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TATE (CO)HOMOLOGY VIA PINCHED COMPLEXES

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Abstract. For complexes of modules we study two new constructions which we call the pinched tensor product and the pinched Hom. They provide new methods for computing Tate homology $\widehat{\mathrm{Tor}}$ and Tate cohomology $\widehat{\mathrm{Ext}}$, which lead to conceptual proofs of balancedness of Tate (co)homology for modules over associative rings.

Another application we consider is in local algebra. Under conditions of the vanishing of Tate (co)homology, the pinched tensor product of two minimal complete resolutions yields a minimal complete resolution.

Introduction

Tate cohomology originated in the study of representations of finite groups. It has been generalized—through works of, in chronological order, Buchweitz [5], Avramov and Martsinkovsky [3], and Veliche [14]—into a cohomology theory for modules with complete resolutions. The parallel theory of Tate homology has been treated in the same generality by Iacob [9].

While these theories function for modules over any associative ring, the central question of balancedness has yet to receive a cogent treatment. The extant literature only solves the problem for modules over special commutative rings. The issue is that if M and N are modules with appropriate complete resolutions, then there are potentially two ways of defining Tate cohomology $\widehat{\mathrm{Ext}}^*(M,N)$; do they yield the same theory? For Tate homology $\widehat{\mathrm{Tor}}_*(M,N)$ one encounters a similar situation, and one goal of this paper is to resolve these balancedness problems.

Proving balancedness of absolute (co)homology, Ext and Tor, boils down to showing that, say, $\mathrm{Tor}_*(M,N)$ can be computed from a complex constructed from resolutions of both variables M and N; namely the tensor product of their projective resolutions. Our approach is similar, but for Tate (co)homology the standard tensor product and Hom complexes fail to do the job, so we introduce two new constructions. We call them the pinched tensor product and the pinched Hom. They resemble the usual tensor product and Hom of complexes, but they are smaller in a sense that is discussed below. The central technical results are Theorems 3.5 and 4.7, which establish that Tate (co)homology can be computed from pinched complexes. The balancedness problems are resolved in Theorems 3.7 and 5.4.

As part of our analysis of the pinched complexes, we establish "pinched versions" of standard isomorphisms for complexes, such as Hom-tensor adjunction. They

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allow us to give criteria—Corollaries 4.10 and 5.9—in terms of the vanishing of Tate (co)homology, for when a pinched Hom complex $\mathrm{Hom}^{\bowtie}(T,U)$ or a pinched tensor product $T \otimes^{\bowtie} U$ of complete resolutions is a complete resolution.

This is of particular interest in local algebra since if one starts with unbounded complexes of finitely generated modules, then the pinched Hom and the pinched tensor product are also complexes of finitely generated modules. Theorem 6.1 gives a criterion, in terms of the vanishing of Tate (co)homology, for a tensor product of minimal complete resolutions to be a minimal complete resolution.

1. Standard constructions with complexes

In this paper R, R', S, and S' are associative unital rings; they are assumed to be algebras over a common commutative unital ring $\Bbbk$. The default $\Bbbk$ is the ring $\mathbb{Z}$ of integers, but in concrete settings other choices may be useful. For example, if the rings are algebras over a field k, then $\Bbbk = k$ is a natural choice. If R is commutative and R', S, and S' are R-algebras, then $\Bbbk = R$ is a candidate.

Modules are assumed to be unitary, and the default action of the ring is on the left. Right modules over R are hence treated as (left) modules over the opposite ring R°. By an R–S°-bimodule we mean a module over the $\Bbbk$-algebra $R \otimes_\Bbbk S^\circ$. Note that every R-module has a natural R–$\Bbbk^\circ$-bimodule structure; in particular they are symmetric $\Bbbk$–$\Bbbk^\circ$-bimodules. Modules over a commutative ring R are tacitly assumed to be symmetric R–R°-bimodules.

Complexes. An R-complex is a (homologically) graded R-module M endowed with a square-zero endomorphism ∂^M of degree -1, which is called the differential. Here is a visualization:

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots .$$

A morphism of complexes $M \to N$ is a degree 0 graded homomorphism $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$ of the underlying graded modules that commutes with the differentials on M and N; i.e. one has $\partial^N\alpha = \alpha\partial^M$. The category of R-complexes is denoted $\mathsf{C}(R)$. If the underlying graded module is an R–S°-bimodule and the differential is a bimodule endomorphism, then the complex is called a complex of R–S°-bimodules;[1] the category of such complexes is denoted $\mathsf{C}(R\text{–}S^\circ)$.

The kernel $\mathrm{Z}(M)$ and the image $\mathrm{B}(M)$ of ∂^M are graded submodules of M and, in fact, subcomplexes, as the induced differentials are trivial. A complex M is called *acyclic* if the homology complex $\mathrm{H}(M) = \mathrm{Z}(M)/\mathrm{B}(M)$ is the zero-complex. We use the notation $\mathrm{C}(M)$ for the cokernel of the differential, i.e. $\mathrm{C}_i(M) = \mathrm{Coker}\,\partial^M_{i+1}$.

The notation $\sup M$ and $\inf M$ is used for the supremum and infimum of the set $\{i \in \mathbb{Z} \mid M_i \neq 0\}$, with the conventions $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$. A complex M is *bounded above* if $\sup M$ is finite, and it is *bounded below* if $\inf M$ is finite.

For $n \in \mathbb{Z}$ the *n-fold shift* of M is the complex $\Sigma^n M$ with $(\Sigma^n M)_i = M_{i-n}$ and $\partial^{\Sigma^n M}_i = (-1)^n \partial^M_{i-n}$. One has $\sup(\Sigma^n M) = \sup M + n$ and $\inf(\Sigma^n M) = \inf M + n$.

Let n be an integer. The *hard truncation above* of M at n is the complex $M_{\leqslant n}$ with $(M_{\leqslant n})_i = 0$ for $i > n$ and $\partial^{M_{\leqslant n}}_i = \partial^M_i$ for $i \leqslant n$. It looks like this:

$$M_{\leqslant n} \;=\; 0 \longrightarrow M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} M_{n-2} \longrightarrow \cdots .$$

[1] The term 'bicomplex' is too close to 'double complex'.

Similarly, $M_{\geqslant n}$ is the complex with $(M_{\geqslant n})_i = 0$ for $i < n$ and $\partial_i^{M_{\geqslant n}} = \partial_i^M$ for $i > n$. Note that $M_{\leqslant n}$ is a subcomplex of M, and $M_{\geqslant n}$ is the quotient complex $M/M_{\leqslant n-1}$. The *soft truncations* of M at n are the complexes

$$M_{\subset n} \;=\; 0 \longrightarrow \mathrm{C}_n(M) \xrightarrow{\overline{\partial_n^M}} M_{n-1} \xrightarrow{\partial_{n-1}^M} M_{n-2} \longrightarrow \cdots$$

and

$$M_{\supset n} \;=\; \cdots \longrightarrow M_{n+2} \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} \mathrm{Z}_n(M) \longrightarrow 0.$$

A morphism of complexes that induces an isomorphism in homology is called a *quasi-isomorphism* and indicated by the symbol '$\simeq$'. A morphism α is a quasi-isomorphism if and only if its mapping cone, the complex $\operatorname{Cone}\alpha$, is acyclic.

The central constructions in this paper, Constructions 3.2 and 4.4, start from the standard constructions of tensor product and Hom complexes, hence we review them in detail.

Tensor product and Hom. Let M be an R°-complex and N be an R-complex. The tensor product $M \otimes_R N$ is the $\Bbbk$-complex whose underlying graded module is given by

$$(M \otimes_R N)_n = \coprod_{i\in\mathbb{Z}} M_i \otimes_R N_{n-i}$$

and whose differential is defined by specifying its action on an elementary tensor of homogeneous elements as follows:

$$\partial^{M\otimes_R N}(x \otimes y) = \partial^M(x) \otimes y + (-1)^{|x|} x \otimes \partial^N(y);$$

here $|x|$ is the degree of x in M. For a morphism of R°-complexes $\alpha\colon M \to M'$ and a morphism of R-complexes $\beta\colon N \to N'$, the map $\alpha \otimes_R \beta\colon M \otimes_R N \to M' \otimes_R N'$, defined by

$$(\alpha \otimes_R \beta)(x \otimes y) = \alpha(x) \otimes \beta(y),$$

is a morphism of $\Bbbk$-complexes. The tensor product yields a functor

$$- \otimes_R -\colon \mathsf{C}(R^\circ) \times \mathsf{C}(R) \;\longrightarrow\; \mathsf{C}(\Bbbk),$$

which is $\Bbbk$-bilinear and right exact in each variable. In case M is a complex of R'–R°-bimodules and N is a complex of R–S°-bimodules, then the tensor product $M \otimes_R N$ is a complex of R'–S°-bimodules. The tensor product yields a functor $\mathsf{C}(R'\text{–}R^\circ) \times \mathsf{C}(R\text{–}S^\circ) \to \mathsf{C}(R'\text{–}S^\circ)$.
F. or R-complexes M and N, the $\Bbbk$-complex $\operatorname{Hom}_R(M,N)$ is given by

$$\operatorname{Hom}_R(M,N)_n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(M_i, N_{i+n})$$

and

$$\partial^{\operatorname{Hom}_R(M,N)}(\varphi) = \partial^N\varphi - (-1)^{|\varphi|}\varphi\partial^M,$$

for a homogeneous φ in $\operatorname{Hom}_R(M,N)$. For morphisms of R-complexes $\alpha\colon M \to M'$ and $\beta\colon N \to N'$, a morphism $\operatorname{Hom}_R(\alpha,\beta)\colon \operatorname{Hom}_R(M',N) \to \operatorname{Hom}_R(M,N')$ of $\Bbbk$-complexes is defined by

$$\operatorname{Hom}_R(\alpha,\beta)(\varphi) = \beta\varphi\alpha.$$

With these definitions, Hom yields a functor,

$$\operatorname{Hom}_R(-,-)\colon \mathsf{C}(R)^{\mathrm{op}} \times \mathsf{C}(R) \;\longrightarrow\; \mathsf{C}(\Bbbk),$$

where the superscript 'op' signifies the opposite category; it is $\Bbbk$-bilinear and left exact in each variable. In case M is a complex of R–R'°-bimodules and N is a complex of R–S°-bimodules, the complex $\operatorname{Hom}_R(M,N)$ is one of R'–S°-bimodules; Hom yields a functor $\mathsf{C}(R\text{–}R'^{\circ})^{\mathrm{op}} \times \mathsf{C}(R\text{–}S^{\circ}) \to \mathsf{C}(R'\text{–}S^{\circ})$.

Resolutions. An R-complex P is called *semi-projective* if each module P_i is projective, and the functor $\operatorname{Hom}_R(P,-)$ preserves quasi-isomorphisms (equivalently, it preserves acyclicity). A bounded below complex of projective R-modules is semi-projective. Similarly, an R-complex I is called *semi-injective* if each module I_i is injective, and the functor $\operatorname{Hom}_R(-,I)$ preserves quasi-isomorphisms (equivalently, it preserves acyclicity). A bounded above complex of injective R-modules is semi-injective. Every R-complex M has a semi-projective resolution and a semi-injective resolution; that is, there are quasi-isomorphisms $\pi\colon P \to M$ and $\iota\colon M \to I$, where P is semi-projective and I is semi-injective; see [2].[2] An R-complex F is called *semi-flat* if each module F_i is flat, and the functor $-\otimes_R F$ preserves quasi-isomorphisms (equivalently, it preserves acyclicity). Every semi-projective complex is semi-flat.

