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THE BRAUER SEMIGROUP OF A GROUPOID AND A SYMMETRIC IMPRIMITIVITY THEOREM

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Abstract. In this paper we define a monoid called the Brauer semigroup for a locally compact Hausdorff groupoid E whose elements consist of Morita equivalence classes of E-dynamical systems. This construction generalizes both the equivariant Brauer semigroup for transformation groups and the Brauer group for a groupoid. We show that groupoid equivalence induces an isomorphism of Brauer semigroups and that this isomorphism preserves the Morita equivalence classes of the respective crossed products, thus generalizing Raeburn's symmetric imprimitivity theorem.

1. Introduction

Let $\mathcal{G}$ and $\mathcal{H}$ be groups with commuting free and proper actions on the left and right, respectively, of a space X. In [22], Rieffel attributed to Green the useful observation that $C_0(X/\mathcal{H}) \rtimes \mathcal{G}$ is Morita equivalent to $C_0(\mathcal{G}\backslash X) \rtimes \mathcal{H}$. One of the many applications of this result is that if $X = \mathcal{G}$ and $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$, then we can induce representations from $\mathcal{H}$ to obtain representations of $\mathcal{G}$. In [19], Raeburn proved the symmetric imprimitivity theorem, a noncommutative version of the result in [22] which gives a Morita equivalence between crossed products of $\mathcal{G}$ and $\mathcal{H}$ on certain C^*-algebras (where $\mathcal{G}$ and $\mathcal{H}$ are groups acting freely and properly on X as above). Again Raeburn's result can be used to construct representations induced from subgroups.

In an effort to study the cohomology of the transformation group $(\mathcal{G}, T)$, the authors in [5] define a group called the equivariant Brauer group $\mathrm{Br}_{\mathcal{G}}(T)$. The elements of this group are $\mathcal{G}$-dynamical systems (A, α), where A is a continuous trace C^*-algebra with spectrum T and the action induced by α on T coincides with the given action. In [14], the authors use the equivariant Brauer group to provide an algebraic setting for Raeburn's symmetric imprimitivity theorem. That is, they show that if $\mathcal{G}$ and $\mathcal{H}$ have commuting free and proper actions on a space X, then there exists an isomorphism $\theta : \mathrm{Br}_{\mathcal{G}}(X/\mathcal{H}) \to \mathrm{Br}_{\mathcal{H}}(\mathcal{G}\backslash X)$ such that if $\theta([A, \alpha]) = [B, \beta]$, then $A \rtimes_\alpha \mathcal{G}$ is Morita equivalent to $B \rtimes_\beta \mathcal{H}$.

These results inspired two distinct generalizations. In 2000, the authors in [12] extended the results in [14] to find a monoid of all separable $\mathcal{G}$ systems (A, α) with A a $C_0(T)$-algebra where the action induced by α on T coincides with a given action. This allows the authors to recover the full power of Raeburn's symmetric imprimitivity theorem. In [13], the authors replaced the transformation group

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$(\mathcal{G}, T)$ in [5] with a groupoid E to define the Brauer group $\mathrm{Br}(E)$ of E consisting of seperable E-dynamical systems. They also indicated that $\mathrm{Br}(E)$ can be used to study groupoid cohomology.

Our goal is to combine these two generalizations into one overarching framework. To that end, for a second countable locally compact Hausdorff groupoid E we define a monoid called the *Brauer semigroup* $S(E)$ consisting of equivariant Morita equivalence classes of E-dynamical systems (Definition 4.5). The Brauer group $\mathrm{Br}(E)$ embeds in $S(E)$ as the set of invertible elements. We show that if G and H are groupoids, X is a (G, H)-equivalence, and $G \ltimes X \rtimes H$ is the associated transformation groupoid, then there exists an isomorphism

$$\nu^{X,H} : S(H) \to S(G \ltimes X \rtimes H)$$

such that if $\nu^{X,H}([B, \beta]) = [A, \omega]$, then $B \rtimes_\beta H$ is Morita equivalent to $A \rtimes_\omega G \ltimes X \rtimes H$ (Theorem 5.2). By symmetry we then get an isomorphism $\nu^X : S(H) \to S(G)$ with the same property.

At first glance, it may appear that the hypothesis that X is a groupoid equivalence in Theorem 5.2 is stronger than the hypotheses used in Raeburn's symmetric imprimitivity theorem. However, if $\mathcal{G}$ and $\mathcal{H}$ are groups with commuting free and proper actions on the left and right, respectively, of a space X, then X provides a groupoid equivalence between the transformation group groupoids $\mathcal{G} \ltimes X/\mathcal{H}$ and $\mathcal{G}\backslash X \rtimes \mathcal{H}$. Furthermore, for a $C_0(X/\mathcal{H})$-algebra A, $A \rtimes \mathcal{G}$ is isomorphic to $A \rtimes (\mathcal{G} \ltimes X/\mathcal{H})$ (similarly for $C_0(\mathcal{G}\backslash X)$-algebras).[1] Thus for $[A, \alpha] \in \mathrm{Br}_\mathcal{G}(X/\mathcal{H})$ and $[B, \beta] \in \mathrm{Br}_\mathcal{H}(\mathcal{G}\backslash X)$, the crossed product $A \rtimes \mathcal{G}$ is Morita equivalent to $B \rtimes \mathcal{H}$ if and only if $A \rtimes (\mathcal{G} \ltimes X/\mathcal{H})$ is Morita equivalent to $B \rtimes (\mathcal{G}\backslash X \rtimes \mathcal{H})$. Hence, Theorem 5.2 recovers Raeburn's symmetric imprimitivity theorem in the group case.

We begin the paper with a review of some preliminary materials, including upper semicontinuous bundles, groupoids, and imprimitivity bundles (Section 2). The generalized fixed point algebra for a groupoid dynamical system, as defined in [2], will play a key role in constructing an inverse for $\nu^{X,H}$. However, the fixed point algebra is defined abstractly in [2], and in order to perform our analysis we need to find a more concrete description. We do this in Section 3. More specifically, for a principal proper groupoid E and (A, α) an E-dynamical system, we define an algebra of continuous sections $\mathrm{Ind}_E^{E^{(0)}}(A, \alpha)$, and show in Proposition 3.6 that $\mathrm{Ind}_E^{E^{(0)}}(A, \alpha)$ is equal to (not just isomorphic to) the generalized fixed point algebra.

Next, we introduce the Brauer semigroup for a groupoid in Section 4. Much of the work in defining the Brauer semigroup was done in [12] and [13], so Section 4 merely outlines the construction. Section 5 contains the statement and proof of the main result of the paper, Theorem 5.2. To prove Theorem 5.2 we follow the outline in [12]. However, the proofs in our setting are substantially different and require significant analysis. In Section 5.1 we show $\nu^{X,H}$ is a homomorphism. In Section 5.2 we use the generalized fixed point algebras described above to construct a map from $S(G \ltimes X \rtimes H)$ to $S(H)$ and in Section 5.3 we show this map is an inverse for $\nu^{X,H}$. We then use the results of [2] to get Morita equivalence as follows. Let (A, ω) be a $G \ltimes X \rtimes H$-dynamical system. The transformation groupoid $G \ltimes X$ is included in $G \ltimes X \rtimes H$ and we can restrict ω to an action ω^G of $G \ltimes X$ on A. Since G acts freely and properly on X, $G \ltimes X$ is a principal and proper groupoid. Let $\mathrm{Fix}_G(A)$ be the

[1]See [9, Example 1] for the proof in the case where A has Hausdorff spectrum.

generalized fixed point algebra for the ω^G action. We will show that there is an action $\mathrm{Fix}_G(\omega)$ of H on $\mathrm{Fix}_G(A)$ and that $(\nu^{X,H})^{-1}([A,\omega]) = ([\mathrm{Fix}_G(A), \mathrm{Fix}_G(\omega)])$. Next, [2] gives an imprimitivity bimodule Z between $A \rtimes_r (G \ltimes X)$ and $\mathrm{Fix}_G(A)$. Since $G \ltimes X$ is principal and proper, it is amenable [1], so Z is an $A \rtimes (G \ltimes X)$ – $\mathrm{Fix}_G(A)$-imprimitivity bimodule. We then analyze Z to show it is equivariant for the H actions so that $\mathrm{Fix}_G(A) \rtimes H$ and $(A \rtimes (G \ltimes X)) \rtimes H$ are Morita equivalent by [18, Section 9.1]. Finally, [3] shows that $(A \rtimes (G \ltimes X)) \rtimes H \cong A \rtimes (G \ltimes X \rtimes H)$, giving the result. We prefer this approach to the one outlined in [18, Section 9.2] (which generalizes the constructions in [13]) because our approach allows for induction in stages. In any case, we show in Section 6 that when restricted to the Brauer group the isomorphism ν^X is equal to the isomorphism of Brauer groups constructed in [13].

In a short appendix we answer a question raised in [2] by giving a fairly general condition which guarantees that the generalized fixed point algebra of a proper groupoid dynamical system is Morita equivalent to an ideal of the reduced crossed product.

2. Preliminaries

We assume all Banach algebras and C^*-algebras (with the exception of multiplier algebras) are separable. For a C^*-algebra A we denote the multiplier algebra of A by $M(A)$ and its center by $Z(A)$.

2.1. Upper semicontinuous bundles.

Let $p : X \to T$ and $q : Y \to T$ be surjections. Throughout we denote the fibered product of X and Y by

$$X * Y := \{(x, y) \in X \times Y : p(x) = q(y)\}.$$

Definition 2.1 ([18, Definition 3.1])**.** Let T be a second countable locally compact Hausdorff space. An *upper semicontinuous Banach bundle over* T is a topological space $\mathscr{Z}$ together with a continuous open surjection $p_{\mathscr{Z}} : \mathscr{Z} \to T$ such that $Z(t) := p_{\mathscr{Z}}^{-1}(t)$ is a Banach space for each $t \in T$ and such that the following axioms hold:

(1) The map $z \mapsto \|z\|$ is upper semicontinuous from $\mathscr{Z}$ to $\mathbb{R}^+$.
(2) The map $\mathscr{Z} * \mathscr{Z} \to \mathscr{Z}$ defined by $(z, w) \mapsto z + w$ is continuous.
(3) For each $\kappa \in \mathbb{C}$ the map $\mathscr{Z} \to \mathscr{Z}$ defined by $z \mapsto \kappa z$ is continuous.
(4) If $\{z_i\}$ is a net in $\mathscr{Z}$ such that $\|z_i\| \to 0$ and $p_{\mathscr{Z}}(z_i) \to t$, then $z_i \to 0_t$, where 0_t is the zero element in $Z(t)$.

An *upper semicontinuous C^*-bundle* over T is an upper semicontinuous Banach bundle $p_{\mathscr{Z}} : \mathscr{Z} \to T$ such that $Z(t)$ is a C^*-algebra for each t and the following additional axioms hold:

(5) The map $\mathscr{Z} * \mathscr{Z} \to \mathscr{Z}$ defined by $(z, w) \mapsto zw$ is continuous.
(6) The map $\mathscr{Z} \to \mathscr{Z}$ defined by $z \mapsto z^*$ is continuous.

Showing that a sequence in an upper semicontinuous Banach bundle converges is often very delicate. Our main tool for this is the following proposition which roughly states that a sequence is convergent if there exists a convergent sequence close to it. Many of our proofs amount to finding a convergent sequence close to a given sequence. The proof is the same as in [27, Proposition C.20], so we omit it.

Proposition 2.2. *Let $p_{\mathscr{Z}} : \mathscr{Z} \to T$ be an upper semicontinuous Banach bundle over T. Let $\{a_i\}_{i \in I}$ be a net in $\mathscr{Z}$ such that $p_{\mathscr{Z}}(a_i) \to p_{\mathscr{Z}}(a)$ for some $a \in \mathscr{Z}$.*

Suppose that for all $\epsilon > 0$ there is a net $\{b_i\}_{i\in I}$ and $b \in \mathbb{Z}$ such that

(1) $b_i \to b$ *in* $\mathscr{Z}$,
(2) $p_{\mathscr{Z}}(b_i) = p_{\mathscr{Z}}(a_i)$ *and* $p_{\mathscr{Z}}(b) = p_{\mathscr{Z}}(a)$, *and*
(3) $\max\{\|a - b\|, \|a_i - b_i\|\} < \epsilon$ *eventually.*

Then $a_i \to a$.

Let $p_{\mathscr{Z}} : \mathscr{Z} \to T$ be an upper semicontinuous Banach bundle over T and $q : X \to T$ be a continuous open surjection. We define the pull back bundle over X by

$$q^*\mathscr{Z} := \{(x, z) \in X \times \mathscr{Z} : q(x) = p_{\mathscr{Z}}(z)\} \quad \text{with } p_{q^*\mathscr{Z}} : (x, z) \to x,$$

where the topology on $q^*\mathscr{Z}$ is given by the relative topology. Note that the fiber of $q^*\mathscr{Z}$ over x is naturally isomorphic to $Z(q(x))$.

Let $p_{\mathscr{Z}} : \mathscr{Z} \to T$ be an upper semicontinuous Banach bundle. A continuous function $f : T \to \mathscr{Z}$ is called a *section* if $p_{\mathscr{Z}} \circ f = \mathrm{id}_T$. We denote the set of continuous sections by $\Gamma(T, \mathscr{Z})$, the continuous bounded sections by $\Gamma^b(T, \mathscr{Z})$, the continuous compactly supported sections by $\Gamma_c(T, \mathscr{Z})$, and the continuous sections that vanish at infinity by $\Gamma_0(T, \mathscr{Z})$. By the Tietze Extension Theorem for upper semicontinuous Banach bundles [17, Proposition A.5], if T is locally compact Hausdorff, then $\{f(t) : f \in \Gamma_0(T, \mathscr{Z})\} = Z(t)$. Since we only consider bundles over locally compact Hausdorff spaces we will use this property without comment.

The equation $\|f\| = \sup_{t\in T} \|f(t)\|$ defines a norm on $Z = \Gamma_0(T, \mathscr{Z})$, and under this norm Z is a Banach algebra. It is a C^*-algebra if $\mathscr{Z}$ is an upper semicontinuous C^*-bundle. In either case we refer to $\Gamma_0(T, \mathscr{Z})$ as the section algebra of $\mathscr{Z}$ and denote it by the corresponding Roman letter Z. For $\phi \in C_0(T)$ and $f \in \Gamma(T, \mathscr{Z})$ define

$$\phi \cdot f(t) = \phi(t)f(t).$$

Observe that if $\phi \in C_c(T)$, then $\phi \cdot f \in \Gamma_c(T, \mathscr{Z})$. Further, the map $\phi \mapsto (f \mapsto \phi \cdot f)$ induces an action of $C_0(T)$ on Z [27, Lemma C.22].

Let $p_{\mathscr{Z}} : \mathscr{Z} \to T$ and $p_{\mathscr{Y}} : \mathscr{Y} \to T$ be upper semicontinuous Banach bundles over T. We say $\Phi : \mathscr{Z} \to \mathscr{Y}$ is a homomorphism if Φ is continuous, $p_{\mathscr{Z}}(z) = p_{\mathscr{Y}}(\Phi(z))$, and Φ is a homomorphism on the fibers. A bundle homomorphism $\Phi : \mathscr{Z} \to \mathscr{Y}$ induces a $C_0(T)$-linear homomorphism $f \mapsto (t \mapsto \Phi(f)(t))$ of the section algebras. Every $C_0(T)$-linear homomorphism $Z \to Y$ induces a bundle homomorphism of $\mathscr{Z} \to \mathscr{Y}$ as well [18, page 18]. We will often convert from bundle homomorphisms to $C_0(T)$-linear homomorphisms without comment.

Definition 2.3. Let T be a second countable locally compact Hausdorff space and A be a (separable) C^*-algebra. We say A is a $C_0(T)$*-algebra* if there exists a nondegenerate $*$-homomorphism of $C_0(T)$ into the center of the multiplier algebra of A.

If $\mathscr{A}$ is an upper semicontinuous C^*-bundle over T, then the section algebra $A = \Gamma_0(T, \mathscr{A})$ is a $C_0(T)$-algebra. In fact all $C_0(T)$-algebras arise in this way.

Proposition 2.4 ([27, Theorem C.26]). *Let T be a second countable locally compact Hausdorff space and A a C^*-algebra with spectrum $\widehat{A}$. Then the following are equivalent:*

(1) A is a $C_0(T)$-algebra.

(2) There exists an upper semicontinuous C^*-bundle $\mathscr{A}$ over T such that $A = \Gamma_0(T, \mathscr{A})$.
(3) There exists a continuous map $\sigma_A : \widehat{A} \to T$.

The next two propositions appear in [7, Corollary II.14.7 and Theorem II.13.18] for Banach bundles and are proven in [27, Proposition C.24 and Theorem C.25] for upper semicontinuous C^*-bundles. We restate them here for the convenience of the reader. The proofs in [27] proceed without change for upper semicontinuous Banach bundles.

Proposition 2.5. *Let $p_{\mathscr{Z}} : \mathscr{Z} \to T$ be an upper semicontinuous Banach bundle and Γ a subspace of $\Gamma_0(T, \mathscr{Z})$. Suppose*
(1) $f \in \Gamma$ and $\phi \in C_0(T)$ implies $\phi \cdot f \in \Gamma$, and
(2) for each $t \in T$, $\{f(t) : f \in \Gamma\}$ is dense in $Z(t)$.
Then Γ is dense in $\Gamma_0(T, \mathscr{Z})$.

