TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 2, February 2014, Pages 611–625
S 0002-9947(2013)05747-9
Article electronically published on July 26, 2013

SOME RESULTS AND OPEN QUESTIONS ON SPACEABILITY IN FUNCTION SPACES

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This work was completed after the passing of the second author. The first and third authors wish to dedicate this article to the loving memory of their friend and colleague, Vladimir I. Gurariy (1935-2005).

Abstract. A subset M of a topological vector space X is called *lineable* (respectively, *spaceable*) in X if there exists an infinite dimensional linear space (respectively, an infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$. In this article we prove that, for every infinite dimensional closed subspace X of $\mathcal{C}[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X. We discuss problems related to these concepts for certain subsets of some important classes of Banach spaces (such as $\mathcal{C}[0,1]$ or Müntz spaces). We also propose several open questions in the field and study the properties of a new concept that we call the *oscillating spectrum* of subspaces of $\mathcal{C}[0,1]$, as well as *oscillating* and *annulling* properties of subspaces of $\mathcal{C}[0,1]$.

1. Introduction

The authors would like to begin by mentioning that, although this article was not fully completed until very recently, most of the results in it were proved shortly before the passing of the second author, Vladimir I. Gurariy, to whom this paper is dedicated. The work presented here is a contribution to an ongoing search for what are often called large linear spaces of functions on $[0,1]$ which have *special* properties. Given such a property, we say that the subset M of functions on $[0,1]$ which satisfies it is *spaceable* if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace. The set M will be called *lineable* if $M \cup \{0\}$ contains an infinite dimensional linear (not necessarily closed) space. These notions of lineability and spaceability were originally coined by Gurariy and they first appeared in [3, 29]. Since this concept appeared, a trend has started in which many authors have become interested in the study of subsets of $\mathcal{C}[0,1]$ enjoying certain special or, as they sometimes are called, *"pathological"* properties. Prior to the publication of [3, 29], some authors, when working with infinite dimensional spaces, have already found large linear structures enjoying these types of special properties (even though they did not explicitly use terms such as lineability or spaceability). One of the first results illustrating this was due to Levine and Milman (1940, [23]):

Received by the editors July 15, 2011 and, in revised form, October 20, 2011.
2010 *Mathematics Subject Classification.* Primary 15A03, 26A15, 46E15.
Key words and phrases. Lineability, spaceability, subspaces of continuous functions, zeros of functions, Müntz spaces.
The second author was supported by the Spanish Ministry of Science and Innovation, grant MTM2009-07848.

Theorem 1.1. *The subset of* $\mathcal{C}[0,1]$ *of all functions of bounded variation is not spaceable.*

Later, the following analogue of this previous result was proved ([16]).

Theorem 1.2. *The set of everywhere differentiable functions on* $[0,1]$ *is not spaceable.*

On the other hand (see also [16]),

Theorem 1.3. *There exist closed infinite dimensional subspaces of* $\mathcal{C}[0,1]$ *all of whose members are differentiable on* $(0,1)$.

Within the context of subsets of continuous functions, in 1966 Gurariy [17] showed that the set of continuous nowhere differentiable functions on $[0,1]$ is lineable. Soon after, Fonf, Gurariy, and Kadeč [10] showed that the set of continuous nowhere differentiable functions on $[0,1]$ is spaceable in $\mathcal{C}[0,1]$. Actually, much more is known about this set. Rodríguez-Piazza [28] showed that the space constructed in [10] can be chosen to be isometrically isomorphic to any separable Banach space. More recently, Hencl [20] showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder. Another set that has also attracted the attention of several authors is the set of differentiable nowhere monotone functions on $\mathbb{R}$, which was proved to be lineable (see, e.g., [3, 11]). We refer the interested reader to [1, 2, 4, 5, 8, 11–14, 19] for recent results and advances in this topic of lineability and spaceability, where many more examples can be found and techniques are developed.

This paper is arranged in four main sections. In Section 2 we shall give several results providing the necessary information about what we shall call *oscillating* and *annulling* properties of subspaces of continuous functions. The results from Section 2, although they might be of independent interest by themselves, shall be needed in Section 3. The goal of Section 3 (and the main result of this paper) is to show that for every infinite dimensional subspace X of $\mathcal{C}[0,1]$ the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X (Corollary 3.8). In order to do this we shall refer to certain known concepts such as the *inclination* of a Banach space ([18]). In Section 4 we study the oscillating spectrum of subspaces of $\mathcal{C}[0,1]$, providing results and characterizations of certain spaces depending on their oscillating spectrum. Section 5 shall deal with new directions of study as well as related problems and open questions in other frameworks of study, such as Müntz spaces. We believe that the questions found here are of interest in the theory of Banach spaces of continuous functions and that their answers (either in the positive or in the negative) would certainly help to develop this fruitful field. Let us recall that, given an interval J and a function $f \in \mathcal{C}(J)$, we shall denote by $\|f\|_J$ the sup norm of f restricted to J. Also, if $a, d > 0$ and $t_0 \in [0,1]$, we denote $I(t_0, d) = [0,1] \cap [t_0 - d, t_0 + d]$. From now on, the word subspace shall stand for a *closed* subspace. Otherwise we shall simply say linear space.

2. Oscillating and annulling points. Preliminary results

Let us begin this section with some definitions that shall be needed throughout this paper. The first one corresponds to the concept of an "annulling point" for a family of functions.

Definition 2.1. Let $x \in \mathcal{C}[0,1]$.

(1) x is said to be an "annulling function" on $[a,b] \subset [0,1]$ if x has infinitely many zeros in $[a,b]$.

