1. Introduction and statements

The classical Borel-Cantelli lemmas are a powerful tool in probability theory and dynamical systems. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((A_n)\) a sequence of measurable sets in \(\mathcal{F}\). These lemmas say that (see [7] for proofs):

(BC1) If \(\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty\), then \(\mathbb{P}(x \in A_n \text{ i.o.}) = 0\).

(BC2) If the sets \(A_n\) are independent and \(\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty\), then \(\mathbb{P}(x \in A_n \text{ i.o.}) = 1\).

(BC3) If the sets \(A_n\) are pairwise independent and \(\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty\), then

\[
\frac{\sum_{i=1}^{n} 1_{A_i}}{\sum_{i=1}^{n} \mathbb{P}(A_i)} \to 1 \quad \text{a.s.}
\]

Here \(1_{A_i}\) is the indicator function of the set \(A_i\). Note that (BC3) implies (BC2), but the proof of (BC3) is more elaborated.
1.1. A Borel-Cantelli lemma.

**Theorem 1.** Let $X_i$ be non-negative random variables and $S_n = \sum_{i=1}^{n} X_i$. If $\sup E X_i < \infty$, $E S_n \to \infty$ and there exists $\gamma > 1$ such that

$$\text{var}(S_n) = O \left( \frac{(E S_n)^2}{(\log E S_n)(\log \log E S_n)^\gamma} \right),$$

then

$$\frac{S_n}{E S_n} \to 1 \quad \text{a.s.}$$

We see that Theorem 1 implies (BC3), because when $A_i$ are pairwise independent sets, $X_i = 1_{A_i}$ and $E S_n \to \infty$, then var$(S_n) = \sum_{i=1}^{n} \text{var}(X_i) \leq E S_n$. A slight modification of Theorem 1 also gives a version of the Strong Law of Large Numbers without assuming the random variables are pairwise independent.

**Corollary 1.** Let $X_i$ be identically distributed random variables with $E X_i = \mu$, $E X_i^2 < \infty$ and $S_n = \sum_{i=1}^{n} X_i$. If $X_i \geq -M$, for some constant $M > 0$, and there exists $\gamma > 1$ such that

$$\sum_{1 \leq i < j \leq n} \left( E(X_i X_j) - \mu^2 \right) = O \left( \frac{n^2}{(\log n)(\log \log n)^\gamma} \right),$$

then

$$\frac{S_n}{n} \to \mu \quad \text{a.s.}$$

1.2. An almost sure local central limit theorem. Let $X_i$ be independent random variables such that each $X_i$ assume the values $+1$ and $-1$ with probabilities $1/2$ and $1/2$. Then $S_n = \sum_{i=1}^{n} X_i$ is the simple random walk on the line. It is well known that the sequence of random variables $1_{\{S_i = 0\}}$ does not obey the law of large numbers. More precisely (see [12]),

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n \log \log n}} \sum_{i=1}^{n} 1_{\{S_i = 0\}} = \sqrt{2} \quad \text{a.s.}$$

and there exists a constant $0 < \gamma_0 < \infty$ such that

$$\liminf_{n \to \infty} \frac{\sqrt{\log \log n}}{\sqrt{n}} \sum_{i=1}^{n} 1_{\{S_i = 0\}} = \gamma_0 \quad \text{a.s.}$$

It is then natural to ask if $1_{\{S_n = 0\}}$ obeys the law of large numbers for some increasing sequence $n_i$ of even positive integers.

More generally, we consider i.i.d. random variables $X_i$, which are $h$-lattice valued, i.e. $\sum_{k \in \mathbb{Z}} P(X_i = kh + b) = 1$, for some $h > 0$ and $b \in \mathbb{R}$ (we assume $h$ with this property is maximal). Let $S_n = \sum_{i=1}^{n} X_i$ and $a \in \mathbb{R}$. By abuse of notation, when we write $S_n = a \sqrt{n}$ we mean $S_n = \lfloor (a \sqrt{n} - nb)/h \rfloor h + nb$.

**Theorem 2.** Let $X_i$ be i.i.d. $h$-lattice valued random variables with $E X_i = 0$, $E X_i^2 = \sigma^2 > 0$ and $E|X_i|^3 < \infty$, and $S_n = \sum_{i=1}^{n} X_i$. Let $n_i$ be an increasing sequence of positive integers and $a \in \mathbb{R}$. Then

(a) If $\sum_{i=1}^{\infty} n_i^{-1/2} < \infty$, then $P(S_{n_i} = a \sqrt{n_i} \ i.o.) = 0$. 

(b) If there exist $A > 0$ and $\gamma > 1$ such that

$$n_{i+1} - n_i \geq A n_i^{1/2} \quad \text{and} \quad n_i \leq A^{-1} i^2 (\log i)(\log \log i)^{-3}(\log \log \log i)^{-2\gamma}$$

for all $i$, then

$$\sum_{i=1}^{n} \frac{1}{n} 1\{S_{n_i} = a \sigma \sqrt{n_i}\} \to \frac{h}{\sqrt{2\pi}} e^{-a^2/2} \quad \text{a.s.}$$

Let $\Delta_n^a$ be the quotient between the left and right hand sides of (2). Then, for every $N > 0$ there exists $C > 0$ such that, for every $\epsilon > 0$,

$$\sup_{a \in [-N,N]} P(|\Delta_n^a - 1| > \epsilon) \leq C \epsilon^{-2} (\log \log n)^{-1}(\log \log \log n)^{-\gamma}.$$

Remark 1. Concerning the divergent case, $\sum_{i=1}^{\infty} n_i^{-1/2} = \infty$ and $a = 0$. In [3] the authors prove that if there exists an integer $m > 0$ such that $n_{i+m} \geq n_i + m$ for every $i, j$, then $P(S_{n_i} = 0 \text{ i.o.}) = 1$. In [5] the same authors claim that if there exist $A > 0$ such that $n_{i+1} - n_i > A n_i^{1/2}$ for every $i$, then $P(S_{n_i} = 0 \text{ i.o.}) = 1$; however it seems their proof is not correct (see p. 184, Equation (21)).

