TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 365, Number 12, December 2013, Pages 6185–6198
S 0002-9947(2013)05872-2
Article electronically published on July 10, 2013

C^1-REGULARITY FOR LOCAL GRAPH REPRESENTATIONS OF IMMERSIONS

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ABSTRACT. We consider immersions admitting uniform graph representations over the affine tangent space over a ball of fixed radius $r > 0$. We show that for sufficiently small C^0-norm of the graph functions, each graph function is smooth with small C^1-norm.

1. INTRODUCTION

An immersion in $\mathbb{R}^n$ is a differentiable function $f : M \to \mathbb{R}^n$ defined on a differentiable manifold M^m, such that for each $q \in M$ the mapping $f_*|T_qM$ is injective. A simple consequence of the implicit function theorem says that any immersion can locally be written as the graph of a function $u : B_r \to \mathbb{R}^k$ over the affine tangent space. Moreover, for a given $\lambda > 0$ we can choose $r > 0$ small enough such that $\|Du\|_{C^0(B_r)} \leq \lambda$. If this is possible at any point of the immersion with the same radius r, we call f an (r, λ)-immersion.

This concept is used in various geometric contexts; as an example and as motivation we consider the following compactness theorem proved by J. Langer [5]: Let $f^i : \Sigma^i \to \mathbb{R}^3$ be a sequence of immersed surfaces with uniformly L^p-bounded second fundamental form, $p > 2$, and uniformly bounded area. Then, after passing to a subsequence, there is a limit immersion $f : \Sigma \to \mathbb{R}^3$ and diffeomorphisms $\phi^i : \Sigma \to \Sigma^i$, such that $f^i \circ \phi^i$ converges in the C^1-topology to f. The result can be generalized to higher dimensions and codimensions; see [2], [3], and [4]. For proving the statement, one uses the Sobolev embedding and shows that a uniform L^p-bound for the second fundamental form with p greater than the dimension implies that for any $\lambda > 0$ there is an $r > 0$ such that every immersion is an (r, λ)-immersion.

This conclusion plays an important role in the proof of the compactness theorem and is just one example of a fundamental principle frequently used in geometric analysis and related fields: For a given global object, that is, a manifold embedded or immersed in $\mathbb{R}^n$ — usually of some specific geometric type, for example, a minimal surface — one investigates the local graph representations in order to derive further characteristics of the given object. For that one uses the global geometric information and derives specific properties satisfied by each of the graph functions, for example, bounds for specific norms, or particular partial differential equations to be satisfied. For each of the graph functions, it is then possible to apply all the well-known results from real analysis like embedding theorems or regularity theory.

Received by the editors July 30, 2011.
2010 *Mathematics Subject Classification.* Primary 53C42, 53B25.
This work was supported by the DFG-Forschergruppe *Nonlinear Partial Differential Equations: Theoretical and Numerical Analysis.* The contents of this paper were part of the author's dissertation, which was written at Universität Freiburg, Germany.

In this paper, we take a slightly different point of view. Instead of deriving special kinds of graph representations from specific geometrical settings, we shall take immersions with specific graph representations as our starting point. More precisely, our concept is the following: We consider an immersion and assume that it can be represented at any point over a ball of fixed radius $r > 0$ as the graph of a function u satisfying some specific properties; now, loosely speaking, we claim that each of the graph functions satisfies much better properties than one would anticipate from the ordinary rules of analysis.

In fact, there is a huge difference between a single graph and a graph coming from an immersion in the way described above. In the latter case, we know that such a graph representation is possible at *any* point of the immersion. In particular, two graphs that are close to each other have overlapping parts and each of the graphs satisfies specific properties, such as a bounded norm. Hence all graphs having one point in common depend on each other. This can be seen as a combinatorial restriction and allows much stronger results than one would expect using only the given properties of each single graph.

Let us first generalize the concept of immersions with bounded norm $\|Du\|_{C^0(B_r)} \leq \lambda$ for the graph functions u to immersions satisfying only a weaker bound. Again we consider C^1-immersions with graph representations $u : B_r \to \mathbb{R}^k$ over the affine tangent space, but this time we only assume that $\|u\|_{C^0(B_r)} \leq r\lambda$. If such a representation is possible at every point for fixed λ and r, we say that f is a C^0-(r, λ)-immersion. The factor r on the right-hand side is necessary for scale-invariance. A graph function of a C^0-(r, λ)-immersion does not need to be differentiable; this explains the notation that we use for this kind of immersion. For the precise definitions and further details, the reader is referred to Section 2.

Of course it is completely impossible to derive Lipschitz estimates for a single function satisfying only a C^0-bound, even if the function is known to be smooth or if the C^0-norm is particularly small. However, as we have claimed above, graph functions coming from immersions in the described way have much better properties than a single function. Denoting by m the dimension of the manifold on which the immersion is defined, we obtain the following theorem:

Theorem 1.1 (Embedding theorem for C^0-(r, λ)-immersions)**.** *For every $m \in \mathbb{N}$ there is a $\Lambda = \Lambda(m) > 0$, such that every C^0-(r, λ)-immersion with $\lambda \leq \Lambda$ is also an $(r, \frac{\lambda}{\Lambda})$-immersion.*

The constant Λ can be given explicitly by $\Lambda(m) := 10^{-5} m^{-2}$.

Hence a sufficiently small C^0-norm implies that each graph function is smooth with small C^0-norm of Du, that is, with small Lipschitz constant. Equivalently, we can say that the space of C^0-(r, λ)-immersions embeds into the space of $(r, \frac{\lambda}{\Lambda})$-immersions. The statement is true in arbitrary codimension and also for noncompact manifolds.

As here we are assuming only a small C^0-norm, we obtain all at once whole classes of new embedding theorems — provided the functions come from graph representations as described above. For example, one can think of the case of Hölder continuous $C^{0,\alpha}$-graphs or the Sobolev border case of $W^{2,m}$-graphs in dimension m.

The question arises whether the result will still be true if we assume graph representations are not over the affine tangent space, but over other appropriately chosen m-spaces. In the appendix we will show that this is not the case.