(2) Given $t_0 \in [0,1]$, we say that t_0 is a "selectively annulling point" for a family $F \subset \mathcal{C}[0,1]$ if there exists a sequence $\{t_j\}_{j\in\mathbb{N}}$, $t_j \neq t_0$, for $j \in \mathbb{N}$, with $\lim_{j\to\infty} t_j = t_0$ and for which there is a non-trivial (i.e., non-constant) function $x \in F$ such that $x(t_j) = 0$ for every $j \in \mathbb{N}$.

(3) Similarly, if $t_0 \in [0,1]$, we say that t_0 is an "annulling point" for a family $F \subset \mathcal{C}[0,1]$ if there exists a sequence $\{t_j\}_{j\in\mathbb{N}}$, $t_j \neq t_0$, for $j \in \mathbb{N}$, with $\lim_{j\to\infty} t_j = t_0$ and such that, for every $x \in F$, we have $x(t_j) = 0$ for every $j \in \mathbb{N}$.

Definition 2.2. (1) The "oscillation" $O_{[a,b]}x$ of $x \in \mathcal{C}[0,1]$ on $[a,b]$ is defined as

$$O_{[a,b]}x = \sup_{t,s\in[a,b]} |x(t) - x(s)|.$$

(2) The "normalized oscillation" $\tilde{O}_{[a,b]}x$ of $x \in \mathcal{C}[0,1]$ on $[a,b]$ is defined as

$$\tilde{O}_{[a,b]}x = \frac{O_{[a,b]}x}{\|x\|_{[a,b]}}.$$

Now, and with the previous notions at hand, we can give the definition of an *oscillating point* for a family of functions, a concept that shall be crucial in what follows.

Definition 2.3. Let $a > 0$ and $t_0 \in [0,1]$. We say that t_0 is a-oscillating for a family of functions $F \subset \mathcal{C}[0,1]$ if for every $d > 0$ there is $x \in F$ such that $O_{I(t_0,d)}x > a$. For short, we shall say that t_0 is oscillating if it is $a-$oscillating for some $a > 0$. By writing "$a-$n-oscillating" we shall mean that the normalized oscillation on $I(t_0,d)$ is greater than a, that is, $\tilde{O}_{I(t_0,d)}x > a$.

Remark 2.4. Let us recall that the notion of oscillation given here is related to the concept of Szlenk index in general Banach space theory (see, e.g., [22]). These two notions measure non-uniform convergence. However, it is far from the purpose of this paper (which deals with subspaces of $\mathcal{C}[0,1]$) to go deeper into the relations between them.

The following remark gives a list of nice properties enjoyed by the oscillation and that shall be used throughout this paper.

Remark 2.5. Some basic properties and inequalities that the oscillation verifies are:

$$O_{[a,b]}\lambda x = |\lambda| O_{[a,b]}x, \text{ for every } \lambda \in \mathbb{R}. \tag{2.1}$$

$$O_{[a,b]}(x+y) \leq O_{[a,b]}x + O_{[a,b]}y. \tag{2.2}$$

$$O_{[a,b]}x = 0 \text{ if and only if } x \text{ is constant on } [a,b]. \tag{2.3}$$

$$O_{[a,b]}x \leq 2\|x\|_{[a,b]}. \tag{2.4}$$

$$O_{[a,b]}(x+y) \geq O_{[a,b]}x - 2\|y\|_{[a,b]}. \tag{2.5}$$

$$\text{If } [c,d] \subset [a,b], \text{ then } \quad O_{[c,d]}x \leq O_{[a,b]}x. \tag{2.6}$$

$$\lim_{d\to 0} O_{I(t,d)}x = 0 \text{ for every } t \in [0,1]. \tag{2.7}$$

Next, defining the "*oscillating spectrum*" of a family $F \subset \mathcal{C}[0,1]$ is in order. As we mentioned in the introduction we shall have a full section (Section 4) devoted to its properties. In general, the set of all oscillating points ($a-$oscillating points) of a given family $F \subset \mathcal{C}[0,1]$ shall be called the oscillating spectrum of F, and we shall denote it by $\Omega(F)$ ($\Omega_a(F)$, respectively). In this section we shall prove that for every infinite dimensional subspace X of $\mathcal{C}[0,1]$, every oscillating point is a selectively annulling point and, moreover, it is also an annulling point for some infinite dimensional subspace of X. Oscillating points also enjoy certain properties related to their stability, as we show in the following proposition. Recall that, at times, we shall be working with normalized functions, so we shall have $O_{[a,b]}x = \tilde{O}_{[a,b]}x$.

Proposition 2.6. *Let $t_0 \in [0,1]$ be a normalized oscillating point for an infinite dimensional linear space L in $\mathcal{C}[0,1]$ and let $L(\overline{t}) \subset L$ be the linear space consisting of all elements of L that vanish at every element of a given finite set $\overline{t} = \{t_j\}_{j=1}^k$. Then t_0 is an oscillating point for $L(\overline{t})$. Moreover, if t_0 is $a-$oscillating for L, then t_0 is $q-n$-oscillating for $L(\overline{t})$ where $q > \frac{a}{8^k}$.*

Proof. It suffices to show this result for a singleton, $\overline{t} = \{t_1\}$ ($k = 1$). By hypothesis, there is $a > 0$ such that, for every $d > 0$, there exists $x \in L$ (which can be taken such that $\|x\| = 1$) with $O_{I(t_0,d)}x \geq a$. Thus, by (2.7), there is $y \in L$, $\|y\| = 1$, and $h \in (0, d)$ such that $O_{I(t_0,h)}y \geq a$ and $O_{I(t_0,h)}x < \frac{a}{8}$. Now take $\lambda, \mu \in \mathbb{R}$ such that $\max\{|\lambda|, |\mu|\} = 1$, and consider the function $z = \lambda x + \mu y$ for which $z \in L(\overline{t})$, and thus $z(t_1) = 0$. Let us consider two cases.

