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CONVERGENCE OF GENERAL INVERSE σ_k-FLOW ON KÄHLER MANIFOLDS WITH CALABI ANSATZ

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ABSTRACT. We study the convergence behavior of the general inverse σ_k-flow on Kähler manifolds with initial metrics satisfying the Calabi ansatz. The limiting metrics can be either smooth or singular. In the latter case, interesting conic singularities along negatively self-intersected subvarieties are formed as a result of partial blow up.

1. INTRODUCTION

Geometric flows are powerful tools to study the metric, algebraic and topological properties of the underlying manifold. An important example is the Ricci flow introduced by Hamilton [H] three decades ago and its Kählarian version, the Kähler-Ricci flow [Cao]. Since then, both have developed into significant research fields. See [T, SW4] for more complete surveys and further references on the Kähler-Ricci flow. Another metric flow, the so-called J-flow, appears naturally in the study of Kähler geometry and is extensively studied by [Chen, Do, We1, We2, SW1].

In [FLM, FL], we have introduced the general inverse σ_k-flow on compact Kähler manifolds, which is a generalization of the J-flow. In this paper, we study some concrete examples to explore several applications of the general inverse σ_k-flow in algebraic geometry and fully non-linear partial differential equations.

We recall the definition of the general inverse σ_k-flow. Let (X, ω) be a compact Kähler manifold of complex dimension n, endowed with a fixed Kähler form ω. Let χ be another Kähler form. For a fixed integer $k \in [1, n]$, let

$$c_k = \binom{n}{k} \frac{\int_X \chi^{n-k} \wedge \omega^k}{\int_X \chi^n}$$

and define

$$\sigma_k(\chi) = \binom{n}{k} \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

We are interested in $\sigma_k(\chi)$, which is a function defined globally on X. Locally, it is the k-th elementary symmetric polynomial of eigenvalues of χ with respect to ω, $\{\lambda_1, \cdots, \lambda_n\}$. In other words:

$$\sigma_k(\chi) = \sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

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The general inverse σ_k-flow is defined in the space of Kähler potential of χ:

$$\chi_\varphi = \chi + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi \in \mathcal{P}_\chi = \{\varphi \in C^\infty(X) | \chi_\varphi := \chi + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi > 0\},$$

by

$$\begin{cases} \frac{\partial}{\partial t}\varphi(x,t) & = F(\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)}) - F(c_k), \\ \varphi(0) & = 0, \end{cases} \tag{1.1}$$

where $x \in X, t \in [0,\infty)$, and $F \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfies the following conditions:

$$F'(x) < 0, \qquad F''(x) \geq 0, \qquad F''(x) + \frac{F'(x)}{x} \leq 0. \tag{1.2}$$

Note that (1.2) implies the ellipticity and strong concavity of the flow (1.1). See [FL] for details. If we take $F(x) = -x^{1/k}$ (F satisfies (1.2)), the flow is the inverse σ_k flow studied in [FLM], with $k = 1$ being exactly the J-flow.

Any stationary point of (1.1) corresponds to a metric $\tilde{\chi} \in [\chi]$ satisfying the following Kählerian inverse σ_k equation on X:

$$c_k\tilde{\chi}^n = \binom{n}{k}\tilde{\chi}^{n-k} \wedge \omega^k. \tag{1.3}$$

Locally, (1.3) can be written as

$$\frac{\sigma_{n-k}(\tilde{\chi})}{\sigma_n(\tilde{\chi})} = \sigma_k(\tilde{\chi}^{-1}) = c_k, \tag{1.4}$$

where $\tilde{\chi} = \frac{\sqrt{-1}}{2}\tilde{\chi}_{i\bar{j}}dz_1 \wedge d\bar{z_j}$ and $\tilde{\chi}^{-1}$ is the inverse of the matrix $(\tilde{\chi}_{i\bar{j}})$.

Equation (1.4) leads to an obvious necessary condition for (1.3) to admit a smooth solution, which was first formulated in [SW1] for the J-flow. Define

$$\begin{aligned} \mathcal{C}_k(\omega) = &\{[\chi] > 0 \,|\, \exists \chi' \in [\chi] \\ &\text{such that } nc_k\chi'^{n-1} - \binom{n}{k}(n-k)\chi'^{n-k-1} \wedge \omega^k > 0\,\}. \end{aligned} \tag{1.5}$$

Note that for $k = n$, (1.5) holds for any Kähler class. Hence $\mathcal{C}_n(\omega)$ is the entire Kähler cone. If there exists a smooth metric $\tilde{\chi} \in [\chi]$ solving (1.3), it is necessary that

$$[\chi] \in \mathcal{C}_k(\omega). \tag{1.6}$$

In our earlier work with Ma [FLM], we have shown that condition (1.6) is also sufficient for the existence of smooth solutions of (1.4) via the convergence of the flow (1.1) [FL].

Theorem 1.1. *Let (X, ω, χ) be given as above. Then the flow* (1.1) *has long time existence and converges to the critical metric $\tilde{\chi}$ satisfying* (1.3) *if and only if $[\chi] \in \mathcal{C}_k(\omega)$.*

For the particular J-flow case, the smooth convergence of the J-flow has significant geometric implication on the properness of Mabuchi energy (see [SW1]). Another special case of the flow occurs when $k = n$, where the critical equation is a complex Monge-Ampère equation and one can take $F(x) = -\log x$ so the corresponding flow resembles the Kähler-Ricci flow.

Same as the J-flow, the general inverse σ_k-flow (1.1) always has long time existence (see [FL]). It is thus interesting to study convergence properties of the

flow even if condition (1.6) fails to hold. While convergence to smooth metrics is no longer expected, due to the geometric set-up, blow up behavior of the solution along proper subvarieties is expected (cf. [SW1]). In particular, it is our hope that the analytical behavior of the limit metric will reflect the algebro-geometric properties of the original Kähler manifold X.

From an analytical point of view, (1.3) deserves study in its own right. For $k = n$, it is a complex Monge-Ampère equation. If $[\chi]$ is Kähler, by Yau's renowned solution of Calabi conjecture [Y], (1.3) admits a smooth solution unique up to a constant. If $[\chi]$ lies on the boundary of Kähler cone, i.e., $[\chi]$ is merely nef but not Kähler, then (1.3) becomes a degenerate complex Monge-Ampère equation. This has been a subject of intensive study over the past two decades following the pioneering work of Kołodziej [K]. When χ is a big semi-positive form, the result in [EGZ] implies that (1.3) admits a bounded pluri-subharmonic solution. Such a solution of the degenerate complex Monge-Ampère equation is used to produce singular Kähler-Einstein metrics on Kähler manifolds with indefinite anticanonical class [EGZ]. There have been Kähler-Ricci flow approaches for more general canonical singular Kähler-Einstein metrics. See [TZ, ST1, ST2] for details and further references.

For $k \neq n$, we refer to (1.3) as the Kählerian inverse σ_k equation. When $[\chi]$ lies on the boundary of $\mathcal{C}_k(\omega)$, this Monge-Ampère type equation degenerates in an intriguing way. Suggested by the convergence results we obtained on the general inverse σ_k flow, we conjecture that an analogous result of [EGZ] on boundedness of the solution in pluri-potential sense still holds.

In this paper, we shall study the general inverse σ_k-flow assuming certain symmetry of initial data. This is partially inspired by similar results on the Kähler-Ricci flow [SW2, SW3, SY] and on Kähler-Ricci solitons [FIK, L]. It is interesting to compare convergence behaviors of the general inverse σ_k-flow with those of the Kähler-Ricci flow.

Theorem 1.2 (Main Theorem 1)**.** *Let $X = \mathbb{P}^n \# \overline{\mathbb{P}^n}$ be $\mathbb{P}^n$ blown up at one point. $\pi : X \to \mathbb{P}^n$ is the natural map. Let E_0 and E_∞ be the exceptional divisor and the pull-back of the divisor associated to $\mathcal{O}_{\mathbb{P}^n}(1)$, respectively. Assume that χ, ω are Kähler metrics on X satisfying Calabi ansatz (see Section 2 for details) such that*

$$\omega \in \alpha[E_\infty] - [E_0], \quad \chi \in \beta[E_\infty] - [E_0] \quad \textit{with} \quad \alpha > 1 \quad \textit{and} \quad \beta > 1.$$

Let χ_t be the solution of the flow (1.1)*. Then the following convergence behavior of χ_t holds:*

(1) *If $\frac{\alpha^k \beta^{n-k} - 1}{\beta^n - 1} > \frac{n-k}{n}$, then as $t \to \infty$, $\chi_t \xrightarrow{C^\infty(X)} \chi_\infty$, a smooth Kähler metric satisfying* (1.3)*.*

(2) *If $\frac{\alpha^k \beta^{n-k} - 1}{\beta^n - 1} = \frac{n-k}{n}$, then as $t \to \infty$, $\chi_t \xrightarrow{C^\infty_{loc}(X \setminus E_0)} \chi_\infty$, a singular Kähler metric that is smooth away from E_0 and has conic singularity at E_0 of angle π. Furthermore, there is a universal constant C such that the oscillation of the limiting potential φ_∞ satisfies*

$$osc\ \varphi_\infty \leq C.$$

(3) *If* $\frac{\alpha^k\beta^{n-k}-1}{\beta^n-1} < \frac{n-k}{n}$, *then as* $t \to \infty$, $\chi_t \xrightarrow{C^\infty_{loc}(X\setminus E_0)} \chi_\infty + (\lambda-1)[E_0]$, *a Kähler current. Here* $\lambda \in (1,\beta)$ *is unique such that*

$$(n-k)(\frac{\beta}{\lambda})^k + k(\frac{\lambda}{\beta})^{n-k} = n\alpha^k, \tag{1.7}$$

and $[E_0]$ *is the current of integration along the exceptional divisor* E_0. χ_∞ *is the unique limit of the generalized inverse* σ_k *flow with given data* $(X, \omega \in \alpha[E_\infty]-[E_0], \chi \in \beta[E_\infty]-\lambda[E_0])$ *and thereby is a singular Kähler metric with conic singularity with angle* π *transverse to* E_0 *as in case (2).*

Remark 1.3. Case (3) of Theorem 1.2 is interesting in the sense that χ_∞ is obtained as the limit of flow (1.1) for some smooth initial data $(X, \omega \in \alpha[E_\infty]-[E_0], \chi \in \beta[E_\infty]-\lambda[E_0])$. This is an example of partial blow up in the sense of algebraic geometry using analytical tools. See [SW2, SW3, SY] for some corresponding results for the Kähler-Ricci flow.

