 \end{pmatrix} \text{ and } \mathrm{tr}(M_{y'_2}) = 2\sqrt{2}y'_2,$$

where a_2', b_2', d_2', α_2', β_2', and δ_2' are some integers. We calculate their conjugates:

$$\tilde{\Lambda}_{x_2'} = V^{-1}\Lambda_{x_2'}V = \begin{pmatrix} a_2' + x_2' i & \frac{1}{2}((d_2' - a_2') - (b_2' + x_2')i) \\ 2x_2' i & d_2' - x_2' i \end{pmatrix},$$

$$\tilde{M}_{y_2'} = V^{-1}M_m V = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_2' + y_2' i & \frac{1}{2}((\delta_2' - \alpha_2') - (y_2' + \beta_2')i) \\ 2y_2' i & \delta_2' - y_2' i \end{pmatrix}.$$

The $(2,1)$-entries of these equations satisfy the conditions of Definition 5.10. We have already proved that $\tilde{\Lambda}_{x_2'}$ is in $\mathrm{SL}(2,\mathbb{Z}[i])$, so the $(1,2)$-entry of $\tilde{\Lambda}_{x_2'}$ is a Gaussian integer. Since $\tilde{M}_{y_2'}$ is defined as either $\tilde{\Lambda}_{x_1'}\tilde{M}_{y_1'}$ or $\tilde{M}_{y_1'}\tilde{\Lambda}_{x_1'}$ (see also Remark 5.4), by $\tilde{\Lambda}_{x_1'} \in \mathrm{SL}(2,\mathbb{Z}[i])$ and (7.7), we easily know $(1/2)((\delta_2' - \alpha_2') - (y_2' + \beta_2')i)$ is also a Gaussian integer.

Theorem 5.12 is thus proved.

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