(1) $1 = |\lambda| \geq |\mu|$. In this case we need to consider two possibilities:
 (a) $|\mu| < \frac{3a}{8}$. Here, by equations (2.1) and (2.5), we have

$$O_{I(t_0,d)}z = O_{I(t_0,d)}(\lambda x + \mu y) \geq |\lambda| O_{I(t_0,d)}x - 2|\mu|\|y\| > a - \frac{3a}{4} = \frac{a}{4}.$$

 (b) $|\mu| \geq \frac{3a}{8}$. Similarly, and by (2.6), we also obtain

$$O_{I(t_0,d)}z > O_{I(t_0,h)}(\mu y + \lambda x) \geq \frac{a}{4}.$$

(2) $1 = |\mu| \geq |\lambda|$. In this case, we have

$$O_{I(t_0,d)}z \geq O_{I(t_0,h)}z > O_{I(t_0,h)}(\mu y) - O_{I(t_0,h)}(\lambda x) > a - \frac{a}{8} > \frac{a}{4}.$$

Therefore, in all cases we have $O_{I(t_0,d)}z > \frac{a}{4}$. Since $\|z\| \leq 2$, we have $\tilde{O}_{I(t_0,d)}z > \frac{a}{8}$, and this completes the proof. □

Proposition 2.7. *For every infinite dimensional linear space L in $\mathcal{C}[0,1]$ there exists an oscillating point $t_0 \in [0,1]$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of functions in L such that, for every $n \in \mathbb{N}$ and every $k \in \{1, \dots, n\}$,

$$\|x_n\| = 1, \quad x_n(k/n) = 0.$$

This $\{x_n\}_{n\in\mathbb{N}}$ can be given inductively. Next, for every $n \in \mathbb{N}$ take $t_n \in [0,1]$ with $|x_n(t_n)| = 1$ and let $\{t_{n_k}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{t_n\}_{n\in\mathbb{N}}$. Clearly, $t_0 = \lim_{k\to\infty} t_{n_k}$ is an oscillating point for L. □

Theorem 2.8. *Let X be any infinite dimensional subspace of $\mathcal{C}[0,1]$. Every oscillating point for X is also a selectively annulling point.*

Proof. We shall follow an inductive process. Let t_0 be an $a{-}n$-oscillating point for X (for some $a > 0$). Also, let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a positive sequence with $\sum_{k\in\mathbb{N}} \varepsilon_k < \infty$.

In the first step, take $x_0 \in X$ and $d > 0$ with

$$x_0(t_0) = 0, \quad \|x_0\| = 1, \quad \|x_0\|_{I(t_0,d)} < \frac{\varepsilon_1}{16}, \tag{2.8}$$

and let $y_0 \in X$ and $d_1 \in (0, d)$ such that

$$y_0(t_0) = 0, \quad \|y_0\| = 1, \quad O_{I(t_0,d_1)} y_0 > \frac{a}{8}. \tag{2.9}$$

From (2.4) we obtain $\|y_0\|_{I(t_0,d_1)} > \frac{a}{16}$ and, using the continuity of y_0 together with the previous inequality, there is $t_1 \in I(t_0, d_1)$ with $t_0 \neq t_1$ and

$$|y_0(t_1)| > \frac{a}{16}. \tag{2.10}$$

Let us now choose the unique $s_1 \in \mathbb{R}$ for which the function $x_1 = x_0 + s_1 y_0$ is such that $x_1(t_1) = 0$ and (by (2.9)) $x_1(t_0) = 0$. By (2.8) and (2.10), we also have

$$|s_1| = \frac{|x_0(t_1)|}{|y_0(t_1)|} < \frac{\varepsilon_1}{a}$$

and

$$\|x_1\| \leq 1 + \frac{\varepsilon_1}{a}.$$

Next, using Proposition 2.6 and proceeding as before, we can find $d_2 \in (0, d_1)$ and $y_1 \in X$ such that

$$\|x_1\|_{I(t_0,d_2)} < \frac{\varepsilon_2}{2\cdot 8^2},\ \|y_1\| = 1,\ y_1(t_j) = 0 \text{ for } j \in \{0, 1\}, \text{ and } \|y_1\| > \frac{a}{2\cdot 8^2}.$$

We can also find $t_2 \in I(t_0, d_2)$, $t_2 \neq t_0, t_1$ and $s_2 \in \mathbb{R}$ with $|s_2| < \frac{\varepsilon_2}{a}$ such that if $x_2 = x_1 + s_2 y_1$, one has

$$x_2(t_2) = 0 \text{ and, thus, } x_2(t_0) = x_2(t_1) = x_2(t_2) = 0.$$

Proceeding inductively, we can obtain a convergent sequence $\{t_k\}_{k\in\mathbb{N}}$ of limit t_0, a sequence $\{s_k\}_{k\in\mathbb{N}}$ with $|s_k| < \frac{\varepsilon_k}{a}$ for every $k \in \mathbb{N}$, and functions $\{y_k\}_{k\in\mathbb{N}} \subset X$ such that

$$\|y_k\| = 1, \quad y_k(t_j) = 0, \text{ for } j \in \{0, 1, \ldots, k-1\}, \text{ and } k \in \mathbb{N}$$

in such a way that, for $x_k = x_{k-1} + s_{k-1} y_{k-1}$, we have $x_k(t_j) = 0$ for every $k \in \mathbb{N}$ and every $j \in \{0, 1, \ldots, k\}$. Finally, we can conclude that (by construction) the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges in $\mathcal{C}[0, 1]$ to a certain non-trivial function $x \in X$ with $x(t_j) = 0$ for every $j \in \mathbb{N}$. □

Now combining Proposition 2.7 and Theorem 2.8 the following result clearly follows, guaranteeing in this way the existence of a function $f \in \mathcal{C}[0, 1]$ with infinitely many zeros in $[0, 1]$.