Remark 1.4. For the J-flow on Kähler surface $(k=1, n=2)$, the cone condition (1.5) reduces to a simple class condition:

$$[c_1\chi - \omega] > 0. \tag{1.8}$$

Donaldson [Do] noticed that such a condition holds for all Kähler classes $[\chi]$ and $[\omega]$ if there are no curves of negative self-intersection. He also conjectured that if (1.8) fails to hold, then one might expect the flow to blow up over some such curves. This was confirmed in [SW1] in the sense that the quantity $|\varphi| + |\Delta_\omega\varphi|$ blows up, where φ is the solution of the J-flow. In fact, their estimate is on any dimension n when condition (1.5) is violated. They posed a question on improving these estimates. Our results give a partial answer to their question. In particular, case (2) of the theorem asserts that for the given X with initial metrics satisfying Calabi ansatz, $|\varphi|$ stays bounded.

Figure 1 illustrates different cases of Theorem 1.2. In Figure 1, the Kähler cone $\mathcal{K}(X)$ is given by the region $\{b[E_\infty]-a[E_0] | b > a > 0\}$, i.e., the upper half quadrant, while $\mathcal{C}_k(\omega)$ defined by (1.5) is the cone between two dotted lines. β_0 is the constant satisfying $\frac{\beta_0^{n-k}\alpha^k-1}{\beta_0^n-1} = \frac{n-k}{n}$, which defines the boundary of $\mathcal{C}_k(\omega)$ inside of $\mathcal{K}(X)$. Therefore case (2) of Theorem 1.2 corresponds to $[\chi]$ lying on the upper boundary of $\mathcal{C}_k(\omega)$; case (3) of Theorem 1.2 corresponds to $[\chi] \in \mathcal{K}(X)\setminus\overline{\mathcal{C}_k(\omega)}$. The limit χ_∞ of case (3) jumps to the class $\beta[E_\infty]-\lambda[E_0]$ which lies on the upper boundary of $\mathcal{C}_k(\omega)$. The relation (1.7) then follows.

Note that the constant $\frac{\alpha^k\beta^{n-k}-1}{\beta^n-1}$ in Theorem 1.2 is in fact the topological constant

$$\frac{\int_X \chi^{n-k}\wedge\omega^k}{\int_X \chi^n} = \frac{[\chi]^{n-k}[\omega]^k}{[\chi]^n}.$$

Hence by Theorems 1.1 and 1.2, we obtain

Corollary 1.5. *Fix the notation as in Theorem* 1.2. *Let* $\omega \in \alpha[E_\infty]-[E_0]$ *and* $\chi \in \beta[E_\infty]-[E_0]$. *Assume that* ω *satisfies the Calabi ansatz. Then* $[\chi] \in \mathcal{C}_k(\omega)$ *if and only if* $\frac{\int_X \chi^{n-k}\wedge\omega^k}{\int_X \chi^n} = \frac{\alpha^k\beta^{n-k}-1}{\beta^n-1} > \frac{n-k}{n}$. *In particular,* $\mathcal{C}_k(\omega)$ *is a convex open cone.*

Remark 1.6. In general, the cone (1.5) is not easy to calculate, as the definition involved is the positivity of a certain $(n-1, n-1)$ form.

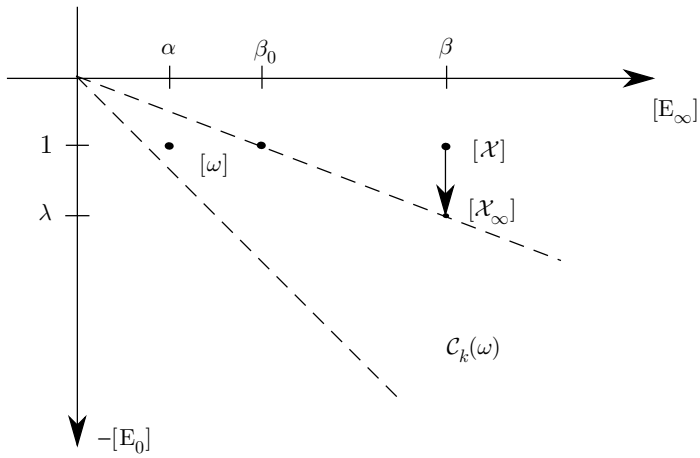

FIGURE 1

In order to get examples involving singularities of higher codimension, we study the flow (1.1) on more complicated manifolds admitting Calabi ansatz.

Let $X_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$ be a projective bundle over $\mathbb{P}^n$ of total dimension $m+n+1$. Let D_∞ be the divisor given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$ and D_H be the pull-back of the divisor on $\mathbb{P}^n$ associated to $\mathcal{O}_{\mathbb{P}^n}(1)$.

For future use, let $\tilde{X}_{m,n}$ be the blow up of $X_{m,n}$ along P_0, where $P_0 \subset X_{m,n}$ is the projectivization of the section $(1, 0, \cdots, 0) \in \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$. Note that P_0 is of dimension n. We denote the resulting exceptional divisor in $\tilde{X}_{m,n}$ by E. Note that $X_{0,n}$ is exactly $\mathbb{CP}^{n+1}$ blown up one point.

Theorem 1.7 (Main Theorem 2)**.** *Let $X_{m,n}$ be as given above. Assume that ω, χ are two Kähler metrics satisfying the Calabi ansatz and*

$$\omega \in [D_H] + b[D_\infty] \quad \textit{and} \quad \chi \in [D_H] + b'[D_\infty], \quad b, b' > 0,$$

where $[D_H]$ and $[D_\infty]$ are two generators of the divisor group of $X_{m,n}$. Let χ_t be the solution of the general inverse σ_k-flow (1.1)*. Then we have the following:*

$k > n$: $\chi_t \xrightarrow{C^\infty(M)} \chi_\infty$*, a smooth Kähler metric solving* (1.3)*, as $t \to \infty$.*

$k \le n$: *Recall* $c_k = \dfrac{\binom{m+n+1}{k}\int_{X_{m,n}} \chi^{m+n+1-k}\wedge\omega^k}{\int_{X_{m,n}} \chi^{m+n+1}}$*. Then we have:*

(1) $c_k > \binom{n}{k}$: $\chi_t \xrightarrow{C^\infty(M)} \chi_\infty$*, a smooth Kähler metric solving* (1.3)*, as $t \to \infty$.*

(2) $c_k = \binom{n}{k}$: $\chi_t \xrightarrow{C^\infty_{loc}(X_{m,n}\setminus P_0)} \chi_\infty$*, a singular Kähler metric with cone singularity of angle π transverse to P_0 as $t \to \infty$. There is a universal contant C such that*

$$osc\ \varphi_\infty \le C.$$

(3) $c_k < \binom{n}{k}$: *Let π be the blown up map: $\pi : \tilde{X}_{m,n} \to X_{m,n}$. Then $\pi^*(\chi_t) \xrightarrow{C^\infty_{loc}(X_{m,n}\setminus P_0)} \chi_\infty + \lambda[E]$ as $t \to \infty$. Here $[E]$ is the current of integration along E, $\lambda \in (0, b')$ is a constant uniquely determined*

by the geometric data, and $\chi_\infty \in [D_H] + b'[D_\infty] - \lambda[E]$ *is a singular Kähler metric on* $\tilde{X}_{m,n}$ *with cone singularity of angle* π *transverse to the fibre direction of* E *such that on* $\tilde{X}_{m,n} \setminus E$,

$$\alpha\chi_\infty^{m+n+1} = \chi_\infty^{m+n+1-k} \wedge \pi^*(\omega)^k,$$

where

$$\alpha = \frac{([D_H] + b'[D_\infty] - \lambda[E])^{m+n+1-k}([D_H] + b'[D_\infty])^k}{([D_H] + b'[D_\infty] - \lambda[E])^{m+n+1}}.$$

Note that in case (3) above, when no confusion arises, D_H and D_∞ are also used to denote the corresponding pulled-back divisors on $\tilde{X}_{m,n}$.

Combining Theorem 1.1 and Theorem 1.7, we also obtain

Corollary 1.8. *On* $X_{m,n}$*, assume that* $\omega \in [D_H]+b[D_\infty]$ *and* $\chi \in [D_H]+b'[D_\infty]$*. If* ω *satisfies the Calabi ansatz, then* $\mathcal{C}_k(\omega)$ *is the entire Kähler cone whenever* $k > n$*, and when* $k \le n$*,* $[\chi] \in \mathcal{C}_k(\omega)$ *if and only if* $c_k = \frac{\binom{m+n+1}{k}\int_{X_{m,n}} \chi^{m+n+1-k}\wedge\omega^k}{\int_{X_{m,n}} \chi^{m+n+1}} > \binom{n}{k}$.

Remark 1.9. For both case (2) of Theorem 1.2 and case (2) of Theorem 1.7, the critical equation (1.3) admits a bounded solution in the sense of pluri-potential theory. This indicates that a general result should hold for this type of equation. Thus, the results of [K] and [EGZ] on solutions for degenerate complex Monge-Ampère equations may be extended to more general complex Monge-Ampère type equations. We would like to discuss this aspect in future works.

Remark 1.10. Similar to Remark 1.3, it is easy to construct a proper inverse σ_k type of flow on $\tilde{X}_{m,n}$ such that the limiting metric under the flow coincides with that of case (3) listed above. The inverse σ_k-flow can thus be viewed as an analytical method to connect birationally equivalent varieties.

In this paper we only discuss some special Kähler manifolds with metrics satisfying strong symmetric conditions. However, the algebraic pictures revealed indicate that the geometric flows that we have studied can be used to transform between (possibly singular) algebraic varieties. In subsequent works, we will discuss general cases and their further applications to birational geometry.