Considering $n_i = i^2$ and the ‘change of variable’ $k = i^2$, we get the following almost sure local central limit theorem.

**Corollary 2.** With the same hypotheses as Theorem 2 (b),

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{n} 1\{S_k = a \sigma \sqrt{k}\} \to \frac{h}{\sqrt{2\pi}} e^{-a^2/2} \quad \text{a.s.}$$

When $a = 0$ this was proved in [4] (see also [6]). For $a \in \mathbb{R}$ this was proved, independently, in [13].

The version for random variables with density follows.

**Theorem 3.** Let $X_i$ be i.i.d. random variables having density function whose Fourier transform (or some positive integer power of it) is integrable, $E X_i = 0$, $E X_i^2 = \sigma^2 > 0$ and $E |X_i|^3 < \infty$, and $S_n = \sum_{i=1}^{n} X_i$. Let $a \in \mathbb{R}$ and $n_i$ be an increasing sequence of positive integers satisfying

$$\frac{n_{i+1}}{n_i} \geq 1 + \frac{A (\log i)(\log \log i)^\alpha}{i} \quad \text{for all } i$$

for some $A > 0$ and $\alpha > 2$. Then

$$\frac{1}{n} \sum_{i=1}^{n} 1\{S_{n_i} = a \sigma \sqrt{n_i}\} \to \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \quad \text{a.s.}$$

For the special case $a = 0$, (3) holds with $n_i = i$.

Let $\Delta_n^a$ be the quotient between the left and right hand sides of (3). Then, for every $N > 0$ there exists $C > 0$ such that, for every $\epsilon > 0$,

$$\sup_{a \in [-N,N]} P(|\Delta_n^a - 1| > \epsilon) \leq C \epsilon^{-2} (\log n)^{-1}(\log \log n)^{1-\alpha}.$$

In the above theorem we can consider sequences of type $n_i = [e^{A(\log i)^2(\log \log i)^\alpha}]$. In particular, considering the sequence $n_i = 2^i$ and the ‘change of variable’ $k = 2^i$, we get the following almost sure local central limit theorem.
Corollary 3. With the same hypotheses as Theorem 3,

\[
\frac{1}{\log n} \sum_{k=1}^{n} 1 \{ S_k = a \alpha \sqrt{k} \} \frac{1}{k} \to \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \quad \text{a.s.}
\]

Remark 2. Using the same techniques (and the Berry-Esseen Theorem) we can prove Theorem 3 with ‘\( \leq \)’ instead of ‘\( = \)’ in (3), for i.i.d. random variables \( X_i \) with finite third moment, thus giving a new proof of the almost sure central limit theorem (for related results, see [11] and the references therein).

1.3. A dynamical Borel-Cantelli lemma. We want to consider the dynamical version of (BC3). Let \((X,d)\) be a metric space, \(\mu\) be a Borel probability measure on \(X\), and \(T\) be a \(\mu\)-preserving transformation on \(X\). Let \((B_n)\) be a sequence of measurable sets in \(X\). In what conditions does

\[(DBC) \quad \frac{\sum_{i=0}^{n-1} 1_{B_i}(T^i x)}{\sum_{i=0}^{n-1} \mu(B_i)} \to 1 \quad \text{for} \; \mu \; \text{a.e.} \; x\]

hold? We can easily find sufficient conditions for (DBC) to hold:

- The sets \(B_n\) are all equal to \(B\), \(\mu(B) > 0\) and \(\mu\) is ergodic.
- The sets \(T^{-n}B_n\) are pairwise independent and \(\sum \mu(B_n) = \infty\).

The first one follows from the Birkhoff ergodic theorem, and the second one by (BC3) since \(1_{B_i}(T^i x) = 1_{T^{-i}B_i}(x)\). However, it is very unlikely for a dynamical system \((T,X,\mu)\) to have \(T^{-n}B_n\) pairwise independent. A far more reasonable condition is to have some sufficiently fast decay of correlations. In this section we extend some results of [10] where the same kind of problem is treated. For related results we refer the reader to [10] and the references therein.

We denote by \(|\cdot|_{Lip}\) the usual Lipschitz norm. We say that \((T,X,\mu,d)\) has polynomial decay of correlations (for Lipschitz observables) if, for every Lipschitz function \(\varphi, \psi: X \to \mathbb{R}\),

\[
\left| \int \varphi \circ T^n \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq c(n) \| \varphi \|_{Lip} \| \psi \|_{Lip},
\]

where \(c(n) \leq Cn^{-\alpha}\) for some constants \(C > 0\) and \(\alpha > 0\) (rate).

We say that \((T,X,\mu,d)\) has \(\beta\)-exponential decay of correlations, \(\beta > 0\), if \(c(n) \leq Ce^{-\alpha n^\beta}\), for some constants \(C, \alpha > 0\), in (4). For \(\beta = 1\) we get the usual definition of exponential decay of correlations. For \(\beta < 1\) this is also known as the stretched exponential decay of correlations.

We will assume the following condition on the measure \(\mu\). There exist \(C > 0\), \(\delta > 0\) (\(\delta < 2\)) and \(r_0 > 0\) such that for all \(x_0 \in X\), \(0 < r \leq r_0\) and \(0 < \epsilon \leq 1\),

\[(A) \quad \mu(\{x : r < d(x,x_0) < r + \epsilon\}) < C \epsilon^\delta.\]

In what follows, if we only consider nested balls \(B_i = B(x_0,r_i)\) \((r_i \to 0)\) centered at a given point \(x_0\), then we only require that (A) holds for the point \(x_0\); in other words, \(r \mapsto \mu(B(x_0,r))\) is Hölder continuous (with exponent \(\delta(x_0) > 0\)).