Corollary 2.9. *In every infinite dimensional subspace X of $\mathcal{C}[0, 1]$ there exists a non-trivial function with infinitely many zeros in* $[0, 1]$.

Definition 2.10. Let X be a subspace of $\mathcal{C}[0, 1]$ and let $t_0 \in [0, 1]$. X is said to be "sequentially cancelable" (SC for short) at t_0 if for any sequence $\{t_n\}_{n\in\mathbb{N}}$, convergent to t_0, there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and a function $x \in X \setminus \{0\}$ such that $x(t_{n_k}) = 0$ for every $k \in \mathbb{N}$. Otherwise (that is, if there exists a sequence with no such subsequence) we shall say that X is NSC at t_0, and the sequence $\{t_n\}_{n\in\mathbb{N}}$ shall be called "non-permissible".

Theorem 2.11. *Let $t_0 \in [0,1]$ and let $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$ be any strictly monotone sequence with $\lim_{k\to\infty} t_k = t_0$. There exists an infinite dimensional subspace X of $\mathcal{C}[0,1]$ such that t_0 is an oscillating point for X, and X is NSC at t_0 with non-permissible sequence $\{t_k\}_{k\in\mathbb{N}}$.*

Proof. It suffices to consider the case $t_0 = 1$, $0 < t_1 < t_2 < \dots$ and $\lim_{k\to\infty} t_k = 1$. We shall choose a double sequence of positive real numbers $\{y_k^{(n)}\}_{k,n\in\mathbb{N}}$ that tends to 0 on both n and k at a certain rate that shall be given later. Let us also define a sequence $\{x_k(t)\}_{k\in\mathbb{N}} \subset \mathcal{C}[0,1]$ as follows:

(1) $x_k(0) = 0$, $x_k(1) = 0$, and for $\overline{t_k} = \frac{t_k + t_{k+1}}{2}$ we have $x_k(\overline{t_k}) = 1$ for every $k \in \mathbb{N}$.
(2) $x_k(t_n) = y_k^{(n)}$ for every $k, n \in \mathbb{N}$.
(3) For every $k \in \mathbb{N}$ the function x_k is linear on the intervals

$$[0, t_1], [t_1, t_2], \dots, [t_k, \overline{t_k}], [\overline{t_k}, t_{k+1}], [t_{k+1}, t_{k+2}], \dots.$$

Now using the Theorem of Stability ([21]) we can assume that $\{x_k\}_{k\in\mathbb{N}}$ is a normalized basic sequence in $\mathcal{C}[0,1]$. Now, let $X = [\{x_k\}_{k\in\mathbb{N}}]$ (a closed linear span of $\{x_k\}_{k\in\mathbb{N}}$). Next, let $\{t_{k_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{t_k\}_{k\in\mathbb{N}}$ for which we have $x(t_{k_j}) = 0$ for every $j \in \mathbb{N}$ and for some normalized function $x \in X$. We shall show that, actually, $x(t) = 0$ for every $t \in [0,1]$. We argue by contradiction. We can write $x = \sum_{k\in\mathbb{N}} \alpha_k x_k$ with $\alpha_1 \neq 0$. Using well-known properties of bases, we have that $\sup_k |\alpha_k| < \infty$ and, for every $j \in \mathbb{N}$,

$$x(t_{k_j}) = \alpha_1 x_1(t_{k_j}) + \sum_{i\geq 2} \alpha_i x_i(t_{k_j}) = y_1^{(k_j)} \left(\alpha_1 + \frac{\sum_{i\geq 2} \alpha_i y_i^{(k_j)}}{y_1^{(k_j)}} \right).$$

As we announced earlier in the proof, let us now settle for the double sequence $\{y_k^{(n)}\}_{k,n\in\mathbb{N}}$ to tend to zero at a rate such that the sequence $\left(\sum_{i\geq 2} \alpha_i y_i^{(k_j)}\right) / y_1^{(k_j)}$ also tends to zero. The latter implies that, for sufficiently large values of j, one would have that $x(t_{k_j}) \neq 0$, which provides the contradiction we needed, as well as the conclusion of the proof. □

The argument we used in the proof of Theorem 2.11 can also be applied to show the following.

Theorem 2.12. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a normalized basis of a subspace X of $\mathcal{C}[0,1]$, $t_0 \in [0,1]$, and $\overline{t} = \{t_j\}_{j\in\mathbb{N}}$ be a sequence of distinct terms in $[0,1]$ of limit t_0. Let us further assume that the sequence $\{y_j^{(k)}\}_{j,k\in\mathbb{N}}$, given by $y_j^{(k)} = x_k(t_j)$, verifies that, for each $k \in \mathbb{N}$,*

(1) *the sequence $\{y_j^{(k)}\}_{j\in\mathbb{N}}$ is zero for, at most, finitely many terms*
(2) *and*

$$\lim_{j\to\infty} \frac{\sum_{m=k+1}^{\infty} |y_j^{(m)}|}{|y_j^{(k)}|} = 0.$$

Then X is NSC at t_0 with respect to $\overline{t}$.

3. The main result. Continuous functions with infinitely many zeros

Finally, and as we announced in the Abstract and Introduction of this paper, this section shall present the main results of this paper (Theorem 3.6, Corollary 3.7, and Corollary 3.8). Our goal shall be to prove that given any (infinite dimensional closed) subspace $X \subset \mathcal{C}[0,1]$, there is a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$ and a further subspace $Y \subset X$ such that for every $f \in Y$ and every k, $f(t_k) = 0$. Before that, let us recall some well-known notions and results from geometry of Banach spaces (see, e.g., [18]).