The rest of the paper is organized as follows. In Section 2, we study the J-flow on $\mathbb{P}^n\#\overline{\mathbb{P}^n}$ as a prototype. In Section 3, we use the same method to treat the general inverse σ_k-flow on $\mathbb{P}^n\#\overline{\mathbb{P}^n}$. In Section 4, we study the convergence behavior on $X_{m,n}$, a class of more generality.

2. J-flow on $\mathbb{P}^n\#\overline{\mathbb{P}^n}$

Let $X = \mathbb{P}^n\#\overline{\mathbb{P}^n}$ be $\mathbb{P}^n$ blowing up at one point. Denote $[E_0]$ and $[E_\infty]$ as the exceptional divisor and the pull-back of the hyper-plane in $\mathbb{P}^n$, respectively. We have $H^{1,1}(X,\mathbb{R}) = \mathrm{span}\{[E_0],[E_\infty]\}$, and any Kähler class Ω on X is of the form

$$\Omega = b[E_\infty] - a[E_0], \quad b > a > 0.$$

We recall the Calabi ansatz [Ca] on the construction of rotational symmetric Kähler metrics on X. For notational convenience, let $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{4\pi}(\partial - \bar{\partial})$; then $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. On $X \setminus (E_0 \cup E_\infty) \cong \mathbb{C}^n \setminus 0$, we can associate a coordinate system $(z_1, \cdots, z_n)$. Define

$$\rho = \ln(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2).$$

For a function $u \in C^\infty(\mathbb{R})$ such that

$$u'(\rho) > 0, \quad u''(\rho) > 0,$$

the (1,1)-form $\omega = dd^c u(\rho)$ is then a Kähler form.

In order to extend ω to a smooth Kähler metric on X, the following asymptotic properties of u are required:

(2.1)

(†) $u_0(r) := u(\ln r) - a \ln r$

is extendable by continuity to a smooth function at $r = 0$, and $u_0'(0) > 0$,

(‡) $u_\infty(r) := u(-\ln r) + b \ln r$

is extendable by continuity to a smooth function at $r = 0$, and $u_\infty'(0) > 0$.

It is easy to see, by the asymptotic behavior of u, that

$$\lim_{\rho \to -\infty} u'(\rho) = a, \quad \lim_{\rho \to \infty} u'(\rho) = b.$$

Moreover, since $u''(\rho) > 0$, $b > a$, a and b characterize the Kähler class of ω in the following manner:

$$\omega \in b[E_\infty] - a[E_0].$$

In this section, we treat the J-flow, which is a special case of general inverse σ_k flows. It is defined as follows:

$$\begin{cases} \frac{\partial}{\partial t}\varphi & = c_1 - \frac{n\chi_\varphi^{n-1} \wedge \omega}{\chi_\varphi^n}, \\ \varphi(0) & = 0. \end{cases} \tag{2.2}$$

After normalization we may assume

$$\omega \in \alpha[E_\infty] - [E_0], \quad \chi \in \beta[E_\infty] - [E_0], \quad \alpha > 1, \quad \beta > 1.$$

If both ω and χ satisfy the Calabi ansatz, on the coordinate patch $X \setminus (E_0 \cup E_\infty)$, we have smooth functions $u, v \in C^\infty(\mathbb{R})$ such that

$$\omega = dd^c u(\rho), \quad \chi = dd^c v(\rho).$$

This leads to

$$\omega = (\frac{u'}{e^\rho}\delta_{ij} + (u'' - u')\frac{\bar{z}_i z_j}{e^{2\rho}})dz_i \wedge d\bar{z}_j,$$

$$\chi = (\frac{v'}{e^\rho}\delta_{ij} + (v'' - v')\frac{\bar{z}_i z_j}{e^{2\rho}})dz_i \wedge d\bar{z}_j.$$

Thus the eigenvalues of χ with respect to ω are

$$\underbrace{\frac{v'}{u'}, \frac{v'}{u'}, \cdots, \frac{v'}{u'}}_{(n-1)\text{-times}} \quad \frac{v''}{u''}.$$

It is easy to see that the J-flow preserves the Calabi ansatz condition. Hence, we may assume that the solution of the flow (2.2) is $\chi_{\varphi(\cdot,t)} = dd^c v(\rho, t)$. Consequently, (2.2) can be written as an evolution equation on $v(\rho, t)$:

$$\begin{cases} \frac{\partial v}{\partial t} & = c_1 - (n-1)\frac{u'}{v'} - \frac{u''}{v''}, \\ v(\rho, 0) & = v_0(\rho). \end{cases} \tag{2.3}$$

The corresponding critical equation (1.3) is

$$c_1 = (n-1)\frac{u'}{v'} + \frac{u''}{v''}. \tag{2.4}$$

Taking a one-time derivative on (1.1) and applying maximum principle, we obtain the bound (cf. (3.1) in [FL])

$$0 < C_1 \leq \frac{\sigma_{n-1}(\chi_\varphi)}{\sigma_n(\chi_\varphi)} \leq C_2, \tag{2.5}$$

for two universal constants C_1 and C_2 depending only on the initial data.

In terms of the potential u and v, (2.5) is

$$0 < C_1 \leq (n-1)\frac{u'}{v'} + \frac{u''}{v''} \leq C_2. \tag{2.6}$$

To study flow (2.3), we regard $(v'(\rho,t), u'(\rho))$ as a family of parametric plane curves. Since $u'' > 0$ and $v'' > 0$, each curve is the graph of a monotone increasing function $f(x,t)$, i.e.,

$$f(v'(\rho,t),t) = u'(\rho). \tag{2.7}$$

We are concerned with the corresponding evolution equation for f.

Proposition 2.1. *The function $f = f(x,t)$ defined by (2.7) is a classical solution of the initial-boundary value problem*

$$\frac{\partial f}{\partial t} = Q(f)[\frac{\partial^2 f}{\partial x^2} + (n-1)\frac{\frac{\partial f}{\partial x}}{x} - (n-1)\frac{f}{x^2}], \tag{2.8}$$

with

$$f(1,t) = 1, \quad f(\beta,t) = \alpha \quad \forall t,$$

where the initial value $f(x,0) = f_0(x)$ is any smooth monotone function satisfying the above boundary condition. $Q \in C^\infty([1,\alpha],\mathbb{R}_{\geq 0})$ is uniquely determined by u such that $Q(y) = 0$ if and only if $y = 1$ and $y = \alpha$.

Proof. Taking one-time derivative of (2.7), we get

$$\frac{\partial f}{\partial x}\frac{\partial v'}{\partial t} + \frac{\partial f}{\partial t} = 0. \tag{2.9}$$

Taking one spacial derivative of (2.3), we get

$$\frac{\partial v'}{\partial t} = -(n-1)(\frac{u'}{v'})' - (\frac{u''}{v''})'. \tag{2.10}$$

Taking spacial derivatives of (2.7) we also have

$$\frac{\partial f}{\partial x} = \frac{u''}{v''}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{(\frac{u''}{v''})'}{v''}. \tag{2.11}$$

Plugging (2.10) and (2.11) in (2.9), we get

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\partial f}{\partial x}\frac{\partial v'}{\partial t}\\ &= \frac{\partial f}{\partial x}[(n-1)(\frac{u''}{v'} - \frac{u'v''}{v'^2}) + (\frac{u''}{v''})']\\ &= \frac{\partial f}{\partial x}[(n-1)(\frac{u''}{v'} - \frac{u'v''}{v'^2}) + \frac{\partial^2 f}{\partial x^2}v'']\\ &= u''[\frac{\partial^2 f}{\partial x^2} + (n-1)\frac{\frac{\partial f}{\partial x}}{x} - (n-1)\frac{f}{x^2}].\end{aligned} \tag{2.12}$$

Here we denote the variable of f, v' by x. For a given u, since $u'' > 0$, then u' is monotone. It follows that we can write u'' as

$$u'' = u''(u'^{-1}(f)) := Q(f).$$

By the asymptotic behavior of u' (2.1), we have $Q(1) = Q(\alpha) = 0$ and $Q(f) > 0$ whenever $1 < f < \alpha$.

The boundary and initial conditions follow directly from the limit behavior of u' and v'. □

In order to study the convergence behavior of (2.3), we first study the convergence behavior of (2.8) and then translate the convergence of f back to convergence of v' and correspondingly of v. (2.8) is a degenerate parabolic equation of one spacial dimension. A priori, we know the long time existence and we also have a uniform C^1 bound on f by (2.6), i.e.,

$$\frac{\partial f}{\partial x} = \frac{u''}{v''} \le C_2.$$

Therefore we have a uniform limit

$$\lim_{t\to\infty} f(x,t) = f_\infty(x). \tag{2.13}$$

It follows that $f_\infty(x)$ is a weak solution of the corresponding stationary problem:

$$Q(f)[\frac{\partial^2 f}{\partial x^2} + (n-1)\frac{\frac{\partial f}{\partial x}}{x} - (n-1)\frac{f}{x^2}] = 0, \quad f(1) = 1,\ f(\beta) = \alpha. \tag{2.14}$$

Equivalently, we can write (2.14) as

$$\Psi_{\{1<f<\alpha\}}(\frac{\partial^2 f}{\partial x^2} + (n-1)\frac{\frac{\partial f}{\partial x}}{x} - (n-1)\frac{f}{x^2}) = 0 \tag{2.15}$$

subject to the boundary condition $f(1) = 1$ and $f(\beta) = \alpha$, where $\Psi_{\cdot}$ is the characteristic function of a set. Standard elliptic theory shows that f_∞ is a strong solution of (2.14) and (2.15).

Since when $1 < f < \alpha$ one can just drop the characteristic function $\Psi_{\{1<f<\alpha\}}$, we are able to write down all possible solutions of (2.15) explicitly. Piece-wisely, they are either constant functions $f = 1$ and $f = \alpha$ or solutions of the differential equation

$$f'' + (n-1)\frac{f'}{x} - (n-1)\frac{f}{x^2} = 0, \tag{2.16}$$

which can be written as $Ax + \frac{B}{x^{n-1}}$, for appropriate constants A and B.

