NON-SIMPLY CONNECTED MINIMAL PLANAR DOMAINS
IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. We prove that any non-simply connected planar domain can be properly and minimally embedded in $\mathbb{H}^2 \times \mathbb{R}$. The examples that we produce are vertical bi-graphs, and they are obtained from the conjugate surface of a Jenkins-Serrin graph.

1. INTRODUCTION

One of the most fruitful methods to obtain minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is by solving the Dirichlet problem for minimal graphs, with possibly infinite boundary values. This method was originally introduced by H. Jenkins and J. Serrin [10] for minimal graphs in $\mathbb{R}^3$, and extended to $\mathbb{H}^2 \times \mathbb{R}$ by B. Nelli and H. Rosenberg [16], P. Collin and H. Rosenberg [4], and L. Mazet, H. Rosenberg and the second author [11].

In [16], Nelli and Rosenberg also constructed vertical catenoids and helicoids. L. Hauswirth [7] generalized these examples by studying all minimal surfaces foliated by horizontal constant curvature curves. In this way, he obtained a 2-parameter family of minimal Riemann-type surfaces, which have genus zero and infinitely many ends.

Very recently, J. Pyo [17], F. Morabito and the second author [15] have constructed minimal surfaces of genus zero and finite total curvature. The method of construction in [15] consists of three steps. First, one solves the Jenkins-Serrin problem in a suitable geodesic polygonal domain with vertices $p_1, \ldots, p_{2n}$, satisfying $p_{2i}$ in $\mathbb{H}^2$ and $p_{2i-1}$ in the infinite boundary of $\mathbb{H}^2$ (which we will denote by $\partial_\infty \mathbb{H}^2$). Secondly, one uses the conjugation introduced by B. Daniel [5] and L. Hauswirth, R. Sa Earp and E. Toubiana [8] to obtain a minimal graph bounded by $n$ planar geodesics of the surface (not ambient geodesics in $\mathbb{H}^2 \times \mathbb{R}$), all of them at the same height. The complete surface is obtained by doubling the previous graph using the Schwarz reflection principle with respect to the horizontal slice that contains the horizontal geodesics (see Figure 1).

The main theorem of this paper shows that it is possible to take limits in the method of construction described in the above paragraph. Moreover, we have an important control of this limit surface, in such a way that we can prescribe the topology of the resulting minimal surface. This control also allows us to guarantee that the limit sets of distinct ends are disjoint. Regarding the conformal structure, the examples can be constructed with parabolic conformal type. This is not rare,
because in some sense the minimal surfaces that we construct are limits of minimal surfaces with finite total curvature.

So, the main result asserts:

**Theorem.** Let \( \Sigma \) be a non-simply connected planar domain. Then there exists a proper minimal embedding \( f : \Sigma \to \mathbb{H}^2 \times \mathbb{R} \). Furthermore, \( f \) satisfies:

1. \( f(\Sigma) \) is a vertical bi-graph symmetric with respect to a horizontal slice.
2. The annular ends of \( f(\Sigma) \) are asymptotic to vertical planes.
3. The embedding \( f \) can be constructed so that for any two distinct ends \( E_1, E_2 \) of \( \Sigma \), the limit sets \( L(E_1), L(E_2) \) in \( \partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \) are disjoint.
4. \( f(\Sigma) \) has parabolic conformal type.

The above theorem, which can be thought of as a generalization of the results in [18], gives a partial answer to a more general question proposed to the authors by A. Ros:

**Question 1.1.** Let \( M \) be an oriented open surface\(^1\). Can \( M \) be properly embedded into \( \mathbb{H}^2 \times \mathbb{R} \) as a minimal surface?

Furthermore, the main theorem says to us that we cannot expect classification theorems for properly embedded minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) just in terms of their topology, as in \( \mathbb{R}^3 \). (Meeks, Pérez and Ros recently proved in [14] that the only planar domains properly embedded in \( \mathbb{R}^3 \) are the plane, the catenoid, the helicoid and Riemann’s minimal surfaces.)

2. **Preliminaries**

We consider the Poincaré disk model for the hyperbolic plane, i.e.

\[
\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}
\]

\(^1\)We say that a surface is open if it is non-compact and without boundary.
with the hyperbolic metric $g_{-1} = \frac{4}{(1-x^2-y^2)^2} g_0$, where $g_0$ is the Euclidean metric in $\mathbb{R}^2$, and let $0 = (0,0)$ be the origin of $\mathbb{H}^2$. In this model, the asymptotic boundary $\partial_{\infty} \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle $\{x^2 + y^2 = 1\}$.

2.1. The existence of simple exhaustions. In this paper we will use that any open orientable surface $M$ has a smooth compact exhaustion $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$, called a simple exhaustion, with the following properties:

1. $M_1$ is a disk.
2. For any $n \in \mathbb{N}$, each component of $M_{n+1} - \text{Int}(M_n)$ has one boundary component in $\partial M_n$ and at least one boundary component in $\partial M_{n+1}$.
3. For any $n \in \mathbb{N}$, $M_{n+1} - \text{Int}(M_n)$ contains a unique non-annular component which topologically is a pair of pants or an annulus.

If $M$ has a finite topology with genus $g$ and $k$ ends, then we call the compact exhaustion simple if properties 1 and 2 hold, property 3 holds for $n \leq g + k$, and when $n > g + k$, all of the components of $M_{n+1} - \text{Int}(M_n)$ are annular.

The reader should note that, for any simple exhaustion of $M$, each component of $M - \text{Int}(M_n)$ is a smooth, non-compact proper subdomain of $M$ bounded by a simple closed curve and, for each $n \in \mathbb{N}$, $M_n$ is connected (see Figure 2).

![Figure 2. A topological representation of the terms $\Sigma_1$ to $\Sigma_3$ in the exhaustion of an open surface $M$ given in Lemma 2.1.](image)

In [6], Ferrer, Meeks and the first author proved the following result:

**Lemma 2.1** ([6]). Every orientable open surface admits a simple exhaustion.

A non-simply connected planar domain $\Sigma$ is a non-compact orientable surface of genus 0. As has been mentioned in the introduction, our main result is already known for minimal planar domains with finite topology. Hence, we are going to focus on planar domains with infinitely many ends. In this case, Lemma 2.1 gives us the following:

**Corollary 2.2.** Let $\Sigma$ be a planar domain with an infinite number of ends. Then $\Sigma$ admits a compact exhaustion $\mathcal{S} = \{\Sigma_1 \subset \Sigma_2 \subset \cdots\}$, satisfying:

1. $\Sigma_1$ is a sphere minus two disks.
2. Each component of $\Sigma_{n+1} - \text{Int}(\Sigma_n)$ has one boundary component in $\partial \Sigma_n$ and at least one boundary component in $\partial \Sigma_{n+1}$.
3. $\Sigma_{n+1} - \text{Int}(\Sigma_n)$ contains a unique non-annular component which topologically is a pair of pants.
We are also interested in the asymptotic behavior of the minimal surfaces we are going to construct. So, we need some background about the limit set of an end. In what follows, we will use the ideal boundary of $\mathbb{H}^2 \times \mathbb{R}$; $\partial_\infty (\mathbb{H}^2 \times \mathbb{R}) = (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \cup (\mathbb{H}^2 \times \{\pm \infty\})$.