Definition 3.1. Given P, Q, two subspaces of a Banach space X, we define the "inclination" of P to Q by

$$(\widehat{P,Q}) = \inf_{x\in P, \|x\|=1} \operatorname{dist}(x, Q).$$

Proposition 3.2 (Banach [6] and Grinblium [15])**.** *A sequence $\overline{e} = \{e_k\}_{k\in\mathbb{N}}$ in a Banach space X is a basic sequence if and only if*

$$\gamma(\overline{e}) := \inf_{n<m} \left(\widehat{[\{e_k\}_{k=1}^n], [\{e_k\}_{k=n+1}^m]}\right) > 0.$$

The following result ("almost orthogonality", [18, Proposition 1.4.4]) shall also be useful in order to obtain the main result in this section.

Proposition 3.3. *Let P be a finite dimensional subspace of an infinite dimensional Banach space X. For every $\varepsilon > 0$ there exists a finite codimensional subspace $Q = Q(P, \varepsilon)$ of X such that $(\widehat{P,Q}) > 1 - \varepsilon$.*

Now we need to state a result on the stability of a basic sequence in a Banach space (which can be found in [21]).

Proposition 3.4. *Let $\overline{e} = \{e_k\}_{k\in\mathbb{N}}$ be any normalized basic sequence in a Banach space X. Assume that, for some $\beta > 0$, $\gamma(\overline{e}) \geq \beta$. There exists a sequence of positive scalars $\{\varepsilon_k\}_{k\in\mathbb{N}}$ (depending only on β) such that every sequence $\{g_k\}_{k\in\mathbb{N}} \subset X$ with $\|g_k - e_k\| \leq \varepsilon_k$ ($k \in \mathbb{N}$) is also a basic sequence in X.*

Notice that, in the previous proposition, the sequence of positive scalars given by $\varepsilon_k = 4^{-k}$ ($k \in \mathbb{N}$) works for $\beta = 1/2$. The following result is well known and readily accessible.

Proposition 3.5. *The intersection of a finite number of finite codimensional subspaces of a Banach space X is a finite codimensional subspace of X.*

Now we are ready to state and prove the main result of this section, whose proof follows an inductive process and is in the same spirit as that of Theorem 2.8.

Theorem 3.6. *For every infinite dimensional subspace X of $\mathcal{C}[0,1]$ we have that each normalized oscillating point for X is also annulling for some infinite dimensional subspace Y of X.*

Proof. Let, for some $a > 0$, $t_0 \in [0,1]$ be an $a-$oscillating point for X. Also, let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers with

$$\sum_{k=n}^{\infty} \varepsilon_k \leq \frac{a}{4^n}, \qquad \text{for every } n \in \mathbb{N}.$$

In the first step, let us take $x_{0,1} \in X$ and $d_0 > 0$ such that

$$\|x_{0,1}\| = 1, \quad x_{0,1}(t_0) = 0, \quad \|x_{0,1}\|_{I(t_0,d_0)} < \frac{\varepsilon_1}{16}.$$

We can find $y_0 \in X$ such that

$$y_0(t_0) = 0, \quad \|y_0\| = 1, \text{ and } \|y_0\|_{I(t_0,d_0)} > \frac{a}{16}.$$

Therefore, for some $t_1 \in I(t_0, d_0)$ we would have $y_0(t_1) > a/16$ and, thus, there would be a unique scalar $s_{1,1}$ for which the function

$$x_{1,1} = x_{0,1} + s_{1,1} y_0$$

verifies that $x_{1,1}(t_1) = 0$, with $|s_{1,1}| = \left| \frac{x_{0,1}(t_1)}{y_0(t_1)} \right| < \frac{\varepsilon_1}{a}$.

Now, and by Propositions 3.3 and 2.6, there exists a finite codimensional subspace Q_1 of X, vanishing at t_0 and t_1 and with

$$(\widehat{[x_{0,1}], Q_1}) > \frac{1}{2}.$$

Secondly, take $x_{0,2} \in Q_1$, with $\|x_{0,2}\| = 1$, and $d_1 \in (0, d_0)$ such that

$$\max\left\{\|x_{1,1}\|_{I(t_0,d_1)}, \|x_{0,2}\|_{I(t_0,d_1)}\right\} < \frac{\varepsilon_2}{2 \cdot 8^2}.$$

By Proposition 2.6, we can choose $y_1 \in X$ such that

$$y_1(t_0) = y_1(t_1) = 0, \quad \|y_1\| = 1, \text{ and } O_{I(t_0,d_1)} y_1 > \frac{a}{8^2},$$

which gives (using equation (2.4))

$$\|y_1\|_{I(t_0,d_1)} > \frac{a}{2 \cdot 8^2}.$$

Thus, for some $t_2 \in I(t_0, d_1)$ we would have $|y_1(t_2)| > \frac{a}{2 \cdot 8^2}$ and, in the same fashion as in the previous step, there would also be scalars $s_{2,1}$ and $s_{1,2}$ for which the functions

$$\begin{aligned} x_{2,1} &= x_{1,1} + s_{2,1} y_1, \\ x_{1,2} &= x_{0,2} + s_{1,2} y_1 \end{aligned}$$

would verify that

$$x_{2,1}(t_i) = x_{1,2}(t_i) = 0 \text{ for } i \in \{0, 1, 2\},$$

with

$$\max\{|s_{2,1}|, |s_{1,2}|\} \leq \frac{\max\{|x_{1,1}(t_2)|, |x_{2,2}(t_2)|\}}{|y_1(t_2)|} < \frac{\frac{\varepsilon_2}{2 \cdot 8^2}}{\frac{a}{2 \cdot 8^2}} = \frac{\varepsilon_2}{a}$$