Proposition 2.6. *Let $\mathscr{Z}$ be a set and $p_{\mathscr{Z}} : \mathscr{Z} \to T$ a surjection onto a second countable locally compact Hausdorff space T such that $Z(t)$ is a Banach space. Suppose Γ is an algebra of sections of $\mathscr{Z}$ such that*
(1) for each $f \in \Gamma$, $t \mapsto \|f(t)\|$ is upper semicontinuous, and
(2) for each $t \in T$, $\{f(t) : f \in \Gamma\}$ is dense in $Z(t)$.
Then there is a unique topology on $\mathscr{Z}$ such that $p_{\mathscr{Z}} : \mathscr{Z} \to T$ is an upper semicontinuous Banach bundle over T with $\Gamma \subset \Gamma(T, \mathscr{Z})$. If we replace "Banach space" with C^*-algebra, then $\mathscr{Z}$ is an upper semicontinuous C^*-bundle.

Proposition 2.4 states that the map $\mathscr{A} \mapsto \Gamma_0(T, \mathscr{A})$ defines a one-to-one correspondence between upper semicontinuous C^*-bundles and $C_0(T)$-algebras. In the following we establish this correspondence for imprimitivity bimodules over T. The proof follows along lines similar to the proof of [27, Theorem C.26].

Suppose A and B are $C_0(T)$-algebras and Z is an $A-B$-imprimitivity bimodule. Then the actions of $C_0(T)$ on A and B induce left and right actions of $C_0(T)$ on Z. If $f \cdot z = z \cdot f$ for all $f \in C_0(T)$ and all $z \in Z$, we say that Z is an $A - B$ imprimitivity bimodule over T.

Let Z be an $A - B$ imprimitivity bimodule over T. Define $C_{0,t}(T) := \{f \in C_0(T) : f(t) = 0\}$ and consider $M_t := C_{0,t}(T) \cdot Z := \overline{\operatorname{span}}\{f \cdot z : f \in C_{0,t}(T), z \in Z\}$. For $z - w \in M_t$ we write $z \sim_t w$, and this turns out to be an equivalence relation on Z. Define $Z(t) := Z/\sim_t$ and let q_t be the quotient map. For $z \in Z$ define $z(t) := q_t(z)$. The quotient $Z(t)$ is an $A(t) - B(t)$-imprimitivity bimodule whose actions and inner products are characterized by

$$
\begin{aligned}
{}_{A(t)}\langle z(t), w(t)\rangle &= {}_{A}\langle z, w\rangle (t), & \langle z(t), w(t)\rangle_{B(t)} &= \langle z, w\rangle_{B} (t),\\
a(t) \cdot z(t) &= (a \cdot z)(t), & z(t) \cdot b(t) &= (z \cdot b)(t),
\end{aligned}
$$

where $a \in A$, $b \in B$ and $z, w \in Z$. Define $\mathscr{Z} := \bigsqcup Z(t)$ and $p_{\mathscr{Z}} : \mathscr{Z} \to T$ to be the obvious map.

Consider the set of functions $\Gamma = \{t \mapsto z(t) : z \in Z\}$. Then for each $t \in T$, $\{z(t) : z \in \Gamma\} = Z(t)$. Furthermore, since $\|z(t)\| := \sqrt{\|{}_{A(t)}\langle z(t), z(t)\rangle\|} = \sqrt{\|{}_{A}\langle z, z\rangle (t)\|}$ and since ${}_{A}\langle z, z\rangle$ is in the $C_0(T)$-algebra A, we have that $t \mapsto \|z(t)\|$ is upper semicontinuous. Thus by Proposition 2.6 there is a unique topology on $\mathscr{Z}$ making it an upper semicontinuous Banach bundle such that $z \mapsto z(t)$ is a continuous section for all $z \in Z$. This section vanishes at infinity since $t \mapsto {}_A\langle z, z\rangle(t)$ does. Note that since ${}_A\langle z, w\rangle \in A$, $t \mapsto {}_A\langle z, w\rangle(t)$ is continuous.

Definition 2.7. Let $p_{\mathscr{A}} : \mathscr{A} \to T$ and $p_{\mathscr{B}} : \mathscr{B} \to T$ be C^*-bundles with section algebras A and B respectively. A Banach bundle $p_{\mathscr{X}} : \mathscr{X} \to T$ is an $\mathscr{A} - \mathscr{B}$-*imprimitivity bimodule bundle* if each fiber $Z(t)$ is an $A(t) - B(t)$-imprimitivity bimodule such that the actions $(a, z) \to a \cdot z$ from $\mathscr{A} * \mathscr{X}$ to $\mathscr{X}$, $(z, b) \to z \cdot b$ from $\mathscr{X} * \mathscr{B}$ to $\mathscr{X}$, and inner products $(z, w) \mapsto {}_{A(p_{\mathscr{X}}(z))}\langle z, w\rangle$ from $\mathscr{X} * \mathscr{X}$ to $\mathscr{A}$, and $(z, w) \mapsto \langle z, w\rangle_{B(p_{\mathscr{X}}(z))}$ from $\mathscr{X} * \mathscr{X}$ to $\mathscr{B}$ are continuous.

Remark 2.8. Definition 2.7 is slightly different from that used in [13, Definition 2.17]. In [13] the authors do not assume that the inner products are continuous. The continuity of the inner products in [13] is implied by the continuity of the norm on the bundle. Since we only have upper semicontinuous C^*-bundles we need to assume that the inner products are continuous. This small difference in our definition means that when showing that a bundle is an imprimitivity bimodule bundle we often only need to check the continuity of inner products, as the other conditions were checked in [13].

The next proposition is a slight generalization of [13, Proposition 2.18]. We proved one direction above; the other follows exactly as it does in [13], so we omit it.

Proposition 2.9. *If $\mathscr{X}$ is an $\mathscr{A} - \mathscr{B}$-imprimitivity bimodule bundle, then $Z = \Gamma_0(T, \mathscr{X})$ is an $A - B$ imprimitivity bimodule over T. Conversely, if Z is an $A - B$ imprimitivity bimodule over T, then there exists a unique $\mathscr{A} - \mathscr{B}$ imprimitivity bimodule bundle $\mathscr{X}$ such that $Z = \Gamma_0(T, \mathscr{X})$.*

2.2. **Groupoids.** A groupoid is a small category in which every morphism is invertible. We say a groupoid E is second countable locally compact Hausdorff if it has a second countable locally compact Hausdorff topology in which composition and inversion are continuous. We assume all groupoids are second countable locally compact and Hausdorff. The objects of E can be identified with the identity morphisms. We refer to the set of identity morphisms as the unit space, denoted $E^{(0)}$, and elements of $E^{(0)}$ as units. There are two natural continuous surjections $r_E, s_E : E \to E^{(0)}$ given by $r_E(\gamma) = \gamma\gamma^{-1}$ and $s_E(\gamma) = \gamma^{-1}\gamma$. We drop the subscript from the notation when the domain is clear from context. For $u \in E^{(0)}$ we denote $E^u := r^{-1}(u)$ and $E_u := s^{-1}(u)$, and for D a subset of $E^{(0)}$ we denote $E|_D := \{\gamma \in E : r(\gamma), s(\gamma) \in D\}$. It is straightforward to check that $E|_D$ is a subgroupoid of E.

We say a groupoid E acts on the left of a space X if there exists a continuous open surjection $r_X : X \to E^{(0)}$ and a continuous map $E * X \to X$ given by $(\gamma, x) \mapsto \gamma x$ such that $r_X(\gamma x) = r_E(\gamma)$ and $\gamma(\eta x) = (\gamma\eta)x$ for composable γ and η.[2] The definition of a right action $X * E \to X : (x, \gamma) \mapsto x\gamma$ is analogous. We will use $E \cdot x$ to denote both the image of x in $E\backslash X$ as well as the orbit of x in X. If r_E is open and E acts on X, then the quotient map $X \to E\backslash X$ is open [16, Lemma 2.1].

An action of E on X is *principal* (or free) if $\gamma x = x$ implies $\gamma = r_X(x)$. An action is *proper* if the set $\{\gamma \in E : \gamma K \cap L \neq \emptyset\}$ is compact for all compact subsets

[2] Since γ and η are composable only if $s(\gamma) = r(\eta)$, the relation $\gamma(\eta x) = (\gamma\eta)x$ shows that E can only act on spaces fibered over $E^{(0)}$.

K and L of X. If the action of E on X is proper, then the quotient space $E\backslash X$ is locally compact Hausdorff [1, Proposition 2.1.12].

Note that r_E is open if and only if s_E is open. In this case E acts on the left and right of $E^{(0)}$ by $\gamma \cdot s(\gamma) := r(\gamma)$ and $r(\gamma) \cdot \gamma = s(\gamma)$. We say E is principal if this action is principal; we say E is proper if this action is proper. The orbit of a unit u under this action is then $E \cdot u := \{r(\gamma) : s(\gamma) = u\}$.

Throughout we assume that our groupoids come equipped with a Haar system. That is, a system of measures $\{\lambda^u\}_{u\in E^{(0)}}$ such that

(1) $\mathrm{supp}(\lambda^u) = E^u$,
(2) $u \mapsto \int_E f(\gamma)\, d\lambda^u(\gamma)$ is continuous for all $f \in C_c(E)$, and
(3) $\int_E f(\eta\gamma)\, d\lambda^{s(\eta)}(\gamma) = \int_E f(\gamma)\, d\lambda^{r(\eta)}(\gamma)$.

If E has a Haar system, then r and s are open [25, Corollary page 118]. Note that condition (2) implies that $\sup_{u\in E^{(0)}} \lambda^u(K) < \infty$ for all compact K in E.

Given a left action of E on X we define the transformation groupoid to be $E \ltimes X := \{(\gamma, x) : r_E(\gamma) = r_X(x)\}$ with unit space X and range and source maps $r(\gamma, x) = x$ and $s(\gamma, x) = \gamma^{-1}x$. If E has a Haar system $\{\lambda^u\}_{u\in E^{(0)}}$, then the set $\{\lambda^{r_X(x)} \times \delta_x\}_{x\in X}$ forms a Haar system for $E \ltimes X$. We can construct a transformation groupoid $X \rtimes E$ from a right action in a similar fashion.

Definition 2.10. Let E be a second countable locally compact Hausdorff groupoid with unit space $E^{(0)}$ and $\mathscr{Z}$ an upper semicontinuous Banach bundle over $E^{(0)}$ with (separable) section algebra Z. We say E acts on Z if for each $\gamma \in E$ there exists a norm preserving isomorphism $V_\gamma : Z(s(\gamma)) \to Z(r(\gamma))$ such that

(1) $V_\gamma V_\eta = V_{\gamma\eta}$ for all $(\gamma, \eta) \in E^{(2)}$ and
(2) the map $E * \mathscr{Z} \to \mathscr{Z}$ defined by $(\gamma, z) \mapsto V_\gamma(z)$ is continuous.

If $\mathscr{Z}$ is an upper semicontinuous C^*-bundle and V_γ is a $*$-isomorphism for all γ, then we refer to the pair (Z, V) as an *E-dynamical system.*

Let (A, α) be an E-dynamical system. Consider $\Gamma_c(E, r^*\mathscr{A})$. By [18, Proposition 4.4] the formulas

$$f * g(\gamma) := \int_E f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))\, d\lambda^{r(\gamma)}(\eta) \quad \text{and} \quad f^*(\gamma) := \alpha_\gamma(f(\gamma^{-1})^*)$$

define a $*$-algebra structure on $\Gamma_c(E, r^*\mathscr{A})$. We define a norm on $\Gamma_c(E, r^*\mathscr{A})$ by

$$\|f\|_I := \max\left\{ \sup_{u\in E^{(0)}} \int_E \|f(\gamma)\| d\lambda^u(\gamma),\ \sup_{u\in E^{(0)}} \int_E \|f(\gamma^{-1})\| d\lambda^u(\gamma) \right\}.$$

Let $\mathrm{Rep}(E, A)$ be the set of I-norm bounded representations of $\Gamma_c(E, r^*\mathscr{A})$. We then define the crossed product $A \rtimes_\alpha E$ to be the completion of $\Gamma_c(E, r^*\mathscr{A})$ under the norm $\|f\| = \sup\{\|\pi(f)\| : \pi \in \mathrm{Rep}(E, A)\}$. The reduced crossed product $A \rtimes_{\alpha,r} E$ is the completion of $\Gamma_c(E, r^*\mathscr{A})$ under the norm induced by "regular representations" [2, Section 2.2]. The crossed products considered in this paper involve "amenable" groupoids, and in this case the reduced crossed product coincides with the crossed product.

Example 2.11. Let E be a groupoid. Then $C_0(E^{(0)})$ is a $C_0(E^{(0)})$-algebra. The corresponding upper semicontinuous C^*-bundle is $E^{(0)} \times \mathbb{C}$. For each $\gamma \in E$ define $\mathrm{lt}_\gamma(s(\gamma), \kappa) = (r(\gamma), \kappa)$. Then lt is a continuous action of E on $C_0(E^{(0)})$ called a *left translation.* The resulting crossed product $C_0(E^{(0)}) \rtimes_{\mathrm{lt}} E$ is isomorphic to $C^*(E)$ [10, Remark 4.22].

3. Induced algebras

Definition 3.1. Let E be a principal and proper groupoid with a Haar system, $\mathscr{X}$ an upper semicontinuous Banach bundle over $E^{(0)}$, $Z = \Gamma_0(E^{(0)}, \mathscr{X})$, and $V = \{V_\gamma\}_{\gamma\in E}$ a continuous action of E on $\mathscr{X}$. Define

$$\operatorname{Ind}_E^{E^{(0)}}(Z,V) := \left\{ f \in \Gamma^b(E^{(0)}, \mathscr{X}) : \begin{array}{l} f(r(\gamma)) = V_\gamma(f(s(\gamma))) \text{ and} \\ E\cdot u \mapsto \|f(u)\| \text{ vanishes at } \infty \end{array} \right\}. \tag{1}$$

We denote $\operatorname{Ind}_E^{E^{(0)}}(Z,V)$ by $\operatorname{Ind}(Z,V)$ or just $\operatorname{Ind}(Z)$ when clear from context.

Throughout this section let E be a principal and proper groupoid with Haar system $\{\lambda^u\}_{u\in E^{(0)}}$, and $p_{\mathscr{X}} : \mathscr{X} \to E^{(0)}$ an upper semicontinuous Banach bundle.

Lemma 3.2. *For $h \in \Gamma_c(E^{(0)}, \mathscr{X})$ the map*

$$\lambda(h)(u) := \int_E V_\gamma(h(s(\gamma)))\, d\lambda^u(\gamma) \tag{2}$$

is a well-defined element of $\operatorname{Ind}_E^{E^{(0)}}(Z,V)$.

Proof. Since E is proper, the set $E^v \cap s^{-1}(\operatorname{supp}(h))$ is compact for each $v \in E^{(0)}$. Hence (2) is defined for all v and

$$\operatorname{supp}(\lambda(h)) \subset E\cdot \operatorname{supp}(h). \tag{3}$$

Because λ^u is a Haar system it is not hard to prove that $\lambda(h)$ is continuous. We show that $\lambda(h) \in \operatorname{Ind}_E^{E^{(0)}}(Z,V)$. It follows from a brief computation that $V_\eta(\lambda(h)(s(\eta))) = \lambda(h)(r(\eta))$. Thus $E\cdot v \mapsto \|\lambda(h)(v)\|$ is well defined and by (3) it has compact support contained in the image of $\operatorname{supp}(h)$ under the quotient map. Since the upper semicontinuous image of a compact set is bounded above, this implies that $\|\lambda(h)(v)\|$ is bounded and therefore $\lambda(h) \in \operatorname{Ind}(Z,V)$. □

Lemma 3.3. *Let $g \in \Gamma^b(E^{(0)}, \mathscr{X})$, $u \in E^{(0)}$, and $\epsilon > 0$.*

1. *Then for any $h \in \Gamma_c(E^{(0)}, \mathscr{X})$ such that $h(u) = g(u)$, there exists $\psi \in C_c(E^{(0)})$ such that $\lambda(\psi\cdot h) \in \operatorname{Ind}(Z,V)$ and $\|\lambda(\psi\cdot h)(u) - g(u)\| < \epsilon$.*
2. *For any $z \in \mathscr{X}$ there exists an $f \in \operatorname{Ind}(Z,V)$ such that $\|f(p_{\mathscr{X}}(z)) - z\| < \epsilon$.*

Proof. For item (1), let $g \in \Gamma^b(E^{(0)}, \mathscr{X})$, $u \in E^{(0)}$, $a = g(u)$, $\epsilon > 0$, and pick $h \in \Gamma_c(E^{(0)}, \mathscr{X})$ such that $h(u) = a$. Then the map

$$\gamma \mapsto V_\gamma(h(s(\gamma))) - h(r(\gamma))$$

is continuous. Since the norm is upper semicontinuous, the set

$$N_\epsilon := \{\gamma \in E : \|V_\gamma(h(s(\gamma))) - h(r(\gamma))\| < \epsilon\}$$

is open. Because E is principal and proper, we can apply [2, Lemma 5.3] to find an open neighborhood $U \subset E^{(0)}$ of u such that $\{\gamma : \gamma\cdot U \cap U \neq \emptyset\} \subset N_\epsilon \cap N_\epsilon^{-1}$. Pick a function $\phi \in C_c(E^{(0)})$ such that $0 \le \phi \le 1$, $\phi(u) = 1$, and $\operatorname{supp}(\phi) \subset U$. Define

$$\psi(v) = \left(\int_E \phi(s(\eta))\, d\lambda^u(\eta)\right)^{-1} \phi(v)$$

and note that

$$\int_E \psi(\gamma)\, d\lambda^u(\gamma) = \int_E \left(\int_E \phi(s(\eta))\, d\lambda^u(\eta)\right)^{-1} \phi(s(\gamma))\, d\lambda^u(\gamma) = 1.$$

By Lemma 3.2, $\lambda(\psi \cdot h) \in \operatorname{Ind}(Z, V)$. It remains to show that $\|\lambda(\psi \cdot h)(u) - a\| < \epsilon$. We compute

$$\|\lambda(\psi \cdot h)(u) - a\| = \left\| \int_E \psi(s(\gamma)) V_\gamma(h(s(\gamma)))\, d\lambda^u(\gamma) - h(u) \right\|$$

where, since $\int_E \psi(s(\gamma))\, d\lambda^u(\gamma) = 1$,

$$\begin{aligned} &= \left\| \int_E \psi(s(\gamma)) V_\gamma(h(s(\gamma)))\, d\lambda^u(\gamma) - \int_E \psi(s(\gamma))\, d\lambda^u(\gamma) h(u) \right\| \\ &\leq \int_E \psi(s(\gamma)) \|V_\gamma(h(s(\gamma))) - h(u)\|\, d\lambda^u(\gamma). \end{aligned}$$

However, $u \in U$ and $s(\gamma^{-1}) \in \operatorname{supp}(\psi) \subset U$ implies that $\gamma \in N_\epsilon$, so

$$\|\lambda(\psi \cdot h)(u) - a\| < \epsilon \int_E \psi(s(\gamma))\, d\lambda^u(\gamma) = \epsilon.$$

For item (2) pick $h \in \Gamma_c(E^{(0)}, \mathscr{Z})$ such that $h(p_{\mathscr{Z}}(z)) = z$. By the first part there exists $\psi \in C_c(E^{(0)})$ such that $\lambda(\psi \cdot h) \in \operatorname{Ind}(Z, V)$ and $\|\lambda(\psi \cdot h)(p_{\mathscr{Z}}(z)) - z\| < \epsilon$. Then $f = \lambda(\psi \cdot h)$ suffices. □

Let (A, α) be an E-dynamical system. For $u \in E^{(0)}$ let $\varepsilon_u : \Gamma^b(E^{(0)}, \mathscr{A}) \to A(u)$ be defined by evaluation: $\varepsilon_u(f) = f(u)$.