Example 1.

1. If \(X\) is a compact manifold and \(\mu\) is absolutely continuous with respect to Lebesgue measure with density in \(L^{1+\alpha}\) for some \(\alpha > 0\), then \(\mu\) satisfies (A).
1.4. Quantitative recurrence. As before, let \((T, X, \mu)\) be a measure-preserving transformation of a probability space \(X\) which is also endowed with a metric \(d\). For \(\alpha > 0\), we denote by \(\mathcal{H}^\alpha\) the \(\alpha\)-Hausdorff measure of \((X, d)\). One of the most beautiful results on the recurrence of dynamical systems is the following.

**Theorem 6** (Boshernitzan [2]). If, for some \(\alpha > 0\), \(\mathcal{H}^\alpha\) is \(\sigma\)-finite on \(X\), then

\[
\liminf_{n \to \infty} n^{-\frac{1}{\alpha}} d(T^n(x), x) < \infty \quad \text{for } \mu \text{ a.e. } x \in X.
\]
If, moreover, $\mathcal{H}^\alpha(X) = 0$, then
$$\liminf_{n \to \infty} n^{\frac{1}{\alpha}} d(T^n(x), x) = 0 \quad \text{for } \mu \text{ a.e. } x \in X.$$ 

Also a very nice result in this direction is as follows. Given $x_0 \in X$, let $\tilde{d}_\mu(x_0)$ be the upper pointwise dimension of $\mu$ at $x_0$ defined by
$$\tilde{d}_\mu(x_0) = \limsup_{r \to 0} \frac{\log \mu(B(x_0, r))}{\log r},$$
where $B(x_0, r)$ is the ball centered at $x_0$ of radius $r$. We also say that $(T, X, \mu, d)$ has superpolynomial decay of correlations (for Lipschitz observables) if it has polynomial decay of correlations with rate $\alpha$ for every $\alpha > 0$.

**Theorem 7** (Galatolo [9]). If $(T, X, \mu, d)$ has superpolynomial decay of correlations, then, for every $x_0 \in X$ and $\alpha > \tilde{d}_\mu(x_0)$,
$$\liminf_{n \to \infty} n^{\frac{1}{\alpha}} d(T^n(x), x_0) = 0 \quad \text{for } \mu \text{ a.e. } x \in X.$$ 

The dynamical Borel-Cantelli lemmas (for nested balls) stated in the previous subsection give, under additional assumptions, quantitative versions of these results. Let $\Theta^\alpha_{\mu}(x_0)$ be the lower $\alpha$-density of $\mu$ at $x_0$ defined by
$$\Theta^\alpha_{\mu}(x_0) = \liminf_{r \to 0} \frac{\mu(B(x_0, r))}{r^\alpha}.$$ 

So $0 \leq \Theta^\alpha_{\mu}(x_0) \leq \infty$.

**Theorem 8.**

(a) Suppose $(T, X, \mu, d)$ has polynomial decay of correlations with rate $\theta > 1$ and $x_0 \in X$ satisfies (A) with $\delta(x_0) > 2/\theta$, $\mu(\{x_0\}) = 0$. Let $\beta = \frac{\theta - 2}{\theta + 1}$. Then, for every $\gamma > 1 + \left(\frac{2 - \delta(x_0) (1 - \beta)}{2 + \delta(x_0)}\right)$, there exists $\kappa(n)$ with $\log \kappa(n)/\log n \to 0$ such that

$$\lim_{N \to \infty} \frac{\# \left\{ 1 \leq n \leq N : n^\beta d(T^n(x), x_0) < \kappa(n) \right\}}{N^{1 - \beta} (\log N)^\gamma} = \theta(1 - \beta)^{-1}$$

for $\mu$ a.e. $x \in X$, where $\alpha = \tilde{d}_\mu(x_0)$ and $\theta = 1$.

If, moreover, $\Theta^\alpha_{\mu}(x_0) > 0$, then (5) holds with $\kappa(n) = (\log n)^{\gamma/\alpha}$ and $\theta = \Theta^\alpha_{\mu}(x_0)$.

In particular, if $(T, X, \mu, d)$ has superpolynomial decay of correlations and $x_0 \in X$ satisfies (A), $\mu(\{x_0\}) = 0$, then, for every $\beta < 1$ and $\alpha$ as before, (5) holds.

(b) Suppose $(T, X, \mu, d)$ has $\beta$-exponential decay of correlations and $x_0 \in X$ satisfies (A), $\mu(\{x_0\}) = 0$. Then, for every $\gamma > 1$, there exists $\kappa(n)$ with $\log \kappa(n)/\log n \to 0$ such that

$$\lim_{N \to \infty} \frac{\# \left\{ 1 \leq n \leq N : n^\beta d(T^n(x), x_0) < \kappa(n) \right\}}{(\log N)^{\beta - 1} (\log \log N)^\gamma} = \theta \beta$$

for $\mu$ a.e. $x \in X$, where $\alpha = \tilde{d}_\mu(x_0)$ and $\theta = 1$.

If, moreover, $\Theta^\alpha_{\mu}(x_0) > 0$, then (5) holds with

$$\kappa(n) = (\log n)^{(\beta - 1)/\alpha} (\log \log n)^{\gamma/\alpha}$$

and $\theta = \Theta^\alpha_{\mu}(x_0)$. 

Corollary 6. Suppose \((T, X, \mu, d)\) has stretched exponential decay of correlations and \(x_0 \in X\) satisfies \((A)\), \(\mu(\{x_0\}) = 0\). Then there exists \(\kappa(n)\) with \(\log \kappa(n)/\log n \to 0\) such that

\[
\liminf_{n \to \infty} \frac{n^\alpha}{\kappa(n)} d(T^n(x), x_0) < \infty \quad \text{for } \mu \text{ a.e. } x \in X, 
\]

where \(\alpha = \bar{d}_\mu(x_0)\).