and

$$x_{2,1} = x_{0,1} + s_{1,1} y_0 + s_{2,1} y_1.$$

Similarly as in the first step, there is a finite codimensional subspace Q_2 of X, vanishing at t_0, t_1, and t_2 with $Q_2 \subset Q_1$ and

$$(\widehat{[x_{0,1}, x_{0,2}], Q_2}) > \frac{1}{2}.$$

We can carry on with this inductive process and obtain double sequences of functions that we arrange in the infinite matrix

$$\begin{pmatrix} x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & \cdots \\ & x_{0,2} & x_{1,2} & x_{2,2} & \cdots \\ & & x_{0,3} & x_{1,3} & \cdots \\ & & & x_{0,4} & \cdots \end{pmatrix}$$

with

$$\|x_{0,k}\| = 1 \text{ for every } k \in \mathbb{N}. \tag{3.1}$$

We shall also obtain sequences $\{d_j\}_{j\in\mathbb{N}} \to 0$, $\{t_j\}_{j\in\mathbb{N}} \to t_0$ with $t_j \in I(t_0, d_{j-1})$ for every $j \in \mathbb{N}$, and such that for each $m \in \mathbb{N}$ one has the following condition on the columns of the previous matrix:

$$\max\left\{\|x_{m-1,1}\|_{I(t_0,d_m)}, \ldots, \|x_{0,m}\|_{I(t_0,d_m)}\right\} < \frac{\varepsilon_m}{2\cdot 8^m}.$$

Moreover, we also have a sequence $\{Q_m\}_{m\in\mathbb{N}}$ of finite codimensional subspaces of X, where Q_m vanishes at $t_0, t_1, \ldots, t_m$, with $x_{0,m} \in Q_{m-1}$, and $Q_{m+1} \subset Q_m$ for all m. Thus, for each $n \in \mathbb{N}$,

$$\left(\widehat{[\{x_{0,j}\}_{j=1}^n], [\{x_{0,j}\}_{j=n+1}^\infty]}\right) \geq \left(\widehat{[\{x_{0,j}\}_{j=1}^n], Q_n}\right) > \frac{1}{2}. \tag{3.2}$$

Again, we can also choose a sequence of normalized functions $\{y_j\}_{j\in\mathbb{N}} \subset X$ such that

$$y_j(t_i) = 0 \text{ for } i \in \{0,1,\ldots,j\} \quad \text{ and } \quad |y_j(t_{j+1})| > \frac{a}{2\cdot 8^{j+1}} \text{ for every } j \in \mathbb{N},$$

and a double sequence of scalars

$$\begin{pmatrix} s_{1,1} & s_{2,1} & s_{3,1} & s_{4,1} & \cdots \\ & s_{1,2} & s_{2,2} & s_{3,2} & \cdots \\ & & s_{1,3} & s_{2,3} & \cdots \\ & & & s_{1,4} & \cdots \end{pmatrix}$$

with

$$\max\{|s_{m,1}|, |s_{m-1,2}|, \ldots, |s_{1,m}|\} \leq \frac{\varepsilon_m}{a}, \tag{3.3}$$

with the corresponding functions

$$x_{l,m} = x_{l-1,m} + s_{l,m}y_{l,m} = x_{0,m} + \sum_{j=1}^{l} s_{j,m}y_{j+m-2}, \text{ for all } l,m \in \mathbb{N}. \tag{3.4}$$

By equations (3.1), (3.3), and (3.4) we have that, for every $m \in \mathbb{N}$, the sequence $\{x_{l,m}\}_{l\in\mathbb{N}}$ converges in $\mathcal{C}[0,1]$ to some $x_m \in X$. So, by (3.2) and Proposition 3.2 we have that the sequence $\{x_{0,m}\}_{m\in\mathbb{N}}$ is a basic sequence in X. Next, by (3.3) and (3.4) we have that, for every $m \in \mathbb{N}$,

$$\|x_{0,m} - x_m\| \leq \sum_{j=m}^{\infty} \frac{\varepsilon_j}{a} \leq 4^{-m}.$$

Finally, by Proposition 3.4, we obtain that $\{x_m\}_{m\in\mathbb{N}}$ is a basis for the infinite dimensional subspace of X given by $Y = [\{x_m\}_{m\in\mathbb{N}}]$. Moreover, by construction,

$$x_m(t_j) = 0 \text{ for all } j,m \in \mathbb{N},$$

which concludes the proof. □

The following are straight forward consequences of the previous result.

Corollary 3.7. *For every infinite dimensional subspace X of $\mathcal{C}[0,1]$ there is an infinite dimensional subspace Y of X and a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$, of pairwise different elements, such that $y(t_k)=0$ for every $k\in\mathbb{N}$ and every $y\in Y$.*

Corollary 3.8. *For every infinite dimensional subspace X of $\mathcal{C}[0,1]$ the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X.*

Remark 3.9. Let us mention that, although we have been working on $\mathcal{C}[0,1]$, all the results we have obtained in this section could be easily reformulated for $\mathcal{C}(K)$ spaces, K being a compact metric space.

4. Some remarks on the oscillating spectrum

This section shall deal with properties of some subspaces X of $\mathcal{C}[0,1]$ in connection with their oscillating spectrum, $\Omega(X)$. Let us begin by observing that the following result easily follows from Arzelà's compactness criterion.

Proposition 4.1. *A uniformly bounded set $F\subset\mathcal{C}[0,1]$ is compact if and only if $\Omega(F)=\varnothing$.*

We can now prove the following result by means of an argument due to Wojtaszczyk.