**Definition 2.3.** Let $f : M \to \mathbb{H}^2 \times \mathbb{R}$ be a proper embedding of a surface $M$ with possibly non-empty boundary. The limit set of $M$ is

$$L(M) = \bigcap_{\alpha \in I} (\overline{f(M)} - f(C_\alpha)),$$

where $\{C_\alpha\}_{\alpha \in I}$ is the collection of compact subdomains of $M$ and the closure $\overline{f(M)} - f(C_\alpha)$ is taken in $\partial_\infty (\mathbb{H}^2 \times \mathbb{R})$. The limit set $L(E)$ of an end $E$ of $M$ is defined to be the intersection of the limit sets of all properly embedded subdomains of $M$ with compact boundary which represent $E$. Notice that $L(M)$ and $L(E)$ are closed sets of $\partial_\infty (\mathbb{H}^2 \times \mathbb{R})$.

2.2. **Minimal graphs.** Given an open domain $\Omega \subset \mathbb{H}^2$ and a smooth function $u : \Omega \to \mathbb{R}$, the graph surface of $u$ is minimal in $\mathbb{H}^2 \times \mathbb{R}$ when

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0,$$

where all terms are calculated with respect to the metric of $\mathbb{H}^2$.

**Definition 2.4.** We say that a domain $\Omega \subset \mathbb{H}^2$ is polygonal when it is bounded by geodesic arcs. A polygonal domain $\Omega \subset \mathbb{H}^2$ with a finite number of vertices (possibly at the infinite boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$) is said to be semi-ideal when no two consecutive vertices are ideal (i.e. they are at $\partial_\infty \mathbb{H}^2$) nor interior (i.e. they lie in $\mathbb{H}^2$).

Let $\Omega$ be a semi-ideal domain with a finite number of edges. In particular, $\Omega$ has an even number of vertices $p_1, \ldots, p_{2k}$ (cyclically ordered), with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$, for any $i = 1, \ldots, k$. We call $A_i$ (resp. $B_i$) the geodesic arc joining $p_{2i-1}, p_{2i}$ (resp. $p_{2i}, p_{2i+1}$); i.e.

$$A_i = (p_{2i-1}, p_{2i})_{\mathbb{H}^2}, \quad B_i = (p_{2i}, p_{2i+1})_{\mathbb{H}^2}.$$

We consider a horocycle $H_{2i-1}$ at each ideal vertex $p_{2i-1}$. Assume $H_{2i-1} \cap H_{2j-1} = \emptyset$ for any $i \neq j$. Given a polygonal domain $\mathcal{P}$ inscribed in $\Omega$ (i.e. a polygonal domain $\mathcal{P} \subset \Omega$ whose vertices are vertices of $\Omega$, possibly at $\partial_\infty \mathbb{H}^2$), we denote by $\Gamma(\mathcal{P})$ the part of $\partial \mathcal{P}$ outside the horocycles. (Observe that $\Gamma(\mathcal{P}) = \partial \mathcal{P}$ in the case that all the vertices of $\mathcal{P}$ are in $\mathbb{H}^2$.) Also let us call

$$\alpha(\mathcal{P}) = \sum_{i=1}^{k} |A_i \cap \Gamma(\mathcal{P})| \quad \text{and} \quad \beta(\mathcal{P}) = \sum_{i=1}^{k} |B_i \cap \Gamma(\mathcal{P})|,$$

where $|\bullet| = \text{length}_{\mathbb{H}^2}(\bullet)$.

**Definition 2.5.** A domain $\Omega \subset \mathbb{H}^2$ is called admissible when:

1. It is a convex semi-ideal polygonal domain with vertices $p_1, \ldots, p_{2k}$, with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$.
2. There exists a choice of disjoint horocycles $H_{2i-1}$ at the ideal vertices $p_{2i-1}$ such that:
   1. $\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$. 
   2. $|\bullet| = \text{length}_{\mathbb{H}^2}(\bullet)$. 

(ii) $2\alpha(P) < |\Gamma(P)|$ and $2\beta(P) < |\Gamma(P)|$, for every polygonal domain $P$ inscribed in $\Omega$, $P \neq \Omega$.

Up to an isometry of $\mathbb{H}^2$, we can assume that the origin $0 = (0,0)$ is contained in $\Omega$. We say that $(\Omega, u)$ is an admissible pair if $\Omega$ is an admissible domain and $u : \Omega \to \mathbb{R}$ is a solution to the minimal graph equation (1) with $u(0) = 0$ and whose boundary values are $+\infty$ on each edge $A_i$ and $-\infty$ on each $B_i$.

We remark that condition (i) in the above definition does not depend on the choice of horocycles, and if the inequalities of condition (ii) are satisfied for some choice of horocycles, then they continue to hold for “smaller” horocycles (see the argument given by Collin and Rosenberg in [4]).

The following lemma is very useful to know when a domain satisfying conditions (1) and (2)-(i) in the above definition is admissible. We will use this characterization in the proof of Lemma 2.1.

**Lemma 2.6** ([18]). Let $\Omega$ be a convex semi-ideal polygonal domain with vertices $p_1, \ldots, p_{2k}$, with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$. Suppose there exists a choice of disjoint horocycles $H_{2i-1}$ at the ideal vertices $p_{2i-1}$ such that $\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$. Then $\Omega$ is admissible if and only if $p_{2j} \in \mathbb{H}^2 - \overline{D_{2i-1}}$ for any $i \neq j, j + 1$, where $D_{2i-1}$ is the horodisk at $p_{2i-1}$ passing through $p_{2i-2}$ and $p_{2i}$.

The following theorem says that, given an admissible domain, there exists a unique solution $u : \Omega \to \mathbb{R}$ to the minimal graph equation (1) on $\Omega$ such that $(\Omega, u)$ is an admissible pair.

**Theorem 2.7** ([4] [11] [15]). Let $\Omega$ be an admissible domain with edges $A_1, B_1, \ldots, A_k, B_k$ (cyclically ordered). Then there exists a solution $u$ for the minimal graph equation (1) in $\Omega$ with boundary values

$$u|_{A_i} = +\infty \quad \text{and} \quad u|_{B_i} = -\infty, \quad \text{for any} \ i = 1, \ldots, k.$$ 

This solution is unique up to an additive constant.

Moreover, if we denote by $\Sigma^*$ the conjugate surface of the graph surface of $u$, then $\Sigma^*$ is a graph of a function $u^*$ over an ideal domain $\Omega^*$ with

$$\partial \Omega^* = \gamma_1^* \cup \delta_1^* \cup \ldots \cup \gamma_k^* \cup \delta_k^* \quad \text{(cyclically ordered),}$$

where:

1. $\delta_1^*, \ldots, \delta_k^*$ are concave curves, with respect to $\Omega^*$,
2. $u^*|_{\delta_i^*} = 0$, for $i = 1, \ldots, k$,
3. $\gamma_1^*, \ldots, \gamma_k^*$ are geodesics and $u^*|_{\gamma_i^*} = +\infty$, for any $i = 1, \ldots, k$,
4. $\delta_i^*$ is a horizontal geodesic curvature line of symmetry of $\Sigma^*$, for $i = 1, \ldots, k$,
5. $\delta_i^*$ and $\gamma_i^*$ (resp. $\delta_i^*$ and $\gamma_i^* + 1$) are asymptotic at their common endpoint at $\partial_\infty \mathbb{H}^2$.