Proposition 3.4. *Let E be a principal and proper groupoid with a Haar system and (A, α) an E-dynamical system.*

(1) For $\phi \in C_0(E\backslash E^{(0)})$ and $f \in \operatorname{Ind}_E^{E^{(0)}}(A, \alpha)$ define

$$Q_\phi(f) : u \mapsto \phi(E \cdot u) f(u).$$

Then $\phi \mapsto Q_\phi$ defines a $C_0(E\backslash E^{(0)})$-algebra structure on $\operatorname{Ind}_E^{E^{(0)}}(A, \alpha)$.

(2) For $f \in \operatorname{Ind}_E^{E^{(0)}}(A, \alpha)$ the map

$$\tilde{R} : f \mapsto f|_{E \cdot u}$$

induces an isomorphism $R : \operatorname{Ind}_E^{E^{(0)}}(A, \alpha)(E \cdot u) \to \operatorname{Ind}_{E|_{E \cdot u}}^{E \cdot u}(\Gamma_0(E \cdot u, \mathscr{A}), \alpha)$.

(3) The evaluation map ε_u factors to an isomorphism of $\operatorname{Ind}_E^{E^{(0)}}(A, \alpha)(E \cdot u)$ with $A(u)$.

Proof. For item (1) first observe that $\operatorname{Ind}(A, \alpha)$ is a closed $*$-subalgebra of the C^*-algebra $\Gamma^b(E^{(0)}, \mathscr{A})$ and hence is also a C^*-algebra. Next we prove that $\operatorname{Ind}(A, \alpha)$ is a $C_0(E\backslash E^{(0)})$-algebra. We show that Q is a nondegenerate $*$-homomorphism from $C_0(E\backslash E^{(0)})$ to $Z(M(\operatorname{Ind}(A, \alpha)))$.

We start by demonstrating that $Q_\phi(f) \in \operatorname{Ind}(A, \alpha)$ for all $f \in \operatorname{Ind}(A, \alpha)$. It follows from straightforward computations that

$$\alpha_\gamma(Q_\phi(f)(s(\gamma))) = Q_\phi(f)(r(\gamma)) \quad \text{and} \quad \|Q_\phi(f)(u)\| \leq \|\phi\|_\infty \|f(u)\|.$$

This shows both that Q_ϕ is bounded by $\|\phi\|_\infty$ and that the map $E \cdot u \mapsto \|Q_\phi(f)(u)\|$ vanishes at infinity. Thus $Q_\phi(f) \in \operatorname{Ind}(A, \alpha)$ and Q_ϕ is a bounded operator on $\operatorname{Ind}(A, \alpha)$. Simple calculations show that Q_ϕ is linear and adjointable with adjoint $Q_{\overline{\phi}}$. Thus $Q_\phi \in M(\operatorname{Ind}(A, \alpha))$. Since ϕ is scalar-valued, $Q_\phi(f)g(u) = \phi(E \cdot u) f(u) g(u) = f Q_\phi(g)(u)$ so $Q_\phi \in Z(M(\operatorname{Ind}(A, \alpha)))$, as desired. More simple computations show that $\phi \mapsto Q_\phi$ is a homomorphism.

To show $\operatorname{Ind}(A,\alpha)$ is a $C_0(E\backslash E^{(0)})$-algebra it remains to show that the map $\phi \mapsto Q_\phi$ is nondegenerate. Pick $f \in \operatorname{Ind}(A,\alpha)$ and let $\epsilon > 0$ be given. Since $E\cdot u \mapsto \|f(u)\|$ vanishes at infinity, the set $K := \{E\cdot u \in E\backslash E^{(0)} : \|f(u)\| \geq \epsilon\}$ is compact. Choose $\phi \in C_c(E\backslash E^{(0)})$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on K. Then

$$\|f(u) - Q_\phi(f)(u)\| = \|f(u) - \phi(E\cdot u)f(u)\| \leq \begin{cases} 0 & \text{if} \quad E\cdot u \in K \\ \|f(u)\| & \text{if} \quad E\cdot u \notin K \end{cases} < \epsilon.$$

Hence the map $\phi \mapsto Q_\phi$ is nondegenerate and so $\operatorname{Ind}(\mathscr{A},\alpha)$ is a $C_0(E\backslash E^{(0)})$-algebra.

For item (2) consider the map $f \mapsto f|_{E\cdot u}$. Routine computations show this defines a $*$-homomorphism $\tilde{R} : \operatorname{Ind}(A,\alpha) \to \operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$. If $f \in \operatorname{Ind}(A,\alpha)$ and $\phi \in C_{E\cdot u,0}(E\backslash E^{(0)})$, then $Q_\phi(f)|_{E\cdot u} = 0$, so

$$I_{E\cdot u} := C_{E\cdot u,0}(E\backslash E^{(0)}) \cdot \operatorname{Ind}(A,\alpha) \subset \ker(\tilde{R}). \tag{4}$$

Thus $\tilde{R}$ induces a $*$-homomorphism R from $\operatorname{Ind}(A,\alpha)(E\cdot u)$ into $\operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$. We show R is injective by proving $\ker(\tilde{R}) = I_{E\cdot u}$. By (4), it remains to show that $\ker(\tilde{R}) \subset I_{E\cdot u}$.

Let $\epsilon > 0$ be given. Suppose that $f \in \ker(\tilde{R})$. Then $f|_{E\cdot u} \equiv 0$. Let $K = \{E\cdot u \in E\backslash E^{(0)} : \|f(u)\| \geq \epsilon\}$ and let U be the complement of K in $E\backslash E^{(0)}$. By assumption $E\cdot u \in U$, and U is open since K is a compact subset of the Hausdorff space $E\backslash E^{(0)}$. Pick functions $\phi, \psi \in C_c(E\backslash E^{(0)})$ such that $0 \leq \phi, \psi \leq 1$, $\operatorname{supp}(\phi) \subset U$, $\phi(E\cdot u) = 1$, and $\psi|_K \equiv 1$. Then $(1-\phi)\psi \in C_{E\cdot u,0}(E\backslash E^{(0)})$ and

$$\|f(v) - Q_{(1-\phi)\psi}(f)(v)\| \leq \begin{cases} 0 & \text{if} \quad E\cdot v \in K \\ \|f(v)\| & \text{if} \quad E\cdot v \notin K \end{cases} < \epsilon.$$

Therefore $f \in I_{E\cdot u}$, giving $I_{E\cdot u} = \ker(\tilde{R})$, and thus $R : \operatorname{Ind}(A,\alpha)(E\cdot u) \to \operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$ is injective.

To show R is an isomorphism, it remains to show it is surjective. Pick a function $F \in \operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$ and consider $F(u) \in A(u)$. Let $\epsilon > 0$ be given. By Lemma 3.3, there exists $f \in \operatorname{Ind}(A,\alpha)$ such that $\|f(u) - F(u)\| < \epsilon$. For $v \in E\cdot u$ there exists $\gamma \in E_u$ such that $v = r(\gamma)$. Thus $\|f(v)-F(v)\| = \|f(r(\gamma))-F(r(\gamma))\| = \|\alpha_\gamma(f(u))-\alpha_\gamma(F(u))\| < \epsilon$. Therefore the image of R is dense in $\operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$ and thus R is surjective.

For item (3), by Lemma 3.3, the map $\varepsilon_u : \operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha) \to A(u)$ given by $f \mapsto f(u)$ has a dense range. It is straightforward to show that ε_u is a $*$-homomorphism. It is isometric since $\|f\|_\infty = \|f(u)\|$ for all $f \in \operatorname{Ind}(\Gamma_0(E\cdot u,\mathscr{A}),\alpha)$, and thus ε_u is an isomorphism, as desired. $\square$

Remark 3.5. To differentiate between the $C_0(E\backslash E^{(0)})$-algebra $\operatorname{Ind}_E^{E^{(0)}}(A,\alpha)$ and the corresponding upper semicontinuous C^*-bundle, we will denote the upper semicontinuous C^*-bundle by $\operatorname{Ind}_E^{E^{(0)}}(\mathscr{A},\alpha)$.

Let E be a principal and proper groupoid and suppose (A,α) is an E-dynamical system. Then [2] defines the generalized fixed point algebra $\operatorname{Fix}(A,\alpha)$ to be the closed span of $\lambda(a^*b)$ in $M(A)$ for $a,b \in \Gamma_c(E^{(0)},\mathscr{A})$.[3] By Lemma 3.2, $\operatorname{Fix}(A,\alpha) \subset \operatorname{Ind}(A,\alpha)$. In the following proposition we show that $\operatorname{Fix}(A,\alpha) = \operatorname{Ind}(A,\alpha)$.

[3] $\operatorname{Fix}(A,\alpha)$ is denoted A^α in [2].

Proposition 3.6. *Let E be a principal and proper groupoid and (A, α) an E-dynamical system. Then the generalized fixed point algebra* $\mathrm{Fix}(A, \alpha)$ *is equal to* $\mathrm{Ind}_E^{E^{(0)}}(A, \alpha)$.

Proof. The proof of [2, Proposition 4.4] shows that $\mathrm{Fix}(A, \alpha)$ is a $C_0(E\backslash E^{(0)})$-subalgebra of $\mathrm{Ind}(A, \alpha)$. By Proposition 2.5, to show $\mathrm{Fix}(A, \alpha)$ is all of $\mathrm{Ind}(A, \alpha)$ it suffices to prove $\mathrm{Fix}(A, \alpha)(E \cdot u)$ is dense in $\mathrm{Ind}(A, \alpha)(E \cdot u)$ for every $u \in E^{(0)}$.

Let $g \in \mathrm{Ind}(A, \alpha)$, $u \in E^{(0)}$, and $\epsilon > 0$ be given. Pick an approximate unit $\{e_i\}$ for $A(u)$ and i_0 large enough so that $\|e_{i_0} g(u) - g(u)\| < \epsilon/2$. Choose $f \in \Gamma_c(E^{(0)}, \mathscr{A})$ such that $f(u) = e_{i_0}$. Then $\|f(u)g(u) - g(u)\| < \epsilon/2$. By Lemma 3.3 there exists $\psi \in C_c(E^{(0)})$ such that $\|\lambda(\psi \cdot (fg))(u) - (fg)(u)\| < \epsilon/2$. Thus $\|\lambda(\psi \cdot (fg))(u) - g(u)\| < \epsilon$. Furthermore, since f and $u \mapsto \psi(u)g(u)$ are both in $\Gamma_c(E^{(0)}, \mathscr{A})$, $\lambda(\psi \cdot (fg)) \in \mathrm{Fix}(A, \alpha)$. □

Remark 3.7. We identify $\mathrm{Fix}(A, \alpha)$ with $\mathrm{Ind}_E^{E^{(0)}}(A, \alpha)$. As with $\mathrm{Ind}_E^{E^{(0)}}(A, \alpha)$, we denote the upper semicontinuous C^*-bundle corresponding to $\mathrm{Fix}(A, \alpha)$ by $\mathrm{Fix}(\mathscr{A}, \alpha)$.

We next study the representations and pure states of $\mathrm{Ind}(A, \alpha)$ by showing they come from representations and pure states of A. We will use this analysis in Proposition 5.8.

Let B be a C^*-algebra and denote by $P(B)$ the pure states on B, $\widehat{B}$ the irreducible representations on B, and $\Lambda_B : P(B) \to \widehat{B}$ the map given by the GNS construction. That is, for $\tau \in P(B)$ there exists a unit vector h such that $\tau(a) = (\Lambda_B(\tau)(a)h, h)$. If B is a $C_0(T)$-algebra, let σ_B be the associated map from $\widehat{B}$ to T. For $t \in T$ let $q_t : B \to B(t)$ be the quotient map. Suppose $\pi \in \widehat{B}$ and $t = \sigma_B(\pi)$. By [27, Proposition C.5] there exists a $\pi_t \in \widehat{B(t)}$ such that $\pi = \pi_t \circ q_t$. Thus $\pi(a)$ depends only on the class of a in $A(t)$. Similarly, for $\tau \in P(B)$ and $t = \sigma_B(\Lambda_B(\tau))$,

$$\tau(a) = (\Lambda_B(\tau)(a)h, h) = (\Lambda_B(\tau)_t(q_t(a))h, h),$$

so that τ depends only on the class of a in $A(t)$. It follows there exists τ_t such that $\tau(a) = \tau_t(q_t(a))$.

Let E be a principal and proper groupoid and (A, α) an E-dynamical system. Recall that A must be a $C_0(E^{(0)})$-algebra. Suppose $\pi \in \widehat{A}$ and let $u = \sigma_A(\pi)$. By [27, Proposition C.5] there exists a $\pi_u \in \widehat{A(u)}$ such that $\pi = \pi_u \circ q_u$. For $\gamma \in E_u$,

$$\gamma \cdot \pi(a) := \pi_u \circ \alpha_\gamma^{-1} \circ q_{r(\gamma)}(a)$$

defines a continuous action of E on $\widehat{A}$ [11, Proposition 1.1].

Recall from Proposition 3.4 that $\mathrm{Ind}(A, \alpha)$ is a $C_0(E\backslash E^{(0)})$-algebra and the evaluation map $\varepsilon_u : \mathrm{Ind}(A, \alpha) \to A(u)$ induces an isomorphism of $\mathrm{Ind}(A, \alpha)(E \cdot u)$ with $A(u)$. We can define a representation $M(\pi)$ of $\mathrm{Ind}(A, \alpha)$ by

$$M(\pi)(f) = \pi_{\sigma_A(\pi)}(\varepsilon_{\sigma_A(\pi)}(f)) \tag{5}$$

for $f \in \mathrm{Ind}(A, \alpha)$. Now $M(\pi)(\mathrm{Ind}(A, \alpha)) = \pi_{\sigma_A(\pi)}(A(\sigma_A(\pi))) = \pi(A)$. Thus, since π is irreducible, so is $M(\pi)$. Similarly for $\tau \in P(A)$, let $v = \sigma_A \circ \Lambda_A(\tau)$. We can define a state on $\mathrm{Ind}(A, \alpha)$ by $N(\tau) = \tau_v(\varepsilon_v(f))$. It is straightforward to show that

$$\Lambda_{\mathrm{Ind}(A,\alpha)}(N(\tau)) = M(\Lambda_A(\tau)), \tag{6}$$

which implies that $N(\tau)$ is pure.

Our goal is to show that M defines a continuous, open bijection from $E\backslash\widehat{A}$ to $(\mathrm{Ind}(A,\alpha))^\wedge$. For this we first show N is continuous.