Theorem 4.2. *Let X be a subspace of $\mathcal{C}[0,1]$. If $\Omega(X)$ is finite, then X is isomorphic to a subspace of c_0.*

Proof. We shall consider $\Omega(X)$ to be a singleton, $\Omega(X)=\{t_0\}$, and $t_0=1$. The proof in the general case is essentially identical. Let $\varepsilon>0$. In order to reach our aim, it suffices to consider

$$X_0=\{g\in X : g(1)=0\},$$

since X_0 is, at most, 1 codimensional in X. Let $f\in X_0$ and define, for every $n\in\mathbb{N}$,

$$f_n=f|_{I_n}, \text{ where } I_n=\left[1-\frac{1}{n},1-\frac{1}{n+1}\right].$$

According to Proposition 4.1 the set $\{f_n : f\in X_0, \|f\|=1\}$ is compact in $\mathcal{C}(I_n)$. Let $\|\cdot\|_n$ stand for the sup norm on I_n. Following similar arguments to those from examples in [18], there exists a projection $P_n:\mathcal{C}(I_n)\to\mathcal{C}(I_n)$ with $\|P_n\|=1$ and $P_n(\mathcal{C}(I_n))\backsimeq\ell_\infty^{k_n}$ (for some sequence $\{k_n\}_{n\in\mathbb{N}}$) such that, for all $f\in X_0$, we have

$$\|P_nf_n-f_n\|\le\varepsilon\|f\|.$$

Now consider the map

$$\begin{array}{rcccl} T & : & X_0 & \longrightarrow & \left(\sum_{n\in\mathbb{N}}\oplus\ell_\infty^{k_n}\right)_{(c_0)} \backsimeq c_0, \\ & & f & \mapsto & \{P_nf_n\}_{n\in\mathbb{N}}. \end{array}$$

Clearly, $\lim_{n\to\infty}P_nf_n=0$ since $\lim_{n\to\infty}f_n=0$. Finally,

$$\|Tf\|=\sup_{n\in\mathbb{N}}\|P_nf_n\|_n\le\sup_{n\in\mathbb{N}}\|f_n\|_n=\|f\|, \text{ so } \|T\|\le1,$$

and

$$\|Tf\| = \sup_{n\in\mathbb{N}} \|P_n f_n\|_n \geq \sup_{n\in\mathbb{N}} \|f_n\|_n - \varepsilon\|f\| \geq (1-\varepsilon)\|f\|, \text{ so } \|T^{-1}\| \leq \frac{1}{1-\varepsilon},$$

concluding the proof. □

The following question still remains open.

Question 4.3. Does Theorem 4.2 hold if $\Omega(X)$ is countable?

The following result tells us, somehow, that any closed set can be the oscillating spectrum for a certain subspace of $\mathcal{C}[0,1]$. Its proof is constructive and we shall give a sketch of it, leaving the calculations to the interested reader.

Theorem 4.4. *For any closed subset $M \subset [0,1]$ there exists a subspace X of $\mathcal{C}[0,1]$ with $M = \Omega(X)$.*

Proof. For any $a, b, c, d \in \mathbb{R}$ with $0 \leq a < b \leq 1$, and given $n \in \mathbb{N}$, let $\Delta_n = \frac{b-a}{2n}$ and consider the continuous function $x \in \mathcal{C}[a,b]$ given by

$$x(t) = x^{(n)}_{a,b,c,d}(t) = \begin{cases} c & \text{if } t = a, \\ d & \text{if } t = b, \\ 0 & \text{if } t \in [a+\Delta_n, b-\Delta_n], \\ \text{linear} & \text{if } t \in [a, a+\Delta_n] \cup [b-\Delta_n, b]. \end{cases}$$

Next, let N be the complement of M in $[0,1]$, which is open. Thus, we can write N as the union of pairwise disjoint intervals,

$$N = \bigcup_{a,b\in M} (a,b). \tag{4.1}$$

Consider the space $\mathcal{C}(M)$ and let us also consider the extension of each $x \in \mathcal{C}(M)$ to $[0,1]$ by means of using the above functions $x^{(n)}_{a,b,c,d}$ on each of the open intervals from (4.1). We leave the details to the interested reader to check that the closed linear span of this set of functions is a subspace of X for which $\Omega(X) = M$. □

Let us note that (in general) the set $\Omega(X)$ might not be closed, as one can see from the following result.

Proposition 4.5. *There exists a subspace $X \subset \mathcal{C}[0,1]$ for which $\Omega(X) = (0,1]$.*

Proof. Let X be the set of all functions $x \in \mathcal{C}[0,1]$ such that

$$x(t/2) = \frac{x(t)}{2} \quad \text{for every } t \in [0,1].$$

X is clearly an infinite dimensional subspace of $\mathcal{C}[0,1]$. Now, let

$$I_n = \left[1/2^n, 1/2^{n-1}\right]$$

for every $n \in \mathbb{N}$. It is easy to see that, for each of these I_n's, one has

$$O_{I_n} x \leq \frac{1}{2^{n-1}} \quad \text{for every } n \in \mathbb{N}, \text{ and every } x \in X,$$

and by equation (2.7), $O_{\{0\}}x = 0$ for every $x \in X$, which completes the proof. □

Question 4.6. Which conditions on $M \subset [0,1]$ shall guarantee that $M = \Omega(X)$ for some subspace $X \subset \mathcal{C}[0,1]$?

The following result clearly holds.

Proposition 4.7. *For every $a > 0$ and any subspace X of $\mathcal{C}[0,1]$, the set $\Omega_a(X)$ is closed.*

Let us finish this section by posing a couple of problems that we believe are of interest in this area.

Question 4.8. What are the properties that $\Omega(X)$ should enjoy in order to obtain that X is uncomplemented in $\mathcal{C}[0,1]$?

Question 4.9. Given X, Y subspaces of $\mathcal{C}[0,1]$, how must $\Omega(X)$ and $\Omega(Y)$ be in order to make X and Y non-isomorphic?