In the following subsections we present some useful tools used in the proof of Theorem 2.7 which will also be used in the present paper.

### 2.2.1. Flux of a minimal graph along a curve

Let $u$ be a minimal graph defined on a domain $\Omega \subset \mathbb{H}^2$. Assume $\partial \Omega$ is piecewise smooth and $u$ extends continuously to $\overline{\Omega}$ (possibly with infinite values). We define the flux of $u$ along a curve $\Gamma \subset \partial \Omega$ as

$$F_u(\Gamma) = \int_\Gamma \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \eta \right) ds,$$
Figure 3. Left: The domain $\Omega^*$. Right: The conjugate graph $\Sigma^*$.

where $\eta$ is the outer normal to $\partial \Omega$ in $\mathbb{H}^2$ and $ds$ is the arc-length of $\partial \Omega$.

In the case $\Gamma \subset \Omega$, we can see $\Gamma$ in the boundary of different subdomains of $\Omega$, with two possible induced orientations. The flux $F_u(\Gamma)$ of $u$ along $\Gamma$ is then well defined up to sign, and $|F_u(\Gamma)|$ is well defined.

Lemma 2.8 ([16]). Let $u$ be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$.

(i) For every subdomain $\Omega' \subset \Omega$ such that $\overline{\Omega'}$ is compact, we have $F_u(\partial \Omega') = 0$.

(ii) Let $\Gamma$ be a piecewise smooth curve contained in the interior of $\Omega$, or a convex curve in $\partial \Omega$ where $u$ extends continuously and takes finite values. Then $|F_u(\Gamma)| < |\Gamma|$.

(iii) If $T \subset \partial \Omega$ is a geodesic arc such that $u$ diverges to $+\infty$ (resp. $-\infty$) as one approaches $T$ within $\Omega$, then $F_u(T) = |T|$ (resp. $F_u(T) = -|T|$).

Lemma 2.9 ([11]). Let $u$ be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$, and $T \subset \partial \Omega$ such that $|F_u(T)| = |T|$ (resp. $|F_u(T)| = -|T|$). Then $u$ goes to $+\infty$ (resp. $-\infty$) as we approach $T$ within $\Omega$.

2.2.2. Divergence lines. Let $\Omega \subset \mathbb{H}^2$ be a domain and $\{u_k\}_k$ a sequence of minimal graphs on $\Omega$. We define the convergence domain of $\{u_k\}_k$ as

$$B = \{ p \in \Omega \mid \{ |\nabla u_k(p)| \} \text{ is bounded} \},$$

and the divergence set of $\{u_k\}_k$ as

$$D = \Omega - B.$$ 

The following proposition describes the convergence domain and the divergence set of a sequence of minimal graphs.

Proposition 2.10 ([11]). Let $\Omega \subset \mathbb{H}^2$ be a domain and $\{u_k\}_k$ be a sequence of minimal graphs on $\Omega$. Then:

1. $D$ is composed of geodesic arcs contained in $\Omega$ (called divergence lines), each one joining two points of $\partial \Omega$ (including the vertices of $\Omega$).

2. Let $L \subset D$ be a divergence line. Passing to a subsequence, $|F_{u_k}(T)| \to |T|$ as $k \to +\infty$, for any geodesic arc $T \subset L$.

3. If $D = \emptyset$, then a subsequence of $\{u_k - u_k(p)\}_k$ converges uniformly on compact subsets of $\Omega$ to a minimal graph, for any $p \in \Omega$. 

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2.3. Conjugate minimal surfaces. Let \( \Sigma \) be a simply connected Riemann surface and \( X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R} \) be a conformal minimal immersion. It is known that \( h \) is a real harmonic function and \( \varphi = \pi \circ X \) is a harmonic map from \( \Sigma \) to \( \mathbb{H}^2 \). Daniel [5] and Hauswirth, Sa Earp and Toubiana [8] proved that there exists a minimal immersion \( X^* = (\varphi^*, h^*) : \Sigma \to \mathbb{H}^2 \times \mathbb{R} \), called a conjugate minimal immersion of \( X \), whose induced metric on \( \Sigma \) coincides with the one induced by \( X \), and such that \( h^* \) is the real harmonic conjugate function of \( h \) and the Hopf differential of \( \varphi^* \) is \(-Q_{\varphi} \), \( Q_{\varphi} \) being the Hopf differential of \( \varphi \). \( X^* \) is well defined up to an isometry of \( \mathbb{H}^2 \times \mathbb{R} \).

If \( N \) (resp. \( N^* \)) denotes the unit normal to \( X \) (resp. \( X^* \)), then \( \langle N, \partial_t \rangle = \langle N^*, \partial_t \rangle \) (i.e. their angle maps coincide). Moreover, the correspondence \( X \leftrightarrow X^* \) maps:

- Vertical geodesics of \( \mathbb{H}^2 \times \mathbb{R} \) to horizontal geodesic curvature lines along which the normal vector field of the surface is horizontal.
- Horizontal geodesics of \( \mathbb{H}^2 \times \mathbb{R} \) to geodesic curvature lines contained in vertical geodesic planes along which the normal vector field is tangent to the plane.

We will consider the conjugate surfaces of minimal graphs defined on convex domains. The surfaces obtained in this way are also minimal graphs (and consequently embedded), as ensured by the following Krust-type theorem given by Hauswirth, Sa Earp and Toubiana.

**Theorem 2.11** [8]. If \( \Sigma \) is a minimal graph over a convex domain \( \Omega \) of \( \mathbb{H}^2 \), then \( \Sigma^* \) is also a minimal graph over a (not necessarily convex) domain of \( \mathbb{H}^2 \).

3. Main theorem

Recall that the purpose of this paper is to show that any domain in the plane which is not simply connected can be properly embedded into \( \mathbb{H}^2 \times \mathbb{R} \) as a minimal bi-graph. Since this fact is known in the case of finite topology [15, 17], then we will focus throughout this section on the construction of examples with infinite topology. The case of surfaces with an uncountable number of ends will be particularly interesting.

The main tool in all this construction is Lemma 3.1 which gives us the approximation of an admissible pair by another admissible pair with an extra ideal vertex. Its proof follows from the ideas of Lemma 3.2 in [18]. Roughly speaking, this means that we are able to increase the topology of the conjugate graph by using surfaces which are close enough on compact regions. This kind of idea has been extensively used in the study of the Calabi-Yau problem for minimal surfaces in \( \mathbb{R}^3 \).