Lemma 3.8. *N is continuous for the weak-$*$ topology.*

Proof. Suppose $\tau_i \to \tau$. By definition $\tau_i \to \tau$ if and only if $\tau_i(a) \to \tau(a)$ for all $a \in A$. Pick $f \in \mathrm{Ind}(A,\alpha)$. Define $v_i = \sigma_A \circ \Lambda_A(\tau_i)$ and $v = \sigma_A \circ \Lambda_A(\tau)$. Since σ_A and Λ_A are continuous, $v_i \to v$. Thus there exists a compact set $K \subset E^{(0)}$ such that $\{v_i, v\} \subset K$. Pick $\phi \in C_c(E^{(0)})$ such that $\phi|_K \equiv 1$; then $\phi \cdot f \in A$. Since $\phi \cdot f(v_i) = f(v_i)$, we have $(\tau_i)_{v_i}(\varepsilon_{v_i}(\phi \cdot f)) = (\tau_i)_{v_i}(\varepsilon_{v_i}(f))$. Hence

$$\begin{aligned} N(\tau_i)(f) &= (\tau_i)_{v_i}(\varepsilon_{v_i}(f)) = (\tau_i)_{v_i}(\varepsilon_{v_i}(\phi \cdot f)) = \tau_i(\phi \cdot f) \\ &\to \tau(\phi \cdot f) = (\tau)_v(\varepsilon_v(f)) = N(\tau)(f), \end{aligned}$$

so N is continuous. $\square$

Lemma 3.9. *For each $\pi \in \widehat{A}$ define $M(\pi) : \mathrm{Ind}_E^{E^{(0)}}(A,\alpha) \to B(\mathcal{H}_\pi)$ as in (5). Then M induces a homeomorphism of $E\backslash\widehat{A}$ to $(\mathrm{Ind}_E^{E^{(0)}}(A,\alpha))^\wedge$.*

Proof. We showed above that $M(\pi)$ is irreducible for $\pi \in \widehat{A}$. Note that M descends to a well-defined map of $E\backslash\widehat{A}$ to $(\mathrm{Ind}_E^{E^{(0)}}(A,\alpha))^\wedge$ since

$$\begin{aligned} M(\gamma \cdot \pi)(f) &= (\gamma \cdot \pi)_{r(\gamma)}(f(r(\gamma))) = \pi_{s(\gamma)}(\alpha_{\gamma^{-1}}(f(r(\gamma)))) \\ &= \pi_{s(\gamma)}(f(s(\gamma))) = M(\pi)(f). \end{aligned}$$

We now show that M induces a bijection of $E\backslash\widehat{A}$ and $(\mathrm{Ind}(A,\alpha))^\wedge$. To see M is surjective, suppose ρ is an irreducible representation on $\mathrm{Ind}(A,\alpha)$. We can factor ρ to an irreducible representation of some fiber $\mathrm{Ind}(A,\alpha)(E \cdot v)$, transport it to $A(v)$ via the isomorphism of Proposition 3.4, and then lift to a representation π of A. Tracing through definitions shows that $M(\pi) = \rho$. To see if M is injective, suppose $M(\pi)$ is unitarily equivalent to $M(\rho)$. Let $u = \sigma_A(\pi)$ and $v = \sigma_A(\rho)$. By definition, $M(\pi) = \pi_u \circ \varepsilon_u$ and $M(\rho) = \rho_v \circ \varepsilon_v$. By [27, Proposition C.5], $M(\pi)$ and $M(\rho)$ both factor to representations of some fiber $\mathrm{Ind}(A,\alpha)(E \cdot w)$. However, since the quotient map onto the fiber is given by restriction, the only way these statements can be compatible is if $u, v \in E \cdot w$. Thus there exists a $\gamma \in E$ such that $u = s(\gamma)$ and $v = r(\gamma)$; that is, $\pi_u \in \widehat{A(s(\gamma))}$ and $\rho_v \in \widehat{A(r(\gamma))}$. Since $M(\pi)$ is equivalent to $M(\rho)$, there exists a unitary U such that

$$\pi_{s(\gamma)} \circ \varepsilon_{s(\gamma)}(f) = U\rho_{r(\gamma)} \circ \varepsilon_{r(\gamma)}(f)U^* = U\rho_{r(\gamma)} \circ f(r(\gamma))U^* = U\rho_{r(\gamma)}(\alpha_\gamma(f(s(\gamma))))U^*,$$

which implies that π is unitarily equivalent to $\gamma^{-1} \cdot \rho$. Thus M induces a bijection.

Since N is continuous by Lemma 3.8 and $\Lambda_{\mathrm{Ind}(A,\alpha)}(N(\tau)) = M(\Lambda_A(\tau))$ by (6), we get that M is continuous as well. Finally we show that M is open. Suppose $M(\pi_i) \to M(\pi)$ in $(\mathrm{Ind}_E^{E^{(0)}}(A,\alpha))^\wedge$. Let $u_i = \sigma_A(\pi_i)$ and $u = \sigma_A(\pi)$. Since $M(\pi_i) \to M(\pi)$, a straightforward argument shows that $E \cdot u_i \to E \cdot u$. The map from $E^{(0)}$ onto $E\backslash E^{(0)}$ is open, so we may pass to a subnet, relabel, and choose γ_i such that $r(\gamma_i) = \gamma_i \cdot u_i \to u$. To prove $\gamma_i \cdot \pi_i \to \pi$ it suffices to show that if J is an ideal in A such that $J \not\subset \ker \pi$, then eventually $J \not\subset \ker \gamma_i \cdot \pi_i$ [21, Corollary A.28]. Choose $a \in J$ such that $\pi(a) = \pi_u(a(u)) \neq 0$. Let $\epsilon = \|\pi(a)\|/4$ and use Lemma 3.3 to find $f \in \mathrm{Ind}(A,\alpha)$ such that $\|f(u) - a(u)\| < \epsilon$. Since the norm on $\mathscr{A}$ is upper semicontinuous, the set $\{b \in \mathscr{A} : \|b\| < \epsilon\}$ is open. Both f and a are continuous as functions on $E^{(0)}$, so $f(r(\gamma_i)) - a(r(\gamma_i)) \to f(u) - a(u)$ and this implies that, eventually, $\|f(r(\gamma_i)) - a(\gamma_i \cdot u_i)\| < \epsilon$. Next, observe that by construction $M(\pi)(f) = \pi_u(f(u))$ and that

$$\big|\|M(\pi)(f)\| - \|\pi(a)\|\big| \leq \|\pi_u(f(u)) - \pi_u(a(u))\| \leq \|f(u) - a(u)\| < \epsilon.$$

Since $\epsilon = \|\pi(a)\|/4$, this implies that $\|M(\pi)(f)\| > \frac{\|\pi(a)\|}{2}$. We know from [21, Lemma A.30] that the map $\rho \to \rho(f)$ is lower semicontinuous on $(\mathrm{Ind}_{E^{(0)}}^{E}(A,\alpha))^\wedge$, so the set $\{\rho : \|\rho(f)\| > \|\pi(a)\|/2\}$ is open. Since we assumed $M(\gamma_i \cdot \pi_i) \to M(\pi)$, this implies that for large i,

$$\|M(\gamma_i \cdot \pi_i)(f)\| = \|(\gamma_i \cdot \pi_i)_{\gamma_i \cdot u_i}(f(\gamma_i \cdot u_i))\| > \|\pi(a)\|/2.$$

However, we also eventually have

$$\begin{aligned}\big|\|M(\gamma_i \cdot \pi_i)(f)\| - \|\gamma_i \cdot \pi_i(a)\|\big| &\leq \|(\gamma_i \cdot \pi_i)_{r(\gamma_i)}(f(r(\gamma_i))) - (\gamma_i \cdot \pi_i)_{r(\gamma_i)}(a(r(\gamma_i)))\| \\ &\leq \|f(r(\gamma_i)) - a(r(\gamma_i))\| < \epsilon.\end{aligned}$$

Therefore, for large enough i, $\|\gamma_i \cdot \pi_i(a)\| > \frac{\|\pi(a)\|}{4}$, and in particular $a \notin \ker \gamma_i \cdot \pi_i$. This suffices to show that eventually $J \not\subset \ker \gamma_i \cdot \pi_i$, and thus M is open. $\square$

Corollary 3.10. *Let E be a principal and proper groupoid and (A, α) be an E-dynamical system. Then $(A \rtimes_\alpha E)^\wedge \cong E\backslash\widehat{A}$.*

Proof. Since E is principal and proper, [2, Theorem 5.2] shows that (A, α) is a "saturated" E-dynamical system. That is, $\mathrm{Fix}(A, \alpha)$ is Morita equivalent to $A \rtimes_{\alpha,r} E$. Now E is proper and hence amenable, so $A \rtimes_{\alpha,r} E \cong A \rtimes_\alpha E$ [1, Corollary 2.1.7, Proposition 6.1.8]. Thus $(A \rtimes_\alpha E)^\wedge \cong (A \rtimes_{\alpha,r} E)^\wedge \cong \mathrm{Fix}(A, \alpha)^\wedge$. But by Proposition 3.6, $\mathrm{Fix}(A, \alpha) \cong \mathrm{Ind}(A, \alpha)$, and by Lemma 3.9, $\mathrm{Ind}(A, \alpha)^\wedge \cong E\backslash\widehat{A}$; therefore $(A \rtimes_\alpha E)^\wedge \cong E\backslash\widehat{A}$. $\square$

Remark 3.11. Note that E being principal implies that the isotropy subgroupoid of E is trivial; further E being proper implies that $E\backslash E^{(0)}$ is Hausdorff and therefore "regular" in the sense of [11]. Thus Corollary 3.10 is a special case of [11, Theorem 2.22].

4. The Brauer semigroup

Let E be a second countable locally compact Hausdorff groupoid. As in [12] and [13] we want to define a commutative binary operation on classes of E-dynamical systems. In those papers the binary operation is induced by a balanced tensor product. In [13] they consider only those E-dynamical systems (A, α) with A a continuous trace, and hence nuclear. Thus [13] does not need to specify a particular tensor product. Since we are considering all (separable) E-dynamical systems we must make a choice, and so we follow [12] and use the maximal balanced tensor product introduced in [4] for compact spaces. The results from [4] can be easily extended to arbitrary locally compact Hausdorff spaces, so we cite them without further comment.

Let T be a second countable locally compact Hausdorff space, and A and B be $C_0(T)$-algebras. Consider the ideal J_T of $A \otimes_{\max} B$ generated by

$$\{(\phi \cdot a) \otimes b - a \otimes (\phi \cdot b) : a \in A,\ b \in B,\ \phi \in C_0(T)\}.$$

We define the $C_0(T)$-balanced tensor product by

$$A \otimes_T B := A \otimes_{\max} B / J_T.$$

The map $\phi\cdot(a\otimes b) := (\phi\cdot a\otimes b)$ defines a $C_0(T)$-structure on $A\otimes_T B$. Furthermore, if $\mathscr{A}\otimes_T\mathscr{B}$ is the upper semicontinuous C^*-bundle over T constructed from $A\otimes_T B$, then we have $(A\otimes_T B)(t) = A(t)\otimes_{\max} B(t)$ [6, Lemma 2.4]. Note that if X is a locally compact Hausdorff space and $q : X\to T$ is a continuous open surjection, then $C_0(X)$ is a $C_0(T)$-algebra and $C_0(X)\otimes_T A\cong\Gamma_0(X, q^*\mathscr{A})$ [20, Proposition 1.3].

Let (A,α) and (B,β) be E-dynamical systems. Since $\alpha_\gamma : A(s(\gamma))\to A(r(\gamma))$ and $\beta_\gamma : B(s(\gamma))\to B(r(\gamma))$ are isomorphisms, we can use [21, Lemma B.31] to show that the map $\alpha_\gamma\otimes\beta_\gamma : A(s(\gamma))\otimes_{\max} B(s(\gamma))\to A(r(\gamma))\otimes_{\max} B(r(\gamma))$ characterized by $\alpha_\gamma\otimes\beta_\gamma(a\otimes b) = \alpha_\gamma(a)\otimes\beta_\gamma(b)$ is an isomorphism. The argument on page 919 of [13] shows that the collection $\{\alpha_\gamma\otimes\beta_\gamma\}$ defines a continuous E-action on $A\otimes_{E^{(0)}} B$. Lemma 2.4 of [12] implies that the $C_0(E^{(0)})$-balanced tensor product is an associative, commutative, binary operation on the set of E-dynamical systems.

Definition 4.1 ([13, Definition 3.1])**.** Two E-dynamical systems (A,α) and (B,β) are *equivariantly Morita equivalent* if there is an $\mathscr{A}-\mathscr{B}$-imprimitivity bimodule bundle $\mathscr{Z}$ which admits an action V of E by isomorphisms such that

$$ {}_{A(r(\gamma))}\langle V_\gamma(\xi), V_\gamma(\zeta)\rangle = \alpha_\gamma({}_{A(r(\gamma))}\langle\xi,\zeta\rangle) \text{ and } \langle V_\gamma(\xi), V_\gamma(\zeta)\rangle_{B(r(\gamma))} = \beta_\gamma(\langle\xi,\zeta\rangle_{B(r(\gamma))}). $$

In this case we will write $(A,\alpha)\sim_{(Z,V)}(B,\beta)$.

Remark 4.2. If $(\mathscr{Z},V)$ is an $(A,\alpha)-(B,\beta)$ equivariant imprimitivity bimodule bundle, then a computation shows that $V_\gamma(a\cdot z) = \alpha_\gamma(a)\cdot V_\gamma(z)$ and $V_\gamma(z\cdot b) = V_\gamma(z)\cdot\beta_\gamma(b)$. Furthermore, it follows from [18, Section 9.1] that the corresponding crossed products $A\rtimes_\alpha E$ and $B\rtimes_\beta E$ are Morita equivalent.

The proof of the following lemma is similar to the proof of [13, Lemma 3.2] and has been omitted.

Lemma 4.3. *Equivariant Morita equivalence is an equivalence relation. For an E-dynamical system (A,α) we denote its equivariant Morita equivalence class by $[A,\alpha]$.*

Next we show that the balanced tensor product gives us a semigroup operation.

Proposition 4.4. *Let $[A,\alpha]$ and $[B,\beta]$ be equivariant Morita equivalence classes of E-dynamical systems. Then*

$$ [A,\alpha][B,\beta] := [A\otimes_{E^{(0)}} B, \alpha\otimes\beta] \tag{7} $$

is a well-defined commutative binary operation with identity $[C_0(E^{(0)}), \mathrm{lt}]$.

Proof. Since $\otimes_{E^{(0)}}$ is an associative, commutative, binary operation on E-dynamical systems, it suffices to show that the multiplication in (7) is well defined. Suppose $(A,\alpha)\sim_{(X,V)}(C,\vartheta)$ and $(B,\beta)\sim_{(Y,W)}(D,\delta)$. Then as in the proof of [13, Proposition 3.6] we can define an imprimitivity bimodule bundle $\mathscr{Z}$ with fibers given by the external tensor product $X(u)\otimes Y(u)$ under the inner products characterized by

$$ {}_{A(u)\otimes B(u)}\langle x\otimes y, x'\otimes y'\rangle := {}_{A(u)}\langle x,x'\rangle\otimes{}_{B(u)}\langle y,y'\rangle \quad\text{and} $$
$$ \langle x\otimes y, x'\otimes y'\rangle_{C(u)\otimes D(u)} := \langle x,x'\rangle_{C(u)}\otimes\langle y,y'\rangle_{D(u)}. $$

The topology on $\mathscr{Z}$ is characterized by the condition that $u \mapsto f(u) \otimes g(u)$ is continuous for all $f \in X$, $g \in Y$. The continuity of the left and right actions follows as in the proof of [13, Proposition 3.6]. It remains to show continuity of the inner products.

By symmetry it suffices to show the $A \otimes_{E^{(0)}} B$ valued inner product is continuous. Let $z_i \to z$ and $w_i \to w$ in $\mathscr{Z}$ with $p_{\mathscr{Z}}(w_i) = p_{\mathscr{Z}}(z_i) = u_i$ and $p_{\mathscr{Z}}(w) = p_{\mathscr{Z}}(z) = u$. Let $\epsilon > 0$. Pick finite subsets $J, K \subset X(u) \times Y(u)$ such that $\|z - \sum_{(x,y)\in J} x \otimes y\| < \epsilon$ and $\|w - \sum_{(x',y')\in K} x' \otimes y'\| < \epsilon$. Let π_i be the projection onto the i-th factor. For each $x \in \pi_1(J \cup K)$ pick $f_x \in X$ such that $f_x(u) = x$, and similarly for each $y \in \pi_2(J \cup K)$ pick $g_y \in Y$ such that $g_y(u) = y$. By the continuity of f_x, g_y, and the inner products we have ${}_{A(u_i)}\langle f_x(u_i), f_{x'}(u_i)\rangle \to {}_{A(u)}\langle f_x(u), f_{x'}(u)\rangle$ and ${}_{B(u_i)}\langle g_y(u_i), g_{y'}(u_i)\rangle \to {}_{B(u)}\langle g_y(u), g_{y'}(u)\rangle$. By the definition of the topology on $\mathscr{A} \otimes_{E^{(0)}} \mathscr{B}$ we then get

$$\begin{aligned} {}_{A\otimes B(u_i)}\langle f_x \otimes g_y(u_i), f_{x'} \otimes g_{y'}(u_i)\rangle &= {}_{A(u_i)}\langle f_x(u_i), f_{x'}(u_i)\rangle \otimes {}_{B(u_i)}\langle g_y(u_i), g_{y'}(u_i)\rangle \\ \to {}_{A(u)}\langle f_x(u), f_{x'}(u)\rangle \otimes {}_{B(u)}\langle g_y(u), g_{y'}(u)\rangle &= {}_{A\otimes B(u)}\langle f_x \otimes g_y(u), f_{x'} \otimes g_{y'}(u)\rangle\,. \end{aligned}$$

Thus

$$\begin{aligned} &{}_{A\otimes B(u_i)}\left\langle \sum_{(x,y)\in J} f_x \otimes g_y(u_i), \sum_{(x',y')\in K} f_{x'} \otimes g_{y'}(u_i)\right\rangle \\ &\qquad\qquad \to {}_{A\otimes B(u)}\left\langle \sum_{(x,y)\in J} f_x \otimes g_y(u), \sum_{(x',y')\in K} f_{x'} \otimes g_{y'}(u)\right\rangle. \end{aligned}$$

By the definition of the norms on $Z(u_i)$ and $Z(u)$, and the Cauchy-Schwartz inequality, we have

$$\left\| {}_{A\otimes B(u_i)}\left\langle \sum_{(x,y)\in J} f_x \otimes g_y(u_i), \sum_{(x',y')\in K} f_{x'} \otimes g_{y'}(u_i)\right\rangle - {}_{A\otimes B(u_i)}\langle z_i, w_i\rangle \right\| < 2\epsilon(\|z\| + \epsilon).$$

Proposition 2.2 now gives that ${}_{A\otimes B(u_i)}\langle z_i, w_i\rangle \to {}_{A\otimes B(u)}\langle z, w\rangle$. Thus the $\mathscr{A} \otimes_{E^{(0)}} \mathscr{B}$-valued inner product is continuous as desired, and the multiplication in (7) is well defined.