2. Notation and definitions

We begin with some general notation: For $n = m+k$, let $G_{n,m}$ denote the Grassmannian of (nonoriented) m-dimensional subspaces of $\mathbb{R}^n$. Unless stated otherwise, let B_ϱ denote the open ball in $\mathbb{R}^m$ of radius $\varrho > 0$ centered at the origin.

Now let M be an m-dimensional manifold without boundary and $f : M \to \mathbb{R}^n$ a C^1-immersion. Let $q \in M$ and let T_qM be the tangent space at q. Identifying vectors $X \in T_qM$ with $f_*X \in T_{f(q)}\mathbb{R}^n$, we may consider T_qM as an m-dimensional subspace of $\mathbb{R}^n$. In this manner we define the tangent map

$$\begin{array}{rcl} \tau_f : M & \to & G_{n,m}, \\ q & \mapsto & T_qM. \end{array} \tag{2.1}$$

The notion of an (r,λ)-immersion. We call a mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ a *Euclidean isometry*, if there is a rotation $R \in \mathbb{SO}(n)$ and a translation $T \in \mathbb{R}^n$, such that $A(x) = Rx + T$ for all $x \in \mathbb{R}^n$.

For a given point $q \in M$, let $A_q : \mathbb{R}^n \to \mathbb{R}^n$ be a Euclidean isometry, which maps the origin to $f(q)$, and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(q) + \tau_f(q)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the standard projection onto the first m coordinates.

Finally let $U_{r,q} \subset M$ be the q-component of the set $(\pi \circ A_q^{-1} \circ f)^{-1}(B_r)$. Although the isometry A_q is not uniquely determined, the set $U_{r,q}$ does not depend on the choice of A_q.

We come to the central definition (as first defined in [5]):

Definition 2.1. An immersion f is called an (r,λ)-immersion, if for each point $q \in M$, the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a differentiable function $u : B_r \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_r)} \le \lambda$.

Here, for any $x \in B_r$ we have $Du(x) \in \mathbb{R}^{k\times m}$. In order to define the C^0-norm for Du, we have to fix a matrix norm for $Du(x)$. Of course all norms on $\mathbb{R}^{k\times m}$ are equivalent, therefore our results are true for any norm (possibly up to multiplication by some positive constant). Let us agree upon

$$\|A\| = \left(\sum_{j=1}^{m} |a_j|^2 \right)^{\frac{1}{2}}$$

for $A = (a_1, \ldots, a_m) \in \mathbb{R}^{k\times m}$. For this norm we have $\|A\|_{\mathrm{op}} \le \|A\|$ for any $A \in \mathbb{R}^{k\times m}$ and the operator norm $\|\cdot\|_{\mathrm{op}}$. Hence the bound $\|Du\|_{C^0(B_r)} \le \lambda$ directly implies that u is λ-Lipschitz. Moreover the norm $\|Du\|_{C^0(B_r)}$ does not depend on the choice of the isometry A_q.

The notion of a C^0-(r,λ)-immersion. Every (r,λ)-immersion admits a local representation as a graph of a differentiable function u with $\|Du\|_{C^0(B_r)} \le \lambda$. This inequality corresponds to an estimate of the slope of the graph, i.e. to an estimate of the Lipschitz constant of u. It is a natural generalization to consider immersions with graph functions u, which satisfy only a bound for some weaker norm. Any such definition should reasonably be scale-invariant (i.e. if f is an (r,λ)-immersion and $c > 0$, then cf is a (cr,λ)-immersion).