For an R-module M, a projective (injective) resolution in the classic sense is a semi-projective (-injective) resolution. Thus, the following definitions of homological dimensions of an R-complex extend the classic notions for modules:

$$\operatorname{pd}_R M = \inf\{\sup P \mid P \xrightarrow{\simeq} M \text{ is a semi-projective resolution}\},$$

$$\operatorname{id}_R M = \inf\{-\inf I \mid M \xrightarrow{\simeq} I \text{ is a semi-injective resolution}\}, \quad \text{and}$$

$$\operatorname{fd}_R M = \inf\left\{ n \geqslant \sup \mathrm{H}(M) \;\middle|\; \begin{array}{c} P \xrightarrow{\simeq} M \text{ is a semi-projective} \\ \text{resolution and } \mathrm{C}_n(P) \text{ is flat} \end{array} \right\}.$$

The derived tensor product $-\otimes_R^{\mathbf{L}}-$ and the derived Hom functor $\mathbf{R}\operatorname{Hom}_R(-,-)$ for complexes are computed by way of the resolutions described above. Extending the usual definitions of Tor and Ext for modules, set

$$\operatorname{Tor}_i^R(M,N) = \mathrm{H}_i(M \otimes_R^{\mathbf{L}} N) \quad \text{and} \quad \operatorname{Ext}_R^i(M,N) = \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(M,N))$$

for complexes M and N and $i \in \mathbb{Z}$.

2. Complete resolutions and Tate homology

In this section we recall some definitions and facts from works of Iacob [9] and Veliche [14], and we establish some auxiliary results for later use.

2.1. **Complete projective resolutions.** An acyclic complex T of projective R-modules is called *totally acyclic* if the complex $\operatorname{Hom}_R(T,Q)$ is acyclic for every projective R-module Q.

A *complete projective resolution* of an R-complex M is a diagram

$$T \xrightarrow{\tau} P \xrightarrow{\pi} M, \tag{2.1.1}$$

where π is a semi-projective resolution, T is a totally acyclic complex of projective R-modules, and τ_i is an isomorphism for $i \gg 0$.

See [14] for a proof of the following fact.

[2]In this paper the authors use 'DG-' in place of 'semi-'.

2.2. **Fact.** Let $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ and $T' \xrightarrow{\tau'} P' \xrightarrow{\pi'} M'$ be complete projective resolutions. For every morphism $\alpha\colon M \to M'$ there exists a morphism $\bar{\alpha}$ such that the right-hand square in the diagram

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, "\widehat{\alpha}"] & P \arrow[r, "\pi"] \arrow[d, "\bar{\alpha}"] & M \arrow[d, "\alpha"] \\ T' \arrow[r, "\tau'"] & P' \arrow[r, "\pi'"] & M' \end{tikzcd}$$

is commutative up to homotopy. The morphism $\bar{\alpha}$ is unique up to homotopy, and for every choice of $\bar{\alpha}$ there exists a morphism $\widehat{\alpha}$, also unique up to homotopy, such that the left-hand square is commutative up to homotopy. Moreover, if τ' and π' are surjective, then $\widehat{\alpha}$ and $\bar{\alpha}$ can be chosen such that the diagram is commutative. Finally, if one has $M = M'$ and α is the identity map, then $\bar{\alpha}$ and $\widehat{\alpha}$ are homotopy equivalences.

2.3. **Gorenstein projectivity.** An R-module G is called *Gorenstein projective* if there exists a totally acyclic complex T of projective R-modules with $\mathrm{C}_0(T) \cong G$. In that case, the diagram $T \to T_{\geqslant 0} \to G$ is a complete projective resolution, and for brevity we shall often say that T is a complete projective resolution of G.

The *Gorenstein projective dimension* of an R-complex M, written $\operatorname{Gpd}_R M$, is the least integer n such that there exists a complete projective resolution (2.1.1) where τ_i is an isomorphism for all $i \geqslant n$. In particular, $\operatorname{Gpd}_R M$ is finite if and only if M has a complete projective resolution. Notice that $\mathrm{H}(M)$ is bounded above if $\operatorname{Gpd}_R M$ is finite; indeed, there is an inequality

$$\operatorname{Gpd}_R M \geqslant \sup \mathrm{H}(M). \tag{2.3.1}$$

If M is an R-complex of finite projective dimension, then there is a semi-projective resolution $P \xrightarrow{\simeq} M$ with P bounded above, and then $0 \to P \to M$ is a complete projective resolution. In particular, M has finite Gorenstein projective dimension.

2.4. **Tate homology.** Let M be an R°-complex with a complete projective resolution $T \to P \to M$. For an R-complex N, the *Tate homology of M with coefficients in N* is defined as

$$\widehat{\operatorname{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N).$$

It follows from Section 2.2 that this definition is independent (up to isomorphism) of the choice of complete projective resolution; in particular, one has

$$\widehat{\operatorname{Tor}}_i^R(M, N) \cong \operatorname{Tor}_i^R(M, N) \quad \text{for } i > \operatorname{Gpd}_{R^\circ} M + \sup N. \tag{2.4.1}$$

Note that $\widehat{\operatorname{Tor}}_i^R(M, N)$ is a $\Bbbk$-module for every $i \in \mathbb{Z}$. Moreover, if N is an R–S°-bimodule, then each $\widehat{\operatorname{Tor}}_i^R(M, N)$ is an S°-module.

Tate homology $\widehat{\operatorname{Tor}}_*^R(M, N)$ vanishes if M (or N) is a (bounded above) complex of finite projective dimension; this is the content of Proposition 2.5 and Lemma 2.7 below.

The boundedness condition on N in Lemma 2.7 is a manifestation of the fact that Tate homology $\widehat{\operatorname{Tor}}_*^R(M, -)$ is not a functor from the derived category $\mathsf{D}(R)$. Indeed, every R-complex is isomorphic in $\mathsf{D}(R)$ to a semi-projective complex, and for such a complex P one has $\widehat{\operatorname{Tor}}_*^R(M, P) = 0$ for every R°-complex M of finite Gorenstein projective dimension.

Notice, though, that if M and M' are isomorphic in $\mathsf{D}(R^\circ)$ and of finite Gorenstein projective dimension, then it follows from [2, 1.4.P] that every complete projective resolution $T \to P \to M$ yields a complete resolution $T \to P \to M'$, so one has an isomorphism $\widehat{\operatorname{Tor}}^R_*(M,-) \cong \widehat{\operatorname{Tor}}^R_*(M',-)$ of functors from $\mathsf{C}(R)$.

Proposition 2.5. *Let M be an R°-complex of finite Gorenstein projective dimension. Among the conditions*

(i) $\operatorname{pd}_{R^\circ} M < \infty$,
(ii) $\widehat{\operatorname{Tor}}^R_i(M,-) = 0$ *for all* $i \in \mathbb{Z}$,
(iii) $\widehat{\operatorname{Tor}}^R_i(M,-) = 0$ *for some* $i \in \mathbb{Z}$,
(iv) $\operatorname{fd}_{R^\circ} M < \infty$,

the implications $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$ *hold.*

We recall from works of Jensen [10, prop. 6] and Raynaud and Gruson [13, II. thm. 3.2.6] that if R has finite finitistic projective dimension—for example, R is commutative Noetherian of finite Krull dimension—then every flat R-module has finite projective dimension, and it follows that conditions (i)–(iv) are equivalent.

Proof. If $\pi\colon P \xrightarrow{\simeq} M$ is a semi-projective resolution with P bounded above, then $0 \to P \xrightarrow{\pi} M$ is a complete projective resolution, so one has $\widehat{\operatorname{Tor}}^R_i(M,-) = 0$ for all $i \in \mathbb{Z}$. Thus, (i) implies (ii); the implication $(ii) \Longrightarrow (iii)$ is trivial.

Assume now that one has $\widehat{\operatorname{Tor}}^R_i(M,-) = 0$ for some $i \in \mathbb{Z}$. Let $T \to P \to M$ be a complete projective resolution and set $G = \mathrm{C}_{i-1}(T)$. As the functor $\operatorname{Tor}^R_1(G,-) = \mathrm{H}_i(T \otimes_R -) = \widehat{\operatorname{Tor}}^R_i(M,-)$ vanishes, the R°-module G is flat. It follows that the module $\mathrm{C}_j(T) \cong \mathrm{C}_j(P)$ is flat for every $j \geqslant \max\{i-1, \operatorname{Gpd}_{R^\circ} M\}$, whence $\operatorname{fd}_{R^\circ} M$ is finite. □

Remark 2.6. Let $T \to P \to M$ be a complete projective resolution over R°. For every semi-projective resolution $\pi'\colon P' \xrightarrow{\simeq} N$ over R, application of the functor $T \otimes_R -$ to the exact sequence $0 \to N \to \operatorname{Cone}\pi' \to \Sigma P' \to 0$ yields a short exact sequence, as T is a complex of projective R°-modules. The associated exact sequence in homology yields an isomorphism

$$\mathrm{H}(T \otimes_R N) \cong \mathrm{H}(T \otimes_R \operatorname{Cone}\pi'), \tag{2.6.1}$$

as one has $\mathrm{H}(T \otimes_R P') = 0$ because P' is semi-flat. If N is bounded above and of finite projective dimension, then one can assume that P' and, therefore, $\operatorname{Cone}\pi'$ are bounded above, and then [6, lem. 2.13] yields $\mathrm{H}(T \otimes_R \operatorname{Cone}\pi') = 0$. Thus, we record the following result.

Lemma 2.7. *Let M be an R°-complex of finite Gorenstein projective dimension. For every bounded above R-complex N of finite projective dimension, one has $\widehat{\operatorname{Tor}}^R_i(M,N) = 0$ for all $i \in \mathbb{Z}$.* □

Proposition 2.8. *Let M be an R°-complex of finite Gorenstein projective dimension. For every exact sequence $0 \to N' \to N \to N'' \to 0$ of R-complexes, there is an exact sequence of $\Bbbk$-modules*

$$\cdots \to \widehat{\operatorname{Tor}}^R_{i+1}(M,N'') \to \widehat{\operatorname{Tor}}^R_i(M,N') \to \widehat{\operatorname{Tor}}^R_i(M,N) \to \widehat{\operatorname{Tor}}^R_i(M,N'') \to \cdots.$$

Moreover, if the original exact sequence is one of complexes of R–S°-bimodules, then the derived exact sequence is one of S°-modules.

Proof. Let $T \to P \to M$ be a complete projective resolution. The sequence

$$0 \to T \otimes_R N' \to T \otimes_R N \to T \otimes_R N'' \to 0$$

is exact because T is a complex of projective R°-modules. The associated exact sequence in homology is the desired one, and the statement about additional module structures is evident. □

Proposition 2.9. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R°-complexes of finite Gorenstein projective dimension. For every R-complex N there is an exact sequence of $\Bbbk$-modules*

$$\cdots \to \widehat{\operatorname{Tor}}^R_{i+1}(M'', N) \to \widehat{\operatorname{Tor}}^R_i(M', N) \to \widehat{\operatorname{Tor}}^R_i(M, N) \to \widehat{\operatorname{Tor}}^R_i(M'', N) \to \cdots .$$

Moreover, if N is a complex of R–S°-bimodules, then the derived exact sequence is one of S°-modules.