5. Related results and open questions

We would like to finish this paper by giving some directions for new problems and questions similar to those that we have been dealing with so far. Notice that Theorem 3.6 shows that there is a *"big"* amount (in the form of subspaces) of non-trivial continuous functions with infinitely many zeros in $[0,1]$. In 1998 Bernardes obtained some results on the "*big size*" of certain subsets of continuous functions. Namely, he proved ([7]) that "*most*" continuous functions (in the second category sense) from a convex set in $\mathbb{R}^n$ into itself have uncountably many fixed points, and this set of fixed points has Lebesgue measure zero (whereas Brouwer's Fixed Point Theorem can only guarantee the existence of one fixed point). Maybe Theorem 3.6 can be strengthened in this sense and in the spirit of these results.

Also, could the results above be generalized to spaces $\mathcal{C}(M)$, with M any compact metric space, and their corresponding subspaces? We believe that the results that follow (Theorems 5.1, 5.2, 5.3, and 5.4) might be essential for this purpose. Since Theorems 2.11 and 2.12 provide a connection between bases in a Banach space X and certain oscillating properties of X, it would be reasonable to think that if we consider a particular type of basis (monotone, interpolating, etc.) one could get much more information on this connection. Let us now state the results we just mentioned.

Theorem 5.1 ([26])**.** *For any compact metric space M with the cardinality of the continuum, the space $\mathcal{C}(M)$ is isomorphic to $\mathcal{C}[0,1]$.*

Theorem 5.2 ([18,24])**.** *For every compact metric space M the space $\mathcal{C}(M)$ has a basis. Moreover, $\mathcal{C}(M)$ has an interpolating basis $\{e_k(t)\}_{k\in\mathbb{N}}$ for any given sequence of nodes $\{t_k\}_{k\in\mathbb{N}}$. In other words, in the decomposition of any $x(t) = \sum_{k\in\mathbb{N}} a_k e_k(t)$ the sum $\sum_{k=1}^n a_k e_k(t)$ interpolates the function x at the nodes $t_1,\ t_2,\ \ldots, t_n$, for every $n \in \mathbb{N}$.*

Theorem 5.3 ([25])**.** *For every compact metric space M the space $\mathcal{C}(M)$ has a monotone basis $\{e_k\}_{k\in\mathbb{N}}$. That is, for any sequence of scalars $\{a_k\}_{k\in\mathbb{N}}$, the sequence $\{\sum_{k=1}^n a_k e_k\}_{n\in\mathbb{N}}$ is non-decreasing.*

Theorem 5.4 ([24])**.** *For every compact metric space M the space $\mathcal{C}(M)$ has a monotone interpolating basis.*

Theorem 5.4 is far more general than both Theorems 5.2 and 5.3, although the proofs of the former ones are more instructive and revealing.

Let us now consider a different kind of space, namely Müntz spaces. In order to do this, let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of scalars with $\lambda_k \uparrow \infty$. Consider the corresponding Müntz subspace in $\mathcal{C}[0,1]$,

$$M(\Lambda) = [\{t^{\lambda_k}\}_{k\in\mathbb{N}}] \cup \{1\}.$$

We know that, by Müntz's Theorem [27], $M(\Lambda)$ is not dense if and only if $\sum_{k\in\mathbb{N}} \frac{1}{\lambda_k} < \infty$. Also, if $M(\Lambda)$ is not dense, then each function $x \in M(\Lambda)$ has an analytic continuation on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ ([9,30]). An infinite dimensional subspace of $\mathcal{C}[0,1]$ in which each function has an analytical extension to D is said to be of class $\mathcal{A}_D$ (see [18, Definition 11.1.1]). From the above theorems, together with the theorem of uniqueness for analytic functions, we have:

Theorem 5.5. *Let X be an infinite dimensional subspace of class $\mathcal{A}_D$ in $\mathcal{C}[0,1]$. Then:*

(1) *There exists a non-trivial function $x \in X$ with infinitely many zeros in $[0,1]$.*
(2) $\Omega(X) = \{1\}$.
(3) *For every non-trivial function $x \in X$ with infinitely many zeros $\{t_j\}_{k\in\mathbb{N}}$ we have $\lim_{j\to\infty} t_j = 1$.*

Question 5.6. Does every infinite dimensional subspace X of class $\mathcal{A}_D$ have a subspace Y which is NSC at $t_0 = 1$?

Question 5.7. Is every non-dense Müntz space NSC at $t_0 = 1$?

On a totally different framework, and somehow related to the study of the amount of zeros of functions on a given interval, let us recall a question posed by Aron and Gurariy in 2003, where they asked whether there exists an infinite dimensional subspace of ℓ_∞, every non-zero element of which has a finite number of zero coordinates. If we denote by P the set of odd prime numbers, and if (for $p \in P$) we say $x_p = \left(\frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \frac{1}{p^4}, \cdots\right) \in \ell_\infty$, it is easy to see that any finite (and non-trivial) linear combination of $\{x_p : p \in P\}$ verifies the desired property. Some partial answers to the original problem were given recently (for other sequence spaces) in [14], where the authors proved, among other results in this direction, the following:

Theorem 5.8. *Let X be an infinite dimensional Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. There exists a linear space $V \subset X$ such that:*

(1) *If $0 \neq a \in V$, then* $\operatorname{card}\{n \in \mathbb{N} : a[n] = 0\} < \infty$.
(2) *If $a, b \in V$, then $\sum_{n=1}^{\infty} a[n]\, b[n]\, e_n \in V$.*
(3) *V is dense and not barrelled.*

However, the question originally posed by Aron and Gurariy concerning spaceability for the ℓ_∞ case still remains open.

Acknowledgements

The authors express their gratitude to the referee whose thorough analysis and insightful remarks improved the text. They would also like to thank Professors Gámez-Merino, Muñoz-Fernández and Pellegrino for fruitful conversations and comments on this work.

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