Since $f(x,t)$ is monotone for all t, limit f_∞ can only be of the form

$$f_\infty(x) = \begin{cases} 1, & 1 \le x \le s, \\ g(x), & s \le x \le t, \\ \alpha, & t \le x \le \beta, \end{cases} \tag{2.17}$$

where g is the solution of (2.16) on $[s,t]$.

Due to the degeneracy condition on Q, f_∞ may lose regularity at $x = s$ and $x = t$. Nevertheless, on any compact subset in which $\{1 < f_\infty(x) < \alpha\}$, (2.8) is uniform elliptic, and we obtain higher order estimates (cf. [W]) from general theory of nonlinear parabolic equations. Consequently the convergence (2.13) is smooth in region $\{x|1 < f_\infty(x) < \alpha\}$.

Consider the ODE

$$f''(x) + (n-1)\frac{f'(x)}{x} - (n-1)\frac{f(x)}{x^2} = 0 \tag{2.18}$$

with boundary value $f(1) = 1$ and $f(\beta) = \alpha$. It has a unique solution

$$\tilde{f}(x) = ax + \frac{b}{x^{n-1}}, \tag{2.19}$$

with $a = \frac{\alpha\beta^{n-1}-1}{\beta^n-1}$ and $b = \frac{\beta^n-\alpha\beta^{n-1}}{\beta^n-1}$

We first prove a lemma saying the convergence behavior is independent of the choice of the initial data:

Lemma 2.2. *Let $\varphi(x,t), \psi(x,t) \in \mathcal{P}_\chi$ be two solutions of the flow* (1.1) *with initial values φ_0 and ψ_0, respectively. Let $f_t, \bar{f}_t$ be their corresponding functions satisfying* (2.8)*. Then there exists a universal constant C such that*

$$|\varphi(x,t) - \psi(x,t)|_{C^0} \le C, \quad \forall t. \tag{2.20}$$

In particular, for $f_\infty(x) = \lim_{t\to\infty} f_t(x)$ and $\tilde{f}_\infty(x) = \lim_{t\to\infty} \bar{f}_t(x)$, we have

$$f_\infty(x) = \tilde{f}_\infty(x).$$

Proof. Taking the difference of flows with two initial values, we get that $\varphi(x,t) - \psi(x,t)$ satisfies a parabolic equation

$$\frac{\partial[\varphi(x,t) - \psi(x,t)]}{\partial t} = F^{i\bar{j}}((1-s_t)\chi_{\varphi(t)} + s_t\chi_{\psi(t)})(\varphi(t) - \psi(t))_{i\bar{j}},$$

where $0 < s_t < 1$. Inequality (2.20) then follows from the maximum principle.

For the second part of the lemma, we consider the corresponding limits for $v'(\rho,t)$, denoted by v'_∞ and $\tilde{v}'_\infty$, respectively. By (2.17), there exist constants $s, t, \tilde{s}, \tilde{t}$ such that

$$v'_\infty(-\infty) = s, \quad v'_\infty(\infty) = t$$

and

$$\tilde{v}'_\infty(-\infty) = \tilde{s}, \quad \tilde{v}'_\infty(\infty) = \tilde{t}.$$

Thus, $|v_\infty - \tilde{v}_\infty|$ is not uniformly bounded unless $s = \tilde{s}$ and $t = \tilde{t}$. We have thus proved the lemma. □

We may now state the following key result of this section:

Theorem 2.3. *The flow* (2.8) *converges to a unique limit $f_\infty(x)$. For the expression of f_∞, we have the following four cases:*

(1) *$\alpha > \beta$: $f_\infty(x) = \tilde{f}(x)$;*

(2) $\alpha < \beta$ *and* $\frac{\alpha\beta^{n-1}-1}{\beta^n-1} > \frac{n-1}{n}$*:* $f_\infty(x) = \tilde{f}(x)$*;*

(3) $\alpha < \beta$ *and* $\frac{\alpha\beta^{n-1}-1}{\beta^n-1} = \frac{n-1}{n}$*:* $f_\infty(x) = \tilde{f}(x)$*;*

(4) $\alpha < \beta$ *and* $\frac{\alpha\beta^{n-1}-1}{\beta^n-1} < \frac{n-1}{n}$*: Let*

$$\lambda = \inf\{\lambda' \mid \exists g(x) \textit{ satisfies } (2.18),\ g(\lambda') = 1,\ g(\beta) = \alpha \textit{ and } g \geq 1\}.$$

Let g be the corresponding solution of (2.18) *with* $g(\lambda) = 1$ *and* $g(\beta) = \alpha$*. We have in this case*

$$f_\infty(x) = \begin{cases} 1, & 1 \leq x \leq \lambda, \\ g(x), & \lambda \leq x \leq \beta. \end{cases}$$

To better understand the theorem, we illustrate initial and limit functions of four cases by Figure 2. According to (2.7), we remind the readers that the horizontal axis corresponds to v' and the vertical axis corresponds to u'.

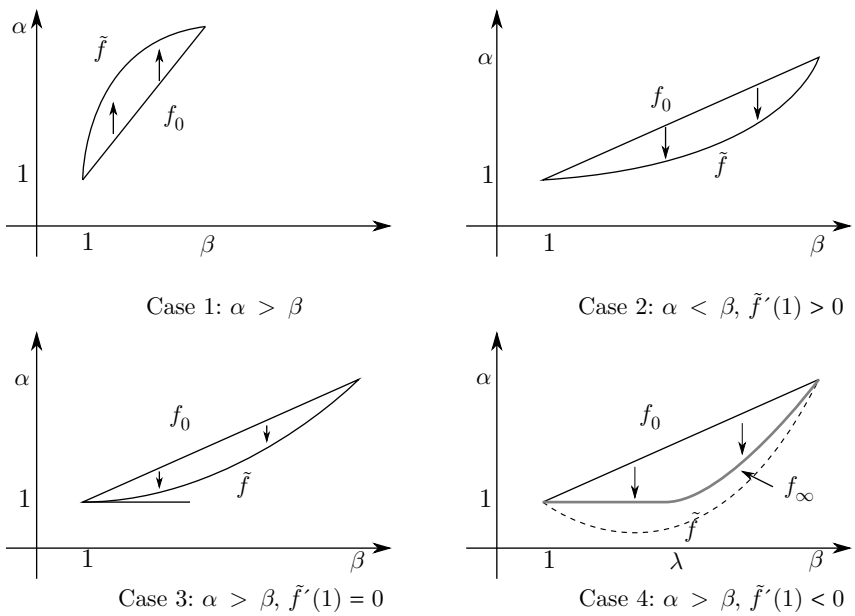

FIGURE 2

Proof. Note that for the first three cases, $1 < f_\infty(x) < \alpha$ if $1 < x < \beta$. In Case 1, $f_\infty(x)$ is a concave function. In Cases 2 and 3, $f_\infty(x)$ is convex. We distinguish Case 2 and Case 3 by the fact that $f'_\infty(1) > 0$ in Case 2 and $f'_\infty(1) = 0$ in Case 3.

Case 1: $\alpha > \beta$. By Lemma 2.2, we may choose a special initial value

$$f_0(x) = \frac{\alpha - 1}{\beta - 1}(x - 1) + 1,\ x \in [1, \beta].$$

Claim 2.4. With conditions given as above,

$$f(x,t) - \tilde{f}(x) < 0, \quad \forall t \in \mathbb{R}, x \in (1, \beta).$$

Proof of the claim. This is a simple application of the strong maximum principle. Since f_∞ is the solution of (2.18) on $[1,\beta]$, thus $h(x,t) := f(x,t) - f_\infty(x)$ satisfies

$$\frac{\partial h}{\partial t} = Q(f)[\frac{\partial^2 h}{\partial x} + (n-1)\frac{\frac{\partial h}{\partial x}}{x} - (n-1)\frac{h}{x^2}].$$

Since $h_0(x) = f_0(x) - f_\infty(x) < 0$ and $h(1,t) = f(1,t) - f_\infty(\beta) = 0$, $h(\beta,t) = f(\beta,t) - f_\infty(\beta) = 0$, it follows from strong maximum principle that $h(x,t) < 0$. □

Claim 2.5. $\frac{\partial f}{\partial t} \geq 0$.

Proof. A direct computation shows that

$$\frac{\partial f}{\partial t}|_{t=0} > 0.$$

Taking the time derivative of (2.8), we get

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{\partial f}{\partial t}) = \frac{\partial f}{\partial t}Q'(f)(f'' + (n-1)\frac{f'}{x} - (n-1)\frac{f}{x^2}) \\ + Q(f)[(\frac{\partial f}{\partial t})'' + (n-1)\frac{(\frac{\partial f}{\partial t})'}{x} - (n-1)\frac{\frac{\partial f}{\partial t}}{x^2}].\end{aligned}$$

Let $h(t) = \min_{1\leq x\leq\beta}\frac{\partial f}{\partial t}(x,t)$. A simple maximum principle argument implies $h'(t) \geq 0$ and hence $\frac{\partial f}{\partial t} \geq 0$. □

Combining Claim 2.4 and Claim 2.5, we find that $f(x,t)$ is monotone increasing to a limiting function $f_\infty(x) = \lim_{t\to\infty} f(x,t)$ and

$$f_\infty(x) \leq \tilde{f}(x). \tag{2.21}$$

However, there is only one solution of the form (2.17) satisfying (2.21), which is exactly $\tilde{f}(x)$. Thus we have proved that $f_\infty(x) = \tilde{f}(x)$ for Case 1.

Case 2: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} > \frac{n-1}{n}$. By Lemma 2.2, we may choose the following initial value for the flow:

$$f_0(x) = \frac{\alpha-1}{\beta-1}(x-1) + 1.$$

Similar to Case 1, we have

Claim 2.6.

$$f(x,t) - \tilde{f}(x) > 0, \quad \forall t \in \mathbb{R}, x \in (1,\beta).$$

Claim 2.7.

$$\frac{\partial f}{\partial t} \leq 0.$$

The proof is in the same fashion as the proof of Claim 2.5; we shall omit it for simplicity.