Proof. By [14, prop. 4.7] there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows such that the columns are complete projective resolutions. The sequence $0 \to T' \to T \to T'' \to 0$ is degreewise split, so the sequence

$$0 \to T' \otimes_R N \to T \otimes_R N \to T'' \otimes_R N \to 0$$

is exact, and the associated sequence in homology is the desired one. The statement about additional module structures is evident. □

As with absolute homology, dimension shifting is a useful technique in dealing with Tate homology.

Lemma 2.10. *Let M be an R°-complex of finite Gorenstein projective dimension and let N be an R-complex. For every complete projective resolution $T \to P \to M$ and for every $m \in \mathbb{Z}$ there are isomorphisms*

$$\text{(a)} \qquad \widehat{\operatorname{Tor}}^R_i(M, N) \cong \widehat{\operatorname{Tor}}^R_{i-m}(\mathrm{C}_m(T), N) \quad \textit{for all } i \in \mathbb{Z}.$$

For every semi-projective resolution $L \xrightarrow{\simeq} N$ and for every integer $n \geqslant \sup N$ there are isomorphisms

$$\text{(b)} \qquad \widehat{\operatorname{Tor}}^R_i(M, N) \cong \widehat{\operatorname{Tor}}^R_{i-n}(M, \mathrm{C}_n(L)) \quad \textit{for all } i \in \mathbb{Z}.$$

Proof. (a) For every $m \in \mathbb{Z}$ the diagram $\Sigma^{-m}T \to \Sigma^{-m}T_{\geqslant m} \to \mathrm{C}_m(T)$ is a complete projective resolution. Hence one has

$$\begin{aligned}
\widehat{\operatorname{Tor}}^R_{i-m}(\mathrm{C}_m(T), N) &= \mathrm{H}_{i-m}((\Sigma^{-m}T) \otimes_R N) \\
&= \mathrm{H}_{i-m}(\Sigma^{-m}(T \otimes_R N)) \\
&\cong \mathrm{H}_i(T \otimes_R N) \\
&= \widehat{\operatorname{Tor}}^R_i(M, N).
\end{aligned}$$

(b) We may assume that N is bounded above; otherwise the statement is void. For every $n \geqslant \sup N$ there is a quasi-isomorphism $\widetilde{\pi}\colon L_{\subset n} \to N$. The acyclic complex $\operatorname{Cone}\widetilde{\pi}$ is bounded above, so $T \otimes_R \operatorname{Cone}\widetilde{\pi}$ is acyclic by [6, lem. 2.13]. An application of Proposition 2.8 to the exact sequence $0 \to N \to \operatorname{Cone}\widetilde{\pi} \to \Sigma L_{\subset n} \to 0$ yields isomorphisms

$$\widehat{\operatorname{Tor}}_i^R(M,N) \cong \widehat{\operatorname{Tor}}_i^R(M,L_{\subset n}) \quad \text{for all } i \in \mathbb{Z}.$$

The first complex in the exact sequence $0 \to L_{\leqslant n-1} \to L_{\subset n} \to \Sigma^n \mathrm{C}_n(L) \to 0$ of R-complexes has finite projective dimension. Indeed, in the exact sequence $0 \to L_{\leqslant n-1} \to L \to L_{\geqslant n} \to 0$, the complexes L and $L_{\geqslant n}$ are semi-projective, so $L_{\leqslant n-1}$ is semi-projective and, moreover, bounded above. Now apply Lemma 2.7 and Proposition 2.8 to get

$$\widehat{\operatorname{Tor}}_i^R(M,L_{\subset n}) \cong \widehat{\operatorname{Tor}}_i^R(M,\Sigma^n \mathrm{C}_n(L)) \quad \text{for all } i \in \mathbb{Z}.$$

The desired isomorphisms follow from these last two displays. □

3. Pinched tensor product complexes

We start by noticing that a very natural approach to the balancedness problem for Tate homology fails.

Example 3.1. Let k be a field and consider the commutative ring $R = k[x,y]/(xy)$. The R-module $R/(x)$ is Gorenstein projective with complete resolution

$$T = \cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} \cdots,$$

where ∂_i^T is multiplication by x for i odd and multiplication by y for i even. As multiplication by y on $R/(x)$ is injective, it is immediate from the definition of Tate homology (see Section 2.4) that one has $\widehat{\operatorname{Tor}}_i^R(R/(x),R/(x)) = 0$ for i even.

The complex $T \otimes_R T$, however, has nonvanishing homology in even degrees. Indeed, for each $n \in \mathbb{Z}$ the module $(T \otimes_R T)_n$ is free with basis $(e_{i,n-i})_{i\in\mathbb{Z}}$. The differential is given by

$$\partial_n^{T\otimes_R T}(e_{i,n-i}) = \begin{cases} xe_{i-1,n-i} - ye_{i,n-i-1}, & n \text{ odd and } i \text{ odd},\\ ye_{i-1,n-i} + xe_{i,n-i-1}, & n \text{ odd and } i \text{ even},\\ xe_{i-1,n-i} - xe_{i,n-i-1}, & n \text{ even and } i \text{ odd},\\ ye_{i-1,n-i} + ye_{i,n-i-1}, & n \text{ even and } i \text{ even}. \end{cases}$$

For n even, the element $xe_{0,n}$ is a cycle and clearly not a boundary. Indeed, since R is graded, the complex $T \otimes_R T$ has an internal grading, and the differential is of degree 1 with respect to this grading. Suppose that $xe_{0,n}$ is a boundary. Since it is an element of internal degree 1, a preimage $\sum_{i\in\mathbb{Z}} \alpha_{i,n+1-i} e_{i,n+1-i}$ of $xe_{0,n}$ under $\partial^{T\otimes_R T}$ may be assumed homogeneous of internal degree zero. That is, we may assume that $\alpha_{i,n+1-i}$ is in k for all i. Let i_0 and i_1 be, respectively, the least and the largest integer i with $\alpha_{i,n+1-i} \neq 0$. With respect to the basis $(e_{i,n-i})_{i\in\mathbb{Z}}$, the element $b = \partial^{T\otimes_R T}(\sum_{i\in\mathbb{Z}} \alpha_{i,n+1-i} e_{i,n+1-i})$ is nonzero in coordinate $(i_0 - 1, n+1-i_0)$, which implies $i_0 = 1$. Similarly, b is nonzero in coordinate $(i_1, n-i_1)$, which implies $i_1 = 0$. Thus one has $i_0 > i_1$, a contradiction.

The isomorphism (2.6.1) shows, nevertheless, that one can compute Tate homology from a tensor product of acyclic complexes. This motivates the next construction; see also the comments before the proof of Theorem 3.5.

Construction 3.2. Let T be an R°-complex and let A be an R-complex. Consider the graded $\Bbbk$-module $T \otimes_R^{\bowtie} A$ defined by

$$(T \otimes_R^{\bowtie} A)_n = \begin{cases} (T_{\geqslant 0} \otimes_R A_{\geqslant 0})_n & \text{for } n \geqslant 0, \\ (T_{\leqslant -1} \otimes_R \Sigma(A_{\leqslant -1}))_n & \text{for } n \leqslant -1. \end{cases}$$

It is elementary to verify that one has

$$(\partial_0^T \otimes_R (\sigma\partial_0^A)) \circ \partial_1^{T_{\geqslant 0} \otimes_R A_{\geqslant 0}} = 0 = \partial_{-1}^{T_{\leqslant -1} \otimes_R \Sigma(A_{\leqslant -1})} \circ (\partial_0^T \otimes_R (\sigma\partial_0^A)),$$

where σ denotes the canonical map $A \to \Sigma A$. Thus, $\partial^{T \otimes_R^{\bowtie} A}$ defined by

$$\partial_n^{T \otimes_R^{\bowtie} A} = \begin{cases} \partial_n^{T_{\geqslant 0} \otimes_R A_{\geqslant 0}} & \text{for } n \geqslant 1, \\ \partial_0^T \otimes_R (\sigma\partial_0^A) & \text{for } n = 0, \\ \partial_n^{T_{\leqslant -1} \otimes_R \Sigma(A_{\leqslant -1})} & \text{for } n \leqslant -1 \end{cases}$$

is a differential on $T \otimes_R^{\bowtie} A$. We refer to this $\Bbbk$-complex as the *pinched tensor product* of T and A.

For morphisms $\alpha\colon T \to T'$ of R°-complexes and $\beta\colon A \to A'$ of R-complexes, it is elementary to verify that the assignment $x \otimes y \mapsto \alpha(x) \otimes \beta(y)$ defines a morphism of $\Bbbk$-complexes

$$\alpha \otimes_R^{\bowtie} \beta\colon T \otimes_R^{\bowtie} A \longrightarrow T' \otimes_R^{\bowtie} A'.$$

Remark 3.3. For every R°-complex T and every R-complex A there are equalities of $\Bbbk$-complexes,

$$(T \otimes_R^{\bowtie} A)_{\geqslant 0} = T_{\geqslant 0} \otimes_R A_{\geqslant 0} \quad \text{and} \tag{3.3.1}$$

$$(T \otimes_R^{\bowtie} A)_{\leqslant -1} = T_{\leqslant -1} \otimes_R \Sigma(A_{\leqslant -1}). \tag{3.3.2}$$

If T is a complex of R'–R°-bimodules and A is a complex of R–S°-bimodules, then $T \otimes_R^{\bowtie} A$ is a complex of R'–S°-bimodules.

The proof of the next proposition is standard, and we omit it.

Proposition 3.4. *The pinched tensor product defined in Construction* 3.2 *yields a functor*

$$- \otimes_R^{\bowtie} -\colon \mathsf{C}(R'\text{–}R^\circ) \times \mathsf{C}(R\text{–}S^\circ) \longrightarrow \mathsf{C}(R'\text{–}S^\circ);$$

in particular, it yields a functor $\mathsf{C}(R^\circ) \times \mathsf{C}(R) \to \mathsf{C}(\Bbbk)$. *Moreover, it is* $\Bbbk$*-bilinear and right exact in each variable.* □

Theorem 3.5. *Let M be an R°-complex with a complete projective resolution $T \to P \to M$. Let A be an acyclic R-complex and set $N = \mathrm{C}_0(A)$. For every $i \in \mathbb{Z}$ there is an isomorphism of $\Bbbk$-modules*

$$\mathrm{H}_i(T \otimes_R^{\bowtie} A) \cong \widehat{\mathrm{Tor}}_i^R(M, N).$$

If A is a complex of R–S°-bimodules, then the isomorphism is one of S°-modules.

Before we proceed with the proof, we point out that if N is an R-module and A is the acyclic complex $0 \to N \xrightarrow{=} N \to 0$ with N in degrees 0 and -1, then one has $T \otimes_R^{\bowtie} A = T \otimes_R N$.