2. Proofs

2.1. Proofs of 1.1.

Proof of Theorem 11: Given \(\sigma > 0\), Chebyshev’s inequality implies

\[
P(|S_n - ES_n| > \sigma ES_n) \leq \frac{\text{var}(S_n)}{(\sigma ES_n)^2},
\]

so

(8) \[
P(|S_n - ES_n| > \sigma ES_n) \leq C\sigma^{-2}(\log ES_n)^{-1}(\log \log ES_n)^{-\gamma}
\]

for some constant \(C > 0\). Note that \(\mathbb{S}\) implies \(S_n/ES_n \to 1\) in probability. To get a.s. convergence we have to take subsequences. Let \(0 < \theta < \gamma - 1\) and \(n_k = \inf\{n : ES_n \geq e^{k/(\log k)^{\theta}}\}\). Let \(U_k = S_{n_k}\) and note that the definition and 

\[
\text{EX} i \leq M \quad \text{imply } e^{k/(\log k)^{\theta}} \leq EU_k < e^{k/(\log k)^{\theta}} + M.
\]

Replacing \(n\) by \(n_k\) in \(\mathbb{S}\) we get

\[
P(|U_k - EU_k| > \sigma EU_k) \leq \bar{C}\sigma^{-2}k^{-1}(\log k)^{\theta-\gamma}
\]

for some constant \(\bar{C} > 0\). So \(\sum_{k=1}^{\infty} P(|U_k - EU_k| > \sigma EU_k) < \infty\), and the Borel-Cantelli lemma (BC1) implies \(P(|U_k - EU_k| > \sigma EU_k \text{ i.o.}) = 0\). Since \(\sigma\) is arbitrary, it follows that \(U_k/\text{EU}_k \to 1\) a.s. To get \(S_n/\text{ES}_n \to 1\) a.s., pick an \(\omega\) so that \(U_k(\omega)/\text{EU}_k \to 1\) and observe that if \(n_k \leq n < n_{k+1}\), then

\[
\frac{U_k(\omega)}{\text{EU}_{k+1}} \leq \frac{S_n(\omega)}{\text{ES}_n} \leq \frac{U_{k+1}(\omega)}{\text{EU}_k}.
\]

To show that the terms at the left and right ends converge to 1, we rewrite the last inequalities as

\[
\frac{\text{EU}_k}{\text{EU}_{k+1}} \frac{U_k(\omega)}{U_{k+1}(\omega)} \leq \frac{S_n(\omega)}{S_{n+1}(\omega)} \leq \frac{\text{EU}_{k+1}}{\text{EU}_k} \frac{U_{k+1}(\omega)}{U_k(\omega)}.
\]

From this we see it is enough to show \(\text{EU}_{k+1}/\text{EU}_k \to 1\). Since

\[
e^{k/(\log k)^{\theta}} \leq \text{EU}_k \leq \text{EU}_{k+1} \leq e^{(k+1)/(\log(k+1))^{\theta}} + M
\]

we must show \(e^{(k+1)/(\log(k+1))^{\theta}}/e^{k/(\log k)^{\theta}}\) converges to 1.

We note that \(e^{k/(\log k)^{\theta}} = k^{h(k)}\), where \(h: (1, \infty) \to (0, \infty), h(x) = x/(\log x)^{1+\theta}\). Then we must show

\[
\frac{(k + 1)^{h(k+1)}}{k^{h(k+1)}} = \left(1 + \frac{1}{k}\right)^{h(k)} e^{(h(k+1) - h(k))\log(k+1)}
\]

converges to 1. Clearly \(h(k) = o(k)\), and so the left hand side of the product above converges to \(e^0 = 1\). We note that \(0 < h''(x) < (\log x)^{-1-\theta}\) and \(h''(x) < 0\) for all sufficiently large \(x\), so the right hand side of the product above is \(\leq \exp(\log(k+1)/(\log k)^{1+\theta})\), which also converges to 1. \(\square\)
2.2. Proofs of 1.2.

Proof of Theorem 2. The simple random walk. We treat this particular case separately because its proof is elementary. Then we say how it extends easily to lattice random walks by using a local central limit theorem.

We will use Stirling’s formula

\[ n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + O \left( \frac{1}{n} \right) \right), \]

where \( O(n^{-1}) > 0 \). In this particular case let \( n_i \) be a sequence of even positive integers and let \( E_i \) denote the event \( S_{n_i} = a\sqrt{n_i} \) whose precise meaning is \( S_{n_i} = 2[a\sqrt{n_i}/2] \) (this eliminates probability zero of \( E_i \)).

(a) We have

\[ \mathbf{P}(E_i) = \left( \frac{n_i}{n_i + 2[a\sqrt{n_i}/2]} \right)^{2-n_i} \sim \sqrt{\frac{2}{\pi}} n_i^{-1/2} e^{-a^2/2} \]

by Stirling’s formula, and the result follows from (BC1).

(b) First of all, we notice that the first condition on the sequence \( n_i \) implies \( (\sqrt{n_j} - \sqrt{n_i})^{-1} \leq \max\{1, 4A^{-1}\} \) for every \( j > i \). This will be important because when we perform approximation to the lattice sometimes we have to use \( a \pm 2(\sqrt{n_j} - \sqrt{n_i})^{-1} \) instead of \( a \). Let \( \bar{a} = |a| + 2\max\{1, 4A^{-1}\} \).

Also, the first condition on the sequence \( n_i \) implies there exists \( A_1 > 0 \) such that \( n_i \geq A_1 i^2 \) for every \( i \), and so

\[ A_2 (\log n)^{1/2} (\log \log n)^{3/2} (\log \log \log n)^7 \leq \sum_{i=1}^{n} n_i^{-1/2} \leq A_2^{-1} \log n \]

for some constant \( A_2 > 0 \).