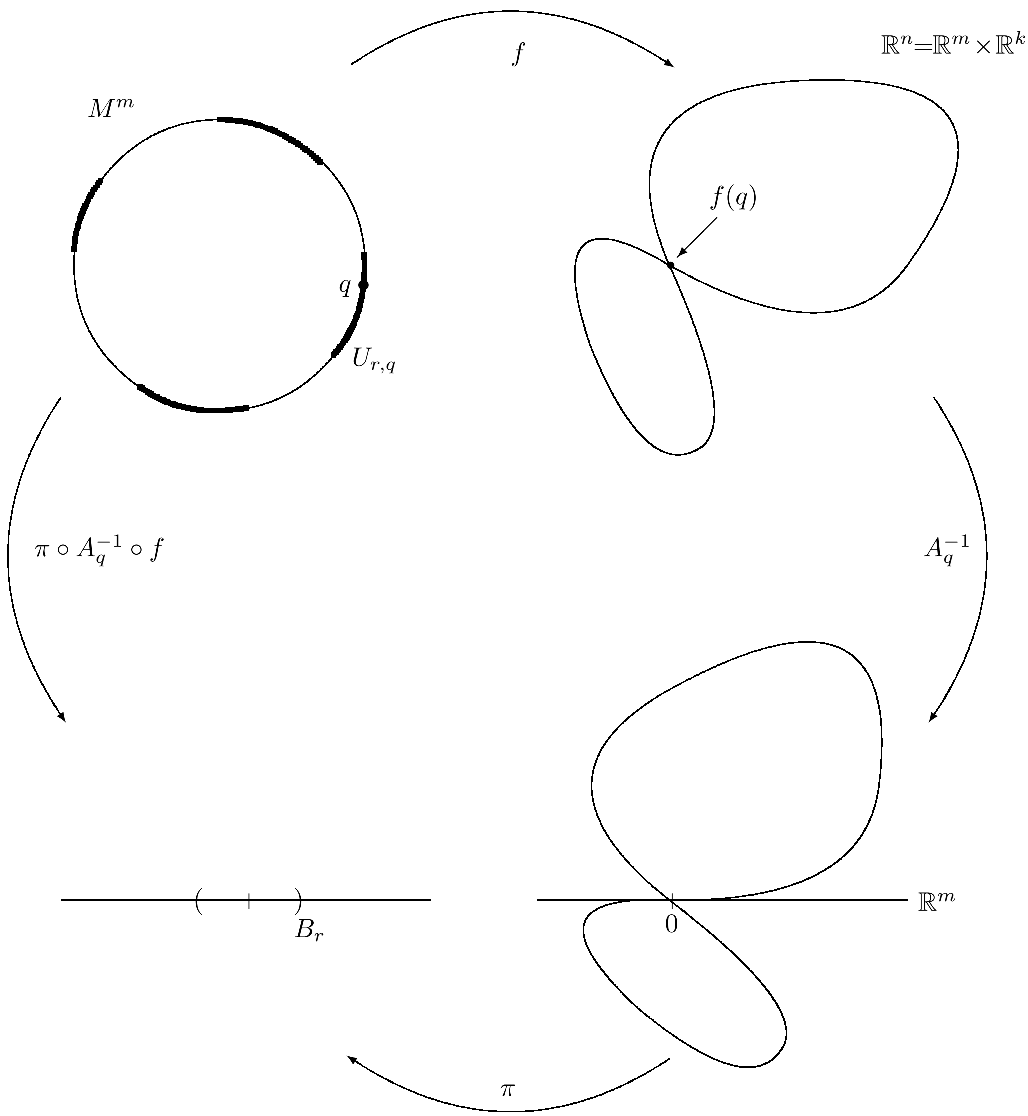

FIGURE 2.1. Local representation as a graph. The subset of M drawn in bold lines represents the pre-image $(\pi \circ A_q^{-1} \circ f)^{-1}(B_r)$.

Assuming only a bound for the C^0-norm yields the notion of a C^0-(r,λ)-immersion:

Definition 2.2. An immersion f is called a C^0-(r,λ)-immersion, if for each point $q \in M$ the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a continuous function $u : B_r \to \mathbb{R}^k$ with $\|u\|_{C^0(B_r)} \leq r\lambda$.

It would not be sensible here to assume $\|u\|_{C^0(B_r)} \leq \lambda$, as the notion of C^0-(r,λ)-immersions would not be scale-invariant then. For that reason we require the bound $r\lambda$.

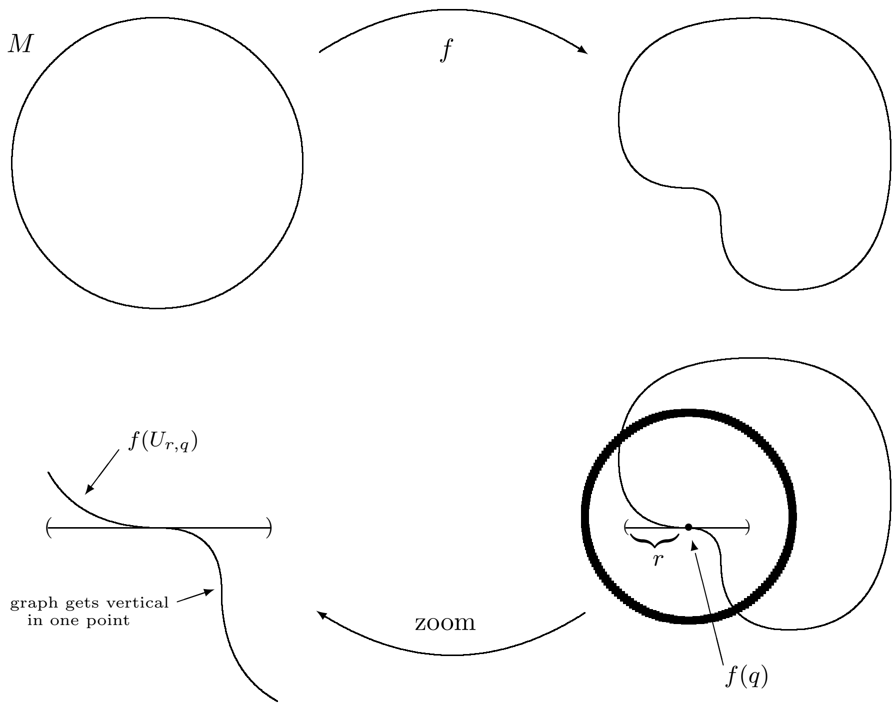

FIGURE 2.2. A simple example which shows how a graph function of a smooth C^0-(r, λ)-immersion fails to be differentiable (here e.g. $\lambda = 2$).

Here we require u only to be a continuous function. Note that the assumption on f to be a smooth immersion does *not* imply that u is differentiable. Surely the implicit function theorem ensures a smooth graph representation over the tangent space. However this representation might only be possible for radii less than r. Over the ball B_r one might have a continuous graph representation with a graph which gets vertical in a point. Hence smoothness of f does not guarantee smoothness of u.

Obviously every (r, λ)-immersion is also a C^0-(r, λ)-immersion. Surprisingly, in some sense the opposite is also true: Every C^0-(r, λ)-immersion is also an $(r, \frac{\lambda}{\Lambda})$-immersion if $\lambda \leq \Lambda = \Lambda(m)$. This is precisely the statement of Theorem 1.1. Moreover, as we have seen above, a graph function u does not need to be smooth in the case of a C^0-(r, λ)-immersion; for that reason we may interpret Theorem 1.1 also as a higher regularity result.

Reformulation of Theorem 1.1. Theorem 1.1 is a statement for C^0-(r, λ)-immersions with fixed r and λ. We like to give an alternative formulation which holds for any immersion.

For an immersion $f : M \to \mathbb{R}^n$, let $r_1(f, \lambda) \geq 0$ be the maximal radius such that for any $q \in M$, the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a C^1-function $u : B_r \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_r)} \leq \lambda$.

Similarly, let $r_0(f,\lambda) \geq 0$ be the maximal radius such that for any $q \in M$, the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a C^0-function with $\|u\|_{C^0(B_r)} \leq r\lambda$.

Obviously

$$r_1(f,\lambda) \;\leq\; r_0(f,\lambda).$$

With this notation Theorem 1.1 reads as follows:

Theorem 2.3 (Reformulation of Theorem 1.1)**.** *For every $m \in \mathbb{N}$ there is a $\Lambda = \Lambda(m) > 0$ such that for every immersion $f : M^m \to \mathbb{R}^n$ and all $\lambda \leq \Lambda$, the inequality $r_1(f,\lambda/\Lambda) \geq r_0(f,\lambda)$ holds.*

The constant Λ can be given explicitly by $\Lambda(m) := 10^{-5}m^{-2}$.