Proof. By definition one has $\widehat{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N)$, so the goal is to establish an isomorphism between $\mathrm{H}(T \otimes_R^{\bowtie} A)$ and $\mathrm{H}(T \otimes_R N)$. The quasi-isomorphisms

$$\pi\colon A_{\geqslant 0} \xrightarrow{\simeq} N \qquad \text{and} \qquad \epsilon\colon N \xrightarrow{\simeq} \Sigma(A_{\leqslant -1}),$$

with $\epsilon_0\pi_0 = \sigma\partial_0^A$, induce quasi-isomorphisms

$$(T \otimes_R^{\bowtie} A)_{\geqslant 0} \xrightarrow{\simeq} T_{\geqslant 0} \otimes_R N \quad \text{and} \quad T_{\leqslant -1} \otimes_R N \xrightarrow{\simeq} (T \otimes_R^{\bowtie} A)_{\leqslant -1};$$

see (3.3.1), (3.3.2), and [6, prop. 2.14]. It follows that there are isomorphisms $\mathrm{H}_i(T \otimes_R^{\bowtie} A) \cong \mathrm{H}_i(T \otimes_R N)$ for all $i \in \mathbb{Z} \setminus \{0, -1\}$. To establish the isomorphism in the remaining two degrees, consider the following diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & T_0 \otimes_R \mathrm{B}_0(A) \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
(T \otimes_R^{\bowtie} A)_1 \arrow[r] \arrow[d, "T_1\otimes\pi_0"'] & (T \otimes_R^{\bowtie} A)_0 \arrow[d, "T_0\otimes\pi_0"'] \arrow[dr] & (T \otimes_R N)_{-1} \arrow[r] \arrow[d, "T_{-1}\otimes\epsilon_0"] & (T \otimes_R N)_{-2} \arrow[d, "T_{-2}\otimes\epsilon_0"] \\
(T \otimes_R N)_1 \arrow[r] \arrow[d] & (T \otimes_R N)_0 \arrow[d] \arrow[ur] & (T \otimes_R^{\bowtie} A)_{-1} \arrow[r] \arrow[d, "T_{-1}\otimes\partial^A_{-1}"] & (T \otimes_R^{\bowtie} A)_{-2} \\
0 & 0 & T_{-1} \otimes_R \mathrm{B}_{-2}(A) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

The identity $\epsilon_0\pi_0 = \sigma\partial_0^A$ shows that the twisted square is commutative. That the other two squares are commutative follows by functoriality of the tensor product.

To see that the homomorphism $T_0 \otimes_R \pi_0$ induces the desired isomorphism in homology, $\mathrm{H}_0(T \otimes_R^{\bowtie} A) \cong \mathrm{H}_0(T \otimes_R N)$, notice first that it maps boundaries to boundaries and then that for x in $\mathrm{Z}_0(T \otimes_R^{\bowtie} A)$ one has

$$(T_{-1} \otimes_R \epsilon_0) \circ \partial_0^{T\otimes_R N} \circ (T_0 \otimes_R \pi_0)(x) = 0$$

by commutativity of the twisted square. As $T_{-1} \otimes_R \epsilon_0$ is injective, it follows that $(T_0 \otimes_R \pi_0)(x)$ is in $\mathrm{Z}_0(T \otimes_R N)$, so there is a well-defined homomorphism

$$\mathrm{H}(T_0 \otimes_R \pi_0)\colon \mathrm{H}_0(T \otimes_R^{\bowtie} A) \to \mathrm{H}_0(T \otimes_R N).$$

It is immediate from the surjectivity of $T_0 \otimes_R \pi_0$ and commutativity of the twisted square that the homomorphism $\mathrm{H}(T_0 \otimes_R \pi_0)$ is surjective. To see that it is injective, let x be an element in $\mathrm{Z}_0(T \otimes_R^{\bowtie} A)$ and assume that there is a y in $(T \otimes_R N)_1$ such that $(T_0 \otimes_R \pi_0)(x) = \partial_1^{T\otimes_R N}(y)$. Choose an element z in $T_1 \otimes_R A_0 \subset (T \otimes_R^{\bowtie} A)_1$ such that $(T_1 \otimes_R \pi_0)(z) = y$. Then the element $x - \partial_1^{T\otimes_R^{\bowtie} A}(z)$ in $(T \otimes_R^{\bowtie} A)_0$ maps to 0 under $T_0 \otimes_R \pi_0$, so it belongs to $T_0 \otimes_R \mathrm{B}_0(A)$. Let w in $T_0 \otimes_R A_1 \subset (T \otimes_R^{\bowtie} A)_1$ be a preimage of $x - \partial_1^{T\otimes_R^{\bowtie} A}(z)$. Then one has

$$\partial_1^{T\otimes_R^{\bowtie} A}(w) = (T_0 \otimes_R \partial_1^A)(w) = x - \partial_1^{T\otimes_R^{\bowtie} A}(z),$$

and so x is a boundary: $\partial_1^{T\otimes_R^{\bowtie} A}(w+z) = x$. Thus, $\mathrm{H}(T_0 \otimes_R \pi_0)$ is an isomorphism.

Similarly, for $i = -1$, it is evident that $T_{-1} \otimes_R \epsilon_0$ maps cycles to cycles. Let x be a boundary in $(T \otimes_R N)_{-1}$, and choose a preimage y of x in $(T \otimes_R N)_0$. By surjectivity of $T_0 \otimes_R \pi_0$ this y has a preimage z in $(T \otimes_R^{\bowtie} A)_0$, and by commutativity of the twisted square one has $\partial_0^{T\otimes_R^{\bowtie} A}(z) = (T_{-1} \otimes_R \epsilon_0)(x)$. Thus, $T_{-1} \otimes_R \epsilon_0$ maps boundaries to boundaries, whence it induces a homomorphism

$$\mathrm{H}(T_{-1} \otimes_R \epsilon_0)\colon \mathrm{H}_{-1}(T \otimes_R N) \to \mathrm{H}_{-1}(T \otimes_R^{\bowtie} A).$$

It follows immediately from the injectivity of $T_{-1} \otimes_R \epsilon_0$ and commutativity of the twisted square that $\mathrm{H}(T_{-1} \otimes_R \epsilon_0)$ is injective. To see that it is surjective, let x be an element in $\mathrm{Z}_{-1}(T \otimes_R^{\bowtie} A)$. Then, in particular, one has

$$0 = (T_{-1} \otimes_R \partial_0^{\Sigma(A_{\leqslant -1})})(x) = -(T_{-1} \otimes_R \partial_{-1}^{A})(x).$$

Therefore, x is in $T_{-1} \otimes_R \mathrm{Z}_{-1}(A) = \mathrm{Im}\,(T_{-1} \otimes \epsilon_0)$, and it follows by injectivity of $T_{-2} \otimes_R \epsilon_0$ that the preimage of x is a cycle in $T_{-1} \otimes_R N$. Thus, $\mathrm{H}(T_{-1} \otimes_R \epsilon_0)$ is surjective and hence an isomorphism.

The claim about S°-module structures is immediate from Construction 3.2. □

Proposition 3.6. *Let T be an R°-complex and let A be an R-complex. The map*

$$\varpi \colon T \otimes_R^{\bowtie} A \longrightarrow A \otimes_{R^\circ}^{\bowtie} T$$

given by

$$\begin{aligned} \varpi_n(t \otimes a) &= (-1)^{|t||a|} a \otimes t && \text{for } n \geqslant 0, \\ \varpi_n(t \otimes \sigma(a)) &= (-1)^{(|t|+1)(|a|+1)} a \otimes \sigma(t) && \text{for } n \leqslant -1 \end{aligned}$$

is an isomorphism of $\Bbbk$-complexes.

Moreover, if T is a complex of R'–R°-bimodules and A is a complex of $R - S^\circ$-bimodules, then ϖ is an isomorphism of complexes of R'–S°-bimodules.

Proof. The map ϖ is clearly an isomorphism of graded $\Bbbk$-modules, and it is straightforward to verify that it commutes with the differentials. The assertions about additional module structures are immediate from Construction 3.2. □

If M is an R°-module of finite Gorenstein projective dimension and N is an R-module of finite Gorenstein projective dimension, then one could also define Tate homology of the pair (M, N) in terms of the complete projective resolution of N. Do the two definitions agree? That is, is Tate homology balanced? This is tantamount to asking if one has $\widehat{\mathrm{Tor}}_*^R(M, N) \cong \widehat{\mathrm{Tor}}_*^{R^\circ}(N, M)$. Iacob [9] gave a positive answer for modules over commutative Noetherian Gorenstein rings. The next theorem settles the question over any associative ring.

Theorem 3.7. *Let M be an R°-complex and let N be an R-complex, both of which are both bounded above and of finite Gorenstein projective dimension. For every $i \in \mathbb{Z}$ there is an isomorphism of $\Bbbk$-modules:*

$$\widehat{\mathrm{Tor}}_i^R(M, N) \cong \widehat{\mathrm{Tor}}_i^{R^\circ}(N, M).$$

Proof. Choose complete projective resolutions $T \to P \to M$ and $T' \to P' \to N$. Set $m = \max\{\sup M, \mathrm{Gpd}_{R^\circ} M\}$ and $n = \max\{\sup N, \mathrm{Gpd}_R N\}$. The modules $\mathrm{C}_m(P) \cong \mathrm{C}_m(T)$ and $\mathrm{C}_n(P') \cong \mathrm{C}_n(T')$ are Gorenstein projective with complete projective resolutions

$$\Sigma^{-m} T \to \Sigma^{-m} P_{\geqslant m} \to \mathrm{C}_m(P) \qquad \text{and} \qquad \Sigma^{-n} T' \to \Sigma^{-n} P'_{\geqslant n} \to \mathrm{C}_n(P').$$

Lemma 2.10, Theorem 3.5, and Proposition 3.6 now conspire to yield the desired isomorphism,

$$\begin{aligned}\widehat{\operatorname{Tor}}_i^R(M,N) &\cong \widehat{\operatorname{Tor}}_{i-m-n}^R(\mathrm{C}_m(P),\mathrm{C}_n(P'))\\ &\cong \mathrm{H}_{i-m-n}((\Sigma^{-m}T)\otimes_R^{\bowtie}\Sigma^{-n}T')\\ &\cong \mathrm{H}_{i-n-m}((\Sigma^{-n}T')\otimes_{R^\circ}^{\bowtie}\Sigma^{-m}T)\\ &\cong \widehat{\operatorname{Tor}}_{i-n-m}^{R^\circ}(\mathrm{C}_n(P'),\mathrm{C}_m(P))\\ &\cong \widehat{\operatorname{Tor}}_i^{R^\circ}(N,M).\end{aligned}$$

□

Remark 3.8. In [9] Iacob considers a variation of Tate homology based on complete flat resolutions. The proof of Theorem 3.5 applies, mutatis mutandis, to also show that these homology groups can be computed from a pinched tensor product. From a result parallel to Lemma 2.10 it, therefore, follows that this version of Tate homology is also balanced.

4. Pinched Hom complexes and Tate cohomology

Tate cohomology was studied in detail by Veliche [14]; we recall the definition.

4.1. Let M be an R-complex with a complete projective resolution $T \to P \to M$. For an R-complex N, the *Tate cohomology of M with coefficients in N* is defined as

$$\widehat{\operatorname{Ext}}_R^i(M,N) = \mathrm{H}_{-i}(\operatorname{Hom}_R(T,N)).$$

This definition is independent (up to isomorphism) of the choice of complete resolution; cf. Section 2.2. In particular, one has

$$\widehat{\operatorname{Ext}}_R^i(M,N) \cong \operatorname{Ext}_R^i(M,N) \quad \text{for } i > \operatorname{Gpd}_R M - \inf N. \tag{4.1.1}$$

Note that $\widehat{\operatorname{Ext}}_R^i(M,N)$ is a $\Bbbk$-module for every $i \in \mathbb{Z}$. Moreover, if N is an R–S°-bimodule, then each $\widehat{\operatorname{Ext}}_R^i(M,N)$ is an S°-module.