We want to apply Theorem 1 to the random variables \( \tilde{X}_i = 1_{E_i} \). Let \( \tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i \). We have \( \mathbf{E} \tilde{S}_n \sim (2/\pi)^{1/2} e^{-a^2/2} \sum_{i=1}^{n} n_i^{-1/2} \rightarrow \infty \). To verify condition 1 in Theorem 1 we have \( \sum_{i=1}^{n} \text{var}(\tilde{X}_i) \leq \mathbf{E} \tilde{S}_n \) and, for \( i < j \),

\[ \mathbf{P}(E_i \cap E_j) = \mathbf{P}(S_{n_i} = a\sqrt{n_i}) \mathbf{P}(S_{n_j} = 2[a\sqrt{n_j}/2] - 2[a\sqrt{n_i}/2]) \]

\[ = \mathbf{P}(S_{n_i} = a\sqrt{n_i}) \mathbf{P}(S_{n_j} = a\sqrt{n_j}) \left( \frac{n_j}{n_j - n_i} \right)^{1/2} R \]

\[ = \mathbf{P}(E_i) \mathbf{P}(E_j) \left( \frac{n_j}{n_j - n_i} \right)^{1/2} R, \]

where, for \( n_j \geq 2\bar{a}^6 \),

\[ R = e^{\frac{\bar{a}^2}{\sqrt{n_i} + \sqrt{n_j}}} \left( 1 + O \left( \frac{1}{n_j - n_i} + \frac{\bar{a}^3}{\sqrt{n_j}} \right) \right) \]

(for \( a = 0 \) this holds with \( \bar{a} = 0 \)) is obtained using Stirling’s formula and

\[ (1 + k/m)^m \leq e^k \leq (1 + k/m)^m (1 + k^2/m), \]

\[ (1 + k/m)^m \leq (1 - k/m)^{-m} \leq (1 + k/m)^m (1 - k^2/m)^{-1}, \]
where \( k \geq 0, m > k^2 \) and, for the second line of the inequalities, we also assume \( m \geq 1, m > k \). To estimate

\[
(10) \quad \left| \sum_{1 \leq i < j \leq n} (\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i) \mathbb{P}(E_j)) \right|
\]

we separate the sum into two cases. Let \( \sqrt{\nu_n} = (\log \mathbb{E} \tilde{S}_n)(\log \log \mathbb{E} \tilde{S}_n)^\gamma \). In all cases we restrict ourselves to \( n_j \geq 2a^6 \) and \( \nu_n \geq a^4 \). Here \( C_1, C_2, \ldots \) denote appropriate absolute constants (which do not depend on \( a \)).

**Case 1:** \( n_j > \nu_n n_i \). Then we see that

\[
\left( \frac{n_j}{n_j - n_i} \right)^{1/2} \leq 1 + \frac{1}{\nu_n} \quad \text{and} \quad R = 1 + O \left( \frac{a^2}{\sqrt{\nu_n}} + \frac{1}{n_j - n_i} + \frac{a^3}{\sqrt{\nu_n}} \right).
\]

Since \( \sum_{i=1}^{\infty} \mathbb{P}(E_i) / \sqrt{\nu_n} \leq C_1 < \infty \), this implies that this case contribution of (10) is less than some constant \( C_2 \) times

\[
\frac{(\mathbb{E} \tilde{S}_n)^2}{\nu_n} + a^2 \frac{(\mathbb{E} \tilde{S}_n)^2}{\sqrt{\nu_n}} + \mathbb{E} \tilde{S}_n a^3 + \frac{\mathbb{E} \tilde{S}_n}{\sqrt{\nu_n}}.
\]

**Case 2:** \( n_j \leq \nu_n n_i \). Clearly this case contribution of (10) is less than

\[
C_3 e^{a^2/2} \sum_{i,j} \mathbb{P}(E_i) \mathbb{P}(E_j) \left( \frac{n_j}{n_j - n_i} \right)^{1/2}
\]

\[
\leq C_4 \sum_{i=1}^{\infty} \mathbb{P}(E_i) \sum_j (n_j - n_i)^{-1/2}.
\]

Given \( i \), let \( N \) be the number of \( j \)'s satisfying \( n_i \leq n_j \leq \nu_n n_i \). Then \( n_{i+N} \leq \nu_n n_i \) and \( N + i \leq C_5 \nu_n (\log i)^{1/2} \). The first condition on the sequence \( n_i \) implies \( n_j - n_i \geq C_6 (j^2 - i^2) \) for all \( i < j \), so applying the Cauchy-Schwarz inequality we get

\[
\sum_{j=i+1}^{N} (n_j - n_i)^{-1/2} \leq C_7 \left( \sum_{j=i+1}^{N+i} (j + i)^{-1/2} \right)^{1/2} \left( \sum_{j=i+1}^{N+i} (j - i)^{-1} \right)^{1/2},
\]

which is less than \( C_8 (\log \log n)^{1/2} (\log n)^{1/2} \). Then (11) is less than

\[
C_9 (\log \log n)^{1/2} (\log n)^{1/2} \mathbb{E} \tilde{S}_n = O \left( (\mathbb{E} \tilde{S}_n)^2 / \sqrt{\nu_n} \right),
\]

where we have used (9).