Since $f(x,t)$ is monotone decreasing with respect to t, and $f_\infty(x) \geq \tilde{f}(x)$, f_∞ has to be the unique solution satisfying (2.17), which is $\tilde{f}$. We have thus finished the proof for Case 2.

Case 3: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} = \frac{n-1}{n}$. The proof is exactly the same as that for Case 2.

Case 4: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} < \frac{n-1}{n}$. Define

$$\tilde{g}(x) = \begin{cases} 1, & 1 \le x \le \lambda, \\ g(x), & \lambda \le x \le \beta. \end{cases}$$

It follows easily that Claim 2.6 and Claim 2.7 are still valid with $\tilde{g}$ replacing $\tilde{f}$. Since $f_\infty(x)$ satisfies (2.17) and $f_t(x) \searrow f_\infty(x)$ as $t \to \infty$,

$$f_\infty(x) = \sup\{f(x) | f(x) \text{ satisfies (2.17)}\}.$$

By the characterization of λ and $g(x)$, we obtain our conclusion.

It is easy to see that λ is the unique solution of

$$(n-1)\frac{\beta}{\lambda} + \frac{\lambda^{n-1}}{\beta^{n-1}} = n\alpha \tag{2.22}$$

such that $\lambda \in (1, \beta)$, and $g'(\lambda) = 0$.

We have proved Theorem 2.3. □

Remark 2.8. It is interesting to remark that $\tilde{g}$ in Case 4 of Theorem 2.3 arises as a solution to an obstacle problem (cf. [C]). In fact, in the convex set

$$K := \{f \in H^1([1, \beta]), f(1) = 1, f(\beta) = \alpha, f \ge 1\}, \tag{2.23}$$

we consider the energy functional $E : K \to \mathbb{R}$

$$E(f) = \frac{1}{2}\int_1^\beta (x^{n-1}f'^2 + (n-1)x^{n-3}f^2)dx. \tag{2.24}$$

The unique minimizer of E in K satisfies

$$\Psi_{\{f>1\}}(f'' + (n-1)\frac{f'}{x} - (n-1)\frac{f}{x^2}) = 0. \tag{2.25}$$

Remark 2.9. It is worth pointing out that for all cases of Theorem 2.3, $f'_\infty(x)$ is continuous. Furthermore, for Case 3 and Case 4, we have

$$f'_\infty(x) = 0, \quad \text{if} \quad f_\infty(x) = 1.$$

This is an important feature of the limiting function that will indicate geometric properties for the geometric flow.

The limiting behavior described for $f(x, t)$ can be used to determine the convergence behavior of metrics following the inverse σ_k flow.

Proof of Main Theorem 1: *J-flow case.* We will divide the proof into four cases, just as in the proof of Theorem 2.3. Bear in mind that once we get the limit f_∞, we can correspondingly get $v'_\infty(\rho)$ by the relation

$$v'_\infty(\rho) = f_\infty^{-1}(u'_\infty(\rho)).$$

Case 1: $\alpha > \beta$. First, we claim that there exists a uniform positive lower bound for $\frac{\partial f}{\partial x}$. Suppose not, then there is a sequence of (x_n, t_n) such that

$$\frac{\partial f}{\partial x}(x_n, t_n) \to 0, \quad t_n \to \infty.$$

Then there must be an accumulation point x_∞ in $[1, \beta]$ for a subsequence of $\{x_n\}$, which is still denoted by $\{x_n\}$ for simplicity. On the other hand,

$$\lim_{n\to\infty} \frac{\partial f}{\partial x}(x_n, t_n) = f'_\infty(x_\infty) \ne 0.$$

Thus this contradiction proves the claim. Therefore there exists a universal constant ϵ such that $\frac{\partial f}{\partial x} \geq \epsilon > 0$. Consequently, there exists a universal constant $C > 0$ such that along the flow (2.2),

$$tr_\omega \chi_\varphi = (n-1)\frac{v'}{u'} + \frac{v''}{u''} = (n-1)\frac{x}{f} + \frac{1}{\frac{\partial f}{\partial x}} < C.$$

A uniform positive lower bound of eigenvalues of χ_φ with respect to ω follows if one couples the above estimate with the bound (2.6). Then it is standard to evoke the Evans-Krylov theory and Schauder estimates to get higher order estimates for χ_φ. Consequently χ_t converges smoothly to χ_∞, which solves the critical equation (1.3).

Case 2: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} > \frac{n-1}{n}$. Similar to Case 1, since $\frac{df_\infty}{dx} \geq \epsilon > 0$, we may obtain a uniform positive lower bound for $\frac{\partial f}{\partial x}$. Thus we have smooth convergence of χ_t to χ_∞ for the J-flow.

Case 3: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} = \frac{n-1}{n}$. In this case, the previous argument fails since $f'_\infty(1) = 0$. A uniform positive lower bound on $\frac{\partial f}{\partial x}$ is not expected. However, we have the following

Claim 2.10. For any $\epsilon > 0$, if $f(x,t) \geq 1+\epsilon$, there exists a positive constant $c = c(\epsilon)$ depending only on ϵ such that

$$\frac{\partial f}{\partial x} \geq c(\epsilon).$$

Proof. Suppose that the claim does not hold. Then there exists a sequence of (x_n, t_n) such that

$$\frac{\partial f}{\partial x}(x_n, t_n) \to 0, \quad t_n \to \infty.$$

We also have that a subsequence of x_n converges to $x_\infty \in [1, \beta]$, for simplicity still denoted by x_n. Since $f(x_n, t_n) \geq 1 + \epsilon$, then $f_\infty(x_\infty) \geq 1 + \epsilon$. Consequently

$$\lim_{n\to\infty} \frac{\partial f}{\partial x}(x_n, t_n) = f'_\infty(x_\infty) \neq 0,$$

we get a contradiction, thus we have proved the claim. □

We continue our proof of Case 3, Main Theorem 1 for J-flow. For any compact subset $K \subset X \setminus E_0$, there exists a constant $\epsilon > 0$ such that $K \subset u'^{-1}([1+\epsilon, \alpha])$. By Claim 2.10, for any $t > 0$, the corresponding $f(x,t)$ defined by (2.7) satisfies $f_x(x,t) \geq c(\epsilon)$ for a given $c(\epsilon) > 0$. We may conclude that χ_t converges smoothly to χ_∞, following arguments given in the proof of Case 1. We have thus established the smooth convergence of the J-flow away from E_0.

The corresponding potential $v_\infty(\rho)$ defines a Kähler metric with singularity along E_0. Indeed, by the explicit expression of f_∞ (2.19), a moment of calculation shows that v_∞ has appropriate asymptotic behavior at $\rho = \infty$ and thus can be extended through E_∞ to a smooth metric.

On the other hand, near $\rho = -\infty$, the resulting Kähler form is not smooth. In fact, $v'_\infty(\rho) \sim 1 + ke^{\frac{\rho}{2}}$ near $\rho = -\infty$. In the local coordinate patch $(z_1, \cdots, z_n)$ centered at any $p \in E_0$, where $E_0 \cap U = \{z_1 = 0\}$, the metric is equivalent to $\frac{1}{|z_1|} dz_1 \wedge d\bar{z_1} + v'_\infty(-\infty)\omega_{FS}$, where ω_{FS} is the Fubini-Study metric on E_0. Thus the metric is singular with cone angle π transverse to E_0.

Case 4: $\alpha < \beta, \frac{\alpha\beta^{n-1}-1}{\beta^n-1} < \frac{n-1}{n}$. Similar to Case 3, we may prove that the flow is convergent smoothly away from E_0. However, now we have

$$\lim_{\rho\to-\infty} v'_\infty(\rho) = \lambda > 1.$$

The convergence

$$v'(\rho,t) \to v'_\infty(\rho)$$

and its resulting metric flow may be understood as two distinct behaviors: one is a Dirac-concentration on E_0 (δ-function at $\rho = -\infty$) with coefficient $(\lambda - 1)$; the other (away from $\rho = -\infty$) is a smooth convergence of metrics away from E_0. From a geometric point of view, the former corresponds to a current of integration on $[E_0]$ with coefficient $(\lambda - 1)$. The latter corresponds to a limit of the J-flow with initial data χ in a different class: for $x \in (\lambda, \beta]$, using the explicit expression for f_∞ (2.19), it is easy to see that v'_∞ is the corresponding limit solution for the J-flow of the triple (X, ω, χ) with

$$\omega \in \alpha[E_\infty] - [E_0], \quad \chi \in \beta[E_\infty] - \lambda[E_0].$$

Since $g'(\lambda) = 0$, this corresponds to Case 3. One can readily check by scaling $\beta[E_\infty] - \lambda[E_0]$ to $\frac{\beta}{\lambda}[E_\infty] - [E_0]$ that the condition of Case 3 is satisfied since

$$\frac{\alpha(\frac{\beta}{\lambda})^{n-1} - 1}{(\frac{\beta}{\lambda})^n - 1} = \frac{n-1}{n}.$$

We have thus finished the proof. □

3. General inverse σ_k-flow on $\mathbb{P}^n\#\overline{\mathbb{P}^n}$

In this section, we discuss the general inverse σ_k-flow (1.1) on $X = \mathbb{P}^n\#\overline{\mathbb{P}^n}$. We follow the discussion of Section 2, however the parabolic equation analogous to (2.8) is more complicated.

3.1. Generalized J-flow case ($k = 1$).

Let (X, ω, χ) be given as before. Assume both ω and χ satisfy Calabi ansatz. The general inverse σ_k-flow (1.1) for $k = 1$ can be written as

$$\frac{\partial v}{\partial t} = F((n-1)\frac{u'}{v'} + \frac{u''}{v''}). \tag{3.1}$$

Again suppose $(v'(\rho,t), u'(\rho))$ is a family of parametric curves implicitly given by

$$f(v'(\rho,t), t) = u'(\rho). \tag{3.2}$$

Then the evolution of f is

$$\frac{\partial f}{\partial t} = -F'Q(f)[\frac{\partial^2 f}{\partial x^2} + (n-1)\frac{\frac{\partial f}{\partial x}}{x} - (n-1)\frac{f}{x^2}]. \tag{3.3}$$

By (1.2), $F' < 0$, the convergence behavior of (3.3) is the same as that of (2.8).