The parallels of Proposition 2.5, Remark 2.6, Lemma 2.7, Proposition 2.8 and Proposition 2.9 are established in [14, sec. 4]. The proof of Lemma 4.3 is similar to the proof of Lemma 2.10. It uses [6, lem. 2.4] and the following fact, which follows from an argument similar to the one given in Remark 2.6.

Lemma 4.2. *Let M be an R-complex of finite Gorenstein projective dimension. For every bounded below R-complex N of finite injective dimension, one has $\widehat{\operatorname{Ext}}_R^i(M,N) = 0$ for all $i \in \mathbb{Z}$.* □

Lemma 4.3. *Let M be an R-complex of finite Gorenstein projective dimension and let N be an R-complex. For every complete projective resolution $T \to P \to M$ and for every integer $m \in \mathbb{Z}$ there are isomorphisms*

$$\widehat{\operatorname{Ext}}_R^i(M,N) \cong \widehat{\operatorname{Ext}}_R^{i-m}(\mathrm{C}_m(T),N) \quad \textit{for all } i \in \mathbb{Z}. \tag{a}$$

For every semi-injective resolution $N \xrightarrow{\simeq} I$ and for every integer $n \geqslant -\inf N$ there are isomorphisms

$$\widehat{\operatorname{Ext}}_R^i(M,N) \cong \widehat{\operatorname{Ext}}_R^{i-n}(M,\mathrm{Z}_{-n}(I)) \quad \textit{for all } i \in \mathbb{Z}. \tag{b}$$

□

Construction 4.4. Let T and A be R-complexes. Consider the graded $\Bbbk$-module $\mathrm{Hom}_R^{\bowtie}(T,A)$ defined by

$$\mathrm{Hom}_R^{\bowtie}(T,A)_n = \begin{cases} \mathrm{Hom}_R(T_{\leqslant -1}, \Sigma^{-1}(A_{\geqslant 1}))_n & \text{for } n \geqslant 1, \\ \mathrm{Hom}_R(T_{\geqslant 0}, A_{\leqslant 0})_n & \text{for } n \leqslant 0. \end{cases}$$

It is elementary to verify that one has

$$\mathrm{Hom}_R(\partial_0^T, \partial_1^A \varsigma) \circ \partial_2^{\mathrm{Hom}_R(T_{\leqslant -1}, \Sigma^{-1}(A_{\geqslant 1}))} = 0 = \partial_0^{\mathrm{Hom}_R(T_{\geqslant 0}, A_{\leqslant 0})} \circ \mathrm{Hom}_R(\partial_0^T, \partial_1^A \varsigma),$$

where ς denotes the canonical map $\Sigma^{-1}A \to A$. Thus, $\partial_n^{\mathrm{Hom}_R^{\bowtie}(T,A)}$ defined by

$$\partial_n^{\mathrm{Hom}_R^{\bowtie}(T,A)} = \begin{cases} \partial_n^{\mathrm{Hom}_R(T_{\leqslant -1}, \Sigma^{-1}(A_{\geqslant 1}))} & \text{for } n \geqslant 2, \\ \mathrm{Hom}_R(\partial_0^T, \partial_1^A \varsigma) & \text{for } n = 1, \\ \partial_n^{\mathrm{Hom}_R(T_{\geqslant 0}, A_{\leqslant 0})} & \text{for } n \leqslant 0 \end{cases}$$

is a differential on $\mathrm{Hom}_R^{\bowtie}(T,A)$. We refer to this $\Bbbk$-complex as the *pinched Hom* of T and A.

For morphisms $\alpha\colon T \to T'$ and $\beta\colon A \to A'$ of R-complexes it is elementary to verify that the assignment $\varphi \mapsto \beta\varphi\alpha$ defines a morphism of $\Bbbk$-complexes:

$$\mathrm{Hom}_R^{\bowtie}(\alpha,\beta)\colon \mathrm{Hom}_R^{\bowtie}(T',A) \longrightarrow \mathrm{Hom}_R^{\bowtie}(T,A').$$

Remark 4.5. For all R-complexes T and A there are equalities of complexes

$$\mathrm{Hom}_R^{\bowtie}(T,A)_{\geqslant 1} = \mathrm{Hom}_R(T_{\leqslant -1}, \Sigma^{-1}(A_{\geqslant 1})) \quad \text{and} \tag{4.5.1}$$

$$\mathrm{Hom}_R^{\bowtie}(T,A)_{\leqslant 0} = \mathrm{Hom}_R(T_{\geqslant 0}, A_{\leqslant 0}). \tag{4.5.2}$$

If T is a complex of R–R'°-bimodules and A is a complex of R–S°-bimodules, then $\mathrm{Hom}_R^{\bowtie}(T,A)$ is a complex of R'–S°-bimodules.

The proof of the next proposition is standard and omitted.

Proposition 4.6. *The pinched Hom defined in Construction* 4.4 *yields a functor*

$$\mathrm{Hom}_R^{\bowtie}(-,-)\colon \mathsf{C}(R\text{–}R'^{\circ})^{\mathrm{op}} \times \mathsf{C}(R\text{–}S^{\circ}) \longrightarrow \mathsf{C}(R'\text{–}S^{\circ});$$

in particular, it yields a functor $\mathsf{C}(R)^{\mathrm{op}} \times \mathsf{C}(R) \to \mathsf{C}(\Bbbk)$. *Moreover, it is* $\Bbbk$*-bilinear and left exact in each variable.* □

Theorem 4.7. *Let M be an R-complex with a complete projective resolution $T \to P \to M$. Let A be an acyclic R-complex and set $N = \mathrm{Z}_0(A)$. For every $i \in \mathbb{Z}$ there is an isomorphism of $\Bbbk$-modules*

$$\mathrm{H}_{-i}(\mathrm{Hom}_R^{\bowtie}(T,A)) \cong \widehat{\mathrm{Ext}}_R^i(M,N).$$

If A is a complex of R–S°-bimodules, then the isomorphism is one of S°-modules.

Notice that if N is an R-module and A is the acyclic complex $0 \to N \xrightarrow{=} N \to 0$ with N in degrees 1 and 0, then one has $\mathrm{Hom}_R^{\bowtie}(T,A) = \mathrm{Hom}_R(T,N)$.

Proof. The quasi-isomorphisms

$$\pi\colon \Sigma^{-1}(A_{\geqslant 1}) \xrightarrow{\simeq} N \qquad \text{and} \qquad \epsilon\colon N \xrightarrow{\simeq} A_{\leqslant 0},$$

with $\epsilon_0\pi_0 = \partial_1^A\varsigma$, yield quasi-isomorphisms

$$\mathrm{Hom}_R^{\bowtie}(T,A)_{\geqslant 1} \xrightarrow{\simeq} \mathrm{Hom}_R(T_{\leqslant -1}, N) \quad \text{and} \quad \mathrm{Hom}_R(T_{\geqslant 0}, N) \xrightarrow{\simeq} \mathrm{Hom}_R^{\bowtie}(T,A)_{\leqslant 0};$$

see (4.5.1), (4.5.2), and [6, prop. 2.6]. It follows that there are isomorphisms $\mathrm{H}_i(\mathrm{Hom}_R^{\bowtie}(T,A)) \cong \mathrm{H}_i(\mathrm{Hom}_R(T,N))$ for all $i \in \mathbb{Z} \setminus \{1,0\}$. To establish the desired isomorphism for $i \in \{0,1\}$, consider the following diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathrm{Hom}_R(T_{-1}, \mathrm{Z}_1(A)) \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_R^{\bowtie}(T,A)_2 \arrow[r] \arrow[d, "\mathrm{Hom}(T_{-2},\pi_0)"'] & \mathrm{Hom}_R^{\bowtie}(T,A)_1 \arrow[d, "\mathrm{Hom}(T_{-1},\pi_0)"'] \arrow[dr] & \mathrm{Hom}_R(T,N)_0 \arrow[r] \arrow[d, "\mathrm{Hom}(T_0,\epsilon_0)"] & \mathrm{Hom}_R(T,N)_{-1} \arrow[d, "\mathrm{Hom}(T_1,\epsilon_0)"] \\
\mathrm{Hom}_R(T,N)_2 \arrow[r] \arrow[d] & \mathrm{Hom}_R(T,N)_1 \arrow[d] \arrow[ur] & \mathrm{Hom}_R^{\bowtie}(T,A)_0 \arrow[r] \arrow[d, "\mathrm{Hom}_R(T_0,\partial_0^A)"] & \mathrm{Hom}_R^{\bowtie}(T,A)_{-1} \\
0 & 0 & \mathrm{Hom}_R(T_0, \mathrm{B}_{-1}(A)) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

The identity $\epsilon_0\pi_0 = \partial_1^A\varsigma$ ensures that the twisted square is commutative; also the other two squares are commutative by functoriality of the Hom functor.

To see that $\mathrm{Hom}_R(T_{-1},\pi_0)$ induces an isomorphism from $\mathrm{H}_1(\mathrm{Hom}_R^{\bowtie}(T,A))$ to $\mathrm{H}_1(\mathrm{Hom}_R(T,N))$, notice first that it maps boundaries to boundaries by commutativity of the left-hand square. For a cycle ζ in $\mathrm{Z}_1(\mathrm{Hom}_R^{\bowtie}(T,A))$ one has

$$\big(\mathrm{Hom}_R(T_0,\epsilon_0) \circ \partial_1^{\mathrm{Hom}_R(T,N)} \circ \mathrm{Hom}_R(T_{-1},\pi_0)\big)(\zeta) = 0,$$

by commutativity of the twisted square. As $\mathrm{Hom}_R(T_0,\epsilon_0)$ is injective, it follows that $\mathrm{Hom}_R(T_{-1},\pi_0)(\zeta)$ is in $\mathrm{Z}_1(\mathrm{Hom}_R(T,N))$, so there is a well-defined homomorphism

$$\mathrm{H}(\mathrm{Hom}_R(T_{-1},\pi_0)\colon\ \mathrm{H}_1(\mathrm{Hom}_R^{\bowtie}(T,A)) \to \mathrm{H}_1(\mathrm{Hom}_R(T,N)).$$

It is immediate by surjectivity of $\mathrm{Hom}_R(T_{-1},\pi_0)$ and commutativity of the twisted square that $\mathrm{H}(\mathrm{Hom}_R(T_{-1},\pi_0))$ is surjective. To see that it is also injective, let ζ be a cycle in $\mathrm{Hom}_R^{\bowtie}(T,A)_1$ and assume that one has $\mathrm{Hom}_R(T_{-1},\pi_0)(\zeta) = \partial_2^{\mathrm{Hom}_R(T,N)}(\alpha)$ for some element α in $\mathrm{Hom}_R(T,N)_2 = \mathrm{Hom}_R(T_{-2},N)$. For any preimage ξ of α in $\mathrm{Hom}_R(T_{-2},A_1) \subset \mathrm{Hom}_R^{\bowtie}(T,A)_2$, the element $\zeta - \partial_2^{\mathrm{Hom}_R^{\bowtie}(T,A)}(\xi)$ in $\mathrm{Hom}_R^{\bowtie}(T,A)_1$ maps to 0 under $\mathrm{Hom}_R(T_{-1},\pi_0)$, so it belongs to $\mathrm{Hom}_R(T_{-1},\mathrm{Z}_1(A))$. As T_{-1} is projective and A is acyclic, there exists a homomorphism ψ in $\mathrm{Hom}_R(T_{-1},A_2) \subset \mathrm{Hom}_R^{\bowtie}(T,A)_2$ such that one has

$$\zeta - \partial_2^{\mathrm{Hom}_R^{\bowtie}(T,A)}(\xi) = \partial_2^A\psi = -\partial_2^{\mathrm{Hom}_R^{\bowtie}(T,A)}(\psi).$$

It follows that ζ is a boundary, $\zeta = \partial_2^{\mathrm{Hom}_R^{\bowtie}(T,A)}(\xi - \psi)$, whence $\mathrm{H}(\mathrm{Hom}_R(T_{-1},\pi_0))$ is an isomorphism.