Given an admissible pair \((\Omega, u)\), we call \( \mathcal{V}_i(\Omega) \) the set of interior vertices of \( \Omega \), and \( \mathcal{V}_\infty(\Omega) \) the set of its ideal vertices. We will finally call \( \mathcal{V}(\Omega) \) the set of vertices of \( \Omega \), i.e.

\[
\mathcal{V}(\Omega) = \mathcal{V}_i(\Omega) \cup \mathcal{V}_\infty(\Omega).
\]

**Lemma 3.1.** Let \( \varepsilon, \delta \) be positive numbers, and \((\Omega, u)\) an admissible pair. For any ideal vertex \( P \) of \( \Omega \) and any \( R > 0 \) such that the hyperbolic disk \( B(R) \) centered at \((0,0)\) of radius \( R \) contains all the interior vertices of \( \Omega \), there exists an admissible pair \((\Omega, \tilde{u})\) satisfying:

1. Each boundary edge of \( \Omega \) that does not have \( P \) as an endpoint is contained in the boundary of \( \Omega \). In particular, \( \mathcal{V}(\Omega) \setminus \{P\} \subset \mathcal{V}(\Omega) \).
Figure 4. An example of an admissible domain with the boundary values for the corresponding admissible function $u$. In this case $\mathcal{V}_i(\Omega) = \{p_2, p_4, p_6\}$ and $\mathcal{V}_\infty(\Omega) = \{p_1, p_3, p_5\}$.

(2) $\tilde{\Omega}$ only contains two ideal vertices and an interior vertex which are not vertices of $\Omega$; this is, $\mathcal{V}_\infty(\tilde{\Omega}) - \mathcal{V}_\infty(\Omega) = \{P_1, P_2\}$ and $\mathcal{V}_i(\tilde{\Omega}) - \mathcal{V}_i(\Omega) = \{P_0\}$.

(3) $\Omega \cap B(R) \subset \tilde{\Omega} \cap B(R)$. In particular, $P_0 \in \mathbb{H}^2 - B(R)$.

(4) $\|\bar{u} - u\|_n < \epsilon$ in $\Omega_\delta \cap B(R)$, for any $n \in \mathbb{N}$, where $\Omega_\delta = \{p \in \Omega \mid \text{dist}_{\mathbb{H}^2}(p, \partial \Omega) > \delta\}$.

Figure 5. The domain $\Omega_\delta$.

Proof. Up to an isometry of $\mathbb{H}^2$, we can assume $P = (1, 0)$. We call $p_1, p_2, \ldots, p_{2k}$ the vertices of $\Omega$, cyclically ordered, so that $p_1 = P$. We consider $P_n^+ = e^{i/n}, P_n^- = e^{-i/n}$.

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$e^{-i/n}$, for any $n \in \mathbb{N}$. It is clear that $P^+_n \to P$ as $n \to +\infty$. We call $C^+_n$ (resp. $C^-_n$) the horocycle at $P^+_n$ (resp. $P^-_n$) passing through $p_2$ (resp. $p_{2k}$). For $n$ big enough, $C^+_n \cap C^-_n \neq \emptyset$. We call $P^0_n$ the intersection point in $C^+_n \cap C^-_n$ which is closer to $P$ (in the sense that the horodisk at $P$ passing through $P^0_n$ is contained in the horodisk at $P$ passing through the other point in $C^+_n \cap C^-_n$). We take $n$ big enough to assure $P^0_n \in \mathbb{H}^2 - B(R)$.

We call $p_1(n) = P^+_n$, $p_2(n) = p_2, \ldots, p_{2k}(n) = p_{2k}$, $p_{2k+1}(n) = P^-_n$, $p_{2k+2}(n) = P^0_n$, and $\Omega_n$ the polygonal domain with vertices $p_1(n), p_2(n), \ldots, p_{2k+2}(n)$. From the fact that $\Omega$ is an admissible domain and using that all the interior vertices of $\Omega_n$ remain fixed except for $p_{2k+2}(n)$, we can deduce that $\Omega_n$ is an admissible domain for $n$ large (here we use Lemma 2.6). Let $u_n : \Omega_n \to \mathbb{R}$ be the solution to the minimal graph equation (1) on $\Omega_n$ such that $(\Omega_n, u_n)$ is an admissible pair (it exists by Theorem 2.7). It is clear that $\Omega_n \to \Omega$ as $n \to +\infty$. Let us prove that $u_n \to u$ uniformly on compact sets of $\Omega$. By Proposition 2.10 it suffices to prove that the sequence $\{u_n\}$ does not have any divergence line.

Suppose by contradiction that $L \subset \Omega$ is a divergence line for $\{u_n\}$. We call $L_n$ the intersection of $\Omega_n$ with the complete geodesic of $\mathbb{H}^2$ containing $L$. Since $\Omega_n$ is convex (by the choice of $P^0_n$), we get that $L_n$ is connected. Let $P_n$ be a component of $\Omega_n - L_n$.

For any $i = 1, \ldots, k + 1$, we call $D_{2i-1}(n)$ the open horodisk at $p_{2i-1}(n)$ passing through $p_{2i-2}(n), p_{2i}(n)$, and we consider a sequence of nested horocycles $H_{2i-1}(n, m)$ at $p_{2i-1}(n)$ contained in $D_{2i-1}(n)$ such that $\text{dist}_{\mathbb{H}^2}(H_{2i-1}(n, m), \partial D_{2i-1}(n)) = m$, for any $m$. In particular, for $m$ large we have $H_{2i-1}(n, m) \cap H_{2j-1}(n, m) = \emptyset$, if $i \neq j$. Let $P_n(m)$ be the polygonal domain bounded by the part of $\partial P_n$ outside the horocycles $H_{2i-1}(n, m)$, together with geodesic arcs joining

![Figure 6](image-url)
the corresponding points in $\partial \mathcal{P}_n \cap (\bigcup_i H_{2i-1}(n, m))$. We also denote

$$\alpha_n(m) = \sum_{i=1}^{k+1} |A_i^n \cap \partial \mathcal{P}_n(m)|, \quad \beta_n(m) = \sum_{i=1}^{k+1} |B_i^n \cap \partial \mathcal{P}_n(m)|,$$

$$f_n(m) = F_{u_n}(\partial \mathcal{P}_n(m) - \partial \mathcal{P}_n),$$

where $A_i^n = (\{p_{2i-1}(n), p_{2i}(n)\})_{\mathbb{H}^2}$ and $B_i^n = (\{p_{2i}(n), p_{2i+1}(n)\})_{\mathbb{H}^2}$. We observe that, for any fixed $n$, $|f_n(m)| < |\partial \mathcal{P}_n(m) - \partial \mathcal{P}_n| \to 0$ as $m \to +\infty$. We can choose $\mathcal{P}_n$ to have

$$\beta_n(m) \geq \alpha_n(m).$$

We consider similar definitions associated to $\Omega$: for any $i = 1, \ldots, k$, let $D_{2i-1}$ be the open horodisk at $p_{2i-1}$ passing through $p_{2i-2}, p_{2i}$ and consider a sequence of nested horocycles $H_{2i-1}(m)$ at $p_{2i-1}$ contained in $D_{2i-1}$ such that $\text{dist}(H_{2i-1}(m), \partial D_{2i-1}) = m$, for any $m$.

We denote by $L(m)$ (resp. $L_n(m)$) the geodesic arc in $L$ (resp. $L_n$) outside the horocycles $H_{2i-1}(m)$ (resp. $H_{2i-1}(n, m)$). By Lemma 2.3

$$F_{u_n}(L_n(m)) = \beta_n(m) - \alpha_n(m) - f_n(m).$$

We observe that $F_{u_n}(L_n(m)) \geq 0$ for $m$ large.