3. Preparations for the proof

The main step of the proof is to compare the position of two tangent spaces at points on the surface that are not too far from each other. For that we have to find a sufficiently large set $U \subset M$, such that $f(U)$ may be written over *both* spaces as a graph with small C^0-norm respectively; this will be done in Lemma 3.3. To compare the spaces with each other, we shall use a finite number of comparison points on each space, constructed by means of the immersion piece $f(U)$. A concrete estimate (in a slightly more general formulation) is deduced in Lemma 3.1. Using this method, we are able to deduce smoothness of the graphs and to estimate the Lipschitz constant. However, due to the limited size of $f(U)$, this estimate holds only on a smaller radius $\varrho < r$. Lemma 3.2 shows how to enlarge the radius, provided the Lipschitz constant is sufficiently small. This enables us to prove the theorem.

We begin with the first statement, the comparison of two spaces by distance bounds of finitely many points. The proof consists of elementary geometry and is carried out here in full detail:

Lemma 3.1. *Let $E \in G_{n,m}$, let $v_1,\ldots,v_m \in E \subset \mathbb{R}^n$ be points on E and $L \leq 1$ a constant. If for the standard basis $\{e_1,\ldots,e_m\}$ of $\mathbb{R}^m$,*

$$|v_j - (e_j,0)| \leq \frac{1}{3\sqrt{m}}\,L \qquad \text{for all } j \in \{1,\ldots,m\}, \tag{3.1}$$

then E is a graph over $\mathbb{R}^m \times \{0\}$, that is, there exists an $A = (a_1,\ldots,a_m) \in \mathbb{R}^{k\times m}$ with

$$E = \operatorname{span}\{(e_1,a_1),\ldots,(e_m,a_m)\},$$

and moreover

$$\|A\| = \left(\sum_{j=1}^{m} |a_j|^2\right)^{\frac{1}{2}} \leq L. \tag{3.2}$$

Proof. First we show that E is a graph over $\mathbb{R}^m \times \{0\}$. Suppose E is not written as a graph over $\mathbb{R}^m \times \{0\}$. If π denotes the standard projection from $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ onto $\mathbb{R}^m$, then

$$0 \leq \dim \pi(E) \leq m-1. \tag{3.3}$$

We split the points v_j into $v_j = (v_j^h, v_j^v) \in \mathbb{R}^m \times \mathbb{R}^k$. Then, on the one hand,

$$v_1^h,\ldots,v_m^h \in \pi(E), \tag{3.4}$$

and on the other hand, with (3.1) and $L \leq 1$ for each j,

$$|v_j^h - e_j| \quad < \quad \frac{1}{\sqrt{m}}. \tag{3.5}$$

The following constructions are carried out within the subspace $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$. By (3.3) there exists an $e \neq 0$ in the orthogonal complement $[\pi(E)]^\perp \subset \mathbb{R}^m$. Set $G := \operatorname{span}\{e\}$. Now consider the cube $Q := [-1,1]^m \subset \mathbb{R}^m$ centered at the origin. Then there is an $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $G \cap \partial Q = \{-s, s\}$ and hence also a $\nu \in \{1, \ldots, m\}$ with $|s_\nu| = 1$. Without loss of generality $s_\nu = 1$, otherwise pass to $-s$.

As long as $s \neq e_\nu$, the points 0, e_ν and s constitute a rectangular triangle with hypotenuse in G. The splitting $e_\nu = e_\nu^\top + e_\nu^\perp \in G \oplus G^\perp$ yields with the Euclidean theorem $|e_\nu|^2 = |e_\nu^\top||s|$, hence

$$|e_\nu^\perp - e_\nu| = \frac{1}{|s|} \geq \frac{1}{\sqrt{m}},$$

also in the case $s = e_\nu$. But then $|w - e_\nu| \geq \frac{1}{\sqrt{m}}$ for *all* $w \in G^\perp$ and as $\pi(E) \subset G^\perp$,

$$|w - e_\nu| \geq \frac{1}{\sqrt{m}} \quad \text{for all} \quad w \in \pi(E). \tag{3.6}$$

But (3.5) is also true for $j = \nu$; a contradiction. This shows that E is a graph over $\mathbb{R}^m \times \{0\}$.

We now estimate the norm of A. For $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$, let $x^j \in \mathbb{R}^n$ be the orthogonal projection of x onto $\operatorname{span}\{(e_j, 0)\} \subset \mathbb{R}^n$. With $L \leq 1$ and (3.1) we have $|v_j^j - (e_j, 0)| \leq |v_j - (e_j, 0)| \leq \frac{1}{3\sqrt{m}} L \leq \frac{1}{3}$, hence $|v_j^j| \geq \frac{2}{3}$ for $1 \leq j \leq m$. Let $w_j := \frac{1}{|v_j^j|} v_j \in E$. The second intercept theorem implies

$$\begin{aligned} |w_j - w_j^j| &= \frac{|w_j|}{|v_j|}|v_j - v_j^j| \\ &\leq \frac{3}{2}|v_j - (e_j, 0)| \\ &\leq \frac{1}{2\sqrt{m}} L. \end{aligned}$$

With $w_j^j = (e_j, 0)$ we obtain

$$|w_j - (e_j, 0)| \leq \frac{1}{2\sqrt{m}} L. \tag{3.7}$$

Next choose ν such that $|a_j| \leq |a_\nu|$ for all j. Without loss of generality $\nu = 1$. There is $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ with $w_1 = \sum_{j=1}^m \lambda_j (e_j, a_j)$. As $w_1^1 = (e_1, 0)$ we have

$\lambda_1 = 1$. It follows that

$$\begin{aligned} |w_1 - (e_1, 0)|^2 &= \left|(0, a_1) + \sum_{j=2}^{m} \lambda_j (e_j, a_j)\right|^2 \\ &= \left|\sum_{j=2}^{m} \lambda_j (e_j, 0) + (0, a_1) + \sum_{j=2}^{m} \lambda_j (0, a_j)\right|^2 \\ &= \sum_{j=2}^{m} \lambda_j^2 + \left|a_1 + \sum_{j=2}^{m} \lambda_j a_j\right|^2 . \end{aligned} \tag{3.8}$$