From the commutativity of the right-hand square, it follows that $\mathrm{Hom}_R(T_0,\epsilon_0)$ maps cycles to cycles. Let β be a boundary in $\mathrm{Hom}_R(T,N)_0$ and choose a preimage α of β in $\mathrm{Hom}_R(T,N)_1$. By surjectivity of $\mathrm{Hom}_R(T_{-1},\pi_0)$ this α has a preimage α' in $\mathrm{Hom}_R^{\bowtie}(T,A)_1$, and by commutativity of the twisted square one has $\mathrm{Hom}_R(T_0,\epsilon_0)(\beta) = \partial_1^{\mathrm{Hom}_R^{\bowtie}(T,A)}(\alpha')$. Thus, $\mathrm{Hom}_R(T_0,\epsilon_0)$ maps boundaries to boundaries, whence it induces a homomorphism

$$\mathrm{H}(\mathrm{Hom}_R(T_0,\epsilon_0))\colon\ \mathrm{H}_0(\mathrm{Hom}_R(T,N)) \to \mathrm{H}_0(\mathrm{Hom}_R^{\bowtie}(T,A)).$$

It follows immediately from injectivity of $\mathrm{Hom}_R(T_0,\epsilon_0)$ and commutativity of the twisted square that $\mathrm{H}(\mathrm{Hom}_R(T_0,\epsilon_0))$ is injective. To see that it is surjective, let ζ be a cycle in $\mathrm{Hom}_R^{\bowtie}(T,A)_0$; one then has $0=\partial_0^{A_{\leqslant 0}}\zeta=\mathrm{Hom}_R(T_0,\partial_0^A)(\zeta)$. By exactness of the second column from the right, it now follows that ζ is in the image of $\mathrm{Hom}_R(T_0,\epsilon_0)$, and by injectivity of $\mathrm{Hom}_R(T_1,\epsilon_0)$ it follows that the preimage of ζ is a cycle in $\mathrm{Hom}_R(T,N)_0$. Thus, $\mathrm{H}(\mathrm{Hom}_R(T_0,\epsilon_0))$ is an isomorphism.

The claim about S°-module structures is immediate from Construction 4.4. $\square$

The next result is a pinched version of Hom-tensor adjunction.

Proposition 4.8. *Let T be an R-complex, let A be a complex of S–R°-bimodules, and let B be an S-module. The map*

$$\varrho\colon\ \mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)\ \longrightarrow\ \mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))$$

given by

$$\varrho_n(\psi)(t)(a)=\begin{cases}\psi(t\otimes\sigma(a)) & \text{for } n\geqslant 1,\\ \psi(t\otimes a) & \text{for } n\leqslant 0\end{cases}$$

is an isomorphism of $\Bbbk$-complexes.

Moreover, if T is a complex of R–R'°-bimodules and B is an S–S'°-bimodule, then ϱ is an isomorphism of complexes of R'–S'°-bimodules.

Proof. For $n\geqslant 1$ one has

$$\begin{aligned}\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)_n&=\mathrm{Hom}_S((T\otimes_{R^\circ}^{\bowtie}A)_{-n},B)\\&=\mathrm{Hom}_S(\textstyle\bigoplus_{i=-n}^{-1}T_i\otimes_{R^\circ}(\Sigma A)_{-n-i},B)\end{aligned}$$

and

$$\begin{aligned}\textstyle\bigoplus_{i=-n}^{-1}\mathrm{Hom}_R(T_i,\mathrm{Hom}_S(A_{-n-i-1},B))&=\textstyle\bigoplus_{i=-n}^{-1}\mathrm{Hom}_R(T_i,\mathrm{Hom}_S(A,B)_{i+n+1})\\&=\textstyle\bigoplus_{i=-n}^{-1}\mathrm{Hom}_R(T_i,(\Sigma^{-1}\mathrm{Hom}_S(A,B))_{i+n})\\&=\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))_n.\end{aligned}$$

The map ϱ_n given by $\varrho_n(\psi)(t)(a)=\psi(t\otimes\sigma(a))$, for $t\in T_i$ and $a\in A_{-n-i-1}$, is, up to σ, just the Hom-tensor adjunction isomorphism of modules. Thus, ϱ_n is an isomorphism of $\Bbbk$-modules. Moreover, still for $n\geqslant 1$, one has

$$\begin{aligned}&\varrho_n\big(\partial_{n+1}^{\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)}(\psi)\big)(t)(a)\\&\qquad=\varrho_n\big(-(-1)^{n+1}\psi\partial_{-n}^{T\otimes_{R^\circ}^{\bowtie}A}\big)(t)(a)\\&\qquad=(-1)^n\psi\partial_{-n}^{T\otimes_{R^\circ}^{\bowtie}A}(t\otimes\sigma(a))\\&\qquad=(-1)^n\psi\big(\partial^T(t)\otimes\sigma(a)+(-1)^{|t|}t\otimes\partial^{\Sigma A}(\sigma(a))\big)\\&\qquad=(-1)^n\psi\big(\partial^T(t)\otimes\sigma(a)-(-1)^{|t|}t\otimes\sigma(\partial^A(a))\big)\\&\qquad=(-1)^{n+|t|+1}\psi(t\otimes\sigma(\partial^A(a)))+(-1)^n\psi(\partial^T(t)\otimes\sigma(a))\end{aligned}$$

and

$$\begin{aligned}\big(\partial_{n+1}^{\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))}&\varrho_{n+1}(\psi)\big)(t)(a)\\ &= \big(\partial^{\Sigma^{-1}\mathrm{Hom}_S(A,B)}\varrho_{n+1}(\psi) - (-1)^{n+1}\varrho_{n+1}(\psi)\partial^T\big)(t)(a)\\ &= (-1)^{|\varrho_{n+1}(\psi)(t)|}\varrho_{n+1}(\psi)(t)(\partial^A(a)) - (-1)^{n+1}\varrho_{n+1}(\psi)(\partial^T(t))(a)\\ &= (-1)^{n+1+|t|}\psi(t\otimes\sigma(\partial^A(a))) + (-1)^n\psi(\partial^T(t)\otimes\sigma(a)).\end{aligned}$$

That is, the identity $\partial_{n+1}^{\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))}\varrho_{n+1} = \varrho_n\partial_{n+1}^{\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)}$ holds for $n \geqslant 1$.

By (3.3.1) and (4.5.1) there are equalities of $\Bbbk$-complexes

$$\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)_{\leqslant 0} = \mathrm{Hom}_S(T_{\geqslant 0}\otimes_{R^\circ}A_{\geqslant 0},B) \quad\text{and}$$
$$\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))_{\leqslant 0} = \mathrm{Hom}_R(T_{\geqslant 0},\mathrm{Hom}_S(A_{\geqslant 0},B)).$$

Thus, for $n \leqslant 0$, the map ϱ_n is the degree n component of the Hom-tensor adjunction isomorphism $\mathrm{Hom}_S(T_{\geqslant 0}\otimes_{R^\circ}A_{\geqslant 0},B) \xrightarrow{\cong} \mathrm{Hom}_R(T_{\geqslant 0},\mathrm{Hom}_S(A_{\geqslant 0},B))$.

To prove that ϱ is an isomorphism of $\Bbbk$-complexes, it remains to verify the identity $\partial_1^{\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))}\varrho_1 = \varrho_0\partial_1^{\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)}$. For $t\in T_0$ and $a\in A_0$ one has

$$\begin{aligned}\varrho_0\big(\partial_1^{\mathrm{Hom}_S(T\otimes_{R^\circ}^{\bowtie}A,B)}(\psi)\big)(t)(a) &= \varrho_0\big(\psi\partial_0^{T\otimes_{R^\circ}^{\bowtie}A}\big)(t)(a)\\ &= \psi(\partial^T(t)\otimes\sigma\partial^A(a))\end{aligned}$$

and

$$\begin{aligned}\big(\partial_1^{\mathrm{Hom}_R^{\bowtie}(T,\mathrm{Hom}_S(A,B))}\varrho_1(\psi)\big)(t)(a) &= \big(\partial_1^{\mathrm{Hom}_S(A,B)}\varsigma\varrho_1(\psi)(\partial^T(t))\big)(a)\\ &= \varrho_1(\psi)(\partial^T(t))(\partial^A(a))\\ &= \psi(\partial^T(t)\otimes\sigma\partial^A(a)),\end{aligned}$$

as required. Finally, the statements about extra module structures are evident in view of Remarks 3.3 and 4.5. □

Proposition 4.9. *Assume that R is commutative. Let M be an R-complex with a complete projective resolution $T\to P\to M$ and let N be a Gorenstein projective R-module with complete projective resolution T'. For every projective R-module Q and every $i\in\mathbb{Z}$ there is an isomorphism of R-modules*

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(T\otimes_R^{\bowtie}T',Q)) \cong \widehat{\mathrm{Ext}}_R^i(M,\mathrm{Hom}_R(N,Q)).$$

Proof. The R-complex $\mathrm{Hom}_R(T',Q)$ is acyclic, and $\mathrm{Hom}_R(N,Q)$ is the kernel of the differential in degree 0. The assertion now follows from Proposition 4.8 and Theorem 4.7. □

Corollary 4.10. *Assume that R is commutative. Let M and N be Gorenstein projective R-modules with complete projective resolutions T and T', respectively. If one has $\widehat{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\in\mathbb{Z}$, then the complex $T\otimes_R^{\bowtie}T'$ of projective R-modules is acyclic, and the following conditions are equivalent:*

(i) The R-complex $T\otimes_R^{\bowtie}T'$ is totally acyclic.

(ii) For every projective R-module Q one has $\widehat{\mathrm{Ext}}_R^i(M,\mathrm{Hom}_R(N,Q))=0$ for all $i\in\mathbb{Z}$.

When these conditions hold, the R-module $M\otimes_R N$ is Gorenstein projective with complete projective resolution $T\otimes_R^{\bowtie}T'$.