Finally, showing that $[C_0(E^{(0)}), \mathrm{lt}]$ acts as an identity for this multiplication follows as in the proof of [13, Proposition 3.6]. □

Definition 4.5. Let E be a locally compact Hausdorff groupoid. The *Brauer semigroup*, denoted $S(E)$, is the Abelian monoid of equivariant Morita equivalence classes of E-dynamical systems with the operation given in Proposition 4.4. The *Brauer group* is the set $\mathrm{Br}(E) = \{[A, \alpha] \in S(E) : A \text{ has continuous trace}\}$.

Remark 4.6. Lemma 6.6 of [12] states that $A \otimes_{E^{(0)}} B$ has continuous trace with spectrum $E^{(0)}$ if and only if A and B have continuous trace with spectrum $E^{(0)}$. Thus if $[A, \alpha]$ is invertible, then A must have continuous trace with spectrum $E^{(0)}$ and therefore be an element of the Brauer group $\mathrm{Br}(E)$. Conversely, if A has continuous trace with spectrum $E^{(0)}$, then $[A, \alpha] \in \mathrm{Br}(E)$ and [13, Theorem 3.7] says $[A, \alpha]$ is invertible. That is, the invertible elements of $S(E)$ are precisely those in $\mathrm{Br}(E)$.

5. Main theorem

Throughout suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems $\{\lambda_G^u\}_{u\in G^{(0)}}$ and $\{\lambda_H^u\}_{u\in H^{(0)}}$, and let X be a (G,H)-equivalence [15, Definition 2.1]. In particular, G and H act freely and properly on the left and right of X, respectively, and $r_X : X \to H^{(0)}$ and $s_X : X \to G^{(0)}$ induce homeomorphisms $G\backslash X \cong H^{(0)}$ and $X/H \cong G^{(0)}$, respectively. We can define a transformation groupoid $G \ltimes X \rtimes H := \{(\gamma, x, \eta) \in G \times X \times H : r_G(\gamma) = r_X(x), s_X(x) = r_H(\eta)\}$ associated to this equivalence whose topology is given by the relative topology and whose operations are defined by

$$r(\gamma,x,\eta) = x, \qquad s(\gamma,x,\eta) = \gamma^{-1}x\eta,$$
$$(\gamma,x,\eta)(\xi,\gamma^{-1}x\eta,\zeta) = (\gamma\xi,x,\eta\zeta), \qquad (\gamma,x,\eta)^{-1} = (\gamma^{-1},\gamma^{-1}x\eta,\eta^{-1}).$$

The transformation groupoids $G \ltimes X$ and $X \rtimes H$ embed naturally in $G \ltimes X \rtimes H$ by $(\gamma, x) \mapsto (\gamma, x, s_X(x))$ and $(x, \eta) \mapsto (r_X(x), x, \eta)$, respectively. We identify $G \ltimes X$ and $X \rtimes H$ with their image under these embeddings.

Suppose (A,ω) is a $G \ltimes X \rtimes H$-dynamical system. Then $\omega^G := \omega|_{G\ltimes X}$ and $\omega^H := \omega|_{X\rtimes H}$ are continuous actions of $G \ltimes X$ and $X \rtimes H$ on A, respectively. Furthermore, since both sides are equal to $\omega_{(\gamma,x,\eta)}$, we have

$$\omega^G_{(\gamma,x)} \circ \omega^H_{(\gamma^{-1}x,\eta)} = \omega^H_{(x,\eta)} \circ \omega^G_{(\gamma,x\eta)}. \tag{8}$$

The next proposition shows that every action of $G \ltimes X \rtimes H$ arises in this way.

Proposition 5.1. *Let A be a $C_0(X)$-algebra and X a (G,H)-equivalence. Then A admits a $G \ltimes X \rtimes H$ action ω if and only if there exist actions ω^G and ω^H of $G \ltimes X$ and $X \rtimes H$ on A that satisfy equation* (8).

Proof. The only if part was shown above. Suppose that A admits actions ω^G and ω^H of $G \ltimes X$ and $X \rtimes H$ that satisfy equation (8). Define $\omega_{(\gamma,x,\eta)} := \omega^G_{(\gamma,x)} \circ \omega^H_{(\gamma^{-1}x,\eta)}$. Using (8) one can show that ω is well defined and respects the groupoid multiplication. It is an isomorphism of the fibers and continuous since ω^G and ω^H are as well. □

Let (A,α) be an H-dynamical system. Then s_X^*A is a $C_0(X)$-algebra and we can define maps

$$s_X^*\alpha_{(\gamma,x,\eta)} : s_X^*A(\gamma^{-1}x\eta) \to s_X^*A(x) \quad \text{by} \quad (\gamma^{-1}x\eta, a) \mapsto (x, \alpha_\eta(a)).$$

We show in Proposition 5.3 that $s_X^*\alpha$ defines a continuous action of $G \ltimes X \rtimes H$ on s_X^*A. Similar statements hold for G-dynamical systems and r_X. Our goal is to prove the following theorem.

Theorem 5.2. *Suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems and X is a (G,H)-equivalence. Then the following statements hold:*

1. *The map $[B,\beta] \mapsto [s_X^*B, s_X^*\beta]$ defines an isomorphism $\upsilon^{X,H} : S(H) \to S(G \ltimes X \rtimes H)$ such that $s_X^*B \rtimes_{s_X^*\beta} (G \ltimes X \rtimes H)$ is Morita equivalent to $B \rtimes_\beta H$.*
2. *The map $[A,\alpha] \mapsto [r_X^*A, r_X^*\alpha]$ defines an isomorphism $\upsilon^{X,G} : S(G) \to S(G \ltimes X \rtimes H)$ such that $r_X^*A \rtimes_{r_X^*\alpha} (G \ltimes X \rtimes H)$ is Morita equivalent to $A \rtimes_\alpha G$.*

(3) The map $\upsilon^X = (\upsilon^{X,G})^{-1} \circ \upsilon^{X,H}$ defines an isomorphism from $S(H)$ to $S(G)$ such that if $\upsilon^X([B,\beta]) = [A,\alpha]$, then $A \rtimes_\alpha G$ is Morita equivalent to $B \rtimes_\beta H$.

To prove Theorem 5.2 it suffices to prove item (1) since the others will follow by symmetry. To do this we will first analyze $\upsilon^{X,H}$ and then define an inverse.

5.1. **The map $\upsilon^{X,H}$ and its properties.** Let $\mathscr{Z}$ be an upper semicontinuous Banach bundle over $H^{(0)}$, $Z = \Gamma_0(H^{(0)}, \mathscr{Z})$, and consider $s_X^*\mathscr{Z}$. Define $s_X^* Z := \Gamma_0(X, s_X^*\mathscr{Z})$.

Proposition 5.3. *Let $\mathscr{Z}$ be an upper semicontinuous Banach bundle over $H^{(0)}$ and V a continuous H action on Z. For $(\gamma, x, \eta) \in G \ltimes X \rtimes H$ and $(\gamma^{-1}x\eta, z) \in s_X^*\mathscr{Z}$ define*

$$(s_X^* V)_{(\gamma,x,\eta)}(\gamma^{-1}x\eta, z) := (x, V_\eta(z)). \tag{9}$$

Then $\{(s_X^*V)_{(\gamma,x,\eta)}\}$ is a continuous $G \ltimes X \rtimes H$ action on $s_X^* Z$. In particular, if (A,α) is an H-dynamical system, then $(s_X^*A, s_X^*\alpha)$ is a $G \ltimes X \rtimes H$-dynamical system.

Proof. This proof is relatively straightforward and the details have been omitted for brevity. Algebraic computations show that $s_X^* V_{(\gamma,x\eta)}$ is an isomorphism and that s_X^*V respects the groupoid operations. The continuity of s_X^*V follows from the continuity of V and an application of Proposition 2.2. $\square$

Hence $\nu^{X,H}$ defines a map from H-dynamical systems to $G \ltimes X \rtimes H$-dynamical systems. We show in the next proposition that $\nu^{X,H}$ descends to a map on equivariant Morita equivalence classes of H-dynamical systems.

Proposition 5.4. *Let X be a (G,H)-equivalence and $(\mathscr{Z}, V)$ an equivariant imprimitivity bimodule bundle between the H-dynamical systems (A,α) and (B,β). Then $(s_X^*\mathscr{Z}, s_X^*V)$ is an equivariant imprimitivity bimodule bundle between the $G \ltimes X \rtimes H$-dynamical systems $(s_X^*A, s_X^*\alpha)$ and $(s_X^*B, s_X^*\beta)$ where the inner products and actions are defined as follows:*

$$\begin{aligned} {}_{s_X^*A(x)}\langle (x,z),(x,w)\rangle &:= (x, {}_{A(s(x))}\langle z,w\rangle), & \langle (x,z),(x,w)\rangle_{s_X^*B(x)} &:= (x, \langle z,w\rangle_{B(s(x))}), \\ (x,a)\cdot(x,z) &:= (x, a\cdot z), & (x,z)\cdot(x,b) &:= (x, z\cdot b) \end{aligned}$$

for $x \in X$, $z,w \in Z(s_X(x))$, $a \in A(s_X(x))$ and $b \in B(s_X(s))$.

Proof. By the definition of $s_X^*\mathscr{Z}$, each fiber of $s_X^*\mathscr{Z}$ is isomorphic as a Hilbert bimodule to a fiber of $\mathscr{Z}$ and therefore is an imprimitivity bimodule. To show that $s_X^*\mathscr{Z}$ is an imprimitivity bimodule bundle it remains to show that the actions and inner products are continuous. However, this follows quickly using the continuity of the actions on $\mathscr{Z}$. Finally, straightforward computations show both

$$\begin{aligned} &(s_X^*\alpha)_{(\gamma,x,\eta)}\big({}_{s_X^*A(\gamma^{-1}x\eta)}\big\langle (\gamma^{-1}x\eta, z),(\gamma^{-1}x\eta,z')\big\rangle\big) \\ &\qquad = {}_{s_X^*A(x)}\big\langle (s_X^*V)_{(\gamma,x,\eta)}(\gamma^{-1}x\eta,z), (s_X^*V)_{(\gamma,x,\eta)}(\gamma^{-1}x\eta,z')\big\rangle \quad \text{and} \\ &(s_X^*\beta)_{(\gamma,x,\eta)}\big(\big\langle (\gamma^{-1}x\eta,z),(\gamma^{-1}x\eta,z')\big\rangle_{s_X^*B(\gamma^{-1}x\eta)}\big) \\ &\qquad = \big\langle (s_X^*V)_{(\gamma,x,\eta)}(\gamma^{-1}x\eta,z), (s_X^*V)_{(\gamma,x,\eta)}(\gamma^{-1}x\eta,z')\big\rangle_{s_X^*B(x)}. \end{aligned}$$

$\square$

Proposition 5.4 shows that $\nu^{X,H}$ descends to a well-defined set map $S(H) \to S(G \rtimes X \ltimes H)$. Next we show that $\nu^{X,H}$ is a semigroup homomorphism.

Proposition 5.5. *Let (A, α) and (B, β) be H-dynamical systems.*

(1) The map $\Phi : s_X^*\mathscr{A} \otimes_X s_X^*\mathscr{B} \to s_X^*(\mathscr{A} \otimes_{H^{(0)}} \mathscr{B})$ characterized by

$$\Phi : (x, a) \otimes (x, b) \mapsto (x, a \otimes b)$$

is an isomorphism intertwining the groupoid actions $(s_X^*\alpha) \otimes_X (s_X^*\beta)$ and $s_X^*(\alpha \otimes_{H^{(0)}} \beta)$.

(2) We have $[s_X^*A, s_X^*\alpha][s_X^*B, s_X^*\beta] = [s_X^*(A \otimes_{H^{(0)}} B), s_X^*(\alpha \otimes_{H^{(0)}} \beta)]$. That is, $\nu^{X,H}$ is a homomorphism of Brauer semigroups.

Proof. Recall from Section 4 that

$$(s_X^*A \otimes_X s_X^*B)(x) = s_X^*A(x) \otimes s_X^*B(x) \quad \text{and}$$
$$A(s_X(x)) \otimes B(s_X(x)) = (A \otimes_{H^{(0)}} B)(s_X(x)).$$

Furthermore, $(x, a) \mapsto a$ is an isomorphism from $s_X^*A(x)$ to $A(s_X(x))$ and similarly $s_X^*B(x) \cong B(s_X(x))$ and $s_X^*(A \otimes_{H^{(0)}} B)(x) \cong (A \otimes_{H^{(0)}} B)(s_X(x))$. Thus for a fixed x the map $(x, a) \otimes (x, b) \mapsto (x, a \otimes b)$ is the composition of isomorphisms

$$\begin{aligned}(s_X^*A \otimes_X s_X^*B)(x) &\to s_X^*A(x) \otimes s_X^*B(x) \to A(s_X(x)) \otimes B(s_X(x)) \\ &\to (A \otimes_{H^{(0)}} B)(s_X(x)) \to s_X^*(A \otimes_{H^{(0)}} B)(x).\end{aligned}$$

Therefore Φ is an isomorphism on the fibers and hence bijective. Thus to show Φ is an isomorphism we need to show that Φ and Φ^{-1} are continuous. The continuity of Φ follows from an application of Proposition 2.2. The argument is similar to the one given below and will not be reproduced here.

To see Φ^{-1} is continuous, suppose $(x_i, z_i) \to (x, z) \in s_X^*(\mathscr{A} \otimes_{H^{(0)}} \mathscr{B})$. Let $\epsilon > 0$ be given and pick a finite subset $I \subset A(s_X(x)) \times B(s_X(x))$ so that we have $\| \sum_{(a,b)\in I} a \otimes b - z\| < \epsilon$. Let π_i be the projection onto the i-th factor. For $a \in \pi_1(I)$ and $b \in \pi_2(I)$ pick functions $f_a \in A$ and $g_b \in B$ such that $f_a(s_X(x)) = a$ and $g_b(s_X(x)) = b$. Now choose a compact neighborhood K of $x \in X$ and a function $\phi \in C_c(X)$ such that $\phi|_K \equiv 1$. The maps $F_{a,b} : y \mapsto (y, \phi(y)f_a(s_X(y))) \otimes (y, \phi(y)g_b(s_X(y)))$ are continuous and compactly supported and thus are in $s_X^*A \otimes_X s_X^*B$. Hence

$$\sum_{(a,b)\in I} F_{a,b}(x_i) \to \sum_{(a,b)\in I} F_{a,b}(x) = \sum_{(a,b)\in I} (x, a) \otimes (x, b).$$

To show $\Phi^{-1}(x_i, z_i) \to \Phi^{-1}(x, z)$ it suffices to show $\Phi^{-1}(x_i, z_i)$ is eventually close to $\sum_{(a,b)\in I} F_{a,b}(x_i)$, by Proposition 2.2. Note that since K is a compact neighborhood of x and $\phi|_K \equiv 1$ we eventually have

$$F_{a,b}(x_i) = (x_i, f_a(s_X(x_i))) \otimes (x_i, g_b(s_X(x_i))) = \Phi^{-1}(x_i, f_a(s_X(x_i)) \otimes g_b(s_X(x_i))).$$

Since f_a and g_b are continuous, for large enough i

$$\left\| \sum_{(a,b)\in I} (x_i, f_a(s_X(x_i)) \otimes g_b(s_X(x_i))) - (x_i, z_i) \right\| < \epsilon.$$

Again since Φ is an isomorphism of the fibers, we eventually have

$$\begin{aligned}\Bigg\| \sum_{(a,b)\in I} F_{a,b}(x_i) - \Phi^{-1}(x_i, z_i)\Bigg\| \\ &= \Bigg\| \sum_{(a,b)\in I} \Phi^{-1}(x_i, f_a(s_X(x_i)) \otimes g_b(s_X(x_i))) - \Phi^{-1}(x_i, z_i)\Bigg\| \\ &= \Bigg\| \sum_{(a,b)\in I} (x_i, f_a(s_X(x_i)) \otimes g_b(s_X(x_i))) - (x_i, z_i)\Bigg\| < \epsilon.\end{aligned}$$

So Proposition 2.2 shows that $\Phi^{-1}(x_i, z_i) \to \Phi^{-1}(x, z)$ and Φ^{-1} is continuous.

It remains to show that Φ intertwines the actions. This follows from a computation on elementary tensors. Part (2) of the proposition follows from part (1) since

$$\begin{aligned}[s_X^* A, s_X^* \alpha][s_X^* B, s_X^* \beta] &= [s_X^* A \otimes_X s_X^* B, s_X^* \alpha \otimes_X s_X^* \beta] \\ &= [s_X^*(A \otimes_{H^{(0)}} B), s_X^*(\alpha \otimes_{H^{(0)}} \beta)]. \qquad \square\end{aligned}$$

5.2. **The generalized fixed point algebra.** The inverse of $\upsilon^{X,H}$ will be constructed using the generalized fixed point algebra. Let (A, ω) be a $G \ltimes X \rtimes H$-dynamical system. Since G acts freely and properly on X, $G \ltimes X$ is a principal and proper groupoid. Thus we may construct the generalized fixed point algebra, $\mathrm{Fix}(A, \omega^G)$ [2, Proposition 4.4, Remark 3.10]. By Proposition 3.6, $\mathrm{Fix}(A, \omega^G)$ is equal to $\mathrm{Ind}(A, \omega^G)$. We denote both by $\mathrm{Fix}_G(A)$. Since $s_X : X \to H^{(0)}$ descends to a homeomorphism of $(G \ltimes X)\backslash X$ with $H^{(0)}$, by Proposition 3.4 $\mathrm{Fix}_G(A)$ is a $C_0(H^{(0)})$-algebra with fibers $\mathrm{Fix}_G(A)(u) = \mathrm{Ind}(\Gamma_0(s_X^{-1}(u), \mathscr{A}), \alpha)$.