- **Suppose $L$ has finite length.** Then $L$ joins a point $q_1 \in [p_{2i}, p_{2i+1}] \cup (p_{2i+1}, p_{2i+2})_{\mathbb{H}^2}$ to a point $q_2 \in [p_{2j}, p_{2j+1}] \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $0 \neq i \neq j$ (see Figure 4). We consider $m$ large enough so that $L(m) = L$ and $L_n(m) = L_n$. The endpoints of $L_n$ are $q_1$ and another point that we are going to call $q_2(n)$ (notice that $q_2(n) = q_2$ when $j \neq 0$). For $n$ large, one has $L \subset L_n$ and $|L_n| = |L| + \delta_n < +\infty$, where $\delta_n \geq 0$ converges to zero as $n \to +\infty$ ($\delta_n = 0$ in the case $j \neq 0$).

In this case, $c_n = \beta_n(m) - \alpha_n(m)$ does not depend on $m$ (it is also constant on $n$ when $j \neq 0$). Taking limits when $m$ goes to $+\infty$, we get $F_{u_n}(L_n) = c_n$. On the other hand, $[F_{u_n}(L_n)] \to |L|$ as $n \to +\infty$. Then $c_n \to |L|$. Let us see this is not possible. We call $C_1$ (resp. $C_2, C_2(n)$) the horocycle at $p_{2i+1}$ (resp. $p_{2j+1}, p_{2j+2}(n)$) passing through $q_1$ (resp. $q_2, q_2(n)$), and

$$d_1 = \text{dist}(C_1, p_{2i}), \quad d_2 = \text{dist}(C_2, p_{2j}), \quad d_2(n) = \text{dist}(C_2(n), p_{2j}(n)).$$

We have that $|d_1 - d_2(n)| = c_n$. Suppose $d_1 > d_2$ (the case $d_2 > d_1$ follows analogously). Thus, $d_1 > d_2(n)$ for $n$ large enough. Taking limits as $n \to +\infty$ we have $d_1 = |L| + d_2$. This implies that $p_{2j}$ (if $q_2 \in [p_{2j}, p_{2j+1}]_{\mathbb{H}^2}$) or $p_{2j+2}$ (if $q_2 \in (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$) lies on $D_{2i+1}$, a contradiction with the fact that $\Omega$ is admissible (see Lemma 2.6).

- **Now suppose that $L$ joins an ideal vertex $p_{2i+1}$, $i \neq 0$, to $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2} \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $j \neq i$.** It follows that $|L(m)| = m + d$, for some constant $d \in \mathbb{R}$, and for $n$ large, $L \subset L_n$ and $|L_n(m)| = |L(m)| + \delta_n$, with $\delta_n \geq 0$ converging to zero ($\delta_n = 0$ if $j \neq 0$).

On the other hand, for $m$ large we have that $c_n = m + \alpha_n(m) - \beta_n(m) \geq 0$ is constant on $m$ ($c_n = 0$ when $q = p_{2j}$). Then

$$|L(m)| - |F_{u_n}(L(m))| = d + c_n + f_n(m) + \lambda_n,$$

where $\lambda_n = F_{u_n}(L_n(m) - L(m))$ converges to zero as $n \to +\infty$ ($\lambda_n = 0$ if $j \neq 0$). Since $|L(m)| - |F_{u_n}(L(m))| \to 0$ as $n \to +\infty$, we conclude that
$c_n \to -d$. That implies that $d \leq 0$ and $p_{2j} \in \overline{D_{2i+1}}$, if $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2}$, or $p_{2j+2} \in \overline{D_{2i+1}}$, if $q \in (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, a contradiction.

- We now consider that $L$ joins two ideal vertices $p_{2i+1}, p_{2j+1}$, with $i \neq j$ both different from zero. Then we have $\alpha_n(m) = \beta_n(m)$ because of the choice of horocycles above. For any compact geodesic arc $T \subset L_n$ and $m$ large, we have $|F_{u_n}(T)| \leq |F_{u_n}(L_n(m))| = |f_n(m)|$. Taking $m \to +\infty$, we get $F_{u_n}(T) = 0$. But this contradicts that $|F_{u_n}(T)| \to |T|$ as $n \to +\infty$.

- If $L$ joins $p_1$ to another ideal vertex $p_{2i+1}$, $i \neq 0$, then $L_n \subset L$ for any $n$. We have $\beta_n(m) - \alpha_n(m) = m - c_n$, with $c_n \geq 0$ independent of $m$ ($c_n = 0$ when $L_n$ finishes at $p_{2k+2}(n)$), and $|L_n(m)| = m + \delta_n$, where $\delta_n \in \mathbb{R}$. Then, $|L_n(m)| - |F_{u_n}(L_n(m))| = \delta_n + c_n + f_n(m) \to \delta_n + c_n$, as $m \to +\infty$.

Since $|L_n(m)| - |F_{u_n}(L_n(m))| \to 0$ as $n \to +\infty$, we conclude that $\delta_n + c_n \to 0$. That implies that, for $n$ big enough, $p_{2k+2}(n) \in \overline{D_{2i+1}}$, a contradiction, as $\Omega_n$ is admissible.

- Finally, let us consider that $L$ joins $p_1$ to a point $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2} \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $j \neq 0$ (excluding the case $q = p_{2j}$). In this case we have $L_n \subset L$, $|L_n| < +\infty$ and $L_n = L_n(m)$ for $m$ large enough. When $n \to +\infty$, $|L_n(m)| - |F_{u_n}(L_n(m))| \to 0$, for any $m$. On the other hand, $|L_n(m)| - |F_{u_n}(L_n(m))| = |L_n| - |F_{u_n}(L_n)| \to |L_n| - c_n$ as $m \to +\infty$, where $c_n = \beta_n(m) - \alpha_n(m)$ for any $m$. The only possibility is $|L_n| - c_n \to 0$ as $n \to +\infty$. That contradicts the fact that $|L_n| \to +\infty$ when $n \to +\infty$ while $c_n$ remains bounded.

Then we get that $\{|\nabla u_n|\}_n$ is uniformly bounded on compact sets of $\Omega$. Then Lemma 3.3 holds for $(\Omega, \overline{u}) = (\Omega_{n_0}, u_{n_0})$ with some $n_0$ big enough, taking $P_0 = P_{n_0}^0$, $P_1 = P_{n_0}^+$ and $P_2 = P_{n_0}^-$.

Using Lemma 3.4 we are able to prove the main result of this paper.

**Theorem 3.2.** Let $\Sigma$ be a non-simply connected planar domain. Then, there exists a proper minimal embedding $f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$. Furthermore, $f$ satisfies:

1. $f(\Sigma)$ is a vertical bi-graph, symmetric with respect to a horizontal slice.
(2) The annular ends of $f(\Sigma)$ are asymptotic to vertical planes.
(3) The embedding $f$ can be constructed so that for any two distinct ends $E_1, E_2$ of $\Sigma$, the limit set $L(E_1), L(E_2)$ in $\partial_{\infty}(H^2 \times \mathbb{R})$ are disjoint.