Proof. By construction, the complex $T \otimes_R^{\bowtie} T'$ consists of projective R-modules, and one has $\mathrm{C}_0(T \otimes_R^{\bowtie} T') \cong M \otimes_R N$. The assumption that the Tate homology $\widehat{\mathrm{Tor}}_*^R(M,N)$ vanishes implies that $T \otimes_R^{\bowtie} T'$ is acyclic; see Theorem 3.5. The equivalence of (i) and (ii) now follows from Proposition 4.9, and the last assertion is then evident. □

5. Tate cohomology is balanced

For R-modules M and N, a potentially different approach to Tate cohomology $\widehat{\mathrm{Ext}}_R^*(M,N)$ uses a resolution of the second argument N. The resulting theory, which is parallel to the one developed in [3, 5, 14], was outlined by Asadollahi and Salarian in [1]. In this section we use the pinched complexes to show that when both approaches apply, they yield the same cohomology theory.

5.1. **Complete injective resolutions.** A complex U of injective R-modules is called *totally acyclic* if it is acyclic, and the complex $\mathrm{Hom}_R(J,U)$ is acyclic for every injective R-module J.

A *complete injective resolution* of an R-complex N is a diagram

$$N \xrightarrow{\iota} I \xrightarrow{\upsilon} U, \tag{5.1.1}$$

where ι is a semi-injective resolution, U is a totally acyclic complex of injective R-modules, and υ_i is an isomorphism for $i \ll 0$.

5.2. **Gorenstein injectivity.** An R-module E is called *Gorenstein injective* if there exists a totally acyclic complex U of injective R-modules with $\mathrm{Z}_0(U) \cong E$. In that case, the diagram $E \to U_{\leqslant 0} \to U$ is a complete injective resolution, and for brevity we shall often say that U is a complete injective resolution of E.

The *Gorenstein injective dimension* of an R-complex N, written $\mathrm{Gid}_R N$, is the least integer n such that there exists a complete injective resolution (5.1.1) where υ_i is an isomorphism for all $i \leqslant -n$. In particular, $\mathrm{Gid}_R N$ is finite if and only if N has a complete injective resolution. Notice that $\mathrm{H}(N)$ is bounded below if $\mathrm{Gid}_R N$ is finite; indeed, there is an inequality

$$\mathrm{Gid}_R N \geqslant -\inf \mathrm{H}(N). \tag{5.2.1}$$

If N is an R-complex of finite injective dimension, then there is a semi-injective resolution $N \xrightarrow{\simeq} I$ with I bounded below, and then $N \to I \to 0$ is a complete injective resolution. In particular, N has finite Gorenstein injective dimension.

Proposition 5.3. *Let N be an R-complex with a complete injective resolution $N \to I \to U$. Let A be an acyclic R-complex and set $M = \mathrm{C}_0(A)$. For every $i \in \mathbb{Z}$ there is an isomorphism of $\Bbbk$-modules*

$$\mathrm{H}_i(\mathrm{Hom}_R^{\bowtie}(A,U)) \cong \mathrm{H}_i(\mathrm{Hom}_R(M,U)).$$

If A is a complex of R–S°-bimodules, then the isomorphism is one of S-modules.

Notice that if M is an R-module and A is the acyclic complex $0 \to M \xrightarrow{=} M \to 0$ with M in degrees 0 and -1, then one has $\mathrm{Hom}_R^{\bowtie}(A,U) \cong \mathrm{Hom}_R(M,U)$.

Proof. The quasi-isomorphisms

$$\pi\colon A_{\geqslant 0} \xrightarrow{\simeq} M \qquad \text{and} \qquad \epsilon\colon M \xrightarrow{\simeq} \Sigma(A_{\leqslant -1}),$$

with $\epsilon_0\pi_0 = \sigma\partial_0^A$, induce quasi-isomorphisms

$$\mathrm{Hom}_R(M, U_{\leqslant 0}) \xrightarrow{\simeq} \mathrm{Hom}_R^{\bowtie}(A, U)_{\leqslant 0} \quad \text{and}$$
$$\mathrm{Hom}_R(\Sigma A_{\leqslant -1}, U_{\geqslant 1}) \xrightarrow{\simeq} \mathrm{Hom}_R(M, U_{\geqslant 1});$$

see (4.5.1) and [6, prop. 2.7]. There is an equality of graded $\Bbbk$-modules

$$\mathrm{Hom}_R(\Sigma A_{\leqslant -1}, U_{\geqslant 1}) = \mathrm{Hom}_R^{\bowtie}(A, U)_{\geqslant 1},$$

and one has $-\partial^{\mathrm{Hom}_R(\Sigma A_{\leqslant -1}, U_{\geqslant 1})} = \partial^{\mathrm{Hom}_R^{\bowtie}(A,U)_{\geqslant 1}}$. It follows that there are isomorphisms $\mathrm{H}_i(\mathrm{Hom}_R^{\bowtie}(A, U)) \cong \mathrm{H}_i(\mathrm{Hom}_R(M, U))$ for $i \neq 1, 0$. To establish the desired isomorphism in the remaining two degrees, consider the following diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathrm{Hom}_R(\mathrm{B}_{-2}(A), U_1) \arrow[d, "\mathrm{Hom}(\partial^A_{-1}, U_1)"'] & 0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_R^{\bowtie}(A, U)_2 \arrow[r] \arrow[d, "-\mathrm{Hom}(\epsilon_0, U_2)"'] & \mathrm{Hom}_R^{\bowtie}(A, U)_1 \arrow[d, "\mathrm{Hom}(\epsilon_0, U_1)"'] \arrow[dr] & \mathrm{Hom}_R(M, U)_0 \arrow[r] \arrow[d, "\mathrm{Hom}(\pi_0, U_0)"] & \mathrm{Hom}_R(M, U)_{-1} \arrow[d, "\mathrm{Hom}(\pi_0, U_{-1})"] \\
\mathrm{Hom}_R(M, U)_2 \arrow[r] \arrow[d] & \mathrm{Hom}_R(M, U)_1 \arrow[d] \arrow[ur] & \mathrm{Hom}_R^{\bowtie}(A, U)_0 \arrow[r] \arrow[d] & \mathrm{Hom}_R^{\bowtie}(A, U)_{-1} \\
0 & 0 & \mathrm{Hom}_R(\mathrm{B}_0(A), U_0) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

The identity $\epsilon_0\pi_0 = \sigma\partial_0^A$ ensures that the twisted square is commutative; also, the other two squares are commutative by standard properties of the Hom functor.

To see that $\mathrm{Hom}_R(\epsilon_0, U_1)$ and $\mathrm{Hom}_R(\pi_0, U_0)$ induce isomorphisms in homology, one proceeds as in the proof of Theorem 4.7. □

If M is a Gorenstein projective R-module with complete projective resolution T and if N is a Gorenstein injective R-module with complete injective resolution U, then Theorem 4.7 and Proposition 5.3 yield

$$\widehat{\mathrm{Ext}}_R^i(M, N) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R^{\bowtie}(T, U)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M, U)).$$

That is, the Tate cohomology of M with coefficients in N can be computed via a complete injective resolution of N. What follows is a balancedness statement that shows that for appropriately bounded complexes—for modules in particular—one can unambiguously extend the notion of Tate cohomology $\widehat{\mathrm{Ext}}_R^*(M, N)$ to the situation where N has a complete injective resolution; see Definition 5.5.

Theorem 5.4. *Let M be a bounded above R-complex with a complete projective resolution and let N be a bounded below R-complex with a complete injective resolution $N \to I \to U$. For every $i \in \mathbb{Z}$ there is an isomorphism*

$$\widehat{\mathrm{Ext}}_R^i(M, N) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M, U)).$$

Proof. Set $n = \sup\{-\inf N, \mathrm{Gid}_R N\}$. Then the module $\mathrm{Z}_{-n}(I) \cong \mathrm{Z}_{-n}(U)$ is Gorenstein injective with complete injective resolution $\mathrm{Z}_{-n}(I) \to \Sigma^n I_{\leqslant -n} \to \Sigma^n U$.

Further, set $m = \operatorname{Gpd}_R M$ and let $T \to P \to M$ be a complete projective resolution. Then the module $\mathrm{C}_m(P) \cong \mathrm{C}_m(T)$ is Gorenstein projective with complete projective resolution $\Sigma^{-m}T \to \Sigma^{-m}P_{\geqslant m} \to \mathrm{C}_m(P)$. In the next chain of isomorphisms, the first one follows from Lemma 4.3, the second and third follow from Theorem 4.7 and Proposition 5.3, and the last one follows by dimension shifting:

$$\begin{aligned}\widehat{\operatorname{Ext}}^i_R(M,N) &\cong \widehat{\operatorname{Ext}}^{i-m-n}_R(\mathrm{C}_m(P), \mathrm{Z}_{-n}(I))\\ &\cong \mathrm{H}_{m+n-i}(\operatorname{Hom}^{\bowtie}_R(\Sigma^{-m}T, \Sigma^n U))\\ &\cong \mathrm{H}_{m+n-i}(\operatorname{Hom}_R(\mathrm{C}_m(P), \Sigma^n U))\\ &\cong \mathrm{H}_{m-i}(\operatorname{Hom}_R(\mathrm{C}_m(P), U)).\end{aligned}$$

Finally, an argument parallel to the one for Lemma 2.10(b) yields isomorphisms

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(M,U)) \cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P_{\subset m}, U)) \cong \mathrm{H}_{m-i}(\operatorname{Hom}_R(\mathrm{C}_m(P), U)).$$

This time it is [6, lem. 2.5] that needs to be invoked. □

Definition 5.5. Let N be a bounded below R-complex with a complete injective resolution $N \to I \to U$. For every bounded above R-complex M, the *Tate cohomology of M with coefficients in N* is given by

$$\widehat{\operatorname{Ext}}^i_R(M,N) = \mathrm{H}_{-i}(\operatorname{Hom}_R(M,U)).$$

Remark 5.6. A fact parallel to Section 2.2 guarantees that the definition above is independent (up to isomorphism) of the choice of complete resolution. In particular, one has the following parallel of (4.1.1):

$$\widehat{\operatorname{Ext}}^i_R(M,N) \cong \operatorname{Ext}^i_R(M,N) \quad \text{for } i > \operatorname{Gid}_R N + \sup M. \tag{5.6.1}$$

Other standard results similar to the information from Proposition 2.5 through Proposition 2.9 in this paper are established in [1]. In that paper, the notation $\widehat{\operatorname{ext}}^*_R(M,N)$ is used for the cohomology defined in Definition 5.5, and it is shown to agree with the notion from [3, 5, 14] (see Section 4.1) over commutative Noetherian local Gorenstein rings.

More generally, for a module N with a complete injective resolution, Nucinkis' [12] notion of I-complete cohomology agrees with Tate cohomology as defined in Definition 5.5. Similarly, for a module M with a complete projective resolution, the P-complete cohomology of Benson and Carlson [4], Vogel/Goichot [8], and Mislin [11] agrees with Tate cohomology in the sense of Section 4.1. Nucinkis proves [12, thm. 5.2, 6.6, 7.9] that P- and I-complete cohomology agree over rings where every module has a complete projective resolution and a complete injective resolution.

The next result establishes a pinched version of the Hom swap isomorphism. It is proved in the same fashion as Proposition 4.8.

Proposition 5.7. *Let T be an R-complex, let B be an S°-module, and let U be a complex of R–S°-bimodules. The map*

$$\vartheta\colon \operatorname{Hom}_{S^\circ}(B, \operatorname{Hom}^{\bowtie}_R(T,U)) \longrightarrow \operatorname{Hom}^{\bowtie}_R(T, \operatorname{Hom}_{S^\circ}(B,U))$$

given by

$$\vartheta_n(\psi)(t)(b) = \psi(b)(t)$$

is an isomorphism of $\Bbbk$-complexes.