More generally, let $\mathscr{Z}$ be an upper semicontinuous Banach bundle over X endowed with a continuous $G \ltimes X \rtimes H$ action $V = \{V_{(\gamma,x,\eta)}\}$. We can define actions V^G and V^H of $G \ltimes X$ and $X \rtimes H$ on $\mathscr{Z}$ by restriction. Let $Z = \Gamma_0(X, \mathscr{Z})$. Define $\mathrm{Fix}_G(Z) := \mathrm{Ind}(Z, V^G)$. Consider the sets

$$\mathrm{Fix}_G(Z)(u) := \{f \in \Gamma^b(s_X^{-1}(u), \mathscr{Z}) : V^G_{(\gamma,x)}(f(\gamma^{-1}x)) = f(x)\}.$$

For $x \in s_X^{-1}(u)$ the evaluation map $\varepsilon_x : \mathrm{Fix}_G(Z)(u) \to Z(x)$ is isometric since $s_X^{-1}(u) = (G \ltimes X) \cdot x$ for some $x \in X$ and $\|f(\gamma x)\| = \|V^G_{(\gamma,x)}(f(x))\| = \|f(x)\|$. Consequently, ε_x has a closed range and it then follows from Lemma 3.3 that ε_x is surjective. In other words, $\varepsilon_x : \mathrm{Fix}_G(Z)(u) \to Z(x)$ is a norm preserving isomorphism. We can then put a topology on $\bigsqcup_{u\in H^{(0)}} \mathrm{Fix}_G(Z)(u)$ using Proposition 2.6 and the sections $u \mapsto F|_{s_X^{-1}(u)}$ for $F \in \mathrm{Fix}_G(Z)$. Denote $\bigsqcup_{u\in H^{(0)}} \mathrm{Fix}_G(Z)(u)$ equipped with this topology by $\mathrm{Fix}_G(\mathscr{Z})$. Using Proposition 2.5 and Lemma 3.3, $\mathrm{Fix}_G(Z) = \Gamma_0(H^{(0)}, \mathrm{Fix}_G(\mathscr{Z}))$.

Proposition 5.6. *Let $\mathscr{Z}$ be an upper semicontinuous Banach bundle over X and V a continuous action of $G \ltimes X \rtimes H$ on Z. For $\eta \in H$ and $f \in \mathrm{Fix}_G(Z)(s(\eta))$ define*

$$\mathrm{Fix}_G(V)_\eta(f)(x) := V^H_{(x,\eta)}(f(x\eta)) = V_{(r_X(x),x,\eta)}(f(x\eta)). \tag{10}$$

Then $\mathrm{Fix}_G(V)$ is a well-defined continuous action of H on $\mathrm{Fix}_G(Z)$. In particular, if (A, ω) is a $G \ltimes X \rtimes H$-dynamical system, then $(\mathrm{Fix}_G(A), \mathrm{Fix}_G(\omega))$ is an H-dynamical system.

Proof. Suppose $\eta \in H$ and $f \in \mathrm{Fix}_G(Z)(s(\eta))$. We first show $\mathrm{Fix}_G(V)_\eta(f) \in \mathrm{Fix}_G(Z)(r_H(\eta))$. Since $\mathrm{supp}(f) \subset s_X^{-1}(s_H(\eta))$ we have $\mathrm{supp}(\mathrm{Fix}_G(V)_\eta(f)) \subset s_X^{-1}(r_H(\eta))$. We know $\mathrm{Fix}_G(V)_\eta(f)$ is continuous since V and f are continuous. It is bounded since f is bounded and $V_{(r_X(x),x,\eta)}$ is a norm preserving isomorphism for all $x \in s_X^{-1}(r_H(\eta))$. Lastly,

$$\begin{aligned} V^G_{(\gamma,x)}(\mathrm{Fix}_G(V)_\eta(f)(\gamma^{-1}x)) &= V^G_{(\gamma,x)}V^H_{(\gamma^{-1}x,\eta)}(f(\gamma^{-1}x\eta)) = V^H_{(x,\eta)}V^G_{(\gamma,x\eta)}(f(\gamma^{-1}x\eta)) \\ &= V^H_{(x,\eta)}(f(x\eta)) = \mathrm{Fix}_G(V)_\eta(f)(x). \end{aligned}$$

Thus $\mathrm{Fix}_G(V)_\eta(f) \in \mathrm{Fix}_G(Z)(r_H(\eta))$. It follows from routine computations that each $\mathrm{Fix}_G(V)_\eta$ is an isometric isomorphism and that $\mathrm{Fix}_G(V)$ preserves the groupoid operations.

It remains to show that $\mathrm{Fix}_G(V)$ is continuous. Suppose $\eta_i \to \eta_0$ and $f_i \in \mathrm{Fix}(Z)(s(\eta_i))$ such that $f_i \to f_0$. We need to show $\mathrm{Fix}_G(V)_{\eta_i}(f_i) \to \mathrm{Fix}_G(V)_{\eta_0}(f_0)$. It suffices to show that every subnet has a subnet converging to $\mathrm{Fix}_G(V)_{\eta_0}(f_0)$. Pass to a subnet, relabel, and pick $x_0 \in s_X^{-1}(r_H(\eta_0))$ and $F \in \mathrm{Fix}_G(Z)$ such that $F|_{s_X^{-1}(r(\eta_0))} = \mathrm{Fix}_G(V)_{\eta_0}(f_0)$. Since s_X is open, by passing to a subnet, we can choose $x_i \in s_X^{-1}(r_H(\eta_i))$ such that $x_i \to x_0$. It follows from an application of Proposition 2.2 that $f_i(x_i\eta_i) \to f_0(x_0\eta_0)$ in $\mathscr{Z}$. Thus by the continuity of V,

$$\begin{aligned} \mathrm{Fix}_G(V)_{\eta_i}(f_i)(x_i) = V_{(r_X(x_i),x_i,\eta_i)}(f(x_i\eta_i)) &\to \\ V_{(r_X(x_0),x_0,\eta_0)}(f(x_0\eta_0)) &= \mathrm{Fix}_G(V)_{\eta_0}(f_0)(x_0). \end{aligned}$$

Let $\epsilon > 0$. It follows from the definition of the topology on $\mathscr{Z}$ and the continuity of F that eventually $\|F(x_i) - \mathrm{Fix}_G(V)_{\eta_i}(f_i)(x_i)\| < \epsilon$. Since $V^G_{(\gamma,x_i)}$ is a norm preserving isomorphism we have, for large i and for all $\gamma \in r_G^{-1}(r_X(x_i))$, that

$$\|V^G_{(\gamma,x_i)}(F(x_i)) - V^G_{(\gamma,x_i)}(\mathrm{Fix}_G(V)_{\eta_i}(f_i)(x_i))\| < \epsilon.$$

Because $F \in \mathrm{Fix}_G(Z)$ and the $\mathrm{Fix}_G(V)_{\eta_i}(f_i)$ are in $\mathrm{Fix}_G(Z)(r(\eta_i))$, this implies that eventually $\|F|_{s_X^{-1}(r_H(\eta_i))} - \mathrm{Fix}_G(V)_{\eta_i}(f_i)\|_\infty < \epsilon$. Another application of Proposition 2.2 now shows $\mathrm{Fix}_G(V)_{\eta_i}(f_i) \to \mathrm{Fix}_G(V)_{\eta_0}(f_0)$, as desired. $\square$

We need to show that Fix_G induces a well-defined map on equivariant Morita equivalence classes. For this we use the next proposition.

Proposition 5.7. *Let X be a (G, H)-equivalence and $(\mathscr{Z}, V)$ an equivariant imprimitivity bimodule bundle for $G \ltimes X \rtimes H$-dynamical systems (A, α) and (B, β). Then $(\mathrm{Fix}_G(\mathscr{Z}), \mathrm{Fix}_G(V))$ is an equivariant imprimitivity bimodule bundle for H-dynamical systems $(\mathrm{Fix}_G(A), \mathrm{Fix}_G(\alpha))$ and $(\mathrm{Fix}_G(B), \mathrm{Fix}(\beta))$, where the left and right actions and inner products are defined by*

$$\begin{aligned} {}_{\mathrm{Fix}_G(A)(u)}\langle z, w\rangle &:= x \mapsto {}_{A(x)}\langle z(x), w(x)\rangle, & \langle z, w\rangle_{\mathrm{Fix}_G(B)(u)} &:= x \mapsto \langle z(x), w(x)\rangle_{B(x)}, \\ a \cdot z &:= x \mapsto a(x) \cdot z(x), & z \cdot b &:= x \mapsto z(x) \cdot b(x) \end{aligned}$$

for $u \in H^{(0)}$, $x \in s_X^{-1}(u)$, $a \in A$, $b \in B$ and $z, w \in Z$.

Proof. First note that $\mathrm{Fix}_G(V)$ is a continuous action on $\mathrm{Fix}_G(Z)$ by Proposition 5.6. Also $\mathrm{Fix}_G(Z)(u) \cong Z(x)$ for any $x \in s_X^{-1}(u)$, and the Hilbert bimodule structure on $\mathrm{Fix}_G(Z)(u)$ is the one pulled back from $Z(x)$ under this isomorphism. Thus each fiber of $\mathrm{Fix}_G(\mathscr{Z})$ is an imprimitivity bimodule. To show that $\mathrm{Fix}_G \mathscr{Z}$ is an imprimitivity bimodule bundle it suffices to show that the operations are continuous. By symmetry it suffices to show that the $\mathrm{Fix}_G(\mathscr{A})$ action and $\mathrm{Fix}_G(\mathscr{A})$

inner product are continuous. We show only the continuity of the $\mathrm{Fix}_G(\mathscr{A})$ action. The proof of continuity for the inner product is similar.

Suppose $a_i \to a_0$ and $z_i \to z_0$ in $\mathrm{Fix}_G(\mathscr{A})$ and $\mathrm{Fix}_G(\mathscr{Z})$, respectively. Let $\epsilon > 0$ and define $u_i = p(a_i) = p(z_i)$ for all i. Pick $a \in \mathrm{Fix}_G(A)$ such that $a|_{s_X^{-1}(u_0)} = a_0$ and $z \in \mathrm{Fix}_G(Z)$ such that $z|_{s_X^{-1}(u_0)} = z_0$. Since $a|_{s_X^{-1}(u_i)}$ and a_i both converge to a_0 in $\mathrm{Fix}_G(\mathscr{A})$ we must eventually have $\|a_i - a|_{s_X^{-1}(u_i)}\| < \epsilon$. Similarly, we eventually have $\|z_i - z|_{s_X^{-1}(u_i)}\| < \epsilon$. Notice this also implies that for large i

$$\|a_i\| \leq \|a_i - a|_{s_X^{-1}(u_i)}\| + \|a|_{s_X^{-1}(u_i)}\| \leq \epsilon + \|a\|,$$

so that $\{\|a_i\|\}$ must be bounded by some M. Finally, observe that

$$a|_{s_X^{-1}(u_i)} \cdot z|_{s_X^{-1}(u_i)} = (a \cdot z)|_{s_X^{-1}(u_i)} \to (a \cdot z)|_{s_X^{-1}(u_0)} = a_0 \cdot z_0.$$

We may now compute for $x \in s_X^{-1}(u_i)$,

$$\begin{aligned} &\|(a \cdot z)|_{s_X^{-1}(u_i)}(x) - a_i \cdot z_i(x)\| \\ &\qquad \leq \|a(x) \cdot z(x) - a_i(x) \cdot z(x)\| + \|a_i(x) \cdot z(x) - a_i(x) \cdot z_i(x)\| \\ &\qquad \leq \|a|_{s_X^{-1}(u_i)} - a_i\|\|z|_{s_X^{-1}}\| + \|a_i\|\|z|_{s_X^{-1}(u_i)} - z_i\| \leq \epsilon\|z\| + M\epsilon. \end{aligned}$$

Hence $\|(a\cdot z)|_{s_X^{-1}(u_i)} - a_i \cdot z_i\|$ is eventually small and we can now use Proposition 2.2 to conclude that $a_i \cdot z_i \to a \cdot z$. Finally, the following identities can be verified with a brief computation:

$$\begin{aligned} \mathrm{Fix}_G(\alpha)_\eta({}_{\mathrm{Fix}_G(A)(s(\eta))}\langle z, w\rangle)(x) &= {}_{\mathrm{Fix}_G(A)(r(\eta))}\langle \mathrm{Fix}_G(V)_\eta(z), \mathrm{Fix}_G(V)_\eta(w)\rangle (x), \\ \mathrm{Fix}_G(\beta)_\eta(\langle z, w\rangle_{\mathrm{Fix}_G(B)(s(\eta))})(x) &= \langle \mathrm{Fix}_G(V)_\eta(z), \mathrm{Fix}_G(V)_\eta(w)\rangle_{\mathrm{Fix}_G(B)(r(\eta))} (x). \end{aligned}$$

□

5.3. **An isomorphism of Brauer semigroups.** In this section we show that $\nu^{X,H}$ and Fix_G are inverses. We begin by showing $\nu^{X,H} \circ \mathrm{Fix}_G = \mathrm{id}$.

Proposition 5.8. *Let (A, ω) be a $G \ltimes X \rtimes H$-dynamical system. Then the map characterized by*

$$\Upsilon(x, f) := f(x) \quad \textit{for } f \in \mathrm{Fix}_G(A)(s_X(x))$$

defines an isomorphism from $s_X^* \mathrm{Fix}_G(\mathscr{A})$ to $\mathscr{A}$. Furthermore, Υ intertwines the action $s_X^* \mathrm{Fix}_G(\omega)$ with ω.

Proof. Suppose (A, ω) is a $G \ltimes X \rtimes H$-dynamical system. By [20, Proposition 1.3]

$$s_X^* \mathrm{Fix}_G(A) = \Gamma_0(X, s_X^* \mathrm{Fix}_G(\mathscr{A})) = C_0(X) \otimes_{G\backslash X} \mathrm{Fix}_G(A).$$

For the first statement, we define a $C_0(X)$-linear isomorphism $\tilde{\Upsilon} : s_X^* \mathrm{Fix}_G(A) \to A$ whose associated isomorphism of upper semicontinuous C^*-bundles is Υ. Consider the map

$$\tilde{\Upsilon} : C_0(X) \otimes_{G\backslash X} \mathrm{Fix}_G(A) \to A \quad \text{characterized by} \quad \tilde{\Upsilon}(\phi \otimes f)(x) = \phi(x) f(x).$$

Then $\tilde{\Upsilon}$ defines a $C_0(X)$-linear $*$-homomorphism. By comparing with elementary tensors we see that $\tilde{\Upsilon}(F)(x) = F(x)(x) = \Upsilon(x, F(x))$ for $F \in \Gamma_0(X, s_X^* \mathrm{Fix}_G(\mathscr{A}))$. Therefore the map of $s_X^* \mathrm{Fix}_G(A)$ induced by Υ is $\tilde{\Upsilon}$.

Let B be the image of $\tilde{\Upsilon}$. By definition $C_0(X) \cdot B \subset B \subset A$. Pick $x \in X$ and $a \in A(x)$. Given $\epsilon > 0$, Lemma 3.3 implies that there exists an $F \in \mathrm{Fix}_G(A)$ with

$\|F(x) - a\| < \epsilon$. Pick $\phi \in C_c(X)$ such that $\phi(x) = 1$; then $\tilde{\Upsilon}(\phi \otimes F)(x) = F(x)$. Thus Proposition 2.5 implies B is dense in A and therefore $\tilde{\Upsilon}$ is onto.

To show that $\tilde{\Upsilon}$ is injective we show it preserves norms. If $F \in s_X^* \operatorname{Fix}_G(A)$, then

$$\begin{aligned}\|\tilde{\Upsilon}(F)\| &= \sup_{\pi \in \widehat{A}} \|\pi(\tilde{\Upsilon}(F))\| = \max_{x \in X} \sup_{\pi \in \widehat{A(x)}} \|\pi \circ q_x(\tilde{\Upsilon}(F))\| \\ &= \max_{x \in X} \sup_{\pi \in \widehat{A(x)}} \|\pi(F(x)(x))\| = \max_{x \in X} \sup_{\pi \in \widehat{A(x)}} \|\pi \circ \varepsilon_x(F(x))\| \\ &= \max_{x \in X} \sup_{\pi \in \widehat{A(x)}} \|M(\pi \circ q_x)_{G \cdot x}(F(x))\|.\end{aligned}$$

By Lemma 3.9, $\{M(\pi \circ q_x)_{G\cdot x} : \pi \in \widehat{A(x)}\} = (\operatorname{Fix}_G(A)(s_X(x)))^\wedge$, and therefore $\|\tilde{\Upsilon}(F)\| = \|F\|$. To see Υ intertwines the actions is a computation which we omit. □

Next we show $\operatorname{Fix}_G \circ \nu^{X,H} = \mathrm{id}$.

Proposition 5.9. *Let (A, β) be an H-dynamical system. The map*

$$\Psi : a \mapsto (x \mapsto (x, a(s_X(x))))$$

defines an isomorphism from A to $\operatorname{Fix}_G(s_X^* A)$ that intertwines β and $\operatorname{Fix}_G(s_X^* \beta)$.