3.2. **General case ($k > 1$).**

For general $k > 1$, (1.1) reduces to

$$\frac{\partial v}{\partial t} = F(\binom{n-1}{k-1}(\frac{u'}{v'})^{k-1}\frac{u''}{v''} + \binom{n-1}{k}(\frac{u'}{v'})^k) - F(c_k) \tag{3.4}$$

and then f defined by (3.2) evolves as

$$\frac{\partial f}{\partial t} = -F'u''(\frac{f}{x})^{k-2}\binom{n-1}{k-1}[\frac{\partial^2 f}{\partial x^2}\frac{f}{x} + (k-1)\frac{(\frac{\partial f}{\partial x})^2}{x} \tag{3.5}$$

$$+ (n+1-2k)\frac{f\frac{\partial f}{\partial x}}{x^2} - (n-k)\frac{f^2}{x^3}]. \tag{3.6}$$

Define $g(x,t) := f^k(x,t)$; then the evolution of g is

$$\begin{aligned}\frac{\partial g}{\partial t} &= -\binom{n-1}{k-1}F'(\frac{f}{x})^{k-1}u''[\frac{\partial^2 g}{\partial x^2} + (n+1-2k)\frac{\frac{\partial g}{\partial x}}{x} - k(n-k)\frac{g}{x^2}] \\ &:= -F'Q(g)[\frac{\partial^2 g}{\partial x^2} + (n+1-2k)\frac{\frac{\partial g}{\partial x}}{x} - k(n-k)\frac{g}{x^2}].\end{aligned} \tag{3.7}$$

Proof of the Main Theorem 1. As in Section 2, we need to discuss the corresponding ODE:

$$\frac{\partial^2 g}{\partial x^2} + (n+1-2k)\frac{\frac{\partial g}{\partial x}}{x} - k(n-k)\frac{g}{x^2} = 0 \tag{3.8}$$

with boundary values $g(1) = 1$ and $g(\beta) = \alpha^k$.

One can solve it explicitly to get the solution

$$g_\infty(x) = ax^k + \frac{b}{x^{n-k}}, \qquad a = \frac{\alpha^k\beta^{n-k} - 1}{\beta^n - 1}, \quad b = \frac{\beta^n - \alpha^k\beta^{n-k}}{\beta^n - 1}. \tag{3.9}$$

It follows that

$$1 < g_\infty(x) < \alpha^k \quad \text{when } x \in (1,\beta),$$

except the cases $\alpha < \beta$ and $\frac{\alpha^k\beta^{n-k}-1}{\beta^n-1} < \frac{n-k}{n}$.

Then in that case define

$$\lambda = \inf\{\lambda' \mid g \text{ satisfies (3.8) }, g(\lambda') = 1, g(\beta) = \alpha^k \text{ and } g \geq 1\}.$$

Let g be the corresponding solution of (3.8) with $g(\lambda) = 1$ and $g(\beta) = \alpha^k$. Then the unique limit in this case is

$$g_\infty(x) = \begin{cases} 1, & 1 \leq x \leq \lambda, \\ g(x), & \lambda \leq x \leq \beta. \end{cases}$$

By the definition of λ, we have $g'(\lambda) = 0$. Then it is easy to check that $\lambda \in (1,\beta)$ is the unique solution of

$$(n-k)(\frac{\beta}{\lambda})^k + k(\frac{\lambda}{\beta})^{n-k} = n\alpha^k. \tag{3.10}$$

Finally, notice we still have the universal constants $C_1, C_2 > 0$ such that

$$0 < C_1 \leq \sigma_k(\underbrace{\frac{u'}{v'}, \cdots \frac{u'}{v'}}_{n-1\text{-times}}, \frac{u''}{v''}) \leq C_2. \tag{3.11}$$

Since

$$\frac{u'}{v'} \geq \frac{1}{\beta},$$

from (3.11) we get a uniform upper bound for $\frac{u''}{v''}$.

The rest of the proof follows that of Section 2. □

4. Flows on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$

In this section, we consider the general inverse σ_k-flow on a family of projective bundles over $\mathbb{P}^n$ and its convergence behavior under the assumption that both ω, χ satisfy the Calabi ansatz. A new geometric limit phenomenon occurs.

Let $E = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$ be a vector bundle over a projective space $\mathbb{P}^n$, where $\mathcal{O}_{\mathbb{P}^n}$ is the trivial line bundle and $\mathbb{O}_{\mathbb{P}^n}(-1)$ is the tautological line bundle. Let

$$X_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$$

be the projectivization of E. $X_{m,n}$ is a $\mathbb{P}^{m+1}$ bundle over $\mathbb{P}^n$ with $\pi : X_{m,n} \to \mathbb{P}^n$ being the bundle map. In particular, $X_{0,n}$ is $\mathbb{P}^{n+1}$ blown up at one point. Let D_∞ be the divisor in $X_{n,m}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$ and D_0 be the divisor in $X_{m,n}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})$. In fact, the additive divisor group $N^1(X_{m,n})$ is spanned by $[D_0]$ and $[D_\infty]$. We also define the divisor D_H by the pull-back of the divisor on $\mathbb{P}^n$ associated to $\mathcal{O}_{\mathbb{P}^n}(1)$. Then

$$[D_\infty] = [D_0] + [D_H].$$

Moveover, D_∞ is a big and semi-ample divisor and any divisor $a[D_H] + b[D_\infty]$ is ample if and only if $a > 0$ and $b > 0$.

To consider the Calabi ansatz (see [C, SY]), let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^n$. Let h be the hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$ such that $Ric(h) = -\omega_{FS}$. Under local trivialization of E, we write

$$e^\rho = h(z)|\xi|^2, \xi = (\xi_1, \xi_2, \cdots, \xi_{m+1}),$$

where $h(z)$ is a local representation of h. In particular, if we choose an inhomogeneous coordinate $z = (z_1, z_2, \cdots, z_n)$ on $\mathbb{P}^n$, we have

$$h(z) = 1 + |z|^2.$$

We consider Kähler metrics of the following type on $X_{m,n}$:

$$\omega = a\pi^*\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(\rho). \tag{4.1}$$

According to Calabi [Ca], (4.1) is Kähler if and only if $a > 0$, $u' > 0$, $u'' > 0$, and the asymptotic behavior of u satisfies:

(4.2)

(†) $u_0(r) := u(\ln r)$

is extendable by continuity to a smooth function at $r = 0$, and $u_0'(0) > 0$.

(‡) $u_\infty(r) := u(-\ln r) + b \ln r$

is extendable by continuity to a smooth function at $r = 0$

for some b, and $u_\infty'(0) > 0$.

Thus we have $\lim_{\rho\to-\infty} u'(\rho) = 0$ and $\lim_{\rho\to\infty} u'(\rho) = b$. Here $\rho = -\infty$ corresponds P_0 and $\rho = \infty$ corresponds to D_∞. Furthermore,

$$\omega \in a[D_H] + b[D_\infty]. \tag{4.3}$$

Note

$$\omega = (a+u')\omega_{FS} + \frac{\sqrt{-1}}{2\pi} h e^{-\rho}(u'\delta_{ij} + he^{-\rho}(u''-u')\bar{\xi_i}\xi_j)\nabla\xi_i \wedge \nabla\bar{\xi_j}, \tag{4.4}$$

where $\nabla\xi_i = d\xi_i + h^{-1}\partial h\xi_i$.

We may now discuss the general inverse σ_k-flow (1.1) on $X_{m,n}$. If ω, χ are of the form (4.1), without loss of generality we may normalize them so that

$$\begin{aligned} &\omega \in [D_H] + b[D_\infty], b > 0, \\ &\chi \in [D_H] + b'[D_\infty], b' > 0. \end{aligned} \tag{4.5}$$

Hence we can assume that

$$\omega = \omega_{FS} + b\partial\bar{\partial}u(\rho), \quad \chi = \omega_{FS} + b'\partial\bar{\partial}v(\rho), \tag{4.6}$$

with u, v satisfying the criterion (4.2). The general inverse σ_k-flow preserves the Calabi ansatz (4.1). We consider the function $f(x,t)$ determined by

$$f(v'(\rho,t),t) = u'(\rho). \tag{4.7}$$

Proposition 4.1. *Consider the general inverse σ_k-flow on the triple $(X_{m,n},\omega,\chi)$. If ω, χ are given as above, then the evolution of $f(x,t)$ defined via (4.7) is*

$$\frac{\partial f}{\partial t} = -F'u''(\sigma_k(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f'))'. \tag{4.8}$$

Proof. From (4.4), we may calculate the eigenvalues of χ with respect to ω to be

$$\underbrace{\frac{1+v'}{1+u'},\cdots,\frac{1+v'}{1+u'}}_{n\text{-times}}, \underbrace{\frac{v'}{u'},\cdots,\frac{v'}{u'}}_{m\text{-times}}, \frac{v''}{u''}. \tag{4.9}$$

Taking the time derivative of (4.7), we get

$$\frac{\partial f}{\partial t} = -f'\frac{\partial v'}{\partial t}. \tag{4.10}$$

Taking the spacial derivative of (1.1), we get

$$\frac{\partial v'}{\partial t} = F'(\sigma_k(\underbrace{\frac{1+u'}{1+v'}}_{n\text{-copies}}, \underbrace{\frac{u'}{v'}}_{m\text{-copies}}, \frac{u''}{v''}))'. \tag{4.11}$$

Bearing in mind the relation (4.7), i.e., $u' = f$ and $v' = x$, we have

$$\begin{aligned} f'(\frac{1+u'}{1+v'})' &= f'[\frac{u''}{1+v'} - \frac{(a+u')v''}{(a+v')^2}] \\ &= u''[\frac{f'}{1+v'} - \frac{1+u'}{(1+v')^2}] = u''(\frac{1+f}{1+x})'. \end{aligned} \tag{4.12}$$

Similarly,

$$f'(\frac{u'}{v'})' = u''(\frac{f}{x})'. \tag{4.13}$$

We also have

$$f'(\frac{u''}{v''})' = f'[\frac{u'''}{v''} - \frac{u''v'''}{v''^2}] \tag{4.14}$$
$$= f'[\frac{f''v''^2 + f'v'''}{v''} - \frac{u''v'''}{v''^2}] = u''f''.$$

Using (4.11), (4.12), (4.13) and (4.14) and (4.10), we get

$$\frac{\partial f}{\partial t} = -F'u''(\sigma_k(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f'))'. \tag{4.15}$$

□

Following our method developed in Section 2, we are mainly concerned with the ODE on the right hand side:

$$(\sigma_k(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f'))' = 0,\ x \in [0, b'], \tag{4.16}$$

with the boundary condition $f(0) = 0$ and $f(b') = b$. Note this ODE is also the critical equation for the flow.