With (3.8), (3.7) and $L \leq 1$ we estimate

$$\begin{aligned} \sum_{j=2}^{m} |\lambda_j| &\leq \sqrt{m} \left(\sum_{j=2}^{m} \lambda_j^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{m}\, |w_1 - (e_1, 0)| \\ &\leq \frac{1}{2}, \end{aligned} \tag{3.9}$$

and

$$\left|a_1 + \sum_{j=2}^{m} \lambda_j a_j\right| \leq \frac{1}{2\sqrt{m}} L. \tag{3.10}$$

With (3.9), with consideration of $|a_j| \leq |a_1|$ for all j, it follows

$$\left|\sum_{j=2}^{m} \lambda_j a_j\right| \leq \left(\sum_{j=2}^{m} |\lambda_j|\right) |a_1| \leq \frac{1}{2} |a_1|. \tag{3.11}$$

From (3.11) and (3.10) we deduce by means of absorption

$$|a_j| \leq |a_1| \leq \frac{1}{\sqrt{m}} L \qquad \text{for all } j \tag{3.12}$$

and finally $\|A\| = \left(\sum_{j=1}^{m} |a_j|^2\right)^{\frac{1}{2}} \leq L$. □

If $f : M \to \mathbb{R}^n$ is an immersion and $q \in M$, then the Euclidean isometry A_q is not uniquely determined (as remarked in Section 2). We say that a Euclidean isometry is *admissible for the point* $q \in M$, if the origin is mapped to $f(q)$ and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(q) + \tau_f(q)$.

If a statement is true for *one* admissible isometry, it is often also true for *any* admissible isometry. This will be used in the proof of the following lemma. Although the statement of the lemma is not very surprising, its proof is quite complex as we have to use the precise Definition 2.1 in the conclusion. Of course the numbers in the lemma are not optimal, but they suffice to prove Theorem 1.1. In Step 2 below we shall apply Lemma 3.1, however the main application of this lemma will be in the proof of Theorem 1.1.

Lemma 3.2. *Every* (r, λ)*-immersion with* $\lambda \leq \frac{1}{8\sqrt{m}}$ *is also a* $(\frac{7}{4} r, 8\sqrt{m}\lambda)$*-immersion.*

Proof. Let $f : M^m \to \mathbb{R}^n$ be an (r, λ)-immersion with $\lambda \leq \frac{1}{8\sqrt{m}}$.

Step 1. Let $q \in M$, $p \in U_{r,q}$ and $\varphi_q := \pi \circ A_q^{-1} \circ f$, where A_q is an arbitrary but fixed admissible isometry as explained above. Then $B_{\frac{4}{5}r}(\varphi_q(p)) \subset \varphi_q(U_{r,p})$.

Proof of Step 1. Without loss of generality we may assume $A_q = \mathrm{Id}_{\mathbb{R}^n}$. The set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a C^1-function $u : B_r \to \mathbb{R}^k$. We set $w := \varphi_q(p) \in B_r$. After a suitable rotation, we may assume that $\{v_1, \ldots, v_m\}$ with $v_j := \frac{(e_j, \partial_j u(w))}{\sqrt{1+|\partial_j u(w)|^2}}$ for $1 \leq j \leq m$ is an orthonormal basis of $\tau_f(p)$ (and still may assume $A_q = \mathrm{Id}_{\mathbb{R}^n}$). Let $R \in \mathbb{SO}(n)$ be a rotation with $R(e_j, 0) = v_j$ for all $j \in \{1, \ldots, m\}$. In particular the mapping $A_p : \mathbb{R}^n \to \mathbb{R}^n$, $A_p(x) := Rx + f(p)$, is an admissible Euclidean isometry for the point $p \in M$. Therefore $A_p^{-1} \circ f(U_{r,p})$ is the graph of a C^1-function $\tilde{u} : B_r \to \mathbb{R}^k$ with $\tilde{u}(0) = 0$ and $\|D\tilde{u}\|_{C^0(B_r)} \leq \lambda$. We define a mapping

$$\begin{aligned} g : \overline{B}_{\frac{29}{30}r}(0) &\to \mathbb{R}^m, \\ y &\mapsto y - \pi \circ R(y, \tilde{u}(y)). \end{aligned}$$

For $y, z \in \overline{B}_{\frac{29}{30}r}(0)$ we estimate

$$\begin{aligned} |g(y) - g(z)| &\leq |(y - z) - \pi \circ R(y - z, 0)| + |\pi \circ R(0, \tilde{u}(y) - \tilde{u}(z))| \\ &\leq \left| \sum_{j=1}^{m} (y_j - z_j)(e_j - \pi v_j) \right| + |\tilde{u}(y) - \tilde{u}(z)| \\ &\leq \left| \sum_{j=1}^{m} (y_j - z_j) \left(1 - \frac{1}{\sqrt{1 + |\partial_j u(w)|^2}} \right) e_j \right| + \lambda |y - z| \\ &\leq \left(1 - \frac{1}{\sqrt{1 + \lambda^2}} \right) |y - z| + \lambda |y - z| \\ &\leq (\lambda^2 + \lambda)|y - z| \\ &< \frac{1}{6}|y - z|, \end{aligned}$$

where we used in the last line $\lambda \leq \frac{1}{8}$. As $g(0) = 0$ we have in particular $g(y) \in B_{\frac{1}{6}r}(0)$ for all $y \in \overline{B}_{\frac{29}{30}r}(0)$.

Now let $x \in \mathbb{R}^m$ be a point in $B_{\frac{4}{5}r}(\varphi_q(p))$. We set $x' := x - \varphi_q(p)$. Then we have $x' \in B_{\frac{4}{5}r}(0)$ and by the considerations above, the mapping

$$\begin{aligned} g + x' : \overline{B}_{\frac{29}{30}r}(0) &\to \overline{B}_{\frac{29}{30}r}(0), \\ y &\mapsto g(y) + x' \end{aligned}$$

is a contraction of the set $\overline{B}_{\frac{29}{30}r}(0)$. By the Banach fixed point theorem there is exactly one $y' \in \overline{B}_{\frac{29}{30}r}(0)$ with $g(y') + x' = y'$, that is, with $\pi \circ R(y', \tilde{u}(y')) = x'$. Furthermore, as $y' \in B_r(0)$, there exists a $p' \in U_{r,p}$ with $f(p') = A_p(y', \tilde{u}(y'))$. Using $A_q = \mathrm{Id}_{\mathbb{R}^n}$, we obtain

$$\begin{aligned} \varphi_q(p') &= \pi \circ A_q^{-1} \circ A_p(y', \tilde{u}(y')) \\ &= \pi \circ R(y', \tilde{u}(y')) + \pi \circ A_q^{-1} \circ f(p) \\ &= x' + \varphi_q(p) \\ &= x. \end{aligned}$$

As $x \in B_{\frac{4}{5}r}(\varphi_q(p))$ is an arbitrary point, it follows $B_{\frac{4}{5}r}(\varphi_q(p)) \subset \varphi_q(U_{r,p})$.