Applying Theorem 11 we get

\[
\sum_{i=1}^{\infty} \frac{\mathbb{P}(S_n = a \sqrt{\nu_n})}{n_i^{-1/2}} \to \frac{\sqrt{2}}{\pi} e^{-a^2/2} \quad \text{a.s.}
\]

**Lattice random walks.** Since \( \mathbb{E} |X_i|^3 < \infty \), we have the following local central limit theorem with rates (see [8]):

\[
(12) \quad \mathbb{P}(S_n = a \sigma \sqrt{n}) = \frac{h}{\sqrt{2\pi n}} e^{-a^2/2} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right).
\]

Here, \( C_1, C_2, \ldots \) denote appropriate absolute constants that might depend (continuously) on \( a, \sigma, h \) and on the distribution of \( X_i \) (this includes the constants in \( O(\cdot) \)).
Let $E_i$ denote the event $S_{n_i} = a\sigma\sqrt{n_i}$, and $	ilde{X}_i = 1_{E_i}$, $	ilde{S}_n = \sum_{i=1}^n \tilde{X}_i$. By (12) we have $E\tilde{S}_n \sim n(\sqrt{2\pi}\sigma)^{-1}e^{-a^2/2} \sum_{i=1}^n n_i^{-1/2} \to \infty$, $\sum_{i=1}^n \text{var}(\tilde{X}_i) \leq E\tilde{S}_n$ and, for $i < j$,

$$P(E_i \cap E_j) = P(S_{n_i} = a\sigma\sqrt{n_i}) P(S_{n_j-n_i} = a\sigma(\sqrt{n_j} - \sqrt{n_i}) + O(1))$$

$$\quad = P(S_{n_i} = a\sigma\sqrt{n_i}) P(S_{n_j} = a\sigma\sqrt{n_j}) \left( \frac{n_j}{n_j-n_i} \right)^{1/2} R$$

$$\quad = P(E_i) P(E_j) \left( \frac{n_j}{n_j-n_i} \right)^{1/2} R,$$

where, for $n_j \geq C_1$,

$$R = e^{\frac{a^2}{2\sqrt{n_j} + \sqrt{n_i}}} \left( 1 + O \left( \frac{1}{\sqrt{n_j-n_i}} \right) \right).$$

Notice that here we also used the first condition on the sequence $n_i$ because we need $a + O(1)(\sqrt{n_j} - \sqrt{n_i})^{-1}$ to be uniformly bounded in order to apply (12). Now the rest of the proof is similar to the simple random walk.

The uniform convergence in probability is an immediate consequence of (8) and the fact that $C$ can be chosen uniformly in $a \in [-N,N]$. If we use more restrictive sequences $n_i$, then we can improve this uniform convergence. For example, if, moreover, $n_i \leq A\sqrt{i}^\alpha$ for some $0 \leq \alpha < 1$, then, for every $N > 0$ and $\gamma > 1$ there exists $C > 0$ such that, for every $\epsilon > 0$,

$$\sup_{a \in [-N,N]} P( |\Delta_n^a| > \epsilon) \leq C \epsilon^{-2}(\log \log n)^{-\gamma},$$

(just use $\sqrt{\nu_n} = (\log E\tilde{S}_n)^\gamma$).

Proof of Theorem 3. Since $E|X_i|^3 < \infty$, we have the following local central limit theorem with rates (see [8]):

$$P(S_n = a\sigma\sqrt{n}) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right).$$

Here, $C_1, C_2, \ldots$ denote appropriate absolute constants that might depend (continuously) on $a$ and on the density of $X_i$ (this includes the constants in $O(\cdot)$). Let $E_i$ denote the event $S_{n_i} = a\sigma\sqrt{n_i}$ and consider the random variables $\tilde{X}_i = 1_{E_i}$ and $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$. As before, we want to apply Theorem 1 to the random variables $\tilde{X}_i$. Clearly (13) implies $E\tilde{S}_n \sim n(\sqrt{2\pi})^{-1}e^{-a^2/2}$. Also $\sum_{i=1}^n \text{var}(\tilde{X}_i) \leq E\tilde{S}_n$ and, for $i < j$,

$$P(E_i \cap E_j) = P(S_{n_i} = a\sigma\sqrt{n_i}) P(S_{n_j-n_i} = a\sigma(\sqrt{n_j} - \sqrt{n_i}))$$

$$\quad = P(S_{n_i} = a\sigma\sqrt{n_i}) P(S_{n_j} = a\sigma\sqrt{n_j}) R$$

$$\quad = P(E_i) P(E_j) R,$$

where, for $n_j \geq C_1$,

$$R = e^{\frac{a^2}{2\sqrt{n_j} + \sqrt{n_i}}} \left( 1 + O \left( \frac{1}{\sqrt{n_j-n_i}} \right) \right).$$
To estimate
\[(14) \quad \left| \sum_{1 \leq i < j \leq n} (\mathbf{P}(E_i \cap E_j) - \mathbf{P}(E_i) \mathbf{P}(E_j)) \right| \]
we separate the sum in two cases. Let \( \sqrt{\nu_n} = (\log \tilde{E}_n)(\log \log \tilde{E}_n)^{\alpha-1} \).

Case 1: \( n_i > \nu_n n_i \). Then \( R = 1 + O(\nu_n^{-1/2}) \) and this case contribution of (14) is \( O((\tilde{E}_n^2)/\sqrt{\nu_n}) \).

Case 2: \( n_j \leq \nu_n n_i \). Given \( i \), let \( N \) be the number of \( j > i \) satisfying \( n_j \leq \nu_n n_i \). Then \( \nu_{i+N} \leq \nu_n n_i \) and, using the hypothesis on the sequence \( n_i \), we get
\[
\sum_{k=i}^{N+i-1} (\log i)(\log \log i)^{\alpha}/i \leq C_2 \log \nu_n.
\]

Then, simple calculus shows that, for all sufficiently large \( n \),
\[
N \leq \exp \left( \left( (\log n)^2 + C_3 \log \nu_n/(\log \log n)^{\alpha/2} \right) - n \right)
\leq C_4 n \left( (\log n)(\log \log n)^{\alpha-1} \right)^{-1}.
\]

Then this case contribution of (14) is less than \( C_5 n^2 \left( (\log n)(\log \log n)^{\alpha-1} \right)^{-1} = O \left( (\tilde{E}_n^2)/\sqrt{\nu_n} \right) \).