Proposition 4.2. *The equation* (4.16) *admits a unique positive monotone increasing solution with $f(0) = 0$ and $f(b') = b$ if and only if*

- $c_k \geq \binom{n}{k}$ *for $k \leq n$. Furthermore the corresponding unique solution satisfies $f'(0) > 0$ provided the strict inequality holds.*
- $c_k > 0$ *for $k > n$. The corresponding unique solution satisfies $f'(0) > 0$.*

Here c_k is the topological constant

$$\frac{\binom{n+m+1}{k}\int_{X_{m,n}} \chi^{m+n+1-k} \wedge \omega^k}{\int_{X_{m,n}} \chi^{m+n+1}}.$$

Proof. Define

$$G_a^{m,n}(x) = \int_0^x t^m (t+a)^n dt. \tag{4.17}$$

Then

$$\begin{aligned}\frac{d[G_{1+t}^{m,n}(x+tf)]}{dx} &= (x+tf)^m(1+t+x+tf)^n(1+tf') \\ &= x^m(1+x)^n(1+t\frac{f}{x})^m(1+t\frac{f+1}{x+1})^n(1+tf') \\ &= [G_1^{m,n}(x)]'[1+t\sigma_1(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f') \\ &+ t^2\sigma_2(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f') \\ &+ \cdots + t^{m+n+1}\sigma_{m+n+1}(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f')].\end{aligned} \tag{4.18}$$

Let

$$G_1^{m,n,k}(f,x) = \frac{1}{k!}\frac{d^k}{dt^k}|_{t=0}(G_{1+t}^{m,n}(x+tf)).$$

Taking the k-th t derivative of (4.18) and evaluating at $t=0$, we find ODE (4.16) is equivalent to

$$G_1^{m,n,k}(f,x) = \alpha G_1^{m,n}(x) + \beta, \tag{4.19}$$

for two constants α and β.

Claim 4.3.

$$G_1^{m,n,k}(f,x) = a_k(x)f^k + a_{k-1}(x)f^{k-1} + \cdots + a_1(x)f + a_0(x), \tag{4.20}$$

where $a_i(x)$ are polynomials of x. In particular, $a_0(x) = \binom{n}{k}G_1^{m,n-k}(x)$ when $k \le n$ and $a_0 = 0$ when $k > n$.

Proof. The proof is a straightforward computation based on the explicit form of $G_1^{m,n}$ via (4.17). □

From (4.19) and (4.20), the boundary value of f implies

$$\alpha = \frac{G_1^{m,n,k}(b,b')}{G_1^{m,n}(b')} \quad \text{and} \quad \beta = 0.$$

Notice that $\frac{G_1^{m,n,k}(b,b')}{G_1^{m,n}(b')}$ is actually the topological constant c_k. This follows from direct computation using metrics ω and χ of the form given here. It is also easy to verify that each $a_i(x)$ has positive coefficients as a polynomial of x. Thus for each fixed x view $G_1^{m,n,k}(f,x)$ as a polynomial of f, it admits a unique positive solution if and only if

$$\alpha G_1^{m,n}(x) \ge a_0(x),$$

with equality holding only at $x=0$. By definition,

$$\alpha G_1^{m,n}(x) - a_0(x) = \int_0^x t^m(t+1)^{n-k}(\alpha(t+1)^k - \binom{n}{k})dt, \tag{4.21}$$

from which $c_k \ge \binom{n}{k}$ follows.

It is clear that $f'(x) \ge 0$. We claim that the function f is strictly increasing. If not, there exists a $x_0 \in [0,b']$ such that $f'(x_0) = 0$. At x_0, we have

$$(\frac{f}{x})'(x_0) = \frac{-f(x_0)}{x_0^2} < 0 \quad \text{and} \quad (\frac{1+f(x)}{1+x})'(x_0) = \frac{-f(x_0)-1}{(1+x_0)^2} < 0. \tag{4.22}$$

Both $\frac{f}{x}$ and $\frac{1+f}{1+x}$ are strictly decreasing near x_0. Note that

$$\sigma_k(\underbrace{\frac{1+f}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f}{x}}_{m\text{-copies}}, f') = \alpha.$$

Therefore, $f''(x_0) > 0$. Hence the points where $f' = 0$ are discrete. It follows that f is strictly increasing.

Finally, to calculate $f'(0)$, we expand (4.19) near $x=0$ and compare the lowest order terms of both sides. Assume

$$f(x) = Ax + \text{higher order terms}.$$

We may derive that

$$\frac{1}{k!}\frac{d^k}{dt^k}|_{t=0}\frac{(x+tf)^{m+1}(1+t)^n}{m+1} = \frac{\binom{n}{k}}{m+1}x^{m+1} + \text{ higher order terms.} \tag{4.23}$$

Hence $A = f'(0) > 0$ if and only if $\alpha > \binom{n}{k}$ when $k \leq n$ and $\alpha > 0$ when $k > n$. □

If $c_k < \binom{n}{k}$, the solution of (4.16) with the boundary condition $f(0) = 0$ and $f(b') = b$ is not positive near $x = 0$. We define

$$\lambda := \inf\{\lambda' \mid f(x) \text{ solves (4.16) s.t. } f(\lambda') = 0, f(b') = b \text{ and } f \geq 0\}$$

and let f be the corresponding solution of (4.16) with $f(\lambda) = 0$ and $f(b') = b$. By the definition of λ, $f'(\lambda) = 0$.

Proposition 4.4. *There exists a unique parameter-triple (α, β, λ), all positive with $\lambda \in (0, b')$, such that*

$$G_1^{m,n,k}(f, x) = \alpha G_1^{m,n}(x) + \beta \tag{4.24}$$

implicitly defines a unique positive monotone increasing solution f of (4.16) *on* $[\lambda, b']$ *satisfying*

$$f(\lambda) = 0, \quad f(b') = b, \quad f'(\lambda) = 0. \tag{4.25}$$

Proof. By (4.20), $f(x)$ satisfies (4.25) if and only if

$$\begin{cases} a_0(\lambda) & = \alpha G_1^{m,n}(\lambda) + \beta, \\ a_0'(\lambda) & = \alpha (G_1^{m,n})'(\lambda), \\ G_1^{m,n,k}(b, b') & = \alpha G_1^{m,n}(b') + \beta. \end{cases} \tag{4.26}$$

Note we have explicit formulae for a_0, $G_1^{m,n}$ and $G_1^{m,n,k}$. It is then easy to get a unique solution (α, β, λ) to (4.26). □

Following the discussion in Sections 2 and 3, we conclude that the limit of f_t in this case is

$$f_\infty(x) = \begin{cases} 0, & 0 \leq x \leq \lambda, \\ f(x), & \lambda \leq x \leq b', \end{cases} \tag{4.27}$$

where $f(x)$ is defined in Proposition 4.4.

Now we are in a position to prove Main Theorem 2.

Proof of Main Theorem 2. For simplicity, we denote $\frac{\partial f(x,t)}{\partial x}$ by f_t' and $f(\cdot, t)$ by f_t. We first prove the following

Claim 4.5. There is a universal constant C depending only on initial values such that

$$f_t' \leq C.$$

Proof. The two-sided bound (2.5) leads to

$$C_1 \leq \sigma_k(\underbrace{\frac{1+f_t}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f_t}{x}}_{m\text{-copies}}, f_t') \leq C_2. \tag{4.28}$$

We separate the proof into the following two cases.

Case 1: $k \leq n$. In this case, there is always a term $(\frac{1+f_t}{1+x})^{k-1} f_t'$ in σ_k. Hence from (4.28) we have

$$(\frac{1+f_t}{1+x})^{k-1} f_t' \leq C_2.$$

Since f_t takes value in $[0, b]$, the term $\frac{1+f_t}{1+x}$ is bounded from below, from which a uniform upper for f_t' follows.

Case 2: $k > n$.

Claim 4.6. $\frac{f_t(x)}{x} \geq C(C_1, k, m, n, b, b') > 0$ for some universal constant C depending only on C_1, k, m, n, b and b'.

Proof of Claim 4.6. For a fixed t, let x_0 be the point where $\frac{f_t(x)}{x}$ achieves its minimum. Such a point exists since $\frac{f_t(x)}{x}$ is a continuous function on $[0, b']$. At x_0, we have

$$f_t'(x_0) = \frac{f_t(x_0)}{x_0}. \tag{4.29}$$

Indeed, if $x_n = 0$, then (4.29) is trivially true by the fact $f_t(0) = 0$. If x_n is an interior point, then

$$(\frac{f_t(x)}{x})' = \frac{f_t' - \frac{f}{x}}{x}.$$

(4.29) follows as well. If $x_0 = b'$, then $\frac{f_t(b')}{b'} = \frac{b}{b'}$. We are done for the lower bound of $\frac{f_t(x)}{x}$. Concerning the lower bound, we can also assume that $\frac{f_t(x_0)}{x_0} \leq 1$. Then $\frac{1+f_t}{1+x} \leq 1$ as well. Take $x = x_0$ in $\sigma_k(\underbrace{\frac{1+f_t}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f_t}{x}}_{m\text{-copies}}, f_t')$. By (4.29) and lower bound in (4.28), we get a uniform lower bound C on $\frac{f_t(x)}{x}$ depending only on C_1, k, m, n, b, b' not on t. We have finished the proof of Claim 4.6. □

We continue our proof of Claim 4.5, Case 2. By Claim 4.5, both $\frac{f_t(x)}{x}$ and $\frac{1+f_t(x)}{1+x}$ are bounded uniformly from below. Using the upper bound in (4.28) again, we get a uniform upper bound for f_t'.