Proof. For $a \in A$, the map $u \mapsto a(u)$ is continuous and bounded into $\mathscr{A}$ so the map $x \mapsto (x, a(s_X(x)))$ is continuous and bounded into $s_X^* \mathscr{A}$. Furthermore,

$$\begin{aligned}(s_X^*\beta)^G_{(\gamma,x)}(\gamma^{-1}x, a(s_X(\gamma^{-1}x))) &= s_X^*\beta_{(\gamma,x,s_X(x))}(\gamma^{-1}x, a(s_X(\gamma^{-1}x))) \\ &= (x, a(s_X(\gamma^{-1}x))) = (x, a(s_X(x))).\end{aligned}$$

Since $u \mapsto a(u)$ vanishes at infinity and $\|(x, a(s_X(x)))\| = \|a(s_X(x))\|$, the map $G \cdot x \mapsto \|(x, a(s_X(x)))\|$ vanishes at infinity too. That is, $x \mapsto (x, a(s_X(x))) \in \operatorname{Fix}_G(s_X^* A)$. By definition the map Ψ is $C_0(H^{(0)})$-linear and maps onto the fibers. Thus by Proposition 2.5 Ψ is onto. The map Ψ is isometric since both norms are supremum norms. Thus Ψ is an isomorphism, as desired.

Since Ψ is an isomorphism of the section algebras it induces an isomorphism of the upper semicontinuous C^*-bundles. From the definition of Ψ, the corresponding bundle isomorphism sends $a \in A(u)$ to the map $s_X^{-1}(u) \to \mathscr{A}$ given by $x \mapsto (x, a)$. It follows from a brief computation that the isomorphism is equivariant. □

5.4. **Morita equivalence.** Let E be a principal and proper groupoid and (A, α) an E-dynamical system. Theorem 5.2 of [2] says that (A, α) is saturated with respect to the subalgebra $C_c(E^{(0)}) \cdot A = \Gamma_c(E^{(0)}, \mathscr{A})$. By [2, Definition 5.1] this means that $\Gamma_c(E^{(0)}, \mathscr{A})$ with actions and pre-inner products given by

$$\begin{aligned}{}_{A \rtimes_r E}\langle f, g\rangle(\gamma, x) &:= f(r(\gamma))\alpha_\gamma(g(s(\gamma))^*), \\ \langle f, g\rangle_{\operatorname{Fix}(A,\alpha)}(u) &:= \int_E \alpha_\gamma(f(s(\gamma))^* g(s(\gamma)))\, d\lambda_E^u(\gamma), \\ F \cdot f(u) &:= \int_E F(\gamma)\alpha_\gamma(f(s(\gamma)))\, d\lambda_E^u(\gamma), \\ f \cdot m(u) &:= f(u)m(u)\end{aligned}$$

for $F \in \Gamma_c(E, r^*\mathscr{A})$, $f, g \in \Gamma_c(E^{(0)}, \mathscr{A})$, and $m \in \operatorname{Fix}(A, \alpha)$ completes to an $A \rtimes_r E - \operatorname{Fix}(A, \alpha)$ imprimitivity bimodule $\operatorname{IMP}(A, E, \alpha)$. We will denote $\operatorname{IMP}(A, E, \alpha)$

by IMP(A) when the action is clear from context. Note that since E acts properly on $E^{(0)}$, E is topologically amenable [1, Corollary 2.1.7] and thus measurewise amenable by [1, Proposition 3.3.5] so that $A \rtimes_{r,\alpha} E = A \rtimes_\alpha E$ [1, Proposition 6.1.8]. Thus we only need to consider the full crossed products.

If C is an invariant closed subspace of $E^{(0)}$, then $E|_C$ is also a principal and proper groupoid. So $(A(C), E|_C, \alpha)$ is a saturated proper dynamical system and we get that IMP($A(C)$) is an $A(C) \rtimes E|_C - \mathrm{Fix}(A(C), \alpha)$ imprimitivity bimodule as above. Define

$$\mathscr{X} := \bigsqcup_{E\cdot u \in E\backslash E^{(0)}} \mathrm{IMP}(A(E\cdot u))$$

and let $p_{\mathscr{X}} : \mathscr{X} \to E\backslash E^{(0)}$ be the obvious map. Since $\mathrm{IMP}(A(E\cdot u))$ is the completion of the section algebra $\Gamma_c(E\cdot u, \mathscr{A})$, IMP($A$) is the completion of $\Gamma_c(E^{(0)}, \mathscr{A})$, and the map $f \mapsto f|_{E\cdot u}$ from $\Gamma_c(E^{(0)}, \mathscr{A}) \to \Gamma_c(E\cdot u, \mathscr{A})$ is onto, we can consider $\Gamma_c(E^{(0)}, \mathscr{A})$ as a dense subalgebra of sections of $\mathscr{X}$ and use $\Gamma_c(E^{(0)}, \mathscr{A})$ as in Proposition 2.6 to define an upper semicontinuous Banach bundle structure on $\mathscr{X}$. In the next proposition we reconcile the imprimitivity bimodules $\mathrm{IMP}(A(E\cdot u))$ and IMP(A) by showing $\Gamma_0(E\backslash E^{(0)}, \mathscr{X}) \cong \mathrm{IMP}(A)$.

Proposition 5.10. *Suppose E is a principal and proper groupoid, (A, α) an E-dynamical system, and $\mathscr{X}$ as above. Then $\Gamma_0(E\backslash E^{(0)}, \mathscr{X})$ is an $A \rtimes E - \mathrm{Fix}(A, \alpha)$ imprimitivity bimodule and the map $\iota : \mathrm{IMP}(A) \to \Gamma_0(E\backslash E^{(0)}, \mathscr{X})$ characterized by*

$$f \mapsto f|_{E\cdot u} \quad \text{for } f \in \Gamma_c(E^{(0)}, \mathscr{A}) \quad \text{and} \quad u \in E^{(0)}$$

defines an isomorphism of IMP(A) *and* $\Gamma_0(E\backslash E^{(0)}, \mathscr{X})$ *as imprimitivity bimodules.*

Proof. By [9, Proposition 4.2], $A \rtimes_\alpha E$ is a $C_0(E\backslash E^{(0)})$-algebra, the map $F \mapsto F|_{E\cdot u}$ extends to a surjective homomorphism from $A \rtimes E$ to $A(E\cdot u) \rtimes E|_{E\cdot u}$, and $A(E\cdot u) \rtimes E|_{E\cdot u}$ is isomorphic to $(A \rtimes_\alpha E)(E\cdot u)$. Furthermore, by Proposition 3.6 we know $\mathrm{Fix}(A(E\cdot u)) = \mathrm{Ind}(A(E\cdot u))$, which by Proposition 3.4 is isomorphic to $\mathrm{Fix}(A)(E\cdot u)$. By construction $\mathscr{X}$ is an imprimitivity bimodule bundle. Under the above identifications, Proposition 2.9 implies that $\Gamma_0(E\backslash E^{(0)}, \mathscr{X})$ is an $A \rtimes E$ − Fix(A) imprimitivity bimodule with actions and inner products given by

$$\begin{aligned}
{}_{A\rtimes_\alpha E}\langle f, g\rangle (E\cdot u)(\gamma) &:= f(E\cdot r(\gamma))(r(\gamma))\alpha_\gamma(g(E\cdot r(\gamma))(s(\gamma))^*),\\
\langle f, g\rangle_{\mathrm{Fix}(A)} (E\cdot u)(u) &:= \int_E \alpha_\gamma(f(E\cdot u)(s(\gamma))^* g(E\cdot u)(s(\gamma)))\, d\lambda_E^u(\gamma),\\
F\cdot f(E\cdot u)(u) &:= \int_E F|_{E\cdot u}(\gamma)\alpha_\gamma(f(E\cdot u)(s(\gamma)))\, d\lambda_E^u(\gamma),\\
f\cdot m(E\cdot u)(u) &:= f(E\cdot u)(u) m|_{E\cdot u}(u).
\end{aligned}$$

By the definition of the topology on $\mathscr{X}$, $\iota(f) \in \Gamma_0(E\backslash E^{(0)}, \mathscr{X})$ for all $f \in \Gamma_c(E^{(0)}, \mathscr{A})$. Using the Tietze Extension Theorem for Banach bundles [17, Proposition A.5], ι maps onto each fiber of $\Gamma_0(E\backslash E^{(0)}, \mathscr{X})$, and therefore ι is onto by Proposition 2.5. Furthermore,

$${}_{A\rtimes E}\langle f, g\rangle (E\cdot u)(\gamma) := f(r(\gamma))\alpha_\gamma(g(s(\gamma))^*) = {}_{A(E\cdot u)\rtimes E|_{E\cdot u}}\langle \iota(f), \iota(g)\rangle (\gamma).$$

Thus ι preserves left inner products and therefore is norm preserving. It follows that ι is injective and hence bijective. Showing ι preserves the actions and the right inner product is similar. Thus ι is an isomorphism of imprimitivity bimodules. □

Let (A,ω) be a $G\ltimes X\rtimes H$-dynamical system. Then $A\rtimes_{\omega^G}(G\ltimes X)$ is a $C_0(H^{(0)})$-algebra by [3, Proposition 3.5] and there exists an action $\check{\omega}^H$ of H on $A\rtimes_{\omega^G}(G\ltimes X)$ [3, Proposition 3.7] characterized by

$$\check{\omega}^H_\eta(f)(\gamma,x):=\omega_{(r_X(x),x,\eta)}(f(\gamma,x\eta)).$$

We use Proposition 5.10 in the next lemma to define an action on $\mathrm{IMP}(A)$ that implements an equivariant Morita equivalence between $(A\rtimes_{\omega^G}(G\ltimes X),\check{\omega}^H)$ and $(\mathrm{Fix}_G(A),\mathrm{Fix}_G(\omega))$. First, recall that $G\backslash X$ is homeomorphic to $H^{(0)}$ so that for each $u\in H^{(0)}$ there exists $x\in X$ such that $s_X^{-1}(u)=G\cdot x$.

Lemma 5.11. *For each $\eta\in H$, the map $V_\eta:\Gamma_c(G\cdot x\eta,\mathscr{A})\to\Gamma_c(G\cdot x,\mathscr{A})$ given by $V_\eta(f)(y)=\omega_{(r(y),y,\eta)}(f(y\eta))$ extends to an isomorphism of $\mathrm{IMP}(A(G\cdot x\eta))\to\mathrm{IMP}(A(G\cdot x))$. Furthermore, $\{V_\eta\}$ defines a continuous action of H on $\mathrm{IMP}(A)$ such that*

$$\begin{aligned}
&{}_{(A\rtimes(G\rtimes X))(G\cdot x)}\langle V_\eta(f),V_\eta(g)\rangle=\check{\omega}^H_\eta({}_{(A\rtimes(G\rtimes X))(G\cdot x)}\langle f,g\rangle)\quad\textit{and}\\
&\langle V_\eta(f),V_\eta(g)\rangle_{\mathrm{Fix}_G(A)(G\cdot x)}=\mathrm{Fix}_G(\omega)_\eta(\langle f,g\rangle_{\mathrm{Fix}_G(A)(G\cdot x\eta)}).
\end{aligned}\tag{11}$$

Proof. Since f is continuous and compactly supported and ω is a continuous action, $V_\eta(f)$ is continuous and compactly supported and is thus in $\Gamma_c(G\cdot x,\mathscr{A})$. The two algebraic conditions in (11) follow from some mostly painless computations which we omit for brevity. It follows from (11) that

$$\begin{aligned}
\|V_\eta(f)\|^2&=\|{}_{(A\rtimes(G\rtimes X))(G\cdot x)}\langle V_\eta(f),V_\eta(f)\rangle\|=\|\check{\omega}^H_\eta({}_{(A\rtimes(G\rtimes X))(G\cdot x\eta)}\langle f,f\rangle)\|\\
&=\|{}_{(A\rtimes(G\rtimes X))(G\cdot x\eta)}\langle f,f\rangle\|=\|f\|^2
\end{aligned}$$

so that V_η preserves the norm on $\Gamma_c(G\cdot x\eta,\mathscr{A})$ and therefore extends to a $*$-homomorphism of $\mathrm{IMP}(A(G\cdot x\eta))$ into $\mathrm{IMP}(A(G\cdot x))$. Finally, some more algebra shows that V_η is an isomorphism and it preserves the groupoid operations.

To show that V_η is an action we need to show that it is continuous. Suppose that $\eta_i\to\eta_0$ and $z_i\to z_i$ in $\mathscr{X}$. Let $v_i=r(\eta_i)$ and choose x_i so that $G\cdot x_i=s_X^{-1}(v_i)$. To show that $V_{\eta_i}(z_i)\to V_{\eta_0}(z_0)$ it suffices to show that every subsequence of $V_{\eta_i}(z_i)$ has a subsequence converging to $V_{\eta_0}(z_0)$. It follows from (yet another) application of Proposition 2.2 that, after passing to a subsequence and relabeling, it suffices to prove $V_{\eta_i}(F|_{G\cdot x_i})\to V_{\eta_0}(F|_{G\cdot x_0})$ for all $F\in\Gamma_c(X,\mathscr{A})$.

So let $F\in\Gamma_c(X,\mathscr{A})$. We first suppose that $r(\eta_i)=v_0$ eventually. Then for any $y\in s_X^{-1}(v_0)$,

$$\begin{aligned}
&\|V_{\eta_i}(F|_{G\cdot x_i})-V_{\eta_0}(F|_{G\cdot x_0})\|^2\\
&=\|\langle V_{\eta_i}(F|_{G\cdot x_i})-V_{\eta_0}(F|_{G\cdot x_0}),V_{\eta_i}(F|_{G\cdot x_i})-V_{\eta_0}(F|_{G\cdot x_0})\rangle_{\mathrm{Fix}_G(A)(v_0)}(y)\|\\
&=\Big\|\int_G\omega^G_{(\gamma,y)}((V_{\eta_i}(F|_{G\cdot x_i})(\gamma^{-1}y)-V_{\eta_0}(F|_{G\cdot x_0})(\gamma^{-1}y))^*(V_{\eta_i}(F|_{G\cdot x_i})(\gamma^{-1}y)\\
&\qquad\qquad-V_{\eta_0}(F|_{G\cdot x_0})(\gamma^{-1}y)))\,d\lambda^y_G(\gamma)\Big\|\\
&=\Big\|\int_G\omega_{(\gamma,y,s_X(y))}(\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i)^*F(\gamma^{-1}y\eta_i))\\
&\qquad\qquad-\omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))^*\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i))\\
&\qquad\qquad-\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i)^*)\omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))\\
&\qquad\qquad+\omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0)^*F(\gamma^{-1}y\eta_0)))\,d\lambda^y_G(\gamma)\Big\|
\end{aligned}$$

$$\leq \int_G \|\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i)) - \omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))\|\|F(\gamma^{-1}y\eta_i)\|$$
$$+ \|\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i))$$
$$- \omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))\|\|F(\gamma^{-1}y\eta_0)\| \, d\lambda_G^y(\gamma).$$

The integrand is zero unless either $\gamma^{-1}y\eta_i$ or $\gamma^{-1}y\eta_0$ is in $\operatorname{supp}(F)$. Since $\{y\eta_i\}$ is compact and the action of G on X is proper, $K = \{\gamma : \{\gamma^{-1}y\eta_i\} \cap \operatorname{supp}(F) \neq \emptyset\}$ is compact; thus

$$\|V_{\eta_i}(F|_{G\cdot x_i}) - V_{\eta_0}(F|_{G\cdot x_0})\|^2$$
$$\leq \int_G 2\|\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i)) - \omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))\|\|F\|\chi_K(\gamma) \, d\lambda_G^y(\gamma).$$

The integral goes to zero since the continuity of ω implies

$$\|\omega_{(s(\gamma),\gamma^{-1}y,\eta_i)}(F(\gamma^{-1}y\eta_i)) - \omega_{(s(\gamma),\gamma^{-1}y,\eta_0)}(F(\gamma^{-1}y\eta_0))\| \to 0$$

and $\lambda^y(K) < \infty$. So in this case $V_{\eta_i}(F|_{G\cdot x_i}) \to V_{\eta_0}(F|_{G\cdot x_0})$.

Next suppose that $r(\eta_i) \neq v_0$ frequently. Since $r(\eta_i) \to v_0$, we can choose a subsequence and relabel to assume that $r(\eta_i) \neq r(\eta_j)$ for $\eta_i \neq \eta_j$. Let $C = \{r(\eta_i)\}$ and $D = s_X^{-1}(C)$. Note that both C and D are closed since C is compact and s_X is continuous. Define a function $\iota : D \to \mathbb{N}$ by $\iota(x) = i$ if and only if $s_X(x) = r(\eta_i)$. Standard arguments show that

$$F_0(x) := \omega_{(r(x),x,\eta_{\iota(x)})}(F(x\eta_{\iota(x)}))$$

is in $\Gamma_c(D, \mathscr{A})$. By [17, Proposition A.5] there exists an $\mathcal{F} \in \Gamma_c(X, \mathscr{A})$ such that $\mathcal{F}|_D = F_0$. By the definition of the topology on $\mathscr{X}$, $\mathcal{F}$ is a continuous section. So

$$V_{\eta_i}(F|_{G\cdot x_i}) = F_0|_{s_X^{-1}(r(\eta_i))} = \mathcal{F}|_{s_X^{-1}(r(\eta_i))} \to \mathcal{F}|_{s_X^{-1}(r(\eta_0))} = V_{\eta_0}(F|_{G\cdot x_0}),$$

and thus V is continuous. □

The payoff of Lemma 5.11 is the following theorem, which gives us an "imprimitivity" type result for the map Fix_G.