Proof. In what follows, we are going to assume that $\Sigma$ has an infinite number of ends. Otherwise, we refer to [15]. From Corollary 2.2, the domain $\Sigma$ admits a simple exhaustion $\{\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots\}$. We are going to give a labeling of the boundary components of the simple exhaustion that will give us a description of the set of ends of $\Sigma$.

The boundary components of $\Sigma_1$ will be denoted by $\partial_0$ and $\partial_1$. The difference $\Sigma_2 \setminus \Sigma_1$ consists of a pair of pants $P_2$ and a cylinder $C_2$. If the cylinder has $\partial_i$ as a common boundary with $\Sigma_1$, then we denote by $\partial_{i,0}$ the other boundary component of $C_2$. On the other hand, if $\partial_j$ is the boundary component of $P_2$ that touches $\Sigma_1$, then we label $\partial_{j,0}$ and $\partial_{j,1}$ as the other two boundary components of $P_2$.

Now, assume we have already labeled the boundary components of $\Sigma_n$. We are going to label the connected components of $\partial \Sigma_{n+1}$. We know that $\Sigma_{n+1} \setminus \Sigma_n$ consists of cylinders $C_{n+1}^1, \ldots, C_{n+1}^k$ and just one pair of pants $P_{n+1}$. For a cylinder $C_{n+1}^i$, if the boundary component of $C_{n+1}^i$ which touches $\Sigma_n$ is labeled as $\partial_{i_1,\ldots,i_n,0}$, then we represent by $\partial_{i_1,\ldots,i_n,0}$ the other boundary component. In the case of the pair of pants $P_{n+1}$, if the boundary component of $P_{n+1}$ which touches $\Sigma_n$ is labeled as $\partial_{j_1,\ldots,j_n}$, then we denote by $\partial_{j_1,\ldots,j_n,0}$ and $\partial_{j_1,\ldots,j_n,1}$ the other two connected components of $\partial P_{n+1}$.

At this point, we are going to construct a sequence of admissible pairs $(\Omega_n, u_n)$, where $\Omega_n$ is an admissible domain with $2(n+1)$ edges, and a sequence of radii $\{R_n\}_{n \geq 2}$ and positive constants $\{\varepsilon_n\}_{n \geq 2}, \{\delta_n\}_{n \geq 2}$, satisfying:

(a) $\varepsilon_n, \delta_n \in (0, 1/2^n)$. In particular, $\sum_{n \geq 2} \varepsilon_n < +\infty$, and $\sum_{n \geq 2} \delta_n < +\infty$.
(b) $\Omega_n$ contains all the vertices of $\Omega_n$, except for an ideal vertex $p$.
(c) $\Omega_{n+1}$ only contains two ideal vertices and an interior vertex which are not vertices of $\Omega_n$. In particular, each boundary edge of $\Omega_n$ that does not contain $p$ is contained in $\partial \Omega_{n+1}$.
(d) $\Omega_n \cap B(R_{n+1}) \subset \Omega_{n+1} \cap B(R_{n+1})$.
(e) For any $k \in \mathbb{N}$, we have $\|u_{n+1} - u_n\|_k < \varepsilon_{n+1}$ in the domain $\Delta_n \overset{\text{def}}{=} \Omega_n(\delta_{n+1}) \cap B(R_{n+1})$. We recall that $\Omega_n(\delta_{n+1}) = \{p \in \Omega_n \mid \text{dist}_{H^2}(p, \partial \Omega_n) > \delta_{n+1}\}$.
(f) If $G_n$ denotes the graph of $u_n$, then the surface $S_n$ obtained by doubling the conjugate graph $G_n^* = \{u(x) \mid (x, u(x)) \in G_n\}$ has the same topological type as $\Sigma_n$.
(g) If $q_{i_1,\ldots,i_n}$ is an interior vertex of $\Omega_n$ and $x_{i_1,\ldots,i_n}$ is a point in $\partial \Omega_n(\delta_{n+1})$ with $\text{dist}_{H^2}(q_{i_1,\ldots,i_n}, x_{i_1,\ldots,i_n}) = \delta_{n+1}$, then the third coordinate of $(x_{i_1,\ldots,i_n}, u(x_{i_1,\ldots,i_n}))^*$ is less than $1/n$, where $(x_{i_1,\ldots,i_n}, u(x_{i_1,\ldots,i_n}))^*$ means the conjugate point in the conjugate graph $G_n^*$ corresponding to $(x_{i_1,\ldots,i_n}, u(x_{i_1,\ldots,i_n}))$.

The existence of such a sequence is obtained by using Lemma 3.1 in a recursive way: First, we take $(\Omega_1, u_1)$ as an admissible pair, where $\Omega_1$ is a 2-nice geodesic quadrilateral. We call $q_0, p_0, q_1, p_1$ the vertices of $\Omega_1$, with $p_0, p_1 \in \partial_{\infty}H^2$. For the sake of clarity, we are going to construct the admissible domain $\Omega_2$. Take $R_2 \geq 0$ such that $B(R_2)$ contains $q_0, q_1$, and $\varepsilon_2 \in (0, 1/4)$. We choose $\delta_2 \in (0, 1/4)$

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2See Definition 3.1 for the definition of the limit set of an end of a surface in a three-manifold. Recall that $\partial_{\infty}(H^2 \times \mathbb{R}) = (\partial_{\infty}H^2 \times \mathbb{R}) \cup (\mathbb{H}^2 \times \{\pm \infty\})$.  

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small enough so that \( \Omega_1(\delta_2) \cap \partial B(R_2) \) has two components. According to the notation we have introduced for the exhaustion \( \Sigma_n \), \( n \in \mathbb{N} \), we should add an interior vertex and two new ideal vertices around \( p_j \): We apply Lemma 3.1 to \( \Omega_1, \varepsilon_2, \delta_2, R_2 \) and \( p_j \). We call them \( q_{j,1} \) and \( p_{j,0}, p_{j,1} \), respectively. The other vertices \( q_i, p_i, q_j \) of \( \Omega_1 \) remain fixed, and we call them \( q_{i,0}, p_{i,0}, q_{j,0} \). The vertices of \( \Omega_2 \) are then \( q_{i,0}, p_{i,0}, q_{j,0}, p_{j,0}, q_{j,1}, p_{j,1} \), consecutively ordered. Note that this action has the topological effect of adding a pair of pants to the surface obtained by doubling the conjugate graph. In order to see this, we call \( \Gamma^* \) vertical geodesic planes small enough so that \( \Omega^* \) has two components. According to the notation we have introduced for the exhaustion \( \Sigma_1 \), \( \Sigma_2 \), \( \Sigma_n \), \( n \in \mathbb{N} \), we reflect \( \Gamma^* \) with respect to the slice \( \{ t = 0 \} \) to obtain a properly embedded minimal surface \( S_1 \) with genus zero and two ends. The ends are asymptotic to the vertical geodesic planes \( \gamma^*_{p_i} \times \mathbb{R} \). In this sense, we could say that there exists a natural correspondence between the ends of \( S_1 \) and the ideal vertices of \( \Omega_1, p_0 \) and \( p_1 \). After the application of Lemma 3.1 we are substituting the end associated to \( p_j \) by two new ends, the ones associated to \( p_{j,0} \) and \( p_{j,1} \), respectively. These two new ends are linked by the horizontal curve of symmetry \( \Gamma^*_{q_{j,1}} \) (see Figure 8).