Step 2. The set $U := U_{r,p} \cap \varphi_q^{-1}(B_{\frac{4}{5}r}(\varphi_q(p)))$ is connected and $A_q^{-1} \circ f(U)$ is the graph of a C^1-function $\hat{u} : B_{\frac{4}{5}r}(\varphi_q(p)) \to \mathbb{R}^k$ with $\|D\hat{u}\|_{C^0(B_{\frac{4}{5}r}(\varphi(q)))} \le 8\sqrt{m}\lambda$.

Proof of Step 2. By Step 1 we have $\pi \circ A_q^{-1} \circ f(U) = B_{\frac{4}{5}r}(\varphi_q(p))$. Moreover, as one can replace $\overline{B}_{\frac{29}{30}r}(0)$ in Step 1 by $\overline{B}_{r-\varepsilon}(0)$ for any sufficiently small $\varepsilon > 0$, we deduce with the fixed point argument of Step 1 that $A_q^{-1} \circ f(U)$ is a graph over $B_{\frac{4}{5}r}(\varphi_q(p))$. Now let $p' \in U_{r,p}$. We write $A_p^{-1} \circ f(U_{r,p})$ (where A_p is as in Step 1) as a graph of the C^1-function $\tilde{u} : B_r \to \mathbb{R}^k$. Then there is a unique $x \in B_r$ with $A_p^{-1} \circ f(p') = (x, \tilde{u}(x))$. With the rotation R of Step 1 we have

$$R^{-1}(\tau_f(p')) = \operatorname{span}\{(e_1, \partial_1\tilde{u}(x)), \dots, (e_m, \partial_m\tilde{u}(x))\}.$$

In particular

$$R(e_j, \partial_j\tilde{u}(x)) \quad \in \quad \tau_f(p') \qquad \text{for all } j \in \{1, \dots, m\}.$$

Let v_j and w be as in Step 1. We note that $R(e_j, 0) = v_j$ and estimate

$$\begin{aligned} |R(e_j, \partial_j\tilde{u}(x)) - (e_j, 0)| &\le |R(e_j, \partial_j\tilde{u}(x)) - R(e_j, 0)| + |R(e_j, 0) - (e_j, \partial_j u(w))| \\ &\quad + |(e_j, \partial_j u(w)) - (e_j, 0)| \\ &= |\partial_j\tilde{u}(x)| + \left(\sqrt{1 + |\partial_j u(w)|^2} - 1\right) + |\partial_j u(w)| \\ &\le 2\lambda + \left(\sqrt{1 + \lambda^2} - 1\right) \\ &\le \frac{5}{2}\lambda \\ &< \frac{1}{3\sqrt{m}} 8\sqrt{m}\lambda. \end{aligned}$$

We apply Lemma 3.1 with $E := \tau_f(p') \in G_{n,m}$, $v_j := R(e_j, \partial_j\tilde{u}(x))$, $L := 8\sqrt{m}\lambda$ and conclude that $\tau_f(p')$ may be written as a graph over $\mathbb{R}^m \times \{0\}$. As this is true for any $p' \in U_{r,p}$, an argument similar to the one in the paragraph preceding (4.6) together with the considerations at the beginning of the proof of Step 2 allows us to conclude that $A_q^{-1} \circ f(U)$ is the graph of a C^1-function $\hat{u} : B_{\frac{4}{5}r}(\varphi_q(p)) \to \mathbb{R}^k$ with $\|D\hat{u}\|_{C^0(B_{\frac{4}{5}r}(\varphi(q)))} \le 8\sqrt{m}\lambda$. In particular $\varphi_q : U \to B_{\frac{4}{5}r}(\varphi_q(p))$ is a diffeomorphism; hence U is connected.

Step 3. The function f is a $(\frac{7}{4}r, 8\sqrt{m}\lambda)$-immersion.

Proof of Step 3. Let φ_q and A_q be as in Step 1. For every $x \in \partial B_{\frac{19}{20}r}$ there is exactly one $p_x \in U_{r,q}$ with $\varphi_q(p_x) = x$. For each $x \in \partial B_{\frac{19}{20}r}$ set $U_x := U_{r,p_x} \cap \varphi_q^{-1}(B_{\frac{4}{5}r}(x))$. Moreover set $V_q := U_{r,q} \cup \bigcup_{x \in \partial B_{\frac{19}{20}r}} U_x$. By Step 1 we have

$$\begin{aligned} \varphi_q(V_q) &= B_r(0) \cup \bigcup_{x \in \partial B_{\frac{19}{20}r}} (\varphi_q(U_{r,p_x}) \cap B_{\frac{4}{5}r}(x)) \\ &= B_r(0) \cup \bigcup_{x \in \partial B_{\frac{19}{20}r}} B_{\frac{4}{5}r}(x) \\ &= B_{\frac{7}{4}r}(0). \end{aligned}$$

Each set U_x is connected and we have $p_x \in U_{r,q} \cap U_x$. Therefore also V_q is connected, and we have $q \in V_q$.