Theorem 5.12. *Suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems and X is a (G,H)-equivalence. Suppose (A,ω) is a $G \rtimes X \ltimes H$-dynamical system. Then $A \rtimes_\omega (G \rtimes X \ltimes H)$ is Morita equivalent to $\operatorname{Fix}_G(A) \rtimes_{\operatorname{Fix}_G(\omega)} H$.*

Proof. By Lemma 5.11, V is an action on $\mathrm{IMP}(A)$ implementing an equivariant Morita equivalence between the H-dynamical systems $(A \rtimes_{\omega^G} G \ltimes X, \breve{\omega}^H)$ and $(\operatorname{Fix}_G(A), \operatorname{Fix}_G(\omega))$. Now, [18, Section 9.1] shows that $(A \rtimes_{\omega^G} (G \ltimes X)) \rtimes_{\breve{\omega}^H} H$ is Morita equivalent to $\operatorname{Fix}_G(A) \rtimes_{\operatorname{Fix}_G(\omega)} H$. However, [3, Theorem 4.1] gives that $(A \rtimes_{\omega^G} (G \ltimes X)) \rtimes_{\breve{\omega}^H} H \cong A \rtimes_\omega (G \rtimes X \ltimes H)$ so that $A \rtimes_\omega (G \rtimes X \ltimes H)$ is Morita equivalent to $\operatorname{Fix}_G(A) \rtimes_{\operatorname{Fix}_G(\omega)} H$, as desired. □

The main result of the paper now follows quickly.

Proof of Theorem 5.2. For item (1), by Proposition 5.5, $\upsilon^{X,H}$ is a semigroup homomorphism. By Propositions 5.8 and 5.9, $\upsilon^{X,H}$ is invertible and hence an isomorphism. Theorem 5.12 shows that $A \rtimes_\omega (G \rtimes X \ltimes H)$ is Morita equivalent to $\operatorname{Fix}_G(A) \rtimes_{\operatorname{Fix}_G(\omega)} H$, and since Fix_G is the inverse of $\upsilon^{X,H}$, this gives the result. More precisely, given an H-dynamical system (B,β), let $A = s_X^* B$ and $\omega = s_X^* \beta$. Then by Proposition 5.8, $B \rtimes_\beta H$ is isomorphic to $\operatorname{Fix}_G A \rtimes_{\operatorname{Fix}_G \omega} H$. However this algebra is Morita equivalent to $A \rtimes_\omega (G \ltimes X \rtimes H) = s_X^*B \rtimes_{s_X^*\beta} (G \ltimes X \rtimes H)$ by Theorem 5.12. Parts (2) and (3) now follow by symmetry. ☐

6. The construction from [13]

In this section we reconcile our construction with the one used in [13]. In particular we show that the isomorphism υ^X described in Theorem 5.2 restricts to the isomorphism $\phi^X : \mathrm{Br}(H) \to \mathrm{Br}(G)$ described by [13, Theorem 4.1].

We define the isomorphism ϕ^X here for the convenience of the reader. Suppose (A, β) is an H-dynamical system with associated bundle $\mathscr{A}$. Then $(x\eta, a) \sim (x, \beta_\eta(a))$ characterizes an equivalence relation on $s_X^*\mathscr{A}$. Let $\mathscr{A}^X := s_X^*\mathscr{A}/H$ be the quotient of $s_X^*\mathscr{A}$ by this equivalence relation. Then $\mathscr{A}^X$ is an upper semicontinuous C^*-bundle over $G^{(0)}$. Denote the image of (x, a) under this equivalence relation by $[x, a]$, and set $A^X = \Gamma^0(G^{(0)}, \mathscr{A}^X)$. Now $\beta_\gamma^X([x, a]) := [\gamma x, a]$ defines an action of G on A^X [13, Proposition 2.15]. They define

$$\phi^X([A, \beta]) := [A^X, \beta^X].$$

Proposition 6.1. *Let (A, β) be an H-dynamical system. For $F \in \mathrm{Fix}_G(s_X^*A)$ define $\Theta(F)(u) = [x, F(x)]$, where $x \in r_X^{-1}(u)$. Then Θ is a well-defined isomorphism from $\mathrm{Fix}_G(s_X^*A)$ to A^X. Moreover, Θ intertwines the actions $\mathrm{Fix}_G(s_X^*\beta)$ and β^X.*

Proof. To see Θ is well defined note that if $x, y \in r_X^{-1}(u)$, then there exists $\eta \in H$ such that $x\eta = y$; therefore $[y, F(y)] = [x\eta, F(x\eta)] = [x\eta, \beta_{\eta^{-1}}(F(x))] = [x, F(x)]$. Furthermore, the image of Θ is a $C_0(G^{(0)})$-subalgebra of $s_X^*\mathscr{A}/H$. For all $u \in H^{(0)}$ and $x \in s_X^{-1}(u)$ we have $A^X(u) \cong A(x)$ [13, page 914] and $A(x) \cong \mathrm{Fix}_G(s_X^*\mathscr{A})(u)$ by Proposition 3.4; thus Proposition 2.5 gives that Θ is onto. To see that Θ is injective, note that if $[x, F(x)] = [x, F'(x)]$, then there exists $\eta \in H$ such that $(x, F(x)) = (x\eta, \beta_\eta^{-1}(F'(x)))$. But since the action of H on X is free, $\eta = s_X(x)$, and so $F(x) = F'(x)$. Since this must hold for all x we get $F = F'$.

To show that Θ is an isomorphism it remains to show that Θ is continuous and open as a map of upper semicontinuous C^*-bundles. An application of Proposition 2.2, which we omit, shows that Θ is continuous. To see that Θ is open, suppose $[x_i, a_i] \to [x, a]$. By making use of the fact that the quotient map $s_X^*\mathscr{A} \to s_X^*\mathscr{A}/H$ associated to the continuous action of H on $s_X^*\mathscr{A}$ is open, we can pass to a subnet and find η_i such that $(x_i\eta_i, \beta_{\eta_i^{-1}}(a_i)) \to (x, a)$. Let $\epsilon > 0$. By Proposition 3.6, $\mathrm{Fix}_G(s_X^*A) = \mathrm{Ind}(s_X^*A)$. Thus, using Proposition 3.4, for each i we can pick $f_i \in \mathrm{Fix}_G(s_X^*A)(r_X(x_i))$ such that

$$f_i(x_i) = a_i, \quad \text{and so} \quad f_i(x_i\eta_i) = \beta_{\eta_i^{-1}}(a_i).$$

In a similar fashion we choose $f \in \mathrm{Fix}_G(s_X^*A)(r_X(x))$ such that $f(x) = a$. We want to show that $f_i \to f$. Pick $F \in \mathrm{Fix}_G(s_X^*A)$ such that $F|_{r_X^{-1}(r_X(x))} = f$. Since F is a continuous bounded section of $s_X^*\mathscr{A}$, $F(x_i\eta_i) - \beta_{\eta_i^{-1}}(a) \to F(x) - a = f(x) - a$. Using the fact that the norm is upper semicontinuous we eventually have $\|F(x_i\eta_i) - \beta_{\eta_i^{-1}}(a_i)\| < \epsilon$. Thus eventually we have

$$\|f_i(x_i\eta_i\eta) - F(x_i\eta_i\eta)\| = \|\beta_{\eta^{-1}}(f_i(x_i\eta_i) - F(x_i\eta_i))\| = \|f_i(x_i\eta_i) - F(x_i\eta_i)\| < \epsilon.$$

Hence $\|f_i - F|_{r_X^{-1}(r_X(x_i\eta_i))}\| < \epsilon$ for large i. Using Proposition 2.2 one last time, it follows that $f_i \to f \in \mathrm{Fix}_G(s_X^*\mathscr{A})$ as desired, and thus Θ is open. A straightforward computation shows that Θ intertwines the actions. ☐

Appendix A. General proper dynamical systems

Let G be a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Let (A,α) be a G-dynamical system. For a $*$-subalgebra A_0 of A let

$$M(A_0)^\alpha := \{d \in M(A) : A_0 d \subset A_0,\ \bar{\alpha}_\gamma(d(s(\gamma))) = d(r(\gamma))\ \forall \gamma \in G\}.$$

Recall from [2, Definition 3.1] that (A,α) is *proper* if there is a dense $*$-subalgebra A_0 of A such that

(1) for all $a,b \in A_0$, the function ${}_E\langle a,b\rangle : \gamma \mapsto a(r(\gamma))\alpha_\gamma(b(s(\gamma))^*)$ is integrable, and
(2) for all $a,b\in A_0$, there exists a unique element $\langle a,b\rangle_D \in M(A_0)^\alpha$ such that

$$(c\cdot \langle a,b\rangle_D)(u) = \int_G c(r(\gamma))\alpha_\gamma(a^* b(s(\gamma)))\, d\lambda^u(\gamma) \quad \text{for all } c\in A_0.$$

In this case $E = \overline{\operatorname{span}}\{{}_E\langle a,b\rangle : a,b\in A_0\}$ is a subalgebra of $A\rtimes_{\alpha,r} G$ Morita equivalent to $\operatorname{Fix}(A,\alpha) := \overline{\operatorname{span}}\{\langle a,b\rangle_D : a,b\in A_0\}$ [2, Theorem 3.9]. In [2] the question was raised as to when E is an ideal of $A\rtimes_{\alpha,r} G$. We provide a condition on A_0 in the next proposition, guaranteeing that E is an ideal.

Proposition A.1. *Let (A,α) be a proper G-dynamical system with respect to A_0 and C an inductive limit dense $*$-subalgebra of $C_c(G)$. Suppose that $C\cdot A_0 \subset A_0$, where the action of $C_c(G)$ on A is given by*

$$f\cdot a(u) := \int_G f(\gamma)\alpha_\gamma(a(s(\gamma)))\, d\lambda^u(\gamma). \tag{12}$$

Then the subalgebra $E\subset A\rtimes_{\alpha,r} G$ guaranteed by [2, Theorem 3.9] *is an ideal.*

Proof. Since E is a $*$-subalgebra of $A\rtimes_{\alpha,r} G$, it suffices to show that $B_0 * E_0 \subset E_0$ for a dense subalgebra $B_0 \subset A\rtimes_{\alpha,r} G$. Let $\Omega : C_c(G)\odot A_0 \to \Gamma_c(G, s^*\mathscr{A})$ be characterized by $\Omega(f\otimes a)(\gamma) = f(\gamma)a(s(\gamma))$. By [20, Proof of Proposition 1.3] the image of Ω is dense in $\Gamma_c(G,s^*\mathscr{A})$ with respect to the inductive limit topology. Since $C\subset C_c(G)$ is dense in the inductive limit topology, so is $\Omega(C\odot A_0)$. By [18, Lemma 4.3] the map $f\mapsto (\gamma\mapsto \alpha_\gamma(f(\gamma)))$ defines an isomorphism $\alpha_G : \Gamma_0(G,s^*\mathscr{A}) \to \Gamma_0(G,r^*\mathscr{A})$. Thus $B_0 := \alpha_G\circ\Omega(C\odot A_0)$ is dense in $\Gamma_c(G,r^*\mathscr{A})$ in the inductive limit topology.

It remains to show $B_0 * E_0 \subset E_0$. For $F\in\Gamma_c(G,r^*\mathscr{A})$ and $a\in A$ define $F\cdot a(u) := \int_G F(\gamma)\alpha_\gamma(a(s(\gamma)))\, d\lambda^u(\gamma)$. If $F\cdot a\in A_0$, then $F * {}_E\langle a,b\rangle = {}_E\langle F\cdot a, b\rangle$. Thus it suffices to show $F\cdot a\in A_0$ for all $F\in B_0$ and $a\in A_0$. By the definition of B_0 it suffices to show $\alpha_G\circ\Omega(g\otimes b)\cdot a\in A_0$ for all $g\in C$ and $a,b\in A_0$. But

$$\begin{aligned}\alpha_G\circ\Omega(g\otimes b)\cdot a(u) &= \int_G \alpha_G\circ\Omega(g\otimes b)(\eta)\alpha_\eta(a(s(\eta)))\, d\lambda^u(\eta)\\ &= \int_G g(\eta)\alpha_\gamma(ba(s(\eta)))\, d\lambda^u(\eta) = (g\cdot ba)(u)\in A_0,\end{aligned}$$

since $ba\in A_0$ and $C\cdot A_0\subset A_0$ by assumption. Hence $B_0 * E_0\subset E_0$ and thus E is an ideal in $A\rtimes_{\alpha,r} G$, as desired. □

Remark A.2. Let $\mathcal{G}$ be a group. In [23], if (A,α) is a *proper* $\mathcal{G}$-dynamical system with respect to the dense subalgebra A_0, the condition $\alpha_s(A_0)\subset A_0$ for all $s\in\mathcal{G}$ ensures that E is an ideal in $A\rtimes_{\alpha,r}\mathcal{G}$. As observed in [2], $\alpha_s(A_0)\subset A_0$ does not make sense for groupoids. It is unclear if the condition of Proposition A.1 reduces to the condition that $\alpha_s(A_0) \subset A_0$ for all $s \in \mathcal{G}$ in the group case. However, the examples below show that many proper group dynamical systems satisfy the condition of Proposition A.1.

Example A.3. Suppose $\mathcal{G}$ acts properly on X, (A, α) is a $\mathcal{G}$-dynamical system, and $\theta : C_0(X) \to M(A)$ is equivariant and nondegenerate. Then (A, α) is proper with respect to the subalgebra $A_0 = \theta(C_c(X))A\theta(C_c(X))$ [24, Theorem 5.7]. It is easy to see that $C_c(\mathcal{G}) \cdot A_0 \subset A_0$. Indeed, suppose $f \in C_c(\mathcal{G})$, $h, k \in C_c(X)$ and $a \in A$; then we have

$$f \cdot (\theta(h)a\theta(k)) = \int_{\mathcal{G}} f(s)\alpha_s(\theta(h)a\theta(k))\, ds.$$

Pick $c \in C_c(X)$ such that $c \equiv 1$ on the set $\operatorname{supp}(f) \cdot (\operatorname{supp} h \cup \operatorname{supp} k)$. Then for all $s \in \operatorname{supp}(f)$, $\theta(c)\alpha_s(\theta(h)) = \alpha_s(\theta(h))$ and $\alpha_s(\theta(k))\theta(c) = \alpha_s(\theta(k))$. Thus

$$f \cdot (\theta(h)a\theta(k)) = \theta(c)(f \cdot (\theta(h)a\theta(k)))\theta(c) \in A_0.$$

Example A.4. Let $\mathcal{G}$ be a compactly generated Abelian Lie group. Using [26, no. 11] we know that $\mathcal{G}$ is of the form $\mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^m \times F$ where F is a finite Abelian group. Now let $\mathcal{G}$ act on a C^*-algebra A with action α. Let $\widehat{\mathcal{G}}$ be the Pontryagin dual of $\mathcal{G}$ and let $\hat{\alpha}_\omega(f)(s) = \omega(s)f(s)$ be the dual action in the sense of Takesaki-Takai. It follows from the statement and proof of [23, Theorem 2.2] that the action of $\widehat{\mathcal{G}}$ on $A \rtimes_\alpha \mathcal{G}$ is proper. The role of the dense subalgebra A_0 is played by the collection $S_\alpha(\mathcal{G}, A)$, which is defined as follows. Let β be the strongly continuous action of $\mathcal{G} \times \mathcal{G}$ on $C_0(\mathcal{G}, A)$ by $\beta_{(s,t)}(f)(r) = \alpha_s(f(r-t))$. Then $S_\alpha(\mathcal{G}, A)$ is the space of elements of $C_0(\mathcal{G}, A)$ which are infinitely differentiable for the action β and which vanish more rapidly at infinity than any polynomial on $\mathcal{G}$ grows. Here derivatives are taken in the $\mathbb{R}$ and $\mathbb{T}$ directions of $\mathcal{G}$, whereas polynomials are taken with respect to the $\mathbb{R}$ and $\mathbb{Z}$ directions.

Consider the action of $C_c(\mathcal{G})$ on $S_\alpha(\mathcal{G}, A)$ given by (12). Observe that there is only one fiber and the Haar system is given by the dual Haar measure. After passing an evaluation at $s \in \mathcal{G}$ through the integral we see that for $\phi \in C_c(\mathcal{G})$ and $f \in S_\alpha(\mathcal{G}, A)$,

$$\phi \cdot f(s) = \int_{\widehat{\mathcal{G}}} \phi(\omega)\hat{\alpha}_\omega(f)(s)\, d\omega = \int_{\widehat{\mathcal{G}}} \phi(\omega)\omega(s)\, d\omega f(s) = \hat{\phi}(s)f(s). \tag{13}$$

Here $\hat{\phi}$ denotes the Fourier transform of ϕ from an element of $C_c(\widehat{\mathcal{G}})$ to an element of $C_0(\mathcal{G})$.

Let C be the set of smooth, compactly supported functions in $C_c(\widehat{\mathcal{G}})$. It is not difficult to see that C is dense with respect to the inductive limit topology. We wish to show that $C \cdot S_\alpha(\mathcal{G}, A) \subset S_\alpha(\mathcal{G}, A)$. However, in light of (13) it suffices to show that if $\phi \in C$, then $\hat{\phi}$ is infinitely differentiable in the $\mathbb{R}$ and $\mathbb{T}$ coordinates, and that in the $\mathbb{R}$ and $\mathbb{Z}$ coordinates $\hat{\phi}$ vanishes at infinity faster than any polynomial grows. Since the Pontryagin dual of a product is the product of the Pontryagin duals, and since our notions of smoothness and growth are all taken coordinatewise, we need to prove that

(1) if ϕ is a compactly supported smooth function on $\mathbb{R}$, then $\hat{\phi}$ is a smooth function on $\mathbb{R}$ which vanishes at infinity faster than any polynomial on $\mathbb{R}$ grows,

(2) if ϕ is a smooth function on $\mathbb{T}$, then $\hat{\phi}$ vanishes at infinity faster than any polynomial on $\mathbb{Z}$ grows, and
(3) if ϕ is finitely supported on $\mathbb{Z}$, then $\hat{\phi}$ is smooth on $\mathbb{T}$.

However, these are all standard facts from Fourier analysis [8, Theorem 2.6, Theorem 7.5]. Thus the conditions of Proposition A.1 are satisfied in this example.

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