The conclusion follows by applying Theorem 1.

In the special case \( a = 0 \), we have \( R = 1 + O((n_j - n_i)^{-1/2}) \), for \( n_j \geq C_1 \). Then in Case 2 (where the hypothesis on \( n_i \) was used) we can use \( n_i = i \) to get that this case contribution of (14) is less than
\[
C_6 \sum_{i=1}^{n} \sum_{j=i+1}^{\nu_n i} (j-i)^{-1/2} \leq C_7 \sqrt{\nu_n} n^{3/2} < (\tilde{E}_n^2)/\sqrt{\nu_n}.
\]

The uniform convergence in probability is an immediate consequence of (8) and the fact that \( C \) can be chosen uniformly in \( a \in [-N, N] \). If we use more restrictive sequences \( n_i \), then we can improve this uniform convergence. For example, if \( n_i+1/n_i \geq 1 + A(\log i)^{\alpha}/i \) for some \( \alpha > 1 \), then, for every \( N > 0 \) and \( 1 < \gamma < \alpha \) there exists \( C > 0 \) such that, for every \( \epsilon > 0 \),
\[
\sup_{a \in [-N, N]} \mathbf{P} \left( |\Delta_n^a| > \epsilon \right) \leq C \epsilon^{-2}(\log n)^{-\gamma}
\]
(just use \( \sqrt{\nu_n} = (\log \tilde{E}_n^\gamma) \)).

\[ \Box \]

2.3. Proofs of 1.3.

Proof of Theorem 1. We use some notation of probability. Given two measurable functions \( f, g: X \to \mathbb{R} \) we denote (whenever it makes sense)
\[
\mu(f) = \int f \, d\mu, \quad \text{var}(f) = \mu(f^2) - \mu(f)^2, \quad \text{cov}(f, g) = \mu(fg) - \mu(f)\mu(g).
\]

Given \( f_1, ..., f_n: X \to \mathbb{R} \) measurable functions, if \( \mu(f_i^2) < \infty \), then (see 7)
\[
\text{var}(f_1 + \cdots + f_n) = \text{var}(f_1) + \cdots + \text{var}(f_n) + 2 \sum_{1 \leq i < j \leq n} \text{cov}(f_i, f_j).
\]
Fix $\theta > 0$ such that $\gamma > 1+(2(1+\theta)-\delta)^{1/(2+\delta)}$. For each $n$ let $\tilde{f}_n$ be a Lipschitz function such that $\tilde{f}_n(x) = 1$ if $x \in B_n$, $\tilde{f}_n(x) = 0$ if $d(x, B_n) > (n \log n)^{1+\theta-1/\delta}$, $0 \leq \tilde{f}_n \leq 1$ and $\|\tilde{f}_n\|_{\text{Lip}} \leq (n \log n)^{1+\theta-1/\delta}$. Let $f_n = \tilde{f}_n \circ T^n$ and $S_n = \sum_{i=0}^{n-1} f_i$. Note that $\mu(S_n) = \sum_{i=0}^{n-1} \mu(B_i) + O(1)$, and for $\mu$ a.e. $x$, $f_n(x) = 1_{B_n}(T^n x)$ except for finitely many $n$ by the Borel-Cantelli lemma (BC1), since $\mu(x : f_n(x) \neq 1_{B_n}(T^n x)) = O(1)$, and for $\mu$ a.e. $x$, $f_n(x) = 1_{B_n}(T^n x)$ except for finitely many $n$ by the Borel-Cantelli lemma (BC1), since $\mu(x : f_n(x) \neq 1_{B_n}(T^n x)) = O(1)$.

By (8), there exists $\mu(x : r_n < d(T^n x, p_n) < r_n + (n \log n)^{1+\theta-1/\delta}) < (n \log n)^{1+\theta-1/\delta}$ by assumption (A). So it is enough to prove that $S_n/\mu(S_n) \to 1$ $\mu$ a.e., which we will do by using Theorem [1].

Since $0 \leq f_i \leq 1$, we get that $\text{var}(f_i) \leq \mu(f_i^2) \leq \mu(f_i)$ and [15] implies

$$\text{var}(S_n) \leq \mu(S_n) + 2 \left( \sum_{0 \leq i < j \leq n-1} \text{cov}(f_i, f_j) + \sum_{0 \leq i < j \leq n-1} \text{cov}(f_i, f_j) \right),$$

where $\nu(n) = \mu(S_n)(\log n)^{-1}(\log \log n)^{-\rho}$, for some $\rho > 1$. We easily bound $I$ by

$$I \leq \sum_{j=1}^{n-1} \sum_{i=j-\nu(n)}^{j-1} \mu(f_j) \leq \mu(S_n)\nu(n).$$

We use decay of correlations to bound $II$ by

$$II \leq \sum_{0 \leq i < j \leq n-1} c(j-i)\|\tilde{f}_i\|_{\text{Lip}}\|\tilde{f}_j\|_{\text{Lip}}$$

$$\leq C(n \log n)^{1+\theta-1/\delta} \sum_{i=0}^{n-2} (i(\log i)^{1+\theta-1/\delta} \sum_{j=i+\nu(n)+1}^{n-1} \frac{1}{(j-i)\alpha}$$

$$\leq \frac{2C}{\alpha-1} \frac{(n \log n)^{1+\theta-1/\delta}}{\nu(n)^{\alpha-1}}.$$ 