Moreover, if T is a complex of R–R'°-bimodules and if B is an S'–S°-bimodule, then ϑ is an isomorphism of complexes of R'–S'°-bimodules. □

Proposition 5.8. *Assume that R is commutative. Let M be an R-complex with a complete projective resolution $T \to P \to M$ and let N be a Gorenstein injective R-module with complete injective resolution U. For every injective R-module J and every $i \in \mathbb{Z}$ there is an isomorphism of R-modules*

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(J, \mathrm{Hom}_R^{\bowtie}(T, U))) \cong \widehat{\mathrm{Ext}}_R^i(M, \mathrm{Hom}_R(J, N)).$$

Proof. The complex $\mathrm{Hom}_R(J, U)$ is acyclic, and $\mathrm{Hom}_R(J, N)$ is the kernel of the differential in degree 0. The assertion now follows from Proposition 5.7 and Theorem 4.7. □

Corollary 5.9. *Assume that R is commutative. Let M be a Gorenstein projective R-module with complete projective resolution T and let N be a Gorenstein injective R-module with complete injective resolution U. If one has $\widehat{\mathrm{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$, then the complex $\mathrm{Hom}_R^{\bowtie}(T, U)$ of injective R-modules is acyclic and the following conditions are equivalent:*

(i) The R-complex $\mathrm{Hom}_R^{\bowtie}(T, U)$ is totally acyclic.

(ii) For every injective R-module J one has $\widehat{\mathrm{Ext}}_R^i(M, \mathrm{Hom}_R(J, N)) = 0$ for all $i \in \mathbb{Z}$.

When these conditions hold, the R-module $\mathrm{Hom}_R(M, N)$ is Gorenstein injective with complete injective resolution $\mathrm{Hom}_R^{\bowtie}(T, U)$.

Proof. By construction the complex $\mathrm{Hom}_R^{\bowtie}(T, U)$ consists of injective R-modules, and one has $\mathrm{Z}_0(\mathrm{Hom}_R^{\bowtie}(T, U)) \cong \mathrm{Hom}_R(M, N)$. The assumption that the Tate cohomology $\widehat{\mathrm{Ext}}_R^*(M, N)$ vanishes implies that $\mathrm{Hom}_R^{\bowtie}(T, U)$ is acyclic; see Theorem 4.7. The equivalence of (i) and (ii) now follows from Proposition 5.8, and the last assertion is then evident. □

6. Local algebra

Throughout this section R denotes a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Recall that every projective R-module is free. An acyclic complex T of finitely generated free R-modules is totally acyclic if and only if $\mathrm{Hom}_R(T, R)$ is acyclic. For an R-module M we use the standard notation M^* for the dual module $\mathrm{Hom}_R(M, R)$. A finitely generated R-module G is Gorenstein projective if and only if one has

$$G \cong G^{**} \quad \text{and} \quad \mathrm{Ext}_R^i(G, R) = 0 = \mathrm{Ext}_R^i(G^*, R) \ \text{ for all } i \geqslant 1$$

(see [3]), and following op. cit. we use the term *totally reflexive* for such modules.

A complex F of finitely generated free R-modules is called *minimal* if one has $\partial(F) \subseteq \mathfrak{m}F$; see [3, sec. 8]. A complete projective resolution $T \to P \to M$ is called minimal if T and P are minimal complexes of finitely generated free R-modules. By [3, thm. 8.4] every finitely generated R-module M of finite Gorenstein projective dimension has a minimal complete projective resolution $T \to P \to M$, and it is unique up to isomorphism. The invariants $\widehat{\beta}_n(M) = \mathrm{rank}_R T_n$ are called the *stable Betti numbers* of M; for $n \geqslant \mathrm{Gpd}_R M$ they agree with usual Betti numbers.

Theorem 6.1. *Let M and N be totally reflexive R-modules with complete projective resolutions T and T', respectively. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all*

$i \in \mathbb{Z}$, then the complex $T \otimes_R^{\bowtie} T'$ of finitely generated free R-modules is acyclic with $\mathrm{C}_0(T \otimes_R^{\bowtie} T') \cong M \otimes_R N$, and the following conditions are equivalent:

(i) The R-complex $T \otimes_R^{\bowtie} T'$ is totally acyclic.
(ii) One has $\widehat{\mathrm{Ext}}_R^i(M, N^*) = 0$ for all $i \in \mathbb{Z}$.
(iii) The R-module $M \otimes_R N$ is totally reflexive.

When these conditions hold, $T \otimes_R^{\bowtie} T'$ is a complete projective resolution of $M \otimes_R N$. It is minimal if and only if T and T' are minimal; in particular, one has

$$\widehat{\beta}_i(M \otimes_R N) = \begin{cases} \sum_{0 \leqslant j \leqslant i} \widehat{\beta}_j(M)\widehat{\beta}_{i-j}(N), & \text{for } i \geqslant 0, \\ \sum_{i \leqslant j < 0} \widehat{\beta}_j(M)\widehat{\beta}_{i-j-1}(N), & \text{for } i < 0 \end{cases}$$

Proof. By Construction 3.2 the complex $T \otimes_R^{\bowtie} T'$ consists of finitely generated free R-modules, and the assumption that Tate homology $\widehat{\mathrm{Tor}}_*^R(M, N)$ vanishes implies that $T \otimes_R^{\bowtie} T'$ is acyclic; see Theorem 3.5. To prove equivalence of the three conditions it suffices, in view of Corollary 4.10, to prove the implication $(iii) \Longrightarrow (i)$. Assume that $\mathrm{C}_0(T \otimes_R^{\bowtie} T') = M \otimes_R N$ is totally reflexive. It follows immediately that the syzygies of $M \otimes_R N$, i.e. $\mathrm{C}_i(T \otimes_R^{\bowtie} T')$ for $i \geqslant 1$ are totally reflexive as well. For $i \leqslant -1$ it follows that $\mathrm{C}_i(T \otimes_R^{\bowtie} T')$ has finite Gorenstein projective dimension. The Krull dimension d of R is an upper bound for the Gorenstein projective dimension of any R-module, so $\mathrm{C}_i(T \otimes_R^{\bowtie} T')$ is totally reflexive as it is the dth syzygy of $\mathrm{C}_{i-d}(T \otimes_R^{\bowtie} T')$; see [6, thm. 3.1]. Thus, each module $\mathrm{C}_i(T \otimes_R^{\bowtie} T')$ is totally reflexive, and then $T \otimes_R^{\bowtie} T'$ is totally acyclic by [3, lem. 2.4].

The assertions about minimality follow immediately from Construction 3.2, and so does the equality of stable Betti numbers. □

Corollary 6.2. *Let R be Gorenstein and let M and N be totally reflexive R-modules with (minimal) complete projective resolutions T and T', respectively. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$, then $M \otimes_R N$ is totally reflexive with (minimal) complete resolution $T \otimes_R^{\bowtie} T'$.*

Proof. As R is Gorenstein, every acyclic complex of projective modules is totally acyclic; see [3, lem. 2.4]. □

For modules M and N of finite Gorenstein projective dimension, vanishing of Tate homology $\widehat{\mathrm{Tor}}_*^R(M, N)$ yields information about the complex $M \otimes_R^{\mathbf{L}} N$ that encodes the absolute homology $\mathrm{Tor}_*^R(M, N)$. We pursue this line of investigation in [7]. We close this paper with an interpretation of the Tate homology modules $\widehat{\mathrm{Tor}}_0^R(M, N)$ and $\widehat{\mathrm{Tor}}_{-1}^R(M, N)$ in terms of a natural homomorphism.

Proposition 6.3. *Let M and N be finitely generated R-modules. If M is totally reflexive, then there is an exact sequence of R-modules,*

$$0 \to \widehat{\mathrm{Tor}}_0^R(M, N) \to M \otimes_R N \xrightarrow{\theta_{MN}} \mathrm{Hom}_R(M^*, N) \to \widehat{\mathrm{Tor}}_{-1}^R(M, N) \to 0,$$

where θ_{MN} is the natural homomorphism given by $x \otimes y \longmapsto [\varphi \mapsto \varphi(x)y]$.

Proof. Let $T \to P \to M$ be a minimal complete projective resolution. The natural map $\theta_{FN} \colon F \otimes_R N \to \mathrm{Hom}_R(F^*, N)$ is an isomorphism for every finitely generated free R-module F. Let $\widetilde{\theta} \colon T_0 \otimes_R N \to \mathrm{Hom}_R(T_{-1}^*, N)$ be the homomorphism given

by $t \otimes y \longmapsto [\varphi \mapsto \varphi(\partial^T(t))y]$. Then the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & T_1 \otimes_R N \arrow[r] \arrow[d, "="'] & T_0 \otimes_R N \arrow[r] \arrow[d, "="'] & T_{-1} \otimes_R N \arrow[r] \arrow[d, "\cong"', "\theta"] & T_{-2} \otimes_R N \arrow[r] \arrow[d, "\cong"', "\theta"] & \cdots \\
\cdots \arrow[r] & T_1 \otimes_R N \arrow[r] & T_0 \otimes_R N \arrow[r, "\widetilde{\theta}"] & \operatorname{Hom}_R(T^*_{-1}, N) \arrow[r] & \operatorname{Hom}_R(T^*_{-2}, N) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, Tate homology $\widehat{\operatorname{Tor}}^R_*(M, N)$ can be computed from the bottom complex. Let $\pi\colon T_0 \to M$ and $\epsilon\colon M \to T_{-1}$ be the natural homomorphisms with $\epsilon\pi = \partial^T_0$, and consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_1 \otimes_R N \arrow[r] \arrow[d] & T_0 \otimes_R N \arrow[r, "\widetilde{\theta}"] \arrow[d, "\pi\otimes N"] & \operatorname{Hom}_R(T^*_{-1}, N) \arrow[r] & \operatorname{Hom}_R(T^*_{-2}, N) \arrow[r] & \cdots \\
& 0 \arrow[r] & M \otimes_R N \arrow[r, "\theta"] \arrow[d] & \operatorname{Hom}_R(M^*, N) \arrow[r] \arrow[u, "\operatorname{Hom}(\operatorname{Hom}(\epsilon,R),N)"'] & 0 \arrow[u] & \\
& & 0 & 0 \arrow[u] & &
\end{tikzcd}$$

A straightforward diagram chase shows that the homomorphisms $\pi \otimes_R N$ and $\operatorname{Hom}_R(\operatorname{Hom}_R(\epsilon, R), N)$ induce isomorphisms in homology. □

The next statement is proved similarly; see [3, lem. 5.8.(3)].

Proposition 6.4. *Let M and N be finitely generated R-modules. If M is totally reflexive, then there is an exact sequence of R-modules,*

$$0 \to \widehat{\operatorname{Ext}}^{-1}_R(M, N) \to M^* \otimes_R N \xrightarrow{\nu_{MN}} \operatorname{Hom}_R(M, N) \to \widehat{\operatorname{Ext}}^0_R(M, N) \to 0,$$

where ν_{MN} is the natural homomorphism given by $\varphi \otimes y \longmapsto [x \mapsto \varphi(x)y]$. □

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