Now let $R > r$ be the greatest radius, such that $A_q^{-1} \circ f(U_{R,q})$ is the graph of a C^1-function $u : B_R \to \mathbb{R}^k$. Suppose $R < \frac{7}{4}r$. As $R > r$, we have $\varrho := R - \frac{4}{5}r > 0$. Define sets U_x as above, but here for $x \in \partial B_\varrho$. Set $W_q := U_{r,q} \cup \bigcup_{x \in \partial B_\varrho} U_x$. Analogous to the considerations above, W_q is a connected set containing q, and it holds that $\varphi_q(W_q) = B_R(0)$. We deduce $W_q \subset U_{R,q}$. As we assume here, $A_q^{-1} \circ f(U_{R,q})$ is a graph over $B_R(0)$, and as $\varphi_q(W_q) = B_R(0)$, we conclude that $W_q = U_{R,q}$. As R is maximal, we deduce $\|Du\|_{C^0(B_R)} = \infty$. But this contradicts Step 2, saying that $\|Du\|_{C^0(B_{\frac{4}{5}r}(x))} \leq 8\sqrt{m}\lambda$ for all $x \in \partial B_\varrho$. Hence it holds $R \geq \frac{7}{4}$.

Using the preceding considerations, we conclude $V_q = U_{\frac{7}{4}r,q}$ (in particular V_q does not depend on the choice of A_q) and $A_q^{-1} \circ f(U_{\frac{7}{4}r,q})$ is the graph of a C^1-function $u : B_{\frac{7}{4}r} \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_{\frac{7}{4}r})} \leq 8\sqrt{m}\lambda$.

As this is true for any point $q \in M$, the function f is a $(\frac{7}{4}r, 8\sqrt{m}\lambda)$-immersion. □

We need the following lemma (which was shown in [5] for (r,λ)-immersions):

Lemma 3.3. *Let $f : M \to \mathbb{R}^n$ be a C^0-(r,λ)-immersion and $p, q \in M$.*

(a) *If $0 < \varrho \leq r$ and $p \in U_{\varrho,q}$, then $|f(q) - f(p)| < \varrho + r\lambda$.*
(b) *If $\lambda \leq \frac{1}{10}$ and $p \in U_{\frac{2}{5}r,q}$, then $U_{\frac{2}{5}r,q} \subset U_{r,p}$.*

Proof.

(a) Pass to the graph representation, use the bound on the C^0-norm and the triangular inequality.
(b) Let $x \in U_{\frac{2}{5}r,q}$ and $\varphi_p = \pi \circ A_p^{-1} \circ f$. With part (a) we estimate

$$\begin{aligned} |\varphi_p(x)| &\leq |f(x) - f(p)| \\ &\leq |f(x) - f(q)| + |f(q) - f(p)| \\ &< 2\left(\frac{2}{5}r + \frac{r}{10}\right) \\ &= r. \end{aligned}$$

Hence $U_{\frac{2}{5}r,q} \subset \varphi_p^{-1}(B_r)$. But $U_{\frac{2}{5}r,q}$ is a connected set containing p, hence included in the p-component of $\varphi_p^{-1}(B_r)$ that is in $U_{r,p}$. Hence $U_{\frac{2}{5}r,q} \subset U_{r,p}$. □

4. Proof of the embedding theorem

With Lemmas 3.1, 3.2 and 3.3 we have all the necessary tools for showing our theorem:

Proof of Theorem 1.1. Let $m \in \mathbb{N}$. Define $\Lambda = \Lambda(m) := 10^{-5}m^{-2}$.

Now let $\lambda \leq \Lambda$, $r > 0$ and $f : M^m \to \mathbb{R}^n$ be a given C^0-(r,λ)-immersion. We set $\varrho := \frac{r}{5}$. Moreover let $q \in M$ be an arbitrary point. As $2\varrho < r$, the set $f(U_{2\varrho,q})$ may be written over $f(q) + \tau_f(q)$ as the graph of a function $u : B_{2\varrho} \to \mathbb{R}^k$ with $\|u\|_{C^0(B_{2\varrho})} \leq r\lambda$.

As the arguments of this proof are invariant under rotations and translations, we may assume without loss of generality that $A_q = \mathrm{Id}_{\mathbb{R}^n}$ (where $A_q : \mathbb{R}^n \to \mathbb{R}^n$ is an admissible isometry for the point $q \in M$). In particular $f(q) = 0$ and $\tau_f(q) = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n$.

Now let $x \in B_\varrho$ be an arbitrary point. Then there is exactly one $p \in U_{\varrho,q}$ with

$$f(p) = A_q(x, u(x)) = (x, u(x)). \tag{4.1}$$

As $\lambda \leq \frac{1}{10}$, $2\varrho = \frac{2}{5}r$ and as $p \in U_{\varrho,q} \subset U_{\frac{2}{5}r,q}$, Lemma 3.3 (b) implies

$$U_{2\varrho,q} \subset U_{r,p}.$$

Therefore the set $f(U_{2\varrho,q})$ may also be written over $f(p) + \tau_f(p)$ as a graph of a function with small C^0-norm. More precisely there exists a function $\tilde{u} : B_r \to \mathbb{R}^k$ with $\|\tilde{u}\|_{C^0(B_r)} \leq r\lambda$ and $f(U_{2\varrho,q}) \subset \{A_p(y, \tilde{u}(y)) : y \in B_r\}$.

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$. For $1 \leq j \leq m$ define

$$x_j := x + \varrho e_j. \tag{4.2}$$

As $x \in B_\varrho$ we have $x_j \in B_{2\varrho}$ for each j. Hence for each j there is exactly one $p_j \in U_{2\varrho,q}$ with

$$f(p_j) = A_q(x_j, u(x_j)) = (x_j, u(x_j)). \tag{4.3}$$

As $p_j \in U_{2\varrho,q}$ and $U_{2\varrho,q} \subset U_{r,p}$, there are also unique $y_j \in B_r$ with

$$f(p_j) = A_p(y_j, \tilde{u}(y_j)). \tag{4.4}$$

Now we estimate as follows:

$$\begin{aligned} |A_p(y_j, 0) - f(p) - \varrho(e_j, 0)| &\leq |A_p(y_j, 0) - f(p_j)| + |f(p_j) - f(p) - \varrho(e_j, 0)| \\ &= |A_p(y_j, 0) - A_p(y_j, \tilde{u}(y_j))| \\ &\quad + |(x_j, u(x_j)) - (x, u(x)) - \varrho(e_j, 0)| \\ &= |\tilde{u}(y_j)| + |u(x_j) - u(x)| \\ &\leq 3r\lambda \\ &= 3 \cdot 10^{-5} m^{-2} r \frac{\lambda}{\Lambda} \\ &\leq \frac{\varrho}{3\sqrt{m}} \cdot 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda}. \end{aligned}$$