We have thus finished the proof of Claim 4.5. □

One more thing to mention is the handle of non-linearity in this situation. As a matter of fact we have already seen that

$$\sigma_k(\underbrace{\frac{1+f_t}{1+x}}_{n\text{-copies}}, \underbrace{\frac{f_t}{x}}_{m\text{ -copies}}, f_t') = \frac{G_1^{m,n,k}(f,x)'}{G_1^{m,n}(x)'}. \tag{4.30}$$

Thus the parabolic equation (4.15) on f can be written as

$$\begin{aligned} \frac{\partial G_1^{m,n,k}(f,x)}{\partial t} &= \frac{\partial G_1^{m,n,k}(f,x)}{\partial f}\frac{\partial f}{\partial t} \\ &= \frac{\partial G_1^{m,n,k}(f,x)}{\partial f}(\frac{G_1^{m,n,k}(f,x)'}{G_1^{m,n}(x)'})' \\ &= Q(f,x)(\frac{G_1^{m,n,k}(f,x)''}{G_1^{m,n}(x)'} - \frac{G_1^{m,n,k}(f,x)' G_1^{m,n}(x)''}{(G_1^{m,n}(x)')^2}), \end{aligned} \tag{4.31}$$

which becomes a degenerate parabolic equation for $G_1^{m,n,k}(f,x)$. This is a generalization of the treatment in Section 3 for the general k case. The uniform upper bound on $\frac{\partial f}{\partial x}$ implies the uniform upper bound on $G_1^{m,n,k}(f,x)'$. The rest of the convergence is similar. We omit it for simplicity. Once we get the convergence

$$\lim_{t\to\infty} G_1^{m,n,k}(f,x) = G_1^{m,n,k}(f_\infty,x),$$

we could infer that

$$\lim_{t\to\infty} f(x,t) = f_\infty(x),$$

since $G_1^{m,n,k}(f,x)$ is monotone increasing on the f variable by the explicit formula (4.20).

Finally, let us discuss the geometric behavior.

First of all, if $f'_\infty(0) > 0$, which is the case for $k > n$ and $c_k > \binom{n}{k}$ when $k \le n$, then as previously discussed, we get smooth convergence.

If $f'_\infty(0) = 0$, the convergence is thus away from $\{\rho = -\infty\}$, which corresponds to P_0. Then the corresponding Kähler metric $dd^c v_\infty$ has a conical singularity of cone angle π transverse to P_0. P_0 can also be regarded as the intersection of $m+1$ effective divisors $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})$, which is of codimension $m+1$. Therefore the convergence of general inverse σ_k-flow can produce Kähler metrics which are singular on subvarieties of higher codimension.

If $c_k < \binom{n}{k}$, the limit is given by (4.27). We can still obtain smooth convergence away from P_0. The limit

$$\lim_{\rho\to-\infty} v'_\infty(\rho) = \lambda \neq 0$$

corresponds to a blow-up along P_0 in the following sense which we explain.

Away from P_0, we have the smooth convergence

$$(X_{m,n} \setminus P_0, \chi_t) \to (X_{m,n} \setminus P_0, \chi_\infty := dd^c v_\infty)$$

as $t \to \infty$. We consider the metric completion of $(X_{m,n} \setminus P_0, \chi_\infty)$. If we restrict $dd^c v_\infty$ fibre-wise, we may find that its metric completion is homeomorphic to the blow up of $\mathbb{P}^{n+1}$ at one point $\rho = -\infty$. Thus globally, the metric completion of $(X_{m,n} \setminus P_0, \chi_\infty)$ is homeomorphic to $\tilde{X}_{m,n}$, the blow up of $X_{m,n}$ along P_0. Let $\pi : \tilde{X}_{m,n} \to X_{m,n}$ be the blow up map and let E be the exceptional divisor, which is homeomorphic to $\mathbb{P}^n \times \mathbb{P}^m$. Since $f'_\infty(\lambda) = 0$, it follows that the pull-back metric $\pi^*(dd^c v_\infty)$ on $\tilde{X}_{m,n}$ is a Kähler metric with conical singularity of angle π transverse to the fibre direction of E. Moreover, since $\lim_{\rho\to-\infty} v'_\infty(\rho) = \lambda$, we have

$$\pi^*(dd^c v_\infty) \in [D_H] + b'[D_\infty] - \lambda[E].$$

Since f_∞ satisfies the equation

$$G_1^{m,n,k}(f,x) = \alpha G_1^{m,n}(x) + \beta,$$

for some constants α, β and $x \in [\lambda, b']$, hence we have that equation

$$\alpha\pi^*(\chi_\infty)^{m+n+1} = \pi^*(\chi_\infty)^{m+n+1-k} \wedge \pi^*(\omega)^k \tag{4.32}$$

holds on $\tilde{X}_{m,n}$ away from E. α is the corresponding topological constant

$$\alpha = \frac{[\chi]^{m+n+1-k}[\omega]^k}{[\chi]^{m+n+1}}$$

with $[\chi] \in b'[\pi^*(D_\infty)] - \lambda[E] + [\pi^*(D_H)]$ and $[\omega] \in b[\pi^*(D_\infty)] + [\pi^*(D_H)]$. $\square$

Remark 4.7. Motivated by the last case of Main Theorem 2, we can also study the general inverse σ_k-flow on $\tilde{X}_{m,n}$. $N^1(\tilde{X}_{m,n})$ is spanned by $[D_H]$, $[D_\infty]$ and $[E]$, where we use the same notation $[D_H]$ and $[D_\infty]$ to denote the pull-back of corresponding divisors on $X_{m,n}$. The class

$$p[D_\infty] - q[E] + r[D_H]$$

is Kähler if and only if $p > q > 0$ and $r > 0$. On local coordinates $(z_1, \cdots, z_n, \xi_0, \cdots, \xi_m) = \mathbb{C}^n \times (\mathbb{C}^{m+1} \setminus \{0\})$, let

$$e^\rho = h(z)|\xi|^2,$$

where $h(z) = 1 + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$.

We consider Kähler metrics of the form

$$r\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(\rho),$$

with u satisfying the proper asymptotic behavior (2.1) near $\rho = -\infty$ and $\rho = \infty$. The convergence behavior is very similar to that on $\mathbb{P}^n \# \overline{\mathbb{P}^n}$, so we will omit the details here.

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References

[C] L. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4**, (1998), 383-402 MR1658612 (2000b:49004)

[Ca] E. Calabi, *Extremal Kähler metrics*, Seminar on Differetial Geometry, vol. **102** of Ann. Math. Studies, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259-290 MR645743 (83i:53088)

[Cao] H. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359-372 MR799272 (87d:58051)

[Chen] X. Chen, *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. **12**, no. 4 (2004), 837-852 MR2104078 (2005h:53116)

[Do] S. Donaldson, *Moment maps and diffeomorphisms*, Asian J. Math. **3**, no. 1 (1999), 1-16 MR1701920 (2001a:53122)

[EGZ] P. Eyssidieux, V. Guedj and A. Zeriahi, *Singular Käler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607-639 MR2505296 (2010k:32031)

[FIK] M. Feldman, T. Ilmanen and D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geometry, **65** (2003), 169-209 MR2058261 (2005e:53102)

[FL] H. Fang and M. Lai, *On the geometric flows solving Kählerian inverse σ_k equations*, arXiv:1203.2571, to appear in Pac. J. Math

[FLM] H. Fang, M. Lai and X. Ma, *On a class of fully nonlinear flows in Kähler geomety*, J. Reine Angew. Math. **653** (2011), 189-220 MR2794631 (2012g:53134)

[H] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306 MR664497 (84a:53050)

[K] S. Kołodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69-117 MR1618325 (99h:32017)

[L] C. Li, *On rotationally symmetric Kähler-Ricci solitons*, arXiv:1004.4049

[ST1] J. Song and G. Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. **170** (2007), no. 3, 609-653 MR2357504 (2008m:32044)

[ST2] J. Song and G. Tian, *The Kähler-Ricci flow through singularities*, arXiv:0909.4898

[SW1] J. Song and B. Weinkove, *The convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. **61** (2008), no. 2, 210–229 MR2368374 (2009a:32038)

[SW2] J. Song and B. Weinkove, *The Kähler-Ricci flow on Hirzebruch surfaces*, arXiv:0903.1900v2, to appear in J. Reine Ange. Math. MR2837013 (2012g:53142)

[SW3] J. Song and B. Weinkove, *Contracting exceptional divisors by the Kähler-Ricci flow*, arXiv:1003.0718v2

[SW4] J. Song and B. Weinkove, *Lecture notes on Kähler-Ricci flow*, www.math.rutgers.edu/∼jiansong/publication_files/krflectures112111.pdf

[SY] J. Song and Y. Yuan, *Metric flips with Calabi Ansatz*, Geom. Funct. Anal., DOI: 10.1007/s00039-012-0151-1, 2012 MR2899688

[T] G.Tian, *New results and problems on Kähler-Ricci flow*, Géométrie différentielle, physique mathématique, mathématiques et société. II. Astérisque **322** (2008), 71-92 MR2521654 (2011a:53131)

[TZ] G. Tian and Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179-192 MR2243679 (2007c:32029)

[W] L. Wang, *On the regularity theory of fully nonlinear parabolic equations. II.* Comm. Pure Appl. Math. 45 (1992), no. **2**, 141-178 MR1139064 (92m:35127)

[We1] B. Weinkove, *Convergence of the J-flow on Kähler surfaces*, Comm. Anal. Geom. 12 (2004), no. **4**, 949–965 MR2104082 (2005g:32027)

[We2] B. Weinkove, *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom. 73 (2006), no. **2**, 351–358 MR2226957 (2007a:32026)

[Y] S-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, I, Comm. Pure Appl. Math. **31** (1978), 339-411 MR480350 (81d:53045)

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