Using the hypothesis on the growth of $\mu(S_n)$ and the definition of $\beta, \gamma$ and $\theta$, we get

$$\text{var}(S_n) = O(\mu(S_n)^2(\log n)^{-1}(\log \log n)^{-\rho}).$$

Then we satisfy the hypothesis of Theorem [1] and so $S_n/\mu(S_n) \to 1$ $\mu$ a.e. By [8], there exists $C_1 > 0$ such that, for every $\varepsilon > 0$,

$$\mu(|S_n/\mu(S_n) - 1| > \varepsilon) \leq C_1 \varepsilon^{-2}(\log n)^{-1}(\log \log n)^{-\rho}.$$ 

Also by the proof of (BC1),

$$\mu(x : f_i(x) \neq 1_{B_i}(T^n x) \text{ for some } i \geq n) \leq \sum_{i=n}^{\infty} (i(\log i)^{1+\theta})^{-1} \leq C_\theta (\log n)^{-\theta},$$

for some $C_\theta > 0$. Then, using $\mu(S_n) = \sum_{i=0}^{n-1} \mu(B_i) + O(1)$ and simple inequalities we get

$$\mu(|\Delta_n - 1| > \varepsilon) \leq C_2 (\log n)^{-1}(\log \log n)^{-\rho} + C_\theta (\log n)^{-\theta},$$

where $C_2 > 0$ depends on $\rho, \theta, \varepsilon$. As in the previous subsection, we can get better ‘large deviation’ results if we increase the growth of $\mu(S_n)$. □
Proof of Theorem 5. Let $S_n$ be as in the proof of Theorem 4 (with $\theta = 1$). First we prove that

$$\sum_{i=0}^{n-1} \rho(B_i) + O(1)$$

for all sufficiently large $n$. Let $\mu(S_n) = \rho(n)(\log n)^{\beta}$. By hypothesis (and $\mu(S_n) = \sum_{i=0}^{n-1} \mu(B_i) + O(1)$) we get

$$\rho(n) \geq \frac{1}{2}(\log n)(\log \log n)^{\beta}.$$

Since $x(\log x)^{-1}(\log \log x)^{-\gamma}$ is an increasing function, in order to prove (16) we may assume $\rho(n) = \frac{1}{2}(\log n)(\log \log n)^{\gamma}$. Then

$$\log \mu(S_n) \leq 2\beta \log n, \quad (\log \mu(S_n))^{\gamma} \leq 2^{\gamma} (\log \log n)^{\gamma},$$

which implies (16).

Now we follow the proof of Theorem 4 but with $\nu(n) = A(\log n)^{\beta}$, where $\alpha A^{\beta^{-1}} > 2/\delta + 1$. Then we get

$$I \leq \mu(S_n)\nu(n) \leq \frac{2^{\gamma+2}\beta A\mu(S_n)^2}{(\log \mu(S_n))(\log \log \mu(S_n))^{\gamma}},$$

where we have used (16). Also

$$II \leq C(n(\log n)^2)^{1/\delta} \sum_{i=0}^{n-2} (i(\log i)^2)^{1/\delta} \sum_{j=i+\lfloor\alpha(n)\rfloor+1}^{n-1} e^{-\alpha(j-i)^{\beta^{-1}}}$$

and $\sum_{k=N}^{\infty} e^{-\alpha k^{\beta^{-1}}} \leq \tilde{C}e^{-\alpha N^{\beta^{-1}}} N^{[\beta]/\beta}$ for some $\tilde{C} > 0$, so

$$II = O\left( (n(\log n)^2)^{2/\delta} n(\log n)^{\beta} e^{-\alpha \nu(n)^{\beta^{-1}}} \right).$$

Since $e^{-\alpha \nu(n)^{\beta^{-1}}} = n^{-\alpha A^{\beta^{-1}}}$, the definition of $A$ implies $II \to 0$. Then

$$\text{var}(S_n) = O\left( \frac{\mu(S_n)^2}{(\log \mu(S_n))(\log \log \mu(S_n))^{\gamma}} \right)$$

and we can apply Theorem 4 to get $S_n/\mu(S_n) \to 1$ $\mu$ a.e.

The proof of the ‘large deviation’ result is similar to the one given in the proof of Theorem 4. \qed

2.4. Proofs of 1.4.

Proof of Theorem 8. (a) Note that the hypotheses imply $\alpha = \hat{d}(x_0) > 0$. If we define $C(r) = \mu(B(x_0, r))/r^\alpha$, then $\limsup_{r \to 0} \log C(r)/\log r = 0$. By the hypotheses, there are $r_i \to 0$ such that

$$C(r_i)^{\alpha} = \mu(B(x_0, r_i)) = i^{-\beta}(\log i)^{\gamma} \leq Cr_i^{\delta(x_0)}.$$

Also $r_i^{-1} = i^{\beta}/\kappa(i)$, where

$$\kappa(i) = \max\left\{ 1, C(r_i)^{-1/\alpha}(\log i)^{\gamma/\alpha} \right\}.$$

By (17) we get $\log r_i^{-1}/\log i \leq \beta \delta(x_0)^{-1} + o(1)$ and so $\log \kappa(i)/\log i \to 0$. The first part of the result follows by applying Corollary 4.
Now assume $\Theta^\alpha(x_0) > 0$. Also assume $\Theta^\alpha(x_0) < \infty$ (the other case is similar). Then, given $0 < \epsilon < \Theta^\alpha(x_0)$, there exists $r_0 > 0$ such that for every $0 < r < r_0$ we have $(\Theta^\alpha(x_0) - \epsilon) \leq \mu(B(x_0, r))/r^\alpha \leq (\Theta^\alpha(x_0) + \epsilon)$. Then we set $r^\alpha_i = i^{-\beta}(\log i) ?$ and apply Corollary [4].

(b) The proof is similar to the proof of (a) with the obvious modifications and using Corollary [5].

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Universidade Federal do Rio de Janeiro, Instituto de Matemática, Rio de Janeiro 21945-970, Brazil.
E-mail address: nuno@im.ufrj.br