![Figure 8. The domain \( \Omega_2 \).](image)

Now, assume we have \((\Omega_n, u_n)\) satisfying the conditions above, and let us construct \((\Omega_{n+1}, u_{n+1})\). We fix \( R_{n+1} > 0 \) such that \( B(R_{n+1}) \) contains all the interior vertices of \( \Omega_n \). We choose \( \delta_{n+1} \in (0, 1/2^{n+1}) \) small enough so that \( \Omega_n(\delta_{n+1}) \cap \partial B(R_{n+1}) \) has \( n + 1 \) components. We also take \( \varepsilon_{n+1} \in (0, 1/2^{n+1}) \). As above, the effect of adding a pair of pants to the boundary \( \partial j_1, ..., j_n \) of \( \Sigma_n \) means that we have to substitute the ideal vertex \( p_{j_1, ..., j_n} \) by two new ideal vertices, which we will call \( p_{j_1, ..., j_n, 0} \) and \( p_{j_1, ..., j_n, 1} \). To do this we apply, as before, Lemma 3.1 to: \( \Omega_n, \varepsilon_{n+1}, \delta_{n+1}, R_{n+1} \) and \( p_{j_1, ..., j_n, 1} \). A new interior vertex also appears; we call it \( q_{j_1, ..., j_n, 1} \). Finally, we relabel the other vertices just by adding a 0 in the subindex.
Let us define \( \Omega \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \Delta_n \). It is not hard to prove \( \Omega = \bigcup_{n=1}^{\infty} (\Omega_n \cap B(R_{n+1})) \) and \( \Omega \) is convex.

Taking into account that the sequence \( \{u_n\}_{n \in \mathbb{N}} \) satisfies item (e) and that \( \sum_n \varepsilon_n \) converges, then we obtain that \( \{u_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence, with respect to the smooth convergence on compact sets in \( \Omega \). Ascoli-Arzelà’s theorem implies that \( \{u_n\}_{n \in \mathbb{N}} \) converges to a smooth function \( u \) which is also a solution of (1) on \( \Omega \). Label \( G \) the graph surface of \( u \). As \( \Omega \) is convex, then Theorem 2.11 tells us that \( G^* \) is also a graph over a domain that we call \( \Omega^* \). In particular, \( G^* \) is embedded.

Claim 3.3. The limit graph \( G \) contains vertical straight lines placed over the interior vertices of \( \Omega_n \), for all \( n \in \mathbb{N} \).

In order to prove this claim, we fix \( n_0 \in \mathbb{N} \) and let \( q \) be a (fixed) interior vertex of \( \Omega_{n_0} \). Two geodesics in \( \partial \Omega_{n_0} \) arrive at this point, denoted by \( \gamma^+_{n_0} \) and \( \gamma^-_{n_0} \), with the properties that \( u_{n_0} |_{\gamma^+_n} = \pm \infty \). Recall that \( q \) is an interior vertex of \( \Omega_n \), for all \( n \geq n_0 \). Consider the corresponding boundary geodesics \( \gamma^+_n, \gamma^-_n \) in \( \partial \Omega_n \) with \( u_n |_{\gamma^+_n} = \pm \infty \).

First, we focus on the sequence \( \{\gamma^+_n\}_{n \in \mathbb{N}} \). Notice that, from the way in which we have obtained our sequence \( \{\Omega_n\}_{n \in \mathbb{N}} \), the initial conditions of the geodesic \( \gamma^+_n \) are given by \( \gamma^+_0(0) = q, (\gamma^+_n)'(0) = e^{i \theta_n} \), where the sequence of arguments \( \{\theta_n\}_{n \in \mathbb{N}} \) is monotone and bounded. So, \( \{\theta_n\}_{n \in \mathbb{N}} \) converges to a real number \( \theta \). Let \( \gamma^+ \) be the geodesic starting at \( q \) with \( (\gamma^+)'(0) = e^{i \theta} \). By construction, \( \{\gamma^+_n\}_{n \in \mathbb{N}} \) smoothly converges to \( \gamma^+ \). The geodesic \( \gamma^+ \) joins \( q \) with a point \( p^+ \in \partial_{\infty} \mathbb{H}^2 \). Moreover, \( \gamma^+ \) is part of \( \partial \Omega \). Let \( p^+ \) be the radial geodesic arriving at \( p^+ \). Taking our method of construction into account, we can guarantee that there are no interior vertices of \( \Omega_n, n \geq n_0 \), in the triangle \( R^+ \) whose sides are \( \gamma^+ \), a bounded piece of \( \gamma^-_{n_0} \) starting at \( q \) that we call \( \sigma \) and a convex curve \( \alpha \) (convex with respect to \( R^+ \)) which is asymptotic to \( p^+ \) at \( p^+ \) (see Figure 9). Let \( v \) be the solution to the Dirichlet problem associated to equation (1) on \( R^+ \) with boundary data \( +\infty \) on \( \gamma^+ \), \( -\infty \) on \( \sigma \) and \( \inf_{n \geq n_0} u_n \) on \( \alpha \). Notice that \( \inf_{n \geq n_0} u_n \) is continuous over \( \alpha \) and then solution \( v \) exists by Theorem 4.9 in [11]. Then the generalized maximum principle given by Collin and Rosenberg in [3] Theorem 2 (see also [11] Theorems 4.13 and 4.16) gives us that \( v \leq u_n \) in \( \Omega_n \cap R^+ \), for all \( n \geq n_0 \). This fact implies that \( u|_{\gamma^+} = +\infty \).

![Figure 9. The triangle $R^+$ corresponds to the darkest region.](image-url)
A similar argument gives us that \( u_{\gamma^-} = -\infty \), where \( \gamma^- \) is the limit of the sequence \( \{ \gamma_n \} \). So, the graph of \( u \) extends to a vertical line over the point \( q \). This concludes the proof of Claim 3.3.

Let \( q \) be an interior vertex of \( \Omega \) and \( \Gamma_q \overset{\text{def}}{=} \{ q \} \times \mathbb{R} \) the vertical line contained in the graph of \( u \), called \( M \). Then the conjugate curve \( \Gamma_q^* \subset M^* \) is a horizontal curvature line of symmetry (see Subsection 2.3).

**Claim 3.4.** For any interior vertex \( q \) in \( \Omega \), \( \Gamma_q^* \) is contained in the plane \( \{ t = 0 \} \). In particular, we can see \( \Gamma_q^* \) as a part of \( \partial \Omega^* \). In this sense, \( \Gamma_q^* \) is concave with respect to \( \Omega^* \). Moreover, the endpoints of \( \Gamma_q^* \) in \( \partial_\infty \mathbb{H}^2 \) are distinct.