We divide the inequality by ϱ and obtain

$$\left| \frac{1}{\varrho}[A_p(y_j, 0) - f(p)] - (e_j, 0) \right| \leq \frac{1}{3\sqrt{m}} \cdot 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda}. \tag{4.5}$$

The isometry A_p maps the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(p) + \tau_f(p)$, in particular

$$\frac{1}{\varrho}[A_p(y_j, 0) - f(p)] \in \tau_f(p) \qquad \text{for all } j \in \{1, \ldots, m\}.$$

Furthermore with $\lambda \leq \Lambda$ we have $8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda} \leq 1$. Hence (4.5) allows us to apply Lemma 3.1 with $E := \tau_f(p) \in G_{n,m}$, $v_j := \frac{1}{\varrho}[A_p(y_j, 0) - f(p)]$ and $L := 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda}$. We conclude that $\tau_f(p)$ may be written as a graph over $\mathbb{R}^m \times \{0\}$. As $f(p) = (x, u(x))$, the implicit function theorem implies that u is differentiable in a neighborhood of x and $\tau_f(p) = \operatorname{span}\{(e_1, \partial_1 u(x)), \ldots, (e_m, \partial_m u(x))\}$. With (3.2) it follows

$$\|Du(x)\| \leq 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda}.$$

As $x \in B_\varrho$ was assumed to be an arbitrary point, u is differentiable on all of B_ϱ and

$$\|Du\|_{C^0(B_\varrho)} \leq 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda}. \tag{4.6}$$

Hence, as $\varrho = \frac{r}{5}$, the function f is an $(\frac{r}{5}, 8^{-3} m^{-\frac{3}{2}} \frac{\lambda}{\Lambda})$-immersion.

Now we can iterate the embedding of Lemma 3.2 three times. Hence f is a $((\frac{7}{4})^3 \frac{r}{5}, \frac{\lambda}{\Lambda})$-immersion and $(\frac{7}{4})^3 > 5$ is also an $(r, \frac{\lambda}{\Lambda})$-immersion. This is the desired conclusion. □

5. Appendix

In this appendix we would like to consider immersions with uniform graph representations not over the affine tangent space, but over other appropriately chosen m-spaces. We will show that our theorem does not hold for these types of immersions.

For a given $q \in M$ and a given m-space $E \in G_{n,m}$ let $A_{q,E} : \mathbb{R}^n \to \mathbb{R}^n$ be a Euclidean isometry, which maps the origin to $f(q)$, and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(q) + E$. Let $U^E_{r,q} \subset M$ be the q-component of the set $(\pi \circ A^{-1}_{q,E} \circ f)^{-1}(B_r)$. Again the isometry $A_{q,E}$ is not uniquely determined but the set $U^E_{r,q}$ does not depend on the choice of $A_{q,E}$.

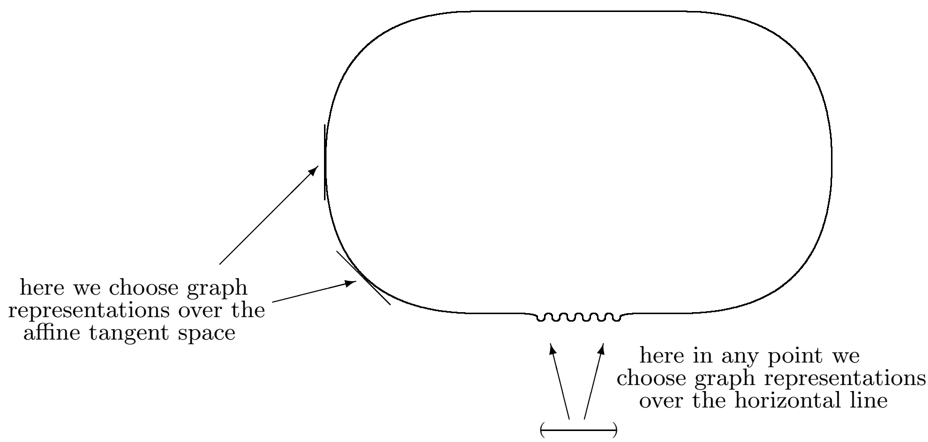

FIGURE 5.1. This example shows that a generalized C^0-(r, λ)-immersion with very small λ does not need to be a generalized $(r, \frac{\lambda}{\Lambda})$-immersion.

The following definition is a natural generalization of Definition 2.1:

Definition 5.1. An immersion f is called a generalized (r, λ)-immersion, if for each point $q \in M$ there is an $E = E(q) \in G_{n,m}$, such that the set $A^{-1}_{q,E} \circ f(U^E_{r,q})$ is the graph of a differentiable function $u : B_r \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_r)} \leq \lambda$.

Obviously every (r,λ)-immersion is a generalized (r,λ)-immersion, as we can choose $E(q)=\tau_f(q)$ for any $q\in M$. As a generalization of Definition 2.2 we have the following definition:

Definition 5.2. An immersion f is called a generalized C^0-(r,λ)-immersion, if for each point $q\in M$ there is an $E=E(q)\in G_{n,m}$, such that the set $A_{q,E}^{-1}\circ f(U_{r,q}^E)$ is the graph of a continuous function $u:B_r\to\mathbb{R}^k$ with $\|u\|_{C^0(B_r)}\leq r\lambda$.

We wonder whether there is a $\Lambda>0$, such that each generalized C^0-(r,λ)-immersion with $\lambda\leq\Lambda$ is also a generalized $(r,\frac{\lambda}{\Lambda})$-immersion. Figure 5.1 shows that this is not the case.

Moreover, in Figure 5.1, the part of the immersion over the horizontal line cannot be represented over any other line (with the same radius). This shows that we require graph representations over the affine tangent space for our theorem.

Acknowledgement

The author thanks his advisor, Ernst Kuwert, for his support. Moreover the author thanks Manuel Breuning for proofreading his dissertation [1], where the result of this paper was first established.

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