In order to prove this claim, we assume that \( q \) is an interior vertex of \( \Omega_k \), for some \( k \in \mathbb{N} \). Then \( q \) is an interior vertex of \( \Omega_n \), for any \( n \geq k \). As a vertex of \( \Omega_n \), \( q \) appears represented as \( q_{i_0, \ldots, i_n} \), with \( i_j \in \{ 0, 1 \}, j = 1, \ldots, n \). Let \( x_{i_0, \ldots, i_n} \) be the corresponding point given by item (g). By construction, the sequence \( \{ x_{i_0, \ldots, i_n} \}_{n \in \mathbb{N}} \) converges to \( q \). So, \( \{ (x_{i_0, \ldots, i_n}, u(x_{i_0, \ldots, i_n}))^* \}_{n \in \mathbb{N}} \) is a sequence of points in \( \mathbb{H}^2 \times \mathbb{R} \) accumulating to \( \Gamma_q^* \). Taking item (g) into account (and using that the intrinsic distance between two vertical geodesics in the boundary of the graphs \( G_n \) remains uniformly bounded), this means that \( \Gamma_q^* \) is contained in the slice \( \{ t = 0 \} \), for any \( q \). The concavity of \( \Gamma_q^* \) with respect to \( \Omega^* \) is a simple consequence of the maximum principle for minimal surfaces, using vertical planes and the fact that \( \Gamma_q^* \) is a curve of symmetry.

Now, we are going to see that the endpoints of \( \Gamma_q^* \) are distinct. We proceed by contradiction. We suppose that both branches of \( \Gamma_q^* \) arrive at the same interior point \( d \in \partial_\infty \mathbb{H}^2 \), and let \( \sigma_\varepsilon \) be the geodesic in \( \mathbb{H}^2 \) whose endpoints \( d_\pm^\varepsilon \) are disposed symmetrically in \( \partial_\infty \mathbb{H}^2 \) with respect to \( d \) and such that \( \text{dist}_{\mathbb{H}^2}(d, d_\pm^\varepsilon) = \varepsilon \). We consider the bounded convex region \( D \) in \( \mathbb{H}^2 \) bounded by \( \Gamma_q^* \) and \( \sigma_\varepsilon \). If we apply the Gauss-Bonnet formula for \( \varepsilon \) small enough, we obtain that

\[
\text{Area}(D) \leq \int_{\Gamma_q^*} k_g - \pi,
\]

where \( k_g \) is the geodesic curvature of \( \Gamma_q^* \) in \( \mathbb{H}^2 \). Since the normal vector field of \( M \) rotates less than \( \pi \) along \( \Gamma_q \), we get \( \int_{\Gamma_q^*} k_g \leq \pi \), which contradicts (2).

We consider the closed set \( D_n = \overline{\Omega_n} \cap B(R_{n+1}) \), where \( \Omega_n' \) is a domain in \( \Omega_n \) with the same vertices as \( \Omega_n \) joined by arcs which are contained in \( \Omega_n \setminus \Omega_n(\delta_n) \). Denote by \( M_n \) the graph of \( u \) over \( D_n \). \( M_n \) is a minimal surface whose boundary contains vertical segments over the interior vertices of \( \Omega_n \). Then the conjugate surface \( M_n^* \) can be reflected with respect to the horizontal slice \( \mathbb{H}^2 \times \{ 0 \} \), and we obtain a surface \( S_n \) which is homeomorphic to \( \Sigma_n \). Furthermore, if we label \( f_n : \Sigma_n \to S_n \) this homeomorphism, we have for all \( i \leq n \) that \( f_n|_{\Sigma_i} \) coincides with the corresponding homeomorphism \( f_i : \Sigma_i \to S_i \), since \( D_i \subset D_n \).

Let \( S \) be the complete surface obtained by gluing together both \( G^* \) and its reflection with respect to \( \mathbb{H}^2 \times \{ 0 \} \). We have that \( S_n \) is a simple exhaustion of \( S \), and the sequence of homeomorphisms \( \{ f_n \}_{n \in \mathbb{N}} \) has a limit \( f : \Sigma \to S \).

In order to prove item (3) in the statement of the theorem, we consider \( E_1 \) and \( E_2 \) to be two different ends of \( f(\Sigma) \). Then there is a first natural \( n \in \mathbb{N} \) so that \( E_1 \) and \( E_2 \) are represented by two different components of \( \Sigma - (\bigcup_{i=1}^n \Sigma_i) \). Then \( \partial_{i_1, \ldots, i_n} \) is the boundary of a component representing both ends \( E_1 \) and \( E_2 \), but \( \partial_{i_1, \ldots, i_n, 0} \) represents \( E_1 \) and \( \partial_{i_1, \ldots, i_n, 1} \) represents \( E_2 \). Consider the points \( q_1 = q_{i_1, \ldots, i_n, 0} \) and
which are interior vertices of $\Omega$. From Claim 3.4 we know that $\Gamma_{q_1}^*$ and $\Gamma_{q_2}^*$ are curves in $\partial \Omega^*$ with distinct endpoints. Moreover, these two curves cannot be asymptotic. Let $\eta_1$ and $\eta_2$ be the geodesics in $\mathbb{H}^2$ joining an endpoint of $\Gamma_{q_1}^*$ to an endpoint of $\Gamma_{q_2}^*$ in such a way that $\eta_1 \cup \Gamma_{q_1}^* \cup \eta_2 \cup \Gamma_{q_2}^*$ bounds an open ideal quadrilateral $Q$. Hence, the limit sets $L(E_1)$ and $L(E_2)$ lie in different components of $\partial_{\infty}((\mathbb{H}^2 \setminus Q) \times \mathbb{R})$. \hfill $\square$

Finally, we would like to discuss the underlying conformal structure of the minimal surfaces we have just constructed. A good reference for the notation and results we are going to use is [1, §6 and §15].

As we have mentioned before, it is important to note that if $\Sigma$ has a finite number of ends, then the examples provided in the above theorem are those already constructed by Morabito and the second author. These examples have total curvature $-4\pi (k-1)$, where $k$ represents the number of ends. Thus, using a classical result by Huber [9], Morabito-Rodríguez’s surfaces are conformally equivalent to a sphere minus $k$ points. In particular, they are parabolic (see the definition below).

The examples with infinite topology given by Theorem 3.2 no longer have finite total curvature. However, we would like to point out that they can be constructed with parabolic conformal type, as explained in Remark 3.6.

**Definition 3.5.** An open Riemann surface $W$ is said to be parabolic if there are no non-constant negative subharmonic functions on $W$.

Among other important characterizations of parabolicity, we know that $W$ is parabolic if and only if one of the following conditions is fulfilled:

- the maximum principle for harmonic maps is valid on $W$;
- the harmonic measure of the ideal boundary of $W$ vanishes;
- there is no Green’s function defined on $W$.

**Remark 3.6.** The embedding $f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ in Theorem 3.2 can be constructed in such a way that $f(\Sigma)$ is parabolic. To do this, we consider the simple exhaustion $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ given in the proof of the theorem. We denote by $\lambda_n$ the extremal length between $\partial S_1$ and $\partial S_n$ and by $\mu_n$ the harmonic modulus $\mu_n \overset{\text{def}}{=} e^{\lambda_n}$. Notice that the surface obtained by doubling the graph $G_n^*$ is parabolic (it has finite total curvature). So, using Lemma 3.1 in a suitable way, we could guarantee in our inductive process that $\mu_n \geq n-1$. This fact implies that $S = f(\Sigma)$ is parabolic (see [1], p. 229